CHAPTER 1
BROUWER'S FIXED POINT THEOREM

In this chapter, we will prove Brouwer's fixed point theorem and draw some of its consequences.

1 Preliminaries on simplicial complexes.

We recall some notions of PL topology which will be useful later.

Definition 1.1.a. **(Simplex in $\mathbf{R}^n$).** Given $P_0, \dots, P_k$ $(k+1)$ affinely independent points in $\mathbf{R}^n$, we will denote by $< P_0, \dots, P_k >$ the convex hull of these $(k+1)$ points. This set $< P_0, \dots, P_k >$ is called the k-simplex generated by $P_0, \dots, P_k$. More generally a k-simplex in $\mathbf{R}^n$ is simply any set of the form described above. We will use the word simplex if we do not want to emphasize the dimension k of a k-simplex.

Definition 1.1.b. **(Faces and boundary of a simplex).** If σ is the k-simplex $< P_0, \dots, P_k >$, a face of σ is any simplex of the form $< P_{i_0}, \dots, P_{i_l} >$, with $i_j \in \{0, \dots, k\}, j = 0, \dots, l$. If we want to precise the dimension of a face τ of σ we will use the term l-face. We use the notation $\tau \preceq \sigma$ to mean that τ is a face of σ; and the notation $\tau \prec \sigma$ to mean that τ is a face of σ which is not σ. The union $\cup\{\tau | \tau \prec \sigma\}$ is called the boundary of σ and is denoted by $\partial\sigma$.

Definition 1.1.c. **(Simplicial complex in $\mathbf{R}^n$).** A simplicial complex in $\mathbf{R}^n$ is a finite collection K of simplexes such that:

(i) $\sigma \in K$ and $\sigma' \preceq \sigma \Rightarrow \sigma' \in K$,

(ii) $\sigma_1, \sigma_2 \in K \Rightarrow \sigma_1 \cup \sigma_2 \preceq \sigma_1$ and $\sigma_1 \cup \sigma_2 \preceq \sigma_2$.

The dimension of K, noted $\dim(K)$, is the maximum of the dimensions of its simplexes. The size of K, noted size(K), is the maximum of the diameters of its simplexes. The geometric support of K, noted $|K|$, is the set $|K| = \cup\{\sigma|\sigma \in K\}$.

Definition 1.1.d. **(Subdivision of a simplicial complex).** A simplicial complex L is a subdivision of the simplicial complex K if we have the following conditions :

(i) $|L| = |K|$,

(ii) $\forall \tau \in L, \exists \sigma \in K$ with $\tau \subset \sigma$.

Exercise 1.2. If L is a subdivision of K, then dim(L)=dim(K). (Hint: a k-simplex cannot be a union of simplexes of dimension $< k$.)

Exercise 1.3. If σ is a simplex show that $\Sigma = \{\tau|\tau \preceq \sigma\}$ is a simplicial complex. If L is a subdivision of Σ and σ' is a face of σ, show that $M = \{\tau \in L|\tau \subset \sigma'\}$ is a subdivision of $\Sigma' = \{\tau|\tau \preceq \sigma'\}$. (Hint: It suffices to consider the case $\dim(\sigma') = \dim(\sigma) - 1$.)

Proposition 1.4. *A given simplicial complex admits a subdivision with an arbitrarily small size.*

To prove proposition 1.4, we will introduce the barycentric subdivision and study some of its properties.

If σ is the simplex $< P_0, \ldots, P_l >$, we will denote by B(σ) its barycenter $(P_0+\cdots+P_l)/(l+1)$. If $\sigma_0 \prec \cdots \prec \sigma_k$ is a sequence, where each simplex is a strict face of the next one, the points B(σ_0), ... , B(σ_k) are affinely independent. This follows from the easily proven fact : if τ is a strict face of σ then B(σ) does not belong to the affine subspace generated by τ. In particular <B(σ_0), ... ,B(σ_k)> is a k-simplex if $\sigma_0 \prec \cdots \prec \sigma_k$.

