10

ELEMENTARY PROPERTIES OF HOLOMORPHIC FUNCTIONS

Complex Differentiation

We shall now study complex functions defined in subsets of the complex plane. It will be convenient to adopt some standard notations which will be used throughout the rest of this book.

10.1 Definitions If $r > 0$ and a is a complex number,

$$D(a; r) = \{z: |z - a| < r\} \tag{1}$$

is the open circular disc with center at a and radius r. $\overline{D}(a; r)$ is the closure of $D(a; r)$, and

$$D'(a; r) = \{z: 0 < |z - a| < r\} \tag{2}$$

is the punctured disc with center at a and radius r.

A set E in a topological space X is said to be *not connected* if E is the union of two nonempty sets A and B such that

$$\overline{A} \cap B = \varnothing = A \cap \overline{B}. \tag{2}$$

If A and B are as above, and if V and W are the complements of $\bar{A}$ and $\bar{B}$, respectively, it follows that $A \subset W$ and $B \subset V$. Hence

(3) $E \subset V \cup W, \quad E \cap V \neq \varnothing, \quad E \cap W \neq \varnothing, \quad E \cap V \cap W = \varnothing.$

Conversely, if open sets V and W exist such that (3) holds, it is easy to see that E is not connected, by taking $A = E \cap W$, $B = E \cap V$.

If E is closed and not connected, then (2) shows that E is the union of two disjoint nonempty closed sets; for if $\bar{A} \subset A \cup B$ and $\bar{A} \cap B = \varnothing$, then $\bar{A} = A$.

If E is open and not connected, then (3) shows that E is the union of two disjoint nonempty closed sets, namely $E \cap V$ and $E \cap W$.

Each set consisting of a single point is obviously connected. If $x \in E$, the family Φ_x of all connected subsets of E that contain x is therefore not empty. The union of all members of Φ_x is easily seen to be connected, and to be a *maximal connected subset* of E. These sets are called the *components* of E. Any two components of E are thus disjoint, and E is the union of its components.

By a *region* we shall mean a nonempty connected open subset of the complex plane. Since each open set Ω in the plane is a union of discs, and since all discs are connected, each component of Ω is open. Every plane open set is thus a union of disjoint regions. The letter Ω will from now on denote a plane open set.

10.2 Definition Suppose f is a complex function defined in Ω. If $z_0 \in \Omega$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

exists, we denote this limit by $f'(z_0)$ and call it the *derivative* of f at z_0. If $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is *holomorphic* (or *analytic*) in Ω. The class of all holomorphic functions in Ω will be denoted by $H(\Omega)$.

To be quite explicit, $f'(z_0)$ exists if to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \qquad \text{for all } z \in D'(z_0; \delta). \tag{2}$$

Thus $f'(z_0)$ is a complex number, obtained as a limit of quotients of complex numbers. Note that f is a mapping of Ω into R^2 and that Definition 8.22 associates with such mappings another kind of derivative, namely, a linear operator on R^2. In our present situation, if (2) is satisfied, this linear operator turns out to be multiplication by $f'(z_0)$ (regarding R^2 as the complex field). We leave it to the reader to verify this.

10.3 Remarks If $f \in H(\Omega)$ and $g \in H(\Omega)$, then also $f+g \in H(\Omega)$ and $fg \in H(\Omega)$, so that $H(\Omega)$ is a ring; the usual differentiation rules apply.

More interesting is the fact that superpositions of holomorphic functions are holomorphic: *If $f \in H(\Omega)$, if $f(\Omega) \subset \Omega_1$, if $g \in H(\Omega_1)$, and if $h = g \circ f$, then $h \in H(\Omega)$, and h' can be computed by the chain rule*

$$h'(z_0) = g'(f(z_0))f'(z_0) \qquad (z_0 \in \Omega). \tag{1}$$

To prove this, fix $z_0 \in \Omega$, and put $w_0 = f(z_0)$. Then

$$f(z) - f(z_0) = [f'(z_0) + \epsilon(z)](z - z_0), \tag{2}$$

$$g(w) - g(w_0) = [g'(w_0) + \eta(w)](w - w_0), \tag{3}$$

where $\epsilon(z) \to 0$ as $z \to z_0$ and $\eta(w) \to 0$ as $w \to w_0$. Put $w = f(z)$, and substitute (2) into (3): If $z \neq z_0$,

$$\frac{h(z) - h(z_0)}{z - z_0} = [g'(f(z_0)) + \eta(f(z))][f'(z_0) + \epsilon(z)]. \tag{4}$$

The differentiability of f forces f to be continuous at z_0. Hence (1) follows from (4).

10.4 Examples For $n = 0, 1, 2, \ldots, z^n$ is holomorphic in the whole plane, and the same is true of every polynomial in z. One easily verifies directly that $1/z$ is holomorphic in $\{z: z \neq 0\}$. Hence, taking $g(w) = 1/w$ in the chain rule, we see that if f_1 and f_2 are in $H(\Omega)$ and Ω_0 is an open subset of Ω in which f_2 has no zero, then $f_1/f_2 \in H(\Omega_0)$.

Another example of a function which is holomorphic in the whole plane (such functions are called *entire*) is the exponential function defined in the Prologue. In fact, we saw there that exp is differentiable everywhere, in the sense of Definition 10.2, and that $\exp'(z) = \exp(z)$ for every complex z.

10.5 Power Series From the theory of power series we shall assume only one fact as known, namely, that to each power series

$$\sum_{n=0}^{\infty} c_n(z - a)^n \tag{1}$$

there corresponds a number $R \in [0, \infty]$ such that the series converges absolutely and uniformly in $\overline{D}(a; r)$, for every $r < R$, and diverges if $z \notin \overline{D}(a; R)$. The "radius of convergence" R is given by the root test:

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}. \tag{2}$$

Let us say that a function f defined in Ω is *representable by power series in* Ω if to every disc $D(a; r) \subset \Omega$ there corresponds a series (1) which converges to $f(z)$ for all $z \in D(a; r)$.

10.6 Theorem *If f is representable by power series in Ω, then $f \in H(\Omega)$ and f' is also representable by power series in Ω. In fact, if*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \tag{1}$$

for $z \in D(a; r)$, then for these z we also have

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}. \tag{2}$$

PROOF If the series (1) converges in $D(a; r)$, the root test shows that the series (2) also converges there. Take $a = 0$, without loss of generality, denote the sum of the series (2) by $g(z)$, fix $w \in D(a; r)$, and choose ρ so that $|w| < \rho < r$. If $z \neq w$, we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=1}^{\infty} c_n \left[\frac{z^n - w^n}{z - w} - nw^{n-1} \right]. \tag{3}$$

The expression in brackets is 0 if $n = 1$, and is

$$(z - w) \sum_{k=1}^{n-1} kw^{k-1} z^{n-k-1} \tag{4}$$

if $n \geq 2$. If $|z| < \rho$, the absolute value of the sum in (4) is less than

$$\frac{n(n-1)}{2} \rho^{n-2} \tag{5}$$

so

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}. \tag{6}$$

Since $\rho < r$, the last series converges. Hence the left side of (6) tends to 0 as $z \to w$. This says that $f'(w) = g(w)$, and completes the proof. ////

Corollary *Since f' is seen to satisfy the same hypothesis as f does, the theorem can be applied to f'. It follows that f has derivatives of all orders, that each derivative is representable by power series in Ω, and that*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) c_n (z-a)^{n-k} \tag{7}$$

if (1) *holds. Hence* (1) *implies that*

(8) $$k!\,c_k = f^{(k)}(a) \qquad (k = 0, 1, 2, \ldots),$$

so that for each $a \in \Omega$ there is a unique sequence $\{c_n\}$ for which (1) *holds.*

We now describe a process which manufactures functions that are representable by power series. Special cases will be of importance later.

10.7 Theorem *Suppose μ is a complex (finite) measure on a measurable space X, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and*

(1) $$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} \qquad (z \in \Omega).$$

Then f is representable by power series in Ω.

PROOF Suppose $D(a; r) \subset \Omega$. Since

(2) $$\left|\frac{z - a}{\varphi(\zeta) - a}\right| \le \frac{|z - a|}{r} < 1$$

for every $z \in D(a; r)$ and every $\zeta \in X$, the geometric series

(3) $$\sum_{n=0}^{\infty} \frac{(z - a)^n}{(\varphi(\zeta) - a)^{n+1}} = \frac{1}{\varphi(\zeta) - z}$$

converges uniformly on X, for every fixed $z \in D(a; r)$. Hence the series (3) may be substituted into (1), and $f(z)$ may be computed by interchanging summation and integration. It follows that

(4) $$f(z) = \sum_{0}^{\infty} c_n (z - a)^n \qquad (z \in D(a; r))$$

where

(5) $$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta) - a)^{n+1}} \qquad (n = 0, 1, 2, \ldots).$$ ////

Note: The convergence of the series (4) in $D(a; r)$ is a consequence of the proof. We can also derive it from (5), since (5) shows that

(6) $$|c_n| \le \frac{|\mu|(X)}{r^{n+1}} \qquad (n = 0, 1, 2, \ldots).$$

Integration over Paths

Our first major objective in this chapter is the converse of Theorem 10.6: *Every* $f \in H(\Omega)$ *is representable by power series in* Ω. The quickest route to this is via Cauchy's theorem which leads to an important integral representation of holomorphic functions. In this section the required integration theory will be developed; we shall keep it as simple as possible, and shall regard it merely as a useful tool in the investigation of properties of holomorphic functions.

