CHAPTER 2
COMPACT SUBSETS OF EUCLIDEAN SPACE
JORDAN'S THEOREM

We will prove in this chapter that if a compact subset A of $\mathbf{R}^n$ disconnects $\mathbf{R}^n$, then any other subset of $\mathbf{R}^n$ homeomorphic to A also disconnects $\mathbf{R}^n$. We will obtain as consequences Jordan's theorem and the topological invariance of the dimension.

1. Some elementary homotopy theory.

Definition 1.1. (Homotopy). Given two continuous maps $f, g : X \to Y$, we say that f is homotopic to g if there exists a continuous map $F : X \times [0,1] \to Y$, such that $F|X \times \{0\} = f$ and $F|X \times \{1\} = g$; we denote this relation by $f \sim g$. The relation of homotopy is an equivalence relation (exercise). The set of equivalence classes of continuous maps from X to Y is denoted by $[X, Y]$. An element of $[X, Y]$ is called a homotopy class.

Lemma 1.2. *If we have continuous maps $f, g : X \to Y$ and $f', g' : Y \to Z$ such that $f \sim g$ and $f' \sim g'$ then $f' \circ f \sim g' \circ g$.*

Proof. Exercise! □

Warning 1.3. If we have continuous maps $f, g : X \to Y$ and $h : Y \to Z$ such that $h \circ f \sim h \circ g$, this does not imply that $f \sim g$. As an example we can take $X = \{0\}, Y = \{0, 1\}$ and $Z = [0, 1]$. The two maps from X to Y are not homotopic, but if we compose them with the inclusion $Y \hookrightarrow Z$ they become homotopic.

Definition 1.4. (null homotopic, essential). A continuous map is said null homotopic if it is homotopic to a constant map. An essential map is a map that is not null homotopic.

Lemma 1.5. *Let X, Y, Z nbe topological spaces.*

(i) *Two null homotopic maps $f, g : X \to Y$ are homotopic if and only if their images are contained in the same path component of Y. In particular, if Y is path connected, there is a well defined null homotopy class;*
(ii) *a space Y is path connected if and only if $[\{0,1\}, Y]$ is reduced to one class;*
(iii) *if we can write a map as a composition $f = f_1 \circ f_2$, where $f_1 : X \to Z$ and $f_2 : Z \to Y$, and either of the two maps f_1, f_2 is null homotopic, then f is also null homotopic. In shorter words, if a map factorizes through a null homotopic map then it is null homotopic.*

Proof. Exercise! □

Definition 1.6. (Contractible space). A space X is said contractible if Id_X is null homotopic.

Lemma 1.7. *Given a space X, the following are equivalent:*

(i) *X is contractible,*
(ii) *$[X, X]$ is reduced to one class,*
(iii) *$\forall Y, [Y, X]$ is reduced to one class,*
(iv) *$\forall Y, [X, Y]$ is reduced to one class.*

Proof. Exercise! □

example 1.8. A convex subset C of $\mathbf{R}^n$ is contractible. If x_0 is a point in C, the map $F : C \times [0,1] \to C, F(x,t) = tx_0 + (1-t)x$ gives the desired contraction.

Proposition 1.9. (Filling criterion). *A map f from the sphere $\mathbf{S}^n$ to a space Y is null homotopic if and only if it extends to a map $\bar{f} : \mathbf{B}^{n+1} \to Y$. In particular $\mathrm{Id}_{\mathbf{S}^n}$ is essential.*

Proof. If a map $f : \mathbf{S}^n \to Y$ extends to $\bar{f} : \mathbf{B}^{n+1} \to Y$, then we can write $f = \bar{f} \circ \mathrm{Id}_{\mathbf{B}^{n+1}} \circ i$, where i is the inclusion of $\mathbf{S}^n$ in $\mathbf{B}^{n+1}$. Since, by 1.8, the ball $\mathbf{B}^{n+1}$ is contractible, this implies that f factorizes through a null homotopic map and hence is itself null homotopic by 1.5.(iii). Suppose now that the map $f : \mathbf{S}^n \to Y$ extends to a map $F : \mathbf{S}^n \times [0,1] \to Y$ such that $F|\mathbf{S}^n \times \{0\}$ is a constant map with value y and $F|\mathbf{S}^n \times \{1\} = f$, then it is easy to check that F factorizes through the surjective map $q : \mathbf{S}^n \times [0,1] \to \mathbf{B}^n, (x,t) \mapsto tx$. If we define $\bar{f}$ by $F = \bar{f} \circ q$, then $\bar{f}$ is indeed a continuous extension of f. Finally, an extension of $\mathrm{Id}_{\mathbf{S}^n}$ to a map $\mathbf{B}^{n+1} \to \mathbf{S}^n$ is precisely a retraction of $\mathbf{B}^{n+1}$ on $\mathbf{S}^n$, this is impossible by the no retraction theorem **1**.4.2. □