Definition 1.5. (Barycentric subdivision). The collection of simplexes $<$ B(σ_0), ..., B(σ_k) $>$ where $\sigma_0 \prec \cdots \prec \sigma_k$ are in the simplicial complex K, is called the barycentric subdivision of K and is denoted by K'.

Lemma 1.6. *The barycentric subdivision of simplicial complex is itself a simplicial complex.*

Proof. It is easy to prove that condition (i) in the definition of a simplicial complex is satisfied by K'. It remains to check that the intersection of two simplexes of K' is a face of both. We will prove this by induction on the number of simplexes in K. If there is only one simplex in K, then it is reduced to a point and $K = K'$. Suppose that we have proven the lemma for each simplicial complex having less simplexes than K. We choose a simplex $\sigma \in K$, with dim(σ)=dim(K). It is easy to check that $K\backslash\{\sigma\}$ is a simplicial complex. By induction $K\backslash\{\sigma\}'$ is also a simplicial complex. It is clear that the simplexes of $K\backslash\{\sigma\}'$ are exactly the simplexes of K' for which B(σ) is not a vertex. Let now τ_1 and τ_2 be two elements of K', and let us check that $\tau_1 \cup \tau_2$ is a face of both. We consider three cases :
First case . Neither τ_1 nor τ_2 admit B(σ) as a vertex, then τ_1 and τ_2 are in $K\backslash\{\sigma\}'$ and this case is proven by induction.
Second case . Only τ_1 admits B(σ) as a vertex. Let us write explicitly $\tau_1 =<$ B(σ_0), ..., B(σ_k), B(σ) $>$ and $\tau_2 =<$ B(σ'_0), ..., B(σ'_l) $>$. Now $\tau_1 \cap \tau_2$ is contained in $\sigma'_l \cap \sigma$ which is

a strict face of σ, since σ is of maximal dimension in K and $\sigma'_l \neq \sigma$. It follows that $\tau_1 \cap \tau_2 =< \mathrm{B}(\sigma_0), \ldots, \mathrm{B}(\sigma_k) > \cap \tau_2$ and we can use the first case.
Third case . $\tau_1 =< \mathrm{B}(\sigma_0), \ldots, \mathrm{B}(\sigma_k), \mathrm{B}(\sigma) >$ and $\tau_2 =< \mathrm{B}(\sigma'_0), \ldots, \mathrm{B}(\sigma'_l), \mathrm{B}(\sigma) >$. By induction $< \mathrm{B}(\sigma_0), \ldots, \mathrm{B}(\sigma_k) > \cap < \mathrm{B}(\sigma'_0), \ldots, \mathrm{B}(\sigma'_l) > =< \mathrm{B}(\sigma''_0), \ldots, \mathrm{B}(\sigma''_m) >$, where $\{\mathrm{B}(\sigma_0), \ldots, \mathrm{B}(\sigma_k)\} \cap \{\mathrm{B}(\sigma'_0), \ldots, \mathrm{B}(\sigma'_l)\} = \{\mathrm{B}(\sigma''_0), \ldots, \mathrm{B}(\sigma''_m)\}$. One can check that $\tau_1 \cap \tau_2 =< \mathrm{B}(\sigma''_0), \ldots, \mathrm{B}(\sigma''_l), \mathrm{B}(\sigma) >$, because for example, each point P of $\sigma \backslash \{\mathrm{B}(\sigma)\}$ can be written in a unique way as $P = \alpha \mathrm{B}(\sigma) + (1-\alpha)P'$, with $\alpha \in [0,1]$ and P' in the boundary of σ. □

Lemma 1.7. *The barycentric subdivision K' of a simplicial complex K is a subdivision of K.*