10.8 Definitions If X is a topological space, a *curve in* X is a continuous mapping γ of a compact interval $[\alpha, \beta] \subset R^1$ into X; here $\alpha < \beta$. We call $[\alpha, \beta]$ the *parameter interval* of γ and denote the range of γ by γ^*. Thus γ is a mapping, and γ^* is the set of all points $\gamma(t)$, for $\alpha \leq t \leq \beta$.

If the *initial point* $\gamma(\alpha)$ of γ coincides with its *end point* $\gamma(\beta)$, we call γ a *closed curve*.

A *path* is a piecewise continuously differentiable curve in the plane. More explicitly, a path with parameter interval $[\alpha, \beta]$ is a continuous complex function γ on $[\alpha, \beta]$, such that the following holds: There are finitely many points s_j, $\alpha = s_0 < s_1 < \cdots < s_n = \beta$, and the restriction of γ to each interval $[s_{j-1}, s_j]$ has a continuous derivative on $[s_{j-1}, s_j]$; however, at the points $s_1, \ldots, s_{n-1}$ the left- and right-hand derivatives of γ may differ.

A *closed path* is a closed curve which is also a path.

Now suppose γ is a path, and f is a continuous function on γ^*. The integral of f over γ is defined as an integral over the parameter interval $[\alpha, \beta]$ of γ:

$$(1) \qquad \int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt.$$

Let φ be a continuously differentiable one-to-one mapping of an interval $[\alpha_1, \beta_1]$ onto $[\alpha, \beta]$, such that $\varphi(\alpha_1) = \alpha$, $\varphi(\beta_1) = \beta$, and put $\gamma_1 = \gamma \circ \varphi$. Then γ_1 is a path with parameter interval $[\alpha_1, \beta_1]$; the integral of f over γ_1 is

$$\int_{\alpha_1}^{\beta_1} f(\gamma_1(t))\gamma_1'(t)\,dt = \int_{\alpha_1}^{\beta_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)\,dt = \int_\alpha^\beta f(\gamma(s))\gamma'(s)\,ds,$$

so that our "reparametrization" has not changed the integral:

$$(2) \qquad \int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz.$$

Whenever (2) holds for a pair of paths γ and γ_1 (and for all f), we shall regard γ and γ_1 as equivalent.

It is convenient to be able to replace a path by an equivalent one, i.e., to choose parameter intervals at will. For instance, if the end point of γ_1 coincides with the initial point of γ_2, we may locate their parameter intervals so that γ_1 and γ_2 join to form one path γ, with the property that

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \tag{3}$$

for every continuous f on $\gamma^* = \gamma_1^* \cup \gamma_2^*$.

However, suppose that $[0,1]$ is the parameter interval of a path γ, and $\gamma_1(t) = \gamma(1-t)$, $0 \le t \le 1$. We call γ_1 the path *opposite* to γ, for the following reason: For any f continuous on $\gamma_1^* = \gamma^*$, we have

$$\int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt = -\int_0^1 f(\gamma(1-t))\gamma'(1-t)\,dt = -\int_0^1 f(\gamma(s))\gamma'(s)\,ds,$$

so that

$$\int_{\gamma_1} f = -\int_\gamma f. \tag{4}$$

From (1) we obtain the inequality

$$\left|\int_\gamma f(z)\,dz\right| \le \|f\|_\infty \int_\alpha^\beta |\gamma'(t)|\,dt, \tag{5}$$

where $\|f\|_\infty$ is the maximum of $|f|$ on γ^* and the last integral in (5) is (by definition) the *length* of γ.

10.9 Special Cases

(*a*) If a is a complex number and $r > 0$, the path defined by

$$\gamma(t) = a + re^{it} \qquad (0 \le t \le 2\pi) \tag{1}$$

is called the *positively oriented circle* with center at a and radius r; we have

$$\int_\gamma f(z)\,dz = ir\int_0^{2\pi} f(a + re^{i\theta})e^{i\theta}\,d\theta, \tag{2}$$

and the length of γ is $2\pi r$, as expected.

(*b*) If a and b are complex numbers, the path γ given by

$$\gamma(t) = a + (b-a)t \qquad (0 \le t \le 1) \tag{3}$$

is the *oriented interval* $[a,b]$; its length is $|b-a|$, and

$$\int_{[a,b]} f(z)\,dz = (b-a)\int_0^1 f[a + (b-a)t]\,dt. \tag{4}$$

If

$$\gamma_1(t) = \frac{a(\beta - t) + b(t - \alpha)}{\beta - \alpha} \qquad (\alpha \le t \le \beta), \tag{5}$$

we obtain an equivalent path, which we still denote by $[a, b]$. The path opposite to $[a, b]$ is $[b, a]$.

(*c*) Let $\{a, b, c\}$ be an ordered triple of complex numbers, let

$$\Delta = \Delta(a, b, c)$$

be the triangle with vertices at a, b, and c (Δ is the smallest convex set which contains a, b, and c), and define

$$\int_{\partial\Delta} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f, \tag{6}$$

for any f continuous on the boundary of Δ. We may regard (6) as the definition of its left side. Or we may regard $\partial\Delta$ as a path obtained by joining $[a, b]$ to $[b, c]$ to $[c, a]$, as outlined in Definition 10.8, in which case (6) is easily proved to be true.

If $\{a, b, c\}$ is permuted cyclically, we see from (6) that the left side of (6) is unaffected. If $\{a, b, c\}$ is replaced by $\{a, c, b\}$, then the left side of (6) changes sign.

We now come to a theorem which plays a very important role in function theory.

10.10 Theorem *Let γ be a closed path, let Ω be the complement of γ^* (relative to the plane), and define*

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} \qquad (z \in \Omega). \tag{1}$$

Then $\operatorname{Ind}_\gamma$ is an integer-valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component of Ω.

We call $\operatorname{Ind}_\gamma(z)$ the *index* of z with respect to γ. Note that γ^* is compact, hence γ^* lies in a bounded disc D whose complement D^c is connected; thus D^c lies in some component of Ω. This shows that Ω has precisely one unbounded component.

PROOF Let $[\alpha, \beta]$ be the parameter interval of γ, fix $z \in \Omega$, then

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s) - z}\, ds. \tag{2}$$

Since $w/2\pi i$ is an integer if and only if $e^w = 1$, the first assertion of the theorem, namely, that $\mathrm{Ind}_\gamma(z)$ is an integer, is equivalent to the assertion that $\varphi(\beta) = 1$, where

$$\varphi(t) = \exp\left\{\int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - z}\,ds\right\} \qquad (\alpha \le t \le \beta). \tag{3}$$

Differentiation of (3) shows that

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\gamma'(t)}{\gamma(t) - z} \tag{4}$$

except possibly on a finite set S where γ is not differentiable. Therefore $\varphi/(\gamma - z)$ is a continuous function on $[\alpha, \beta]$ whose derivative is zero in $[\alpha, \beta] - S$. Since S is finite, $\varphi/(\gamma - z)$ is constant on $[\alpha, \beta]$; and since $\varphi(\alpha) = 1$, we obtain

$$\varphi(t) = \frac{\gamma(t) - z}{\gamma(\alpha) - z} \qquad (\alpha \le t \le \beta). \tag{5}$$

We now use the assumption that γ is a closed path, i.e., that $\gamma(\beta) = \gamma(\alpha)$; (5) shows that $\varphi(\beta) = 1$, and this, as we observed above, implies that $\mathrm{Ind}_\gamma(z)$ is an integer.

By Theorem 10.7, (1) shows that $\mathrm{Ind}_\gamma \in H(\Omega)$. The image of a connected set under a continuous mapping is connected ([26], Theorem 4.22), and since Ind_γ is an integer-valued function, Ind_γ must be constant on each component of Ω.

Finally, (2) shows that $|\mathrm{Ind}_\gamma(z)| < 1$ if $|z|$ is sufficiently large. This implies that $\mathrm{Ind}_\gamma(z) = 0$ in the unbounded component of Ω. ////

Remark: If $\lambda(t)$ denotes the integral in (3), the preceding proof shows that $2\pi\ \mathrm{Ind}_\gamma(z)$ is the net increase in the imaginary part of $\lambda(t)$, as t runs from α to β, and this is the same as the net increase of the argument of $\gamma(t) - z$. (We have not defined "argument" and will have no need for it.) If we divide this increase by 2π, we obtain "the number of times that γ winds around z," and this explains why the term "winding number" is frequently used for the index. One virtue of the preceding proof is that it establishes the main properties of the index without any reference to the (multiple-valued) argument of a complex number.