Proposition 1.10. (Homotopy extension for $\mathbf{S}^n$). *Let X be a metric space, and A be a closed subset of X. If a continuous map $f : A \to \mathbf{S}^n$ is homotopic to the restriction $g|A$ of a continuous map $g : X \to \mathbf{S}^n$, then it can be extended to a continuous map $\bar{f} : X \to \mathbf{S}^n$. In fact, we can extend a homotopy between $g|A$ and f to a map $X \times [0,1] \to \mathbf{S}^n$ whose restriction to $X \times \{0\}$ is g.*

We need a couple of lemmas in order to prove 1.10.

Lemma 1.11. ($\mathbf{S}^n$ is an ANR). *If X is a metric space and A is a closed subset of X, then any map from A to $\mathbf{S}^n$ extends continuously to a map defined on a neighborhood of A in X.*

Proof. By Tietze-Urysohn theorem **1**.4.4, we can extend a continuous map $f : A \to \mathbf{S}^n$ to a continuous map $f : X \to \mathbf{R}^{n+1}$. The set $U = F^{-1}(\mathbf{R}^{n+1} \backslash \{0\})$ is an open neighborhood of A in X, and the map $g : U \to \mathbf{S}^n, x \mapsto F(x)/\|F(x)\|$ is the desired extension. □

Lemma 1.12. (Sandwich lemma). *Let A be a closed subset of the space X, and let Y be a compact space. Any neighborhood of $A \times Y$ in $X \times Y$ contains a neighborhood of the form $U \times Y$, where U is a neighborhood of A in X.*

Proof. Exercise! □

Proof of 1.10. Let $F : A \times [0,1] \to \mathbf{S}^n$ be a homotopy from $g|A$ to f. We can extend F continuously to a map $F_1 : X \times \{0\} \cup A \times [0,1] \to \mathbf{S}^n$ by $F_1|X \times \{0\} = g$. By lemma 1.11, it is possible to extend F_1 to a neighborhood V of $X \times \{0\} \cup A \times [0,1]$ in $X \times [0,1]$, we denote also this extension by F_1. By lemma 1.12, we can find a neighborhood U of A in X such that $U \times [0,1] \subset V$. By Tietze-Urysohn theorem **1**.3.4, there exists a continuous function $\varphi : X \to [0,1]$ such that $\varphi|A = 1$ and $\varphi|X \backslash U = 0$. We define a continuous map $G : X \times [0,1] \to X \times [0,1]$ by $G(x,t) = (x, t\varphi(x))$ (see figure 1.1 below). By construction of G, we have $G(X \times [0,1]) \subset X \times \{0\} \cup U \times [0,1] \subset V$, and $G|A \times [0,1] = \mathrm{Id}$. It is clear that $F_2 = F_1 \circ G$ extends F and that it is a well defined homotopy from g to the desired extension $F_2|X \times \{0\}$. □

Figure 1.1

2.Some elementary constructions of homeomorphisms with compact support.

Definition 2.1. (Support, compact support). The support of a homeomorphism $h : X \tilde{\to} X$, denoted by supp((h)) or supp( )h, is $\overline{\{x \in X \mid h(x) \neq x\}}$, i.e. the closure in X of the set of points that move under h. Of course, h is a homeomorphism with compact support if its support is compact.

Lemma 2.2. *If X is an open subset of the Hausdorff space Y, and $h : X \tilde{\to} X$ is a homeomorphism with compact support, then its extension to Y via the identity is a homeomorphism having the same (compact) support.*

Proof. Exercise! □

Lemma 2.3. *Given two points x, y in the interior of a ball B in $\mathbf{R}^n$, there exists a homeomorphism h with compact support contained in the ball B such that $h(x) = y$.*

Proof. Without loss of generality, we can assume that B is the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$. We define h as the identity outside $\mathbf{B}^n$, and h maps linearly a segment from x to a point z in $\mathbf{S}^{n-1}$ to the segment from y to z (see figure 2.1). In the case $x = 0$, if we denote by h_y this homeomorphism, we have $h_y(v) = (1 - \|v\|)y + v$ for v in $\mathbf{B}^n$, it is easy to check by direct arguments that $h_y|\mathbf{B}^n$ is a bijective continuous map, hence it is a homeomorphism since $\mathbf{B}^n$ is compact. For a general x in $\mathbf{B}^n$, the map h constructed above is in fact $h_y h_x^{-1}$, this proves that it is a homeomorphism. □