Proof. It is clear, by convexity of σ_l that we have $< \mathrm{B}(\sigma_0), \ldots, \mathrm{B}(\sigma_l) > \subset \sigma_l$. It remains to check that $|K| \subset |K'|$. Let $x \in \sigma =< P_0, \ldots, P_k >$, where σ is a simplex in K. Without loss of generality, we can assume that $x = \sum_{i=0}^k \lambda_i P_i$, with $1 \geq \lambda_0, \geq \cdots \geq \lambda_k \geq 0$ and $\sum_{i=0}^k \lambda_i = 1$. Define $\sigma_i =< P_0, \ldots, P_i >$, for $i = 0, \ldots, k$, and $\lambda_{k+1} = 0$, then we have $x = \sum_{i=0}^k (\lambda_i - \lambda_{i+1})(i+1)\mathrm{B}(\sigma_i)$; from this relation we obtain that $x \in < \mathrm{B}(\sigma_0), \ldots, \mathrm{B}(\sigma_k) >$. □

Lemma 1.8. *We have* $\mathrm{size}(K') \leq \dfrac{\dim K}{1+\dim K} \mathrm{size}(K)$.

This lemma is a trivial consequence of the next two sublemmas.

Sublemma 1.9. *If $\sigma =< P_0, \ldots, P_k >$ is a simplex in $\mathbf{R}^n$, then:*

$$\mathrm{diam}(\sigma) = \max_{0 \leq i,j \leq k} ||P_i - P_j||.$$

Proof. Let $R = \max_{0 \leq i,j \leq k} ||P_i - P_j||$. The convexity of $B(P_i, R)$, the closed ball of radius R around P_i shows that $\sigma \subset B(P_i, R)$ since $P_0, \ldots, P_k \in B(P_i, R)$. In particular, if $P \in \sigma$, we have $P_i \in B(P, R)$ for $i = 0, \ldots, k$; the convexity of $B(P, R)$ shows also that $\sigma \subset B(P, R)$. It follows that $\mathrm{diam}(\sigma) = R$. □

Sublemma 1.10. *If τ is a face of σ, then we have:*

$$||\mathrm{B}(\tau) - \mathrm{B}(\sigma)|| \leq \frac{\dim \sigma}{1+\dim \sigma} \mathrm{diam}(\sigma).$$

Proof. We can suppose that $\tau =< P_0, \ldots, P_l >$ and $\sigma =< P_0, \ldots, P_l, P_{l+1}, \ldots, P_k >$. We have $\mathrm{B}(\sigma) - \mathrm{B}(\tau) = \frac{1}{1+k} \sum_{i=0}^k (P_i - \mathrm{B}(\tau))$. Since $\mathrm{B}(\tau)$ is the barycenter of $P_0, \ldots, P_l$, we have $\sum_{i=0}^k (P_i - \mathrm{B}(\tau)) = 0$. This implies $\mathrm{B}(\sigma) - \mathrm{B}(\tau) = \frac{1}{1+k} \sum_{i=l+1}^k (P_i - \mathrm{B}(\tau))$, hence $||\mathrm{B}(\tau) - \mathrm{B}(\sigma)|| \leq \dfrac{k-l}{k+1} \mathrm{diam}(\sigma)$. □

Proof of Proposition 1.4. Since $\dim K' = \dim K$ (Exercise 1.2), we see by lemma 1.8 that if we take the n-th barycentric subdivision $K^{(n)}$, we have:

$$\mathrm{size}\big(K^{(n)}\big) \leq \left(\frac{k}{k+1}\right)^n \mathrm{size}(K)$$

where $k = \dim K$. □

2. Sperner's lemma.

We consider Δ^n the convex hull of $(n+1)$ affinely independant points $P_0, \ldots, P_n$ in $\mathbf{R}^n$. We will also use the notation Δ^n for the simplicial complex defined by Δ^n together with all its faces.

Definition 2.1. (Sperner's labelling). If L is a subdivision of Δ^n, a Sperner's labelling is a map α from L^0, the set of vertices of L, to $\{0, \ldots, n\}$ such that if $v \in L^0$ belongs to the face $< P_{i_0}, \ldots, P_{i_k} >$ of Δ^n, then $\alpha(v) \in \{i_0, \ldots, i_k\}$. We will extend the map α to a map defined on L by sending a simplex to the set of labelling of its vertices, i.e. $\alpha(< v_0, \ldots, v_k >) = \{\alpha(v_0), \ldots, \alpha(v_k)\}$.