10.11 Theorem *If γ is the positively oriented circle with center at a and radius r, then*

$$\mathrm{Ind}_\gamma(z) = \begin{cases} 1 & \text{if } |z - a| < r, \\ 0 & \text{if } |z - a| > r. \end{cases}$$

PROOF We take γ as in Sec. 10.9(*a*). By Theorem 10.10, it is enough to compute $\mathrm{Ind}_\gamma(a)$, and 10.9(2) shows that this equals

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{r}{2\pi}\int_0^{2\pi} (re^{it})^{-1} e^{it}\, dt = 1. \qquad ////$$

The Local Cauchy Theorem

There are several forms of Cauchy's theorem. They all assert that if γ is a closed path or cycle in Ω, and if γ and Ω satisfy certain topological conditions, then the integral of every $f \in H(\Omega)$ over γ is 0. We shall first derive a simple local version of this (Theorem 10.14) which is quite sufficient for many applications. A more general global form will be established later.

10.12 Theorem *Suppose $F \in H(\Omega)$ and F' is continuous in Ω. Then*

$$\int_\gamma F'(z)\, dz = 0$$

for every closed path γ in Ω.

PROOF If $[\alpha, \beta]$ is the parameter interval of γ, the fundamental theorem of calculus shows that

$$\int_\gamma F'(z)\, dz = \int_\alpha^\beta F'(\gamma(t))\gamma'(t)\, dt = F(\gamma(\beta)) - F(\gamma(\alpha)) = 0,$$

since $\gamma(\beta) = \gamma(\alpha)$.

Corollary *Since z^n is the derivative of $z^{n+1}/(n+1)$ for all integers $n \neq -1$, we have*

$$\int_\gamma z^n\, dz = 0$$

for every closed path γ if $n = 0, 1, 2, \ldots$, and for those closed paths γ for which $0 \notin \gamma^*$ if $n = -2, -3, -4, \ldots$. ////

The case $n = -1$ was dealt with in Theorem 10.10.

10.13 Cauchy's Theorem for a Triangle *Suppose Δ is a closed triangle in a plane open set Ω, $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega - \{p\})$. Then*

$$\int_{\partial\Delta} f(z)\, dz = 0. \qquad (1)$$

For the definition of $\partial\Delta$ we refer to Sec. 10.9(*c*). We shall see later that our hypothesis actually implies that $f \in H(\Omega)$, i.e., that the exceptional point p is not really exceptional. However, the above formulation of the theorem will be useful in the proof of the Cauchy formula.

PROOF We assume first that $p \notin \Delta$. Let a, b, and c be the vertices of Δ, let a', b', and c' be the midpoints of $[b, c]$, $[c, a]$, and $[a, b]$, respectively, and consider the four triangles Δ^j formed by the ordered triples

$$\{a, c', b'\}, \qquad \{b, a', c'\}, \qquad \{c, b', a'\}, \qquad \{a', b', c'\}. \tag{2}$$

If J is the value of the integral (1), it follows from 10.9(6) that

$$J = \sum_{j=1}^{4} \int_{\partial\Delta^j} f(z)\,dz. \tag{3}$$

The absolute value of at least one of the integrals on the right of (3) is therefore at least $|J/4|$. Call the corresponding triangle Δ_1, repeat the argument with Δ_1 in place of Δ, and so forth. This generates a sequence of triangles Δ_n such that $\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$, such that the length of $\partial\Delta_n$ is $2^{-n}L$, if L is the length of $\partial\Delta$, and such that

$$|J| \le 4^n \left| \int_{\partial\Delta_n} f(z)\,dz \right| \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

There is a (unique) point z_0 which the triangles Δ_n have in common. Since Δ is compact, $z_0 \in \Delta$, so f is differentiable at z_0.

Let $\epsilon > 0$ be given. There exists an $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \epsilon |z - z_0| \tag{5}$$

whenever $|z - z_0| < r$, and there exists an n such that $|z - z_0| < r$ for all $z \in \Delta_n$. For this n we also have $|z - z_0| \le 2^{-n}L$ for all $z \in \Delta_n$. By the Corollary to Theorem 10.12,

$$\int_{\partial\Delta_n} f(z)\,dz = \int_{\partial\Delta_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,dz, \tag{6}$$

so that (5) implies

$$\left| \int_{\partial\Delta_n} f(z)\,dz \right| \le \epsilon (2^{-n}L)^2, \tag{7}$$

and now (4) shows that $|J| \le \epsilon L^2$. Hence $J = 0$ if $p \notin \Delta$.

Assume next that p is a vertex of Δ, say $p = a$. If a, b, and c are collinear, then (1) is trivial, for any continuous f. If not, choose points $x \in [a, b]$ and $y \in [a, c]$, both close to a, and observe that the integral of f over $\partial\Delta$ is the sum of the integrals over the boundaries of the triangles $\{a, x, y\}$, $\{x, b, y\}$, and $\{b, c, y\}$. The last two of these are 0, since these triangles do not contain p. Hence the integral over $\partial\Delta$ is the sum of the integrals over $[a, x]$, $[x, y]$, and $[y, a]$, and since these intervals can be made arbitrarily short and f is bounded on Δ, we again obtain (1).

Finally, if p is an arbitrary point of Δ, apply the preceding result to $\{a, b, p\}$, $\{b, c, p\}$, and $\{c, a, p\}$ to complete the proof. ////

10.14 Cauchy's Theorem in a Convex Set *Suppose Ω is a convex open set, $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega - \{p\})$. Then $f = F'$ for some $F \in H(\Omega)$. Hence*

$$\int_\gamma f(z)\,dz = 0 \tag{1}$$

for every closed path γ in Ω.

PROOF Fix $a \in \Omega$. Since Ω is convex, Ω contains the straight line interval from a to z for every $z \in \Omega$, so we can define

$$F(z) = \int_{[a,z]} f(\xi)\,d\xi \qquad (z \in \Omega). \tag{2}$$

For any z and $z_0 \in \Omega$, the triangle with vertices at a, z_0, and z lies in Ω; hence $F(z) - F(z_0)$ is the integral of f over $[z_0, z]$, by Theorem 10.13. Fixing z_0, we thus obtain

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z_0,z]} [f(\xi) - f(z_0)]\,d\xi, \tag{3}$$

if $z \neq z_0$. Given $\epsilon > 0$, the continuity of f at z_0 shows that there is a $\delta > 0$ such that $|f(\xi) - f(z_0)| < \epsilon$ if $|\xi - z_0| < \delta$; hence the absolute value of the left side of (3) is less than ϵ as soon as $|z - z_0| < \delta$. This proves that $f = F'$. In particular, $F \in H(\Omega)$. Now (1) follows from Theorem 10.12. ////

10.15 Cauchy's Formula in a Convex Set *Suppose γ is a closed path in a convex open set Ω, and $f \in H(\Omega)$. If $z \in \Omega$ and $z \notin \gamma^*$, then*

$$f(z) \cdot \mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi. \tag{1}$$

The case of greatest interest is, of course, $\mathrm{Ind}_\gamma(z) = 1$.

PROOF Fix z so that the above conditions hold, and define

$$(2) \qquad g(\xi) = \begin{cases} \dfrac{f(\xi) - f(z)}{\xi - z} & \text{if } \xi \in \Omega, \xi \neq z, \\ f'(z) & \text{if } \xi = z. \end{cases}$$

Then g satisfies the hypotheses of Theorem 10.14. Hence

$$(3) \qquad \frac{1}{2\pi i} \int_\gamma g(\xi)\, d\xi = 0.$$

If we substitute (2) into (3) we obtain (1). ////

The theorem concerning the representability of holomorphic functions by power series is an easy consequence of Theorem 10.15, if we take a circle for γ:

10.16 Theorem *For every open set Ω in the plane, every $f \in H(\Omega)$ is representable by power series in Ω.*

PROOF Suppose $f \in H(\Omega)$ and $D(a; R) \subset \Omega$. If γ is a positively oriented circle with center at a and radius $r < R$, the convexity of $D(a; R)$ allows us to apply Theorem 10.15; by Theorem 10.11, we obtain

$$(1) \qquad f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi \qquad (z \in D(a; r)).$$

But now we can apply Theorem 10.7, with $X = [0, 2\pi]$, $\varphi = \gamma$, and $d\mu(t) = f(\gamma(t))\gamma'(t)dt$, and we conclude that there is a sequence $\{c_n\}$ such that

$$(2) \qquad f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \qquad (z \in D(a; r)).$$

The uniqueness of $\{c_n\}$ (see the Corollary to Theorem 10.6) shows that the same power series is obtained for every $r < R$ (as long as a is fixed). Hence the representation (2) is valid for every $z \in D(a; R)$, and the proof is complete. ////

Corollary *If $f \in H(\Omega)$, then $f' \in H(\Omega)$.*

PROOF Combine Theorems 10.6 and 10.16. ////

The Cauchy theorem has a useful converse:

10.17 Morera's Theorem *Suppose f is a continuous complex function in an open set Ω such that*

$$\int_{\partial\Delta} f(z)\, dz = 0$$

for every closed triangle $\Delta \subset \Omega$. *Then* $f \in H(\Omega)$.

PROOF Let V be a convex open set in Ω. As in the proof of Theorem 10.14, we can construct $F \in H(V)$ such that $F' = f$. Since derivatives of holomorphic functions are holomorphic (Theorem 10.16), we have $f \in H(V)$, for every convex open $V \subset \Omega$, hence $f \in H(\Omega)$. ////

The Power Series Representation

The fact that every holomorphic function is locally the sum of a convergent power series has a large number of interesting consequences. A few of these are developed in this section.