Figure 2.1

Theorem 2.4. *Let U be a connected open subset of $\mathbf{R}^n$, and let V be a non empty open subset of U. If $x_1, \ldots, x_k$ are points in U, then there exists a homeomorphism $h : U \tilde{\to} U$ with compact support such that $h(x_1), \ldots, h(x_k)$ are in V.*

Proof. (Induction on k). Consider first the case $k = 1$. Since U is connected and V is non empty, given a point x in U, we can find a sequence of closed balls $B_0, \ldots, B_l$ such that $\mathrm{Int}(B_i) \cap \mathrm{Int}(B_{i+1}) \neq \emptyset, i = 0, \ldots, l-1, x \in \mathrm{Int}(B_0)$ and $V \cap \mathrm{Int}(B_l) \neq \emptyset$. Let us choose points $y_1, \ldots, y_l$ with $y_i \in \mathrm{Int}(B_{i-1}) \cap \mathrm{Int}(B_i), i = 1, \ldots, l$, choose $y_{l+1} \in \mathrm{Int}(V) \cap \mathrm{Int}(B_i)$ and define $y_0 = x$. By 2.3, for $i = 0, \ldots, l$, we can find a homeomorphism h_i with support in B_i such that $h_i(y_i) = y_{i+1}$. The homeomorphism $h = h_l \cdots h_1 h_0$ verifies $h(x) = y_{l+1} \in V$ and has support contained in $B_0 \cup \cdots \cup B_l$, which is a compact subset of U. Suppose now that the theorem is true for k points and that we have to check it for $k+1$ points $x_1, \ldots, x_{k+1}$. Applying the case of k points, it is possible to obtain a homeomorphism $g : U \tilde{\to} U$ with compact support and such that $g(x_1), \dots, g(x_k) \in V$. Of course, we would like to find a homeomorphism with compact support $g' : U \backslash \{g(x_1), \dots, g(x_k)\} \tilde{\rightarrow} U \backslash \{g(x_1), \dots, g(x_k)\}$ such that $g'g(x_{k+1}) \in V$, then we would extend g' by the identity to $\{g(x_1), \dots, g(x_k)\}$. The case $k = 1$ allows us to do that when $U \backslash \{g(x_1), \dots, g(x_k)\}$ is connected. Remark that this is true as soon as the dimension n of $\mathbf{R}^n$ is ≥ 2. Anyway, we can handle the general situation in almost the same way. Let $\widetilde{U}$ be the connected component of $g(x_{k+1})$ in $U \backslash \{g(x_1), \dots, g(x_k)\}$, it is an open set in U different from U. Since U is connected, this implies that $\widetilde{U}$ has a boundary point in U. Such a boundary point cannot be in $U \backslash \{g(x_1), \dots, g(x_k)\}$ because $\widetilde{U}$ is closed in that subset. Hence one of the $g(x_i)$ is a boundary point of $\widetilde{U}$; in particular, since V is a neighborhood of the $g(x_i)$, we have $\widetilde{U} \cap V \neq \emptyset$. We can now apply the case $k = 1$ to $\widetilde{U}, x_{k+1}, \widetilde{U} \cap V$ to obtain a homeomorphism $g' : \widetilde{U} \tilde{\rightarrow} \widetilde{U}$ with compact support in $\widetilde{U}$ and such that $g'(g(x_{k+1})) \in \widetilde{U} \cap V$, we extend this homeomorphism by the identity to a homeomorphism of U, which we still call g'. We define the desired h as $g' \circ g$. □

Exercise 2.5. (i): Let U be an open connected subset of $\mathbf{R}^n$, with $n \geq 2$. If $x_1, \dots, x_k$ (resp. $y_1, \dots, y_k$) are distinct points in U, then there exists a homeomorphism with compact support $h : U \tilde{\rightarrow} U$ such that $h(x_i) = y_i, i = 1, \dots, k$.(Hint: Induction on k. Use the fact $n \geq 2$ to know that the complement of a finite set in U is connected.)

(ii): The statement given above is false for $n = 1$. Why ? Give a correct statement and prove it.

3.Disconnection of Euclidean space by a compact set.

Definition 3.1. (Borsuk map). Let A be a subset in $\mathbf{R}^n$ and $a \notin A$. The map $\theta_{a,A} : A \rightarrow \mathbf{S}^{n-1}, x \mapsto (x-a)/\|x-a\|$, is called the Borsuk map associated with a and A.