Lemma 2.2. (Sperner's lemma). *Let L be a subdivision of Δ^n, and let $\alpha : L^0 \to \{0, \ldots, n\}$ be a Sperner's labelling . There exists at least one n-simplex σ in L labelled $\{0, \ldots, n\}$, i.e. $\alpha(\sigma) = \{0, \ldots, n\}$.*

We will prove Sperner's lemma by induction on n. We first need a sublemma.

Sublemma 2.3. *Let L be a subdivision of Δ^n , and τ be an $(n-1)$-simplex of L, then one of the following two cases happens:*

(i) *$\tau \in \partial\Delta^n$ and τ is contained in exactly one n-simplex of L,*
(ii) *$\tau \notin \partial\Delta^n$, and τ is contained in exactly two n-simplexes of L.*

The proof of this sublemma, left to the reader, rests essentially on the fact that an $(n-1)$-hyperplane of $\mathbf{R}^n$ cuts a convex neighborhood of one of its points in exactly two pieces.

We will prove in fact lemma 2.2 in the following stronger form:

Lemma 2.4. *Under the conditions of lemma 2.2, we have:*

$$\#\{\tau \in L | \alpha(\tau) = \{0, \ldots, n\}\} \equiv 1 \mod 2$$

Proof. We proceed by induction on n. Denote by Δ^{n-1} the $(n-1)$-simplex $< P_0, \ldots, P_{n-1} > \subset < P_0, \ldots, P_n >= \Delta^n$. The simplicial complex $L_1 = \{\tau \in L | \tau \subset \Delta^{n-1}\}$ is a subdivision of Δ^{n-1} (see exercise 1.3) and moreover $\alpha|L_1$ is a Sperner's labelling (with values in $\{0, \ldots, n-1\}$), hence by induction:

$$\#\{\tau \in L^\tau \in \Delta^{n-1}, \alpha(\tau) = \{0, \ldots, n-1\}\} \equiv 1 \mod 2$$

Let us now compute $N = \#\{\sigma \in L | \alpha(\sigma) = \{0, \dots, n\}\} \mod 2$. For σ an n-simplex in L define $N(\sigma) = \#\{\sigma' \preceq \sigma | \sigma'$ an $(n-1)$-simplex with $\alpha(\sigma') = \{0, 1, \dots, n-1\}$. For an n-simplex σ in L, one of the following three cases happens:

(i) $\alpha(\sigma) \not\supset \{0, \dots, n-1\}$ and $N(\sigma) = 0$,
(ii) $\alpha(\sigma) = \{0, \dots, n-1\}$ and $N(\sigma) = 2$,
(iii) $\alpha(\sigma) = \{0, \dots, n\}$ and $N(\sigma) = 1$.

We obtain from this $N \equiv \sum_\sigma N(\sigma) \mod 2$. By sublemma 2.3, the sum $\sum_\sigma N(\sigma)$ is equal to the "number" of $(n-1)$-simplexes of L labelled $\{0, 1, \dots, n-1\}$, where such a simplex τ is counted once if it is included in $\partial\Delta^n$, and twice if it is not. In particular, $\sum_\sigma N(\sigma) \equiv \{\tau \in L | \tau \subset \partial\Delta^n, \alpha(\tau) = \{0, 1, \dots, n-1\}\} \mod 2$. Since α is a Sperner's labelling, the conditions $\tau \subset \partial\Delta^n$ and $\alpha(\tau) = \{0, 1, \dots, n-1\}$ imply $\tau \subset \Delta^{n-1}$. So finally, we get $N \equiv \sum_\sigma N(\sigma) \equiv \{\tau \in L | \tau \subset \partial\Delta^n, \alpha(\tau) = \{0, 1, \dots, n-1\}\} \equiv 1 \mod 2$. □

The following geometric form of Sperner's lemma is due to Knaster, Kuratowski and Mazurkiewicz.