10.18 Theorem *Suppose* Ω *is a region,* $f \in H(\Omega)$, *and*

(1) $$Z(f) = \{a \in \Omega : f(a) = 0\}.$$

Then either $Z(f) = \Omega$, *or* $Z(f)$ *has no limit point in* Ω. *In the latter case there corresponds to each* $a \in Z(f)$ *a unique positive integer* $m = m(a)$ *such that*

(2) $$f(z) = (z - a)^m g(z) \qquad (z \in \Omega),$$

where $g \in H(\Omega)$ *and* $g(a) \neq 0$; *furthermore,* $Z(f)$ *is at most countable.*

(We recall that regions are *connected* open sets.)

The integer m is called the *order* of the zero which f has at the point a. Clearly, $Z(f) = \Omega$ if and only if f is identically 0 in Ω. We call $Z(f)$ the *zero set* of f. Analogous results hold of course for the set of α-points of f, i.e., the zero set of $f - \alpha$, where α is any complex number.

PROOF Let A be the set of all limit points of $Z(f)$ in Ω. Since f is continuous, $A \subset Z(f)$.

Fix $a \in Z(f)$, and choose $r > 0$ so that $D(a; r) \subset \Omega$. By Theorem 10.16,

(3) $$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \qquad (z \in D(a; r)).$$

There are now two possibilities. Either all c_n are 0, in which case $D(a; r) \subset A$ and a is an interior point of A, or there is a smallest integer m [necessarily positive, since $f(a) = 0$] such that $c_m \neq 0$. In that case, define

(4) $$g(z) = \begin{cases} (z - a)^{-m} f(z) & (z \in \Omega - \{a\}), \\ c_m & (z = a). \end{cases}$$

Then (2) holds. It is clear that $g \in H(\Omega - \{a\})$. But (3) implies

$$g(z) = \sum_{k=0}^{\infty} c_{m+k}(z-a)^k \qquad (z \in D(a; r)). \tag{5}$$

Hence $g \in H(D(a; r))$, so actually $g \in H(\Omega)$.

Moreover, $g(a) \neq 0$, and the continuity of g shows that there is a neighborhood of a in which g has no zero. Thus a is an isolated point of $Z(f)$, by (2).

If $a \in A$, the first case must therefore occur. So A is open. If $B = \Omega - A$, it is clear from the definition of A as a set of limit points that B is open. Thus Ω is the union of the disjoint open sets A and B. Since Ω is connected, we have either $A = \Omega$, in which case $Z(f) = \Omega$, or $A = \varnothing$. In the latter case, $Z(f)$ has at most finitely many points in each compact subset of Ω, and since Ω is σ-compact, $Z(f)$ is at most countable. ////

Corollary *If f and g are holomorphic functions in a region Ω and if $f(z) = g(z)$ for all z in some set which has a limit point in Ω, then $f(z) = g(z)$ for all $z \in \Omega$.*

In other words, a holomorphic function in a region Ω is determined by its values on any set which has a limit point in Ω. This is an important uniqueness theorem.

Note: The theorem fails if we drop the assumption that Ω is connected: If $\Omega = \Omega_0 \cup \Omega_1$, and Ω_0 and Ω_1 are disjoint open sets, put $f = 0$ in Ω_0 and $f = 1$ in Ω_1.

10.19 Definition If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then f is said to have an *isolated singularity* at the point a. If f can be so defined at a that the extended function is holomorphic in Ω, the singularity is said to be *removable*.

10.20 Theorem *Suppose $f \in H(\Omega - \{a\})$ and f is bounded in $D'(a; r)$, for some $r > 0$. Then f has a removable singularity at a.*

Recall that $D'(a; r) = \{z : 0 < |z - a| < r\}$.

PROOF Define $h(a) = 0$, and $h(z) = (z-a)^2 f(z)$ in $\Omega - \{a\}$. Our boundedness assumption shows that $h'(a) = 0$. Since h is evidently differentiable at every other point of Ω, we have $h \in H(\Omega)$, so

$$h(z) = \sum_{n=2}^{\infty} c_n(z-a)^n \qquad (z \in D(a; r)).$$

We obtain the desired holomorphic extension of f by setting $f(a) = c_2$, for then

$$f(z) = \sum_{n=0}^{\infty} c_{n+2}(z-a)^n \qquad (z \in D(a; r)). \qquad ////$$

10.21 Theorem *If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then one of the following three cases must occur:*

(*a*) *f has a removable singularity at a.*

(*b*) *There are complex numbers $c_1, \ldots, c_m$, where m is a positive integer and $c_m \neq 0$, such that*

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

(*c*) *If $r > 0$ and $D(a; r) \subset \Omega$, then $f(D'(a; r))$ is dense in the plane.*

In case (*b*), f is said to have a *pole of order m* at a. The function

$$\sum_{k=1}^{m} c_k(z-a)^{-k},$$

a polynomial in $(z-a)^{-1}$, is called the *principal part* of f at a. It is clear in this situation that $|f(z)| \to \infty$ as $z \to a$.

In case (*c*), f is said to have an *essential singularity* at a. A statement equivalent to (*c*) is that to each complex number w there corresponds a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.

PROOF Suppose (*c*) fails. Then there exist $r > 0$, $\delta > 0$, and a complex number w such that $|f(z) - w| > \delta$ in $D'(a; r)$. Let us write D for $D(a; r)$ and D' for $D'(a; r)$. Define

$$(1) \qquad g(z) = \frac{1}{f(z) - w} \qquad (z \in D').$$

Then $g \in H(D')$ and $|g| < 1/\delta$. By Theorem 10.20, g extends to a holomorphic function in D.

If $g(a) \neq 0$, (1) shows that f is bounded in $D'(a; \rho)$ for some $\rho > 0$. Hence (*a*) holds, by Theorem 10.20.

If g has a zero of order $m \geq 1$ at a, Theorem 10.18 shows that

$$(2) \qquad g(z) = (z-a)^m g_1(z) \qquad (z \in D),$$

where $g_1 \in H(D)$ and $g_1(a) \neq 0$. Also, g_1 has no zero in D', by (1). Put $h = 1/g_1$ in D. Then $h \in H(D)$, h has no zero in D, and

(3) $$f(z) - w = (z - a)^{-m} h(z) \qquad (z \in D').$$

But h has an expansion of the form

(4) $$h(z) = \sum_{n=0}^{\infty} b_n (z - a)^n \qquad (z \in D),$$

with $b_0 \neq 0$. Now (3) shows that (*b*) holds, with $c_k = b_{m-k}$, $k = 1, \ldots, m$.

This completes the proof. ////

We shall now exploit the fact that the restriction of a power series $\sum c_n (z - a)^n$ to a circle with center at a is a trigonometric series.

10.22 Theorem *If*

(1) $$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \qquad (z \in D(a; R))$$

and if $0 < r < R$, *then*

(2) $$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 \, d\theta.$$

PROOF We have

(3) $$f(a + re^{i\theta}) = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}.$$

For $r < R$, the series (3) converges uniformly on $[-\pi, \pi]$. Hence

(4) $$c_n r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a + re^{i\theta}) e^{-in\theta} \, d\theta \qquad (n = 0, 1, 2, \ldots),$$

and (2) is seen to be a special case of Parseval's formula. ////

Here are some consequences:

10.23 Liouville's Theorem *Every bounded entire function is constant.*

Recall that a function is *entire* if it is holomorphic in the whole plane.

PROOF Suppose f is entire, $|f(z)| < M$ for all z, and $f(z) = \sum c_n z^n$ for all z. By Theorem 10.22,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} < M^2$$

for all r, which is possible only if $c_n = 0$ for all $n \geq 1$. ////

10.24 The Maximum Modulus Theorem *Suppose Ω is a region, $f \in H(\Omega)$, and $\overline{D}(a; r) \subset \Omega$. Then*

$$|f(a)| \leq \max_{\theta} |f(a + re^{i\theta})|. \tag{1}$$

Equality occurs in (1) *if and only if f is constant in Ω.*

Consequently, $|f|$ has no local maximum at any point of Ω, unless f is constant.

PROOF Assume that $|f(a + re^{i\theta})| \leq |f(a)|$ for all real θ. In the notation of Theorem 10.22 it follows then that

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq |f(a)|^2 = |c_0|^2.$$

Hence $c_1 = c_2 = c_3 = \cdots = 0$, which implies that $f(z) = f(a)$ in $D(a; r)$. Since Ω is connected, Theorem 10.18 shows that f is constant in Ω. ////

10.25 Theorem *If n is a positive integer and*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, \ldots, a_{n-1}$ are complex numbers, then P has precisely n zeros in the plane.

Of course, these zeros are counted according to their multiplicities: A zero of order m, say, is counted as m zeros. This theorem contains the fact that the complex field is algebraically closed, i.e., that every nonconstant polynomial with complex coefficients has at least one complex zero.