Lemma 3.2. *Let A be a closed subset in $\mathbf{R}^n$. If two points a and b are contained in the same connected component of $\mathbf{R}^n \backslash A$, then $\theta_{a,A}$ and $\theta_{b,A}$ are homotopic.*

Proof. Since A is closed, the connected components of $\mathbf{R}^n \backslash A$ are path connected. If $\alpha : [0,1] \rightarrow \mathbf{R}^n \backslash A$ is a path between a and b, the map $A \times [0,1] \rightarrow \mathbf{S}^{n-1}, (x,t) \mapsto (x - \alpha(t))/\|x - \alpha(t)\|$ is a homotopy between $\theta_{a,A}$ and $\theta_{b,A}$. □

Theorem 3.3. *Let A be a compact subset in $\mathbf{R}^n$. The point $a \in \mathbf{R}^n \backslash A$ is contained in a bounded connected component of $\mathbf{R}^n \backslash A$ if and only if the Borsuk map $\theta_{a,A} : A \rightarrow \mathbf{S}^{n-1}$ is essential.*

Proof. Since A is compact, there exists a ball B with center 0 and finite radius containing A in its interior. If $b \in \mathbf{R}^n \backslash B$, the map $\theta_{b,B}$ is an extension of $\theta_{b,A}$ to B, since B is contractible $\theta_{b,A}$ is null homotopic. It follows from 3.2 that $\theta_{a,A}$ is null homotopic if a is contained in an unbounded connected component of $\mathbf{R}^n \backslash A$. Suppose now that a is contained in the bounded connected component G of $\mathbf{R}^n \backslash A$, by the homotopy extension property 1.10, if the map $\theta_{a,A} : A \to \mathbf{S}^{n-1}$ is null homotopic, then $\theta_{a,A}$ extends continuously to the closed set $G \cup A$. We denote by f such an extension. We can extend f to a continuous map $\bar{f} : \mathbf{R}^n \to \mathbf{S}^{n-1}$ by defining it on $\mathbf{R}^n \backslash G$ as the Borsuk map associated with a and $\mathbf{R}^n \backslash G$. Since $G \cup A$ is compact, we can find a ball $B(a, R)$ containing $G \cup A$. The map $\mathbf{B}^n \to \mathbf{S}^{n-1}, x \mapsto \bar{f}(a + Rx)$, is easily seen to be a retraction of $\mathbf{B}^n$ on $\mathbf{S}^{n-1}$. This is impossible by the no retraction theorem **1**.4.2. □

In fact, the end of the previous proof proves also the following theorem:

Theorem 3.4. *If a is contained in the bounded connected component G of $\mathbf{R}^n \backslash A$, where A is a compact subset in $\mathbf{R}^n$, then $\theta_{a,A}$ does not extend continuously to $G \cup A$.*

Corollary 3.5. *Let A be compact subset in $\mathbf{R}^n$. If $\mathbf{R}^n \backslash A$ has a bounded connected component, then there exists an essential map $A \to \mathbf{S}^{n-1}$. In particular, for $n \geq 2$, if A disconnects $\mathbf{R}^n$, there exists an essential map $A \to \mathbf{S}^{n-1}$.*

Proof. The first part is a special case of 3.3. The second part follows from the first, since for $n \geq 2$, $\mathbf{R}^n \backslash A$ has exactly one unbounded connected component. □

Of course, corollary **1**.4.5 is a particular case of corollary 3.5.

As a corollary of theorem 3.4, we obtain:

corollary 3.6. *Let A be a compact subset in $\mathbf{R}^n$ and a be a point in the bounded connected component G of $\mathbf{R}^n \backslash A$. If $b \in \mathbf{R}^n \backslash A$ is such that $\theta_{b,A} \sim \theta_{a,A}$, then $b \in G$. In particular, for $n \geq 2$, two points in $\mathbf{R}^n \backslash A$ have homotopic Borsuk maps if and only if they are contained in the same connected component.*

Proof. If b is not in G, then of course $\theta_{b,G \cup A}$ is well defined and extends $\theta_{a,A}$. By the homotopy extension property 1.10, it follows that if $\theta_{a,A}$ is homotopic to $\theta_{b,A}$ then $\theta_{a,A}$ extends to $G \cup A$. This is impossible by proposition 3.4. Another way of saying what we obtained is the following: if x and y are points in $\mathbf{R}^n \backslash A$ such that $\theta_{x,A} \sim \theta_{y,A}$, then either they are in the same bounded connected component, or they are both contained in unbounded connected components (eventually distinct) of $\mathbf{R}^n \backslash A$. Since, for $n \geq 2$, there is only one unbounded connected component, this proves the second part of the corollary. □

Our goal is to prove the converse of 3.5. We have some preliminary work to do.