Lemma 2.5.(KKM Lemma). *Let* $F_0, \dots, F_n$ *be* $(n+1)$ *closed subsets of the* n*-simplex* $\Delta^n =< P_0, \dots, P_n >$ *verifying* $< P_{i_0}, \dots, P_{i_k} >\subset F_{i_0} \cup \dots \cup F_{i_k}$ *for each subset* $\{i_0, \dots, i_k\} \subset \{0, 1, \dots, n-1\}$ *(in particular* $\Delta^n = F_0 \cup \dots \cup F_n$*), then the intersection* $\bigcap_{i=0}^{i=n} F_i$ *is non-empty.*

Proof. *(By contradiction).* Suppose $\bigcap_{i=0}^{i=n} F_i = \emptyset$. Let us denote by $V_\delta(F_i)$ the closed δ-neighborhood of F_i in Δ^n,i.e. $V_\delta(F_i) = \{x \in \Delta^n | d(x, F_i) \le \delta\}$. Since the F_i's are closed, we have:

$$\bigcap_{\delta>0} \left(\bigcap_{i=0}^{i=n} V_\delta(F_i) \right) = \bigcap_{i=0}^{i=n} \left(\bigcap_{\delta>0} V_\delta(F_i) \right) = \bigcap_{i=0}^{i=n} F_i = \emptyset.$$

The compactness of Δ^n implies that there exists a $\delta > 0$ such that $\bigcap_{i=0}^{i=n} V_\delta(F_i) = \emptyset$. In particular, a subset of Δ^n of diameter less than δ has to miss some F_i. Let now L be a subdivision of Δ^n of size less than δ (see 1.4). We construct a Sperner's labelling α, such that $v \in F_{\alpha(v)}$ for each vertex v of L. To construct such a labelling, for a vertex v we consider $< P_{i_0}, \dots, P_{i_k} >$ the smallest face of Δ^n containing v. Since by hypothesis $< P_{i_0}, \dots, P_{i_k} >\subset F_{i_0} \cup \dots \cup F_{i_k}$, we can find $\alpha(v) \in \{i_0, \dots, i_k\}$ such that $v \in F_{\alpha(v)}$. By Sperner's lemma, there exists a simplex τ in L labelled $\{0, \dots, n\}$. Such a simplex τ has to meet each F_i—at least one vertex of τ is in F_i—but this is impossible by the choice of L. □

3. Brouwer's fixed point theorem.

Theorem 3.1.(Brouwer's fixed point theorem). *Any continuous self map $f : \Delta^n \to \Delta^n$ of the n-simplex has a fixed point.*

Proof. Suppose $\Delta^n =< P_0, \dots, P_n >$. We introduce the barycentric coordinates on Δ^n by the formula $x = \sum_{i=0}^{i=n} \lambda_i(x) P_i$, where $\lambda_i \in [0, 1]$. The maps $\lambda_0, \dots, \lambda_n : \Delta^n \to [0, 1]$ are continuous. For each $i = 0, \dots, n$, the set $F_i = \{x \in \Delta^n | \lambda_i(f(x)) \leq \lambda_i(x)\}$ is closed. Moreover, the sets $F_0, \dots, F_n$ verify the conditions of the KKM lemma. In fact, if $x \in < P_{i_0}, \dots, P_{i_k} >$ we cannot have $\lambda_i < \lambda_i(f(x))$ for each $i \in \{i_0, \dots, i_k\}$, because this would imply the following inequalities: $1 \geq \sum_{i \in \{i_0, \dots, i_k\}} \lambda_i(f(x)) > \sum_{i \in \{i_0, \dots, i_k\}} \lambda_i(x) = 1$. By the KKM lemma, we have $\bigcap_{i=0}^{i=n} F_i \neq \emptyset$. But $\bigcap_{i=0}^{i=n} F_i$ is exactly the set of fixed points of f, since for each x in Δ^n we have: $\sum_{i=0}^{n} \lambda_i(f(x)) = 1 = \sum_{i=0}^{n} \lambda_i(x)$. □

Of course Brouwer's fixed point theorem is valid for any space homeomorphic to Δ^n, in particular it applies to the euclidean n-ball $\mathbf{B}^n = \{x \in \mathbf{R}^n | \|x\| \leq 1\}$, and more generally to any compact convex subset of $\mathbf{R}^n$, because such a set is homeomorphic to $\mathbf{B}^k$ for some $k \leq n$.