PROOF Choose $r > 1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|$. Then

$$|P(re^{i\theta})| > |P(0)| \qquad (0 \leq \theta \leq 2\pi).$$

If P had no zeros, then the function $f = 1/P$ would be entire and would satisfy $|f(0)| > |f(re^{i\theta})|$ for all θ, which contradicts the maximum modulus theorem. Thus $P(z_1) = 0$ for some z_1. Consequently, there is a polynomial Q, of degree $n - 1$, such that $P(z) = (z - z_1)Q(z)$. The proof is completed by induction on n. ////

10.26 Theorem (Cauchy's Estimates) *If $f \in H(D(a; R))$ and $|f(z)| \leq M$ for all $z \in D(a; R)$, then*

(1) $$|f^{(n)}(a)| \le \frac{n!\,M}{R^n} \qquad (n = 1, 2, 3, \dots).$$

PROOF For each $r < R$, each term of the series 10.22(2) is bounded above by M^2. ////

If we take $a = 0$, $R = 1$, and $f(z) = z^n$, then $M = 1$, $F^{(n)}(0) = n!$, and we see that (1) cannot be improved.

10.27 Definition A sequence $\{f_j\}$ of functions in Ω is said to *converge to f uniformly on compact subsets of* Ω if to every compact $K \subset \Omega$ and to every $\epsilon > 0$ there corresponds an $N = N(K, \epsilon)$ such that $|f_j(z) - f(z)| < \epsilon$ for all $z \in K$ if $j > N$.

For instance, the sequence $\{z^n\}$ converges to 0 uniformly on compact subsets of $D(0; 1)$, but the convergence is *not* uniform in $D(0; 1)$.

It is uniform convergence on compact subsets which arises most naturally in connection with limit operations on holomorphic functions. The term "almost uniform convergence" is sometimes used for this concept.

10.28 Theorem *Suppose $f_j \in H(\Omega)$, for $j = 1, 2, 3, \dots$, and $f_j \to f$ uniformly on compact subsets of Ω. Then $f \in H(\Omega)$, and $f'_j \to f'$ uniformly on compact subsets of Ω.*

PROOF Since the convergence is uniform on each compact disc in Ω, f is continuous. Let Δ be a triangle in Ω. Then Δ is compact, so

$$\int_{\partial\Delta} f(z)\,dz = \lim_{j\to\infty} \int_{\partial\Delta} f_j(z)\,dz = 0,$$

by Cauchy's theorem. Hence Morera's theorem implies that $f \in H(\Omega)$.

Let K be compact, $K \subset \Omega$. There exists an $r > 0$ such that the union E of the closed discs $\overline{D}(z; r)$, for all $z \in K$, is a compact subset of Ω. Applying Theorem 10.26 to $f - f_j$, we have

$$|f'(z) - f'_j(z)| \le r^{-1} \|f - f_j\|_E \qquad (z \in K),$$

where $\|f\|_E$ denotes the supremum of $|f|$ on E. Since $f_j \to f$ uniformly on E, it follows that $f'_j \to f'$ uniformly on K. ////

Corollary *Under the same hypothesis, $f_j^{(n)} \to f^{(n)}$ uniformly, as $j \to \infty$, on every compact set $K \subset \Omega$, and for every positive integer n.*

Compare this with the situation on the real line, where sequences of infinitely differentiable functions can converge uniformly to nowhere differentiable functions!

The Open Mapping Theorem

If Ω is a region and $f \in H(\Omega)$, then $f(\Omega)$ is either a region or a point.

This important property of holomorphic functions will be proved, in more detailed form, in Theorem 10.32.

10.29 Lemma *If $f \in H(\Omega)$ and g is defined in $\Omega \times \Omega$ by*

$$g(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

then g is continuous in $\Omega \times \Omega$.

PROOF The only points $(z, w) \in \Omega \times \Omega$ at which the continuity of g is possibly in doubt have $z = w$.

Fix $a \in \Omega$. Fix $\epsilon > 0$. There exists $r > 0$ such that $D(a; r) \subset \Omega$ and $|f'(\zeta) - f'(a)| < \epsilon$ for all $\zeta \in D(a; r)$. If z and w are in $D(a; r)$ and if

$$\zeta(t) = (1 - t)z + tw,$$

then $\zeta(t) \in D(a; r)$ for $0 \leq t \leq 1$, and

$$g(z, w) - g(a, a) = \int_0^1 [f'(\zeta(t)) - f'(a)]\, dt.$$

The absolute value of the integrand is $< \epsilon$, for every t. Thus $|g(z, w) - g(a, a)| < \epsilon$. This proves that g is continuous at (a, a). ////

10.30 Theorem *Suppose $\varphi \in H(\Omega)$, $z_0 \in \Omega$, and $\varphi'(z_0) \neq 0$. Then Ω contains a neighborhood V of z_0 such that*

(*a*) *φ is one-to-one in V,*
(*b*) *$W = \varphi(V)$ is an open set, and*
(*c*) *if $\psi: W \to V$ is defined by $\psi(\varphi(z)) = z$, then $\psi \in H(W)$.*

Thus $\varphi: V \to W$ has a holomorphic inverse.

PROOF Lemma 10.29, applied to φ in place of f, shows that Ω contains a neighborhood V of z_0 such that

$$(1) \qquad |\varphi(z_1) - \varphi(z_2)| \geq \frac{1}{2}|\varphi'(z_0)|\ |z_1 - z_2|$$

if $z_1 \in V$ and $z_2 \in V$. Thus (*a*) holds, and also

(2) $$\varphi'(z) \neq 0 \qquad (z \in V).$$

To prove (*b*), fix $\zeta \in V$. Choose $r > 0$ so that $\overline{D}(\zeta; r) \subset V$. By (1) there exists $\delta > 0$ such that

(3) $$|\varphi(\zeta + re^{i\theta}) - \varphi(\zeta)| > 2\delta \qquad (-\pi \leq \theta \leq \pi).$$

Let α be a complex number which is not in $f(V)$. Then $h = 1/(\alpha - \varphi)$ is holomorphic in V. Since

$$2\delta < |\alpha - \varphi(\zeta)| + |\alpha - \varphi(\zeta + re^{i\theta})|$$

for all θ, the maximum modulus theorem implies

$$\frac{1}{|\alpha - \varphi(\zeta)|} = |h(\zeta)| \leq \sup_\theta |h(\zeta + re^{i\theta})| \leq \frac{1}{2\delta - |\alpha - \varphi(\zeta)|}.$$

Hence $|\alpha - \varphi(\zeta)| \geq \delta$.

This proves that $\varphi(V) \supset D(\varphi(\zeta); \delta)$. Since ζ was an arbitrary point of V, $\varphi(V)$ is open.

To prove (*c*), fix $w_1 \in W$. Then $\varphi(z_1) = w_1$ for a unique $z_1 \in V$. If $w \in W$ and $\psi(w) = z \in V$, we have

(4) $$\frac{\psi(w) - \psi(w_1)}{w - w_1} = \frac{z - z_1}{\varphi(z) - \varphi(z_1)}.$$

By (1), $z \to z_1$ when $w \to w_1$. Hence (2) implies that $\psi'(w_1) = 1/\varphi'(z_1)$. Thus $\psi \in H(W)$. ////

10.31 Definition For $m = 1, 2, 3, \ldots$, we denote the "m^{th} power function" $z \to z^m$ by π_m.

Each $w \neq 0$ is $\pi_m(z)$ for precisely m distinct values of z: If $w = re^{i\theta}$, $r > 0$, then $\pi_m(z) = w$ if and only if $z = r^{1/m}e^{i(\theta + 2k\pi)/m}$, $k = 1, \ldots, m$.

Note also that each π_m is an open mapping: If V is open and does not contain 0, then $\pi_m(V)$ is open by Theorem 10.30. On the other hand, $\pi_m(D(0; r)) = D(0; r^m)$.

Compositions of open mappings are clearly open. In particular, $\pi_m \circ \varphi$ is open, by Theorem 10.30, if φ' has no zero. The following theorem (which contains the more detailed version of the open mapping theorem that was mentioned prior to Lemma 10.29) states a converse: Every nonconstant holomorphic function in a region is locally of the form $\pi_m \circ \varphi$, except for an additive constant.

10.32 Theorem *Suppose Ω is a region, $f \in H(\Omega)$, f is not constant, $z_0 \in \Omega$, and $w_0 = f(z_0)$. Let m be the order of the zero which the function $f - w_0$ has at z_0.*

Then there exists a neighborhood V of z_0, $V \subset \Omega$, and there exists $\varphi \in H(V)$, such that

(*a*) *$f(z) = w_0 + [\varphi(z)]^m$ for all $z \in V$,*
(*b*) *φ' has no zero in V and φ is an invertible mapping of V onto a disc $D(0; r)$.*

Thus $f - w_0 = \pi_m \circ \varphi$ in V. It follows that f is an exactly m-to-1 mapping of $V - \{z_0\}$ onto $D'(w_0; r^m)$, and that each $w_0 \in f(\Omega)$ is an interior point of $f(\Omega)$. Hence $f(\Omega)$ is open.

PROOF Without loss of generality we may assume that Ω is a convex neighborhood of z_0 which is so small that $f(z) \neq w_0$ if $z \in \Omega - \{z_0\}$. Then

$$f(z) - w_0 = (z - z_0)^m g(z) \qquad (z \in \Omega) \tag{1}$$

for some $g \in H(\Omega)$ which has no zero in Ω. Hence $g'/g \in H(\Omega)$. By Theorem 10.14, $g'/g = h'$ for some $h \in H(\Omega)$. The derivative of $g \cdot \exp(-h)$ is 0 in Ω. If h is modified by the addition of a suitable constant, it follows that $g = \exp(h)$. Define

$$\varphi(z) = (z - z_0)\exp\frac{h(z)}{m} \qquad (z \in \Omega). \tag{2}$$

Then (*a*) holds, for all $z \in \Omega$.