Lemma 3.7. *There exists in $\mathbf{R}^n$ a countable dense subset S such that any $n + 1$ points in S are affinely independent.*

Proof. Exercise! □

Lemma 3.8. *Let A be a compact set contained in the interior of $\mathbf{B}^n$. Any continuous map $f : A \to \mathbf{S}^{n-1}$ can be extended to a continuous map $g : \mathbf{B}^n \to \mathbf{R}^n$ such that $g^{-1}(0)$ is finite. In particular, the map $\bar{f} : \mathbf{B}^n \backslash g^{-1}(0) \to \mathbf{S}^{n-1}, x \mapsto g(x)/\|g(x)\|$, is an extension of f to $\mathbf{B}^n$ minus a finite number of points.*

Proof. Since $\mathbf{B}^n$ and Δ^n are homeomorphic, we can prove our lemma for Δ^n instead of $\mathbf{B}^n$. By Tietze-Urysohn theorem **1**.4.4, we can extend f to a map $f_1 : \Delta^n \to \mathbf{R}^n$, but $f_1^{-1}(0)$ is not known to be finite. Since Δ^n is compact, the map f_1 is uniformly continuous; hence we can find a sufficiently fine subdivision K of Δ^n (see **1**.1.4) such that for each simplex σ in K, we have $\operatorname{diam}(f_1(\sigma)) < 1/12$. By lemma 3.7, for each vertex v of K, we can choose a point x_v in $\mathbf{R}^n$ such that $\|f_1(v) - x_v\| < 1/12$, $x_v \neq x_{v'}$, for $v \neq v'$, and any subset of points $x_{v_1}, \ldots, x_{v_k}$, with $k \leq n+1$, is affinely independent. We can define in a unique way a map $f_2 : \Delta^n \to \mathbf{R}^n$, which is affine on each simplex σ in K and sends each vertex v of K to x_v. By the independence property of the points x_v, the map f_2 is injective on each simplex of K. This implies that the preimage of a point under f_2 is finite. If $x = \sum_{i=0}^n t_i v_i, t_i \geq 0, \sum_{i=0}^n t_i = 1$, where the v_i are the vertices of K defining the n-simplex of K containing x, we have:

$$\begin{aligned}\|f_1(x) - f_2(x)\| &\leq \|f_1(x) - \sum_{i=0}^n t_i f_1(v_i)\| + \|\sum_{i=0}^n t_i(f_1(v_i) - x_{v_i})\| \\ &\leq \sum_{i=0}^n t_i\|f_1(x) - f_1(v_i)\| + \sum_{i=0}^n t_i\|f_1(v_i) - x_{v_i}\| \\ &< \frac{1}{6}\end{aligned}$$

Since $f_1(A) = f(A) \subset \mathbf{S}^{n-1}$, we obtain $f_2(A) \cap B(0, 1/2) = \emptyset$. By Tietze-Urysohn theorem **1**.4.4, we can then choose $\varphi : \Delta^n \to [0,1]$ such that $\varphi(A) = 1$ and $\varphi(f_2^{-1}[B(0,1/2)]) = 0$. We define the map $g : \mathbf{B}^n \to \mathbf{R}^n$ by $g(x) = \varphi(x)f_1(x) + (1 - \varphi(x))f_2(x)$. Since $\|f_1(x) - f_2(x)\| < 1/6$, we have $\|f_1(x) - g(x)\| < 1/6$; hence, if $\|f_2(x)\| \geq 1/2$, we have: $\|g(x)\| \geq \|f_2(x)\| - \|g(x) - f_2(x)\| \geq 1/2 - 1/6 > 0$. This implies that $g^{-1}(0)$ is contained in $f_2^{-1}(B(0, 1/2))$, but on this set $g = f_2$ by construction of φ; in particular, $g^{-1}(0) = f_2^{-1}(0)$ is finite. Moreover, we have also $g|A = f_1|A = f$ by construction of φ. □

Corollary 3.9. *Given a compact set A in $\mathbf{R}^n$, and a continuous map $f : A \to \mathbf{S}^{n-1}$, we can find a finite number of points $q_1, \ldots, q_l$ in $\mathbf{R}^n$ and a continuous extension $\bar{f} : \mathbf{R}^n \backslash \{q_1, \ldots, q_l\} \to \mathbf{S}^{n-1}$ of f.*