4. Consequences.

Definition 4.1. (Retract). The subspace Y of X is called a retract of X if there exists a continuous map $r : X \to Y$ such that $r|Y$ is the identity. Such a map is called a retraction.

Theorem 4.2. (No retraction theorem). *The sphere $\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n | \|x\| = 1\}$ is not a retract of the ball $\mathbf{B}^n = \{x \in \mathbf{R}^n | \|x\| \leq 1\}$.*

Proof. (By contradiction). If $r : \mathbf{B}^n \to \mathbf{S}^{n-1}$ is a retraction, then $x \mapsto -r(x)$ is a self map of $\mathbf{B}^n$ with no fixed point. □

Corollary 4.3. *If G is a bounded subset in $\mathbf{R}^n$, then $\mathbf{R}^n \backslash G$ is not a retract of $\mathbf{R}^n$.*

Proof. (By contradiction). Without loss of generality, we can suppose that $0 \in G$. Let r be a retraction of $\mathbf{R}^n$ on $\mathbf{R}^n \backslash G$. Since G is bounded, we can find $R > 0$ such that G is contained in $B(0, R) = \{x \in \mathbf{R}^n | \|x\| \leq R\}$. It is easy to check that the map

$r : \mathbf{B}^n \to \mathbf{S}^{n-1}, x \mapsto r(Rx)/\|r(Rx)\|$ is a retraction. this is impossible by the no retraction theorem. $\square$

The next theorem is well known from general topology.

Theorem 4.4. (Tietze-Urysohn). *Let X be a metric space and let F be a closed subset of X. Any continuous map $F \to \mathbf{I}^k = [0,1]^k$ can be extended to a continuous map $X \to \mathbf{I}^k$.*

Corollary 4.5. *If A is a compact subset in $\mathbf{R}^n$ homeomorphic to $\mathbf{B}^k$, then $\mathbf{R}^n \backslash A$ has no bounded connected component. In particular, for $n \geq 2$, $\mathbf{R}^n \backslash A$ is connected.*

Proof. (By contradiction). Suppose that $\mathbf{R}^n \backslash A$ has a bounded connected component G. We will show how to construct a retraction of $\mathbf{R}^n$ on $\mathbf{R}^n \backslash G$, this will contradict corollary 4.3. Let $f : A \to \mathbf{I}^k$ be a homeomorphism ($\mathbf{B}^k$ is homeomorphic to $\mathbf{I}^k$!); by 4.4, we can extend f to a continuous map $\bar{f} : \mathbf{R}^n \to \mathbf{I}^k$. We define our retraction r by $r|\mathbf{R}^n \backslash G$ =identity and $r|G \cup A = f^{-1} \circ \bar{f}(x)$. These two definitions agree on $(\mathbf{R}^n \backslash G) \cap (G \cup A) = A$ since $\bar{f} = f$ on A. It is clear that r restricted to each one of the two pieces $\mathbf{R}^n \backslash G$ and $G \cup A$ is continuous. Since A is closed, the connected components of $\mathbf{R}^n \backslash A$ are open; in particular $\mathbf{R}^n \backslash G$ and $G \cup A = \mathbf{R}^n \backslash \cup \{C | C$ component of $\mathbf{R}^n \backslash A$, with $C \neq G\}$ are closed. It follows from the previous facts that r is continuous.

If $n \geq 2$, the complement of a ball in $\mathbf{R}^n$ is connected. This implies that the complementary of a bounded set can have only one unbounded component. In particular, if A is homeomorphic to $\mathbf{I}^k$, we obtain that $\mathbf{R}^n \backslash A$ is connected by the first part of the corollary. $\square$