Also, $\varphi(z_0) = 0$ and $\varphi'(z_0) \neq 0$. The existence of an open set V that satisfies (*b*) follows now from Theorem 10.30. This completes the proof. ////

The next theorem is really contained in the preceding results, but it seems advisable to state it explicitly.

10.33 Theorem *Suppose Ω is a region, $f \in H(\Omega)$, and f is one-to-one in Ω. Then $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse of f is holomorphic.*

PROOF If $f'(z_0)$ were 0 for some $z_0 \in \Omega$, the hypotheses of Theorem 10.32 would hold with some $m > 1$, so that f would be m-to-1 in some deleted neighborhood of z_0. Now apply part (*c*) of Theorem 10.30. ////

Note that the converse of Theorem 10.33 is false: If $f(z) = e^z$, then $f'(z) \neq 0$ for every z, but f is not one-to-one in the whole complex plane.

The Global Cauchy Theorem

Before we state and prove this theorem, which will remove the restriction to convex regions that was imposed in Theorem 10.14, it will be convenient to add a little to the integration apparatus which was sufficient up to now. Essentially, it is a matter of no longer restricting ourselves to integrals over single paths, but to consider finite "sums" of paths instead. A simple instance of this occurred already in Sec. 10.9(*c*).

10.34 Chains and Cycles Suppose $\gamma_1, \cdots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Each γ_i induces a linear functional $\tilde{\gamma}_i$ on the vector space $C(K)$, by the formula

$$\tilde{\gamma}_i(f) = \int_{\gamma_i} f(z)\,dz. \tag{1}$$

Define

$$\tilde{\Gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n. \tag{2}$$

Explicitly, $\tilde{\Gamma}(f) = \tilde{\gamma}_1(f) + \cdots + \tilde{\gamma}_n(f)$ for all $f \in C(K)$. The relation (2) suggests that we introduce a "formal sum"

$$\Gamma = \gamma_1 \dotplus \cdots \dotplus \gamma_n \tag{3}$$

and define

$$\int_\Gamma f(z)\,dz = \tilde{\Gamma}(f). \tag{4}$$

Then (3) is merely an abbreviation for the statement

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz \qquad (f \in C(K)). \tag{5}$$

Note that (5) serves as the definition of its left side.

The objects Γ so defined are called *chains*.

If Ω is an open set, if $\gamma_i^* \subset \Omega$ for $1 \le i \le n$, and if Γ is defined by (3), then Γ is a *chain* in Ω.

If (3) holds and each γ_i is a closed path, then Γ is called a *cycle*.

A chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 \dotplus \cdots \dotplus \gamma_n = \delta_1 \dotplus \cdots \dotplus \delta_k$$

means simply that

$$\sum_i \int_{\gamma_i} f(z)\,dz = \sum_j \int_{\delta_j} f(z)\,dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$. In particular, a cycle may very well be represented as a sum of paths that are not closed.

If (3) holds, we define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*. \tag{6}$$

If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the *index* of α with respect to Γ by

$$\mathrm{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-\alpha}, \tag{7}$$

just as in Theorem 10.10. Obviously, (3) implies

$$\mathrm{Ind}_\Gamma(\alpha) = \sum_{i=1}^{n} \mathrm{Ind}_{\gamma_i}(\alpha). \tag{8}$$

If each γ_i in (3) is replaced by its opposite path (see Sec. 10.8), the resulting chain will be denoted by $-\Gamma$. Then

$$\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz \qquad (f \in C(\Gamma^*)). \tag{9}$$

In particular, $\mathrm{Ind}_{-\Gamma}(\alpha) = -\mathrm{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$.

Finally, note that chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals: The statement $\Gamma = \Gamma_1 \dotplus \Gamma_2$ means

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz \tag{10}$$

for every $f \in C(\Gamma_1^* \cup \Gamma_2^*)$.

10.35 Cauchy's Theorem *Suppose $f \in H(\Omega)$, where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω that satisfies*

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \qquad \textit{for every } \alpha \textit{ not in } \Omega, \tag{1}$$

then

$$f(z) \cdot \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw \qquad \textit{for } z \in \Omega - \Gamma^* \tag{2}$$

and

$$\int_\Gamma f(z)\,dz = 0. \tag{3}$$

If Γ_0 and Γ_1 are cycles in Ω such that

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha) \qquad \textit{for every } \alpha \textit{ not in } \Omega, \tag{4}$$

then

(5) $$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

PROOF The function g defined in $\Omega \times \Omega$ by

(6) $$g(z, w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases}$$

is continuous in $\Omega \times \Omega$ (Lemma 10.29). Hence we can define

(7) $$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z, w)\,dw \qquad (z \in \Omega).$$

For $z \in \Omega - \Gamma^*$, the Cauchy formula (2) is clearly equivalent to the assertion that

(8) $$h(z) = 0.$$

To prove (8), let us first prove that $h \in H(\Omega)$. Note that g is uniformly continuous on every compact subset of $\Omega \times \Omega$. If $z \in \Omega$, $z_n \in \Omega$, and $z_n \to z$, it follows that $g(z_n, w) \to g(z, w)$ uniformly for $w \in \Gamma^*$ (a compact subset of Ω). Hence $h(z_n) \to h(z)$. This proves that h is continuous in Ω. Let Δ be a closed triangle in Ω. Then

(9) $$\int_{\partial\Delta} h(z)\,dz = \frac{1}{2\pi i} \int_\Gamma \left(\int_{\partial\Delta} g(z, w)\,dz \right) dw.$$

For each $w \in \Omega$, $z \to g(z, w)$ is holomorphic in Ω. (The singularity at $z = w$ is removable.) The inner integral on the right side of (9) is therefore 0 for every $w \in \Gamma^*$. Morera's theorem shows now that $h \in H(\Omega)$.

Next, we let Ω_1 be the set of all complex numbers z for which $\text{Ind}_\Gamma(z) = 0$, and we define

(10) $$h_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\,dw \qquad (z \in \Omega_1).$$

If $z \in \Omega \cap \Omega_1$, the definition of Ω_1 makes it clear that $h_1(z) = h(z)$. Hence there is a function $\varphi \in H(\Omega \cup \Omega_1)$ whose restriction to Ω is h and whose restriction to Ω_1 is h_1.

Our hypothesis (1) shows that Ω_1 contains the complement of Ω. Thus φ is an entire function. Ω_1 also contains the unbounded component of the complement of Γ^*, since $\text{Ind}_\Gamma(z)$ is 0 there. Hence

$$\lim_{|z|\to\infty} \varphi(z) = \lim_{|z|\to\infty} h_1(z) = 0. \tag{11}$$

Liouville's theorem implies now that $\varphi(z) = 0$ for every z. This proves (8), and hence (2).

To deduce (3) from (2), pick $a \in \Omega - \Gamma^*$ and define $F(z) = (z - a)f(z)$. Then

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z-a}\,dz = F(a)\cdot \text{Ind}_\Gamma(a) = 0, \tag{12}$$

because $F(a) = 0$.

Finally, (5) follows from (4) if (3) is applied to the cycle $\Gamma = \Gamma_1 - \Gamma_0$. This completes the proof.

10.36 Remarks

(*a*) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of Theorem 10.14 to $f(z) = (z - \alpha)^{-1}$ shows that $\text{Ind}_\gamma(\alpha) = 0$. Hypothesis (1) of Theorem 10.35 is therefore satisfied by every cycle in Ω if Ω is convex. This shows that Theorem 10.35 generalizes Theorems 10.14 and 10.15.

(*b*) The last part of Theorem 10.35 shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the value of the integral. For example, let Ω be the plane with three disjoint closed discs D_i removed. If $\Gamma, \gamma_1, \gamma_2, \gamma_3$ are positively oriented circles in Ω such that Γ surrounds $D_1 \cup D_2 \cup D_3$ and γ_i surrounds D_i but not D_j for $j \neq i$, then

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{3}\int_{\gamma_i} f(z)\,dz$$

for every $f \in H(\Omega)$.

(*c*) In order to apply Theorem 10.35, it is desirable to have a reasonably efficient method of finding the index of a point with respect to a closed path. The following theorem does this for all paths that occur in practice. It says, essentially, that the index *increases* by 1 when the path is crossed "from right to left." If we recall that $\text{Ind}_\gamma(\alpha) = 0$ if α is in the unbounded component of the complement W of γ^*, we can then successively determine $\text{Ind}_\gamma(\alpha)$ in the other components of W, provided that W has only finitely many components and that γ traverses no arc more than once.

10.37 Theorem *Suppose γ is a closed path in the plane, with parameter interval $[\alpha, \beta]$. Suppose $\alpha < u < v < \beta$, a and b are complex numbers, $|b| = r > 0$, and*

(i) $\gamma(u) = a - b$, $\gamma(v) = a + b$,

(ii) $|\gamma(s) - a| < r$ *if and only if* $u < s < v$,

(iii) $|\gamma(s) - a| = r$ *if and only if* $s = u$ *or* $s = v$.