Proof. Without loss of generality, we can assume that A is contained in the interior of $\mathbf{B}^n$. By 3.8, there exists $p_1, \ldots, p_k$ in $\mathbf{B}^n$ and an extension $\mathbf{B}^n \backslash \{p_1, \ldots, p_k\} \to \mathbf{S}^{n-1}$. Since A is contained in the interior of $\mathbf{B}^n$ and the set $\{p_1, \ldots, p_k\}$ is finite, we can find $r \in]0,1[$ so that A is contained in the interior of $B(0,r)$ and the boundary $\{x \mid \|x\| = r\}$ of $B(0,r)$ does not contain any of the p_i. We define $\{q_1, \ldots, q_l\}$ as $\{p_1, \ldots, p_k\} \cap B(0,r)$. The extension $\bar{f}$ is given by $\bar{f} = g$ on $B(0,r) \backslash \{q_1, \ldots, q_l\}$ and $\bar{f}(x) = g((r/\|x\|)x)$ on $\mathbf{R}^n \backslash \operatorname{Int}(B(0,r))$. □

Theorem 3.10. *Let A be a compact set in $\mathbf{R}^n$. Let $(U_i)_{i \in I}$ be the (at most countable) family of bounded connected components in $\mathbf{R}^n \backslash A$. For each $i \in I$, we choose some point $x_i \in U_i$. Given any continuous map $f : A \to \mathbf{S}^{n-1}$, we can find a subfamily $\{x_{i_1}, \ldots, x_{i_k}\}$ of the family $(x_i)_{i \in I}$, and an extension to a continuous map $\bar{f} : \mathbf{R}^n \backslash \{x_{i_1}, \ldots, x_{i_k}\} \to \mathbf{S}^{n-1}$. In particular, if $\mathbf{R}^n \backslash A$ has no bounded component each map $A \to \mathbf{S}^{n-1}$ extends to $\mathbf{R}^n$ and is therefore null homotopic.*

Proof. By 3.9, there exists an extension g to $\mathbf{R}^n \backslash \{q_1, \ldots, q_l\}$. Let $U_{i_1}, \ldots, U_{i_k}$ be the finite set of bounded connected components of $\mathbf{R}^n \backslash A$ containing one of the q_i, the rest of the q_i are contained in the unbounded connected components. For each $U_{i_j}, j = 1, \ldots, k$, let us choose a small ball $B_j = B(x_{i_j}, r_j)$ of radius r_j and center x_{i_j} contained in U_{i_j} (see figure 3.1). Let R be big enough so that $B = B(0, R)$ contains $A \cup B_1 \cup \cdots \cup B_k$ in its interior. By the choices done, the connected components of $\mathbf{R}^n \backslash A$ which contain some q_i do intersect $\big(\mathbf{R}^n \backslash B\big) \cup \mathrm{Int}(B_1) \cup \cdots \cup \mathrm{Int}(B_k)$. Using theorem 2.4 in each one of these components and piecing things together, we can find a homeomorphism $h : \mathbf{R}^n \tilde{\rightarrow} \mathbf{R}^n$ with compact support contained in $\mathbf{R}^n \backslash A$ such that $h^{-1}(q_i) \notin B \backslash (\mathrm{Int}(B_1) \cup \cdots \cup \mathrm{Int}(B_k)), i = 1, \ldots, k$. The map $g' = g \circ h$ is well defined on $B \backslash (\mathrm{Int}(B_1) \cup \cdots \cup \mathrm{Int}(B_k))$, it is an extension of f. Since $B \backslash (\mathrm{Int}(B_1) \cup \cdots \cup \mathrm{Int}(B_k))$ is a retract of $\mathbf{R}^n \backslash \{x_{i_1}, \ldots, x_{i_k}\}$, it suffices to compose g' with a retraction to obtain the desired result. □

Figure 3.1

Putting together 3.5 and 3.10, we obtain the following theorem:

Theorem 3.11. *Let A be a compact set in $\mathbf{R}^n$. There exists a bounded connected component in $\mathbf{R}^n \backslash A$ if and only if there exists an essential map $f : A \rightarrow \mathbf{S}^{n-1}$. In particular, if $n \geq 2$, then A disconnects $\mathbf{R}^n$ if and only if there exists an essential map $A \rightarrow \mathbf{S}^{n-1}$.*

Corollary 3.12. *If A and B are compact sets in $\mathbf{R}^n$ which have the same homotopy type, then one of them disconnects $\mathbf{R}^n$ if and only if the other does.*

This corollary is the fundamental result from which we are now going to deduce some of the most important topological properties of Euclidean spaces.

4. Jordan's separation theorem and the invariance of domain.

Theorem 4.1. **(Jordan's theorem).** *Let $j : \mathbf{S}^{n-1} \hookrightarrow \mathbf{R}^n$ be a continuous injective map, then $j(\mathbf{S}^{n-1})$ disconnects $\mathbf{R}^n$ and no proper subset of $j(\mathbf{S}^{n-1})$ disconnects $\mathbf{R}^n$. In particular,* *the boundary in* $\mathbf{R}^n$ *of a connected component of* $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$ *is precisely* $j(\mathbf{S}^{n-1})$.

Proof. Since $\mathbf{S}^{n-1}$ is compact, the map j is a homeomorphism onto $j(\mathbf{S}^{n-1})$. It follows from 3.12, by considering the case of the standard inclusion $\mathbf{S}^{n-1} \subset \mathbf{R}^n$, that $j(\mathbf{S}^{n-1})$ disconnects $\mathbf{R}^n$ and that no proper closed subset of $j(\mathbf{S}^{n-1})$ disconnects $\mathbf{R}^n$. If U is a connected component of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$, then its boundary in $\mathbf{R}^n$ is a closed subset of $j(\mathbf{S}^{n-1})$ that disconnects $\mathbf{R}^n$; hence the boundary of U must be the whole of $j(\mathbf{S}^{n-1})$. It remains to show that if A is a proper not necessarily closed subset of $j(\mathbf{S}^{n-1})$, then $\mathbf{R}^n \backslash A$ is connected. If U is a connected component of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$, then $U \cup (j(\mathbf{S}^{n-1}) \backslash A)$ is connected, since it is contained between the connected set U and its closure $U \cup j(\mathbf{S}^{n-1})$. Since $j(\mathbf{S}^{n-1}) \backslash A$ is non-empty and $\mathbf{R}^n \backslash A$ is the union of connected sets of the form $V \cup (j(\mathbf{S}^{n-1}) \backslash A)$, where V is a connected component of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$, it follows that $\mathbf{R}^n \backslash A$ is connected. □

Remark 4.2. In fact, there is only one bounded connected component in $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$, but our methods do not allow us to prove this fact. We will prove it for n= 2 in the next chapter. Of course, for n= 1, it is a trivial fact.

Theorem 4.3. *Let* $j : \mathbf{S}^{n-1} \hookrightarrow \mathbf{R}^n$ *be a continuous injective map, and let a be a point in* $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$. *Define the Borsuk map* $\theta_{a,j} : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}, x \mapsto (j(x) - a)/\|j(x) - a\|$. *The point a is in a bounded connected component of* $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$ *if and only if* $\theta_{a,j}$ *is essential. Moreover, if n*≥ 2, *two points are in the same connected component of* $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$ *if and only if the corresponding Borsuk maps are homotopic.*

Proof. This is particular case of 3.3 and 3.6. □

Lemma 4.4. *Let* $j : \mathbf{B}^n \hookrightarrow \mathbf{R}^n$ *be a continuous injective map. The only bounded connected component of* $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$ *is* $j(\mathrm{Int}(\mathbf{B}^n))$. *In particular, the set* $j(\mathrm{Int}(\mathbf{B}^n))$ *is open in* $\mathbf{R}^n$.

Proof. Since $\mathbf{B}^n$ is compact, the map j is a homeomorphism onto $j(\mathbf{B}^n)$. By **1**.4.5, the set $\mathbf{R}^n \backslash j(\mathbf{B}^n)$ has only unbounded connected components. In particular, no bounded connected component of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$ can intersect $\mathbf{R}^n \backslash j(\mathbf{B}^n)$. It follows that $j(\mathrm{Int}(\mathbf{B}^n))$ contains the union of the bounded connected components of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$. Since $j(\mathrm{Int}(\mathbf{B}^n))$ is connected, and by Jordan's theorem 4.1 there exists a bounded connected component in $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$, it follows that $j(\mathrm{Int}(\mathbf{B}^n))$ is the only bounded connected component of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$. □

Theorem 4.5. (Invariance of domain). *If* $j : U \hookrightarrow \mathbf{R}^n$ *is an injective continuous map, where* U *is an open subset in* $\mathbf{R}^n$, *then* j *is an open map. In particular* $j(U)$ *is open in* $\mathbf{R}^n$, *and* j *is a homeomorphism of* U *on* $j(U)$.