Assume furthermore that $D(a; r) - \gamma^*$ *is the union of two regions,* D_+ *and* D_-, *labeled so that* $a + bi \in \overline{D}_+$ *and* $a - bi \in \overline{D}_-$. *Then*

$$\text{Ind}_\gamma(z) = 1 + \text{Ind}_\gamma(w)$$

if $z \in D_+$ *and* $w \in D_-$.

As $\gamma(t)$ traverses $D(a; r)$ from $a - b$ to $a + b$, D_- is "on the right" and D_+ is "on the left" of the path.

PROOF To simplify the writing, reparametrize γ so that $u = 0$ and $v = \pi$. Define

$$C(s) = a - be^{is} \qquad (0 \le s \le 2\pi)$$

$$f(s) = \begin{cases} C(s) & (0 \le s \le \pi) \\ \gamma(2\pi - s) & (\pi \le s \le 2\pi) \end{cases}$$

$$g(s) = \begin{cases} \gamma(s) & (0 \le s \le \pi) \\ C(s) & (\pi \le s \le 2\pi) \end{cases}$$

$$h(s) = \begin{cases} \gamma(s) & (\alpha \le s \le 0 \text{ or } \pi \le s \le \beta) \\ C(s) & (0 \le s \le \pi). \end{cases}$$

Since $\gamma(0) = C(0)$ and $\gamma(\pi) = C(\pi)$, f, g, and h are closed paths.

If $E \subset \overline{D}(a; r)$, $|\zeta - a| = r$, and $\zeta \notin E$, then E lies in the disc $D(2a - \zeta; 2r)$ which does not contain ζ. Apply this to $E = g^*$, $\zeta = a - bi$, to see [from Remark 10.36(a)] that $\text{Ind}_g(a - bi) = 0$. Since $\overline{D}_-$ is connected and D_- does not intersect g^*, it follows that

(1) $$\text{Ind}_g(w) = 0 \qquad \textit{if } w \in D_-.$$

The same reasoning shows that

(2) $$\text{Ind}_f(z) = 0 \qquad \textit{if } z \in D_+.$$

We conclude that

$$\begin{aligned} \text{Ind}_\gamma(z) &= \text{Ind}_h(z) = \text{Ind}_h(w) \\ &= \text{Ind}_C(w) + \text{Ind}_\gamma(w) = 1 + \text{Ind}_\gamma(w). \end{aligned}$$

The first of these equalities follows from (2), since $h = \dot{\gamma} + f$. The second holds because z and w lie in $D(a; r)$, a connected set which does not intersect h^*. The third follows from (1), since $h \dotplus g = C \dotplus \gamma$, and the fourth is a consequence of Theorem 10.11. This completes the proof. ////

We now turn to a brief discussion of another topological concept that is relevant to Cauchy's theorem.

10.38 Homotopy Suppose γ_0 and γ_1 are closed curves in a topological space X, both with parameter interval $I = [0, 1]$. We say that γ_0 and γ_1 are *X-homotopic* if there is a continuous mapping H of the unit square $I^2 = I \times I$ into X such that

$$(1) \qquad H(s, 0) = \gamma_0(s), \qquad H(s, 1) = \gamma_1(s), \qquad H(0, t) = H(1, t)$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s, t)$. Then (1) defines a *one-parameter family of closed curves γ_t in X*, which *connects* γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.

If γ_0 is X-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is *null-homotopic in X*. If X is connected and if every closed curve in X is null-homotopic, X is said to be *simply connected*.

For example, every convex region Ω is simply connected. To see this, let γ_0 be a closed curve in Ω, fix $z_1 \in \Omega$, and define

$$(2) \qquad H(s, t) = (1 - t)\gamma_0(s) + tz_1 \qquad (0 \le s \le 1, \quad 0 \le t \le 1).$$

Theorem 10.40 will show that condition (4) of Cauchy's theorem 10.35 holds whenever Γ_0 and Γ_1 are Ω-homotopic closed paths. As a special case of this, note that *condition* (1) *of Theorem* 10.35 *holds for every closed path* Γ *in* Ω *if* Ω *is simply connected.*

10.39 Lemma *If γ_0 and γ_1 are closed paths with parameter interval* $[0, 1]$, *if α is a complex number, and if*

$$(1) \qquad |\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \qquad (0 \le s \le 1)$$

then $\text{Ind}_{\gamma_1}(\alpha) = \text{Ind}_{\gamma_0}(\alpha)$.

PROOF Note first that (1) implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$. Hence one can define $\gamma = (\gamma_1 - \alpha)/(\gamma_0 - \alpha)$. Then

$$(2) \qquad \frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1 - \alpha} - \frac{\gamma_0'}{\gamma_0 - \alpha}$$

and $|1-\gamma| < 1$, by (1). Hence $\gamma^* \subset D(1;1)$, which implies that $\mathrm{Ind}_\gamma(0) = 0$. Integration of (2) over $[0,1]$ now gives the desired result. ////

10.40 Theorem *If Γ_0 and Γ_1 are Ω-homotopic closed paths in a region Ω, and if $\alpha \notin \Omega$, then*

(1) $$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha).$$

PROOF By definition, there is a continuous $H: I^2 \to \Omega$ such that

(2) $$H(s,0) = \Gamma_0(s), \qquad H(s,1) = \Gamma_1(s), \qquad H(0,t) = H(1,t).$$

Since I^2 is compact, so is $H(I^2)$. Hence there exists $\epsilon > 0$ such that

(3) $$|\alpha - H(s,t)| > 2\epsilon \qquad \text{if} \qquad (s,t) \in I^2.$$

Since H is uniformly continuous, there is a positive integer n such that

(4) $$|H(s,t) - H(s',t')| < \epsilon \qquad \text{if} \qquad |s-s'| + |t-t'| \le 1/n.$$

Define polygonal closed paths $\gamma_0, \ldots, \gamma_n$ by

(5) $$\gamma_k(s) = H\left(\frac{i}{n}, \frac{k}{n}\right)(ns + 1 - i) + H\left(\frac{i-1}{n}, \frac{k}{n}\right)(i - ns)$$

if $i - 1 \le ns \le i$ and $i = 1, \ldots, n$. By (4) and (5),

(6) $$|\gamma_k(s) - H(s, k/n)| < \epsilon \qquad (k = 0, \ldots, n; 0 \le s \le 1).$$

In particular, taking $k = 0$ and $k = n$,

(7) $$|\gamma_0(s) - \Gamma_0(s)| < \epsilon, \qquad |\gamma_n(s) - \Gamma_1(s)| < \epsilon.$$

By (6) and (3),

(8) $$|\alpha - \gamma_k(s)| > \epsilon \qquad (k = 0, \ldots, n; 0 \le s \le 1).$$

On the other hand, (4) and (5) also give

(9) $$|\gamma_{k-1}(s) - \gamma_k(s)| < \epsilon \qquad (k = 1, \ldots, n; 0 \le s \le 1).$$

Now it follows from (7), (8), (9), and $n + 2$ applications of Lemma 10.39 that α has the same index with respect to each of the paths $\Gamma_0, \gamma_0, \gamma_1, \ldots, \gamma_n, \Gamma_1$. This proves the theorem. ////

Note: If $\Gamma_t(s) = H(s,t)$ in the preceding proof, then each Γ_t is a closed *curve*, but not necessarily a *path*, since H is not assumed to be differentiable. The paths γ_k were introduced for this reason. Another (and perhaps more satisfactory) way to circumvent this difficulty is to extend the definition of index to closed curves. This is sketched in Exercise 28.

The Calculus of Residues

10.41 Definition A function f is said to be *meromorphic* in an open set Ω if there is a set $A \subset \Omega$ such that

(*a*) A has no limit point in Ω,
(*b*) $f \in H(\Omega - A)$,
(*c*) f has a pole at each point of A.

Note that the possibility $A = \varnothing$ is not excluded. Thus every $f \in H(\Omega)$ is meromorphic in Ω.

Note also that (*a*) implies that no compact subset of Ω contains infinitely many points of A, and that A is therefore at most countable.

If f and A are as above, if $a \in A$, and if

$$Q(z) = \sum_{k=1}^{m} c_k(z-a)^{-k} \tag{1}$$

is the principal part of f at a, as defined in Theorem 10.21 (i.e., if $f - Q$ has a removable singularity a), then the number c_1 is called the *residue* of f at a:

$$c_1 = \operatorname{Res}(f; a). \tag{2}$$

If Γ is a cycle and $a \notin \Gamma^*$, (1) implies

$$\frac{1}{2\pi i}\int_\Gamma Q(z)\,dz = c_1 \operatorname{Ind}_\Gamma(a) = \operatorname{Res}(Q; a)\operatorname{Ind}_\Gamma(a). \tag{3}$$

This very special case of the following theorem will be used in its proof.