Proof. By 4.4, if B is a small ball in U, then $j(\mathrm{Int}(B))$ is open in $\mathbf{R}^n$. □

Corollary 4.6. (Invariance of dimension). *For k<n, there does not exist an injective continuous map of an open non empty subset of* $\mathbf{R}^n$ *into* $\mathbf{R}^k$. *In particular, if an open subset of* $\mathbf{R}^k$ *is homeomorphic to an open subset of* $\mathbf{R}^n$, *then* $n = k$.

Proof. Let U be an open non empty subset in $\mathbf{R}^n$ and $j : U \hookrightarrow \mathbf{R}^k$ an injective continuous map with k<n. Since for k<n, we have $\mathbf{R}^k \subset \mathbf{R}^n$, theorem 4.5 implies that $j(U)$ is open in $\mathbf{R}^n$. This is impossible, since $\mathbf{R}^k$ has no interior in $\mathbf{R}^n$. □

Corollary 4.7. *Let X be a subset in $\mathbf{R}^n$. If x is a point in X, the following conditions are equivalent:*

(i) *x is in the interior of X with respect to $\mathbf{R}^n$,*
(ii) *x is contained in a subset of X homeomorphic to $\mathbf{R}^n$,*
(iii) *x is contained in a subset of X homeomorphic to an open subset in $\mathbf{R}^n$,*
(iv) *x has a neighborhood in X homeomorphic to $\mathbf{R}^n$,*
(v) *x has a neighborhood in X homeomorphic to an open subset of $\mathbf{R}^n$.*

Proof. This is a consequence of invariance of domain. □

Definition 4.8. Let X be a space which is homeomorphic to a subset of $\mathbf{R}^n$. Let k be the maximum of all m$\in \mathbf{N}$ such that X contains a subset homeomorphic to $\mathbf{R}^m$ (by 4.7 k$\leq$n). We call the set $\mathrm{PI}(X) = \{x \in X \mid x \text{ is contained in a subset homeomorphic to } \mathbf{R}^k\}$ the pseudo interior of X and we call the set $\partial(X) = X \backslash \mathrm{PI}(X)$ the pseudo boundary (or even sometimes boundary) of X.

The next two theorems are easy consequences of 4.7.

Theorem 4.9. *Let A and B be two subsets of $\mathbf{R}^n$. If $h : A \tilde{\to} B$ is a homeomorphism, then $h(\mathrm{PI}(A)) = \mathrm{PI}(B)$ and $h(\partial(A)) = \partial(B)$.*

Theorem 4.10. *Let A be a closed subset of $\mathbf{R}^n$ with non empty interior, then $\mathrm{Int}(A) = \mathrm{PI}(A)$ and $\partial(A)$ is the frontier of A in $\mathbf{R}^n$.*

The following corollary is an immediate consequence of the last two theorems.

Corollary 4.11. *Let A and B be two closed subsets in $\mathbf{R}^n$. If $h : A \tilde{\to} B$ is a homeomorphism, then $h(\mathrm{Int}(A)) = \mathrm{Int}(B)$ and $h(\partial(A)) = \partial(B)$.*

5. Accessibility.

Definition 5.1. Let A be a subset of X. A point x in X is accessible from A, if there exists a path $\alpha : [0,1] \to X$ such that $\alpha([0,1[) \subset A$ and $\alpha(1) = x$.

Theorem 5.2. *Let $j : \mathbf{S}^{n-1} \hookrightarrow \mathbf{R}^n$ be an imbedding. If U is a connected component of $\mathbf{R}^n \backslash j(\mathbf{S}^{n-1})$, then the set of points of $j(\mathbf{S}^{n-1})$ accessible from U is dense in $j(\mathbf{S}^{n-1})$.*

Proof. Let $V \subset j(\mathbf{S}^{n-1})$ be an open non empty open subset of $j(\mathbf{S}^{n-1})$. By Jordan's theorem 4.1, the open subset $\mathbf{R}^n \backslash [j(\mathbf{S}^{n-1}) \backslash V]$ of $\mathbf{R}^n$ is connected. Since it contains U and V, we can find a path $\alpha : [0,1] \to \mathbf{R}^n \backslash [j(\mathbf{S}^{n-1}) \backslash V]$ such that $\alpha(0) \in U$ and $\alpha(1) \in V$. If $t_0 = \inf\{t \in [0,1] \mid \alpha(t) \in j(\mathbf{S}^{n-1})\}$, then we have $\alpha(t_0) \in V$ and $\alpha([0,t_0[) \subset U$. This shows that V contains a point accessible from U. □