10.42 The Residue Theorem *Suppose f is a meromorphic function in Ω. Let A be the set of points in Ω at which f has poles. If Γ is a cycle in $\Omega - A$ such that*

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \qquad \textit{for all} \qquad \alpha \notin \Omega, \tag{1}$$

then

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{a\in A} \operatorname{Res}(f; a)\operatorname{Ind}_\Gamma(a). \tag{2}$$

PROOF Let $B = \{a \in A : \operatorname{Ind}_\Gamma(a) \neq 0\}$. Let W be the complement of Γ^*. Then $\operatorname{Ind}_\Gamma(z)$ is constant in each component V of W. If V is unbounded, or if V intersects Ω^c, (1) implies that $\operatorname{Ind}_\Gamma(z) = 0$ for every $z \in V$. Since A has no limit point in Ω, we conclude that B is a finite set.

The sum in (2), though formally infinite, is therefore actually finite.

Let $a_1, \ldots, a_n$ be the points of B, let $Q_1, \ldots, Q_n$ be the principal parts of f at $a_1, \ldots, a_n$, and put $g = f - (Q_1 + \cdots + Q_n)$. (If $B = \varnothing$, a possibility which is not excluded, then $g = f$.) Put $\Omega_0 = \Omega - (A - B)$. Since g has removable singularities at $a_1, \ldots, a_n$, Theorem 10.35, applied to the function g and the open set Ω_0, shows that

(3) $$\int_\Gamma g(z)\,dz = 0.$$

Hence

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \frac{1}{2\pi i}\int_\Gamma Q_k(z)\,dz = \sum_{k=1}^{n} \operatorname{Res}(Q_k; a_k)\operatorname{Ind}_\Gamma(a_k),$$

and since f and Q_k have the same residue at a_k, we obtain (2). ////

We conclude this chapter with two typical applications of the residue theorem. The first one concerns zeros of holomorphic functions, the second is the evaluation of a certain integral.

10.43 Theorem *Suppose γ is a closed path in a region Ω, such that $\operatorname{Ind}_\gamma(\alpha) = 0$ for every α not in Ω. Suppose also that $\operatorname{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in \Omega - \gamma^*$, and let Ω_1 be the set of all α with $\operatorname{Ind}_\gamma(\alpha) = 1$.*

For any $f \in H(\Omega)$ let N_f be the number of zeros of f in Ω_1, counted according to their multiplicities.

(*a*) *If $f \in H(\Omega)$ and f has no zeros on γ^* then*

(1) $$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \operatorname{Ind}_\Gamma(0)$$

where $\Gamma = f \circ \gamma$.

(*b*) *If also $g \in H(\Omega)$ and*

(2) $$|f(z) - g(z)| < |f(z)| \qquad \textit{for all } z \in \gamma^*$$

then $N_g = N_f$.

Part (*b*) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in Ω_1 if they are close together on the boundary of Ω_1, as specified by (2).

PROOF Put $\varphi = f'/f$, a meromorphic function in Ω. If $a \in \Omega$ and f has a zero of order $m = m(a)$ at a, then $f(z) = (z - a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a. In $V - \{a\}$,

$$\varphi(z) = \frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}. \tag{3}$$

Thus

$$\operatorname{Res}(\varphi; a) = m(a). \tag{4}$$

Let $A = \{a \in \Omega_1 : f(a) = 0\}$. If our assumptions about the index of γ are combined with the residue theorem one obtains

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{a \in A} \operatorname{Res}(\varphi; a) = \sum_{a \in A} m(a) = N_f.$$

This proves one half of (1). The other half is a matter of direct computation:

$$\begin{aligned}\operatorname{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Gamma'(s)}{\Gamma(s)}\,ds \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)\,ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.\end{aligned}$$

The parameter interval of γ was here taken to be $[0, 2\pi]$.

Next, (2) shows that g has no zero on γ^*. Hence (1) holds with g in place of f. Put $\Gamma_0 = g \circ \gamma$. Then it follows from (1), (2), and Lemma 10.39 that

$$N_g = \operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_\Gamma(0) = N_f. \qquad ////$$

10.44 Problem *For real t, find the limit, as $A \to \infty$, of*

$$\int_{-A}^{A} \frac{\sin x}{x} e^{ixt}\,dx. \tag{1}$$

SOLUTION Since $z^{-1} \cdot \sin z \cdot e^{itz}$ is entire, its integral over $[-A, A]$ equals that over the path Γ_A obtained by going from $-A$ to -1 along the real axis, from -1 to 1 along the lower half of the unit circle, and from 1 to A along the real axis. This follows from Cauchy's theorem. Γ_A avoids the origin, and we may therefore use the identity

$$2i \sin z = e^{iz} - e^{-iz}$$

to see that (1) equals $\varphi_A(t+1) - \varphi_A(t-1)$, where

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi i}\int_{\Gamma_A} \frac{e^{isz}}{z}\,dz. \tag{2}$$

Complete Γ_A to a closed path in two ways: First, by the semicircle from A to $-Ai$ to $-A$; secondly, by the semicircle from A to Ai to $-A$. The function e^{isz}/z has a single pole, at $z = 0$, where its residue is 1. It follows that

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi}\int_{-\pi}^{0} \exp(isAe^{i\theta})\,d\theta \tag{3}$$

and

$$\frac{1}{\pi}\varphi_A(s) = 1 - \frac{1}{2\pi}\int_{0}^{\pi} \exp(isAe^{i\theta})\,d\theta. \tag{4}$$

Note that

$$|\exp(isAe^{i\theta})| = \exp(-As\sin\theta), \tag{5}$$

and that this is < 1 and tends to 0 as $A \to \infty$ if s and $\sin\theta$ have the same sign. The dominated convergence theorem shows therefore that the integral in (3) tends to 0 if $s < 0$, and the one in (4) tends to 0 if $s > 0$. Thus

$$\lim_{A\to\infty} \varphi_A(s) = \begin{cases} \pi & \text{if } s > 0, \\ 0 & \text{if } s < 0, \end{cases} \tag{6}$$

and if we apply (6) to $s = t + 1$ and to $s = t - 1$, we get

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin x}{x} e^{itx}\,dx = \begin{cases} \pi & \text{if } -1 < t < 1, \\ 0 & \text{if } |t| > 1. \end{cases} \tag{7}$$

Since $\varphi_A(0) = \pi/2$, the limit in (7) is $\pi/2$ when $t = \pm 1$. ////

Note that (7) gives the Fourier transform of $\sin x/x$. We leave it as an exercise to check the result against the inversion theorem.

Exercises

1 The following fact was tacitly used in this chapter: If A and B are disjoint subsets of the plane, if A is compact, and if B is closed, then there exists a $\delta > 0$ such that $|\alpha - \beta| \geq \delta$ for all $\alpha \in A$ and $\beta \in B$. Prove this, with an arbitrary metric space in place of the plane.

2 At the end of Sec. 10.8 occurs a definition of the length of a path. Does this agree with the definition given in Exercise 10, Chap. 8?

3 Suppose f and g are entire functions, and $|f(z)| \leq |g(z)|$ for every z. What conclusion can you draw?

4 Suppose f is an entire function, and

$$|f(z)| \leq A + B|z|^k$$

for all z, where A, B, and k are positive numbers. Prove that f must be a polynomial.

5 Suppose $\{f_n\}$ is a uniformly bounded sequence of holomorphic functions in Ω such that $\{f_n(z)\}$ converges for every $z \in \Omega$. Prove that the convergence is uniform on every compact subset of Ω.

Hint: Apply the dominated convergence theorem to the Cauchy formula for $f_n - f_m$.

6 There is a region Ω such that $\exp(\Omega) = D(1; 1)$. Show that exp is one-to-one in Ω, but that there are many such Ω. Fix one, and define $\log z$, for $|z - 1| < 1$, to be that $w \in \Omega$ for which $e^w = z$. Prove that $\log'(z) = 1/z$. Find the coefficients a_n in

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n(z-1)^n$$

and hence find the coefficients c_n in the expansion

$$\log z = \sum_{n=0}^{\infty} c_n(z-1)^n.$$

In what other discs can this be done?

7 If $f \in H(\Omega)$, the Cauchy formula for the derivatives of f,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \qquad (n = 1, 2, 3, \ldots)$$

is valid under certain conditions on z and Γ. State these, and prove the formula.

8 Suppose P and Q are polynomials, the degree of Q exceeds that of P by at least 2, and the rational function $R = P/Q$ has no pole on the real axis. Prove that the integral of R over $(-\infty, \infty)$ is $2\pi i$ times the sum of the residues of R in the upper half plane. [Replace the integral over $(-A, A)$ by one over a suitable semicircle, and apply the residue theorem.] What is the analogous statement for the lower half plane? Use this method to compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx.$$

9 Compute $\int_{-\infty}^{\infty} e^{itx}/(1+x^2)\,dx$ for real t, by the method described in Exercise 8. Check your answer against the inversion theorem for Fourier transforms.

10 Let γ be the positively oriented unit circle, and compute

$$\frac{1}{2\pi i}\int_{\gamma} \frac{e^z - e^{-z}}{z^4}\,dz.$$

11 Suppose α is a complex number, $|\alpha| \neq 1$, and compute

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$$

by integrating $(z-\alpha)^{-1}(z-1/\alpha)^{-1}$ over the unit circle.

12 Compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{itx}\,dx \qquad (\text{for real } t).$$

13 Compute

$$\int_0^{\infty} \frac{dx}{1+x^n} \qquad (n = 2, 3, 4, \ldots).$$

[For even n, the method of Exercise 8 can be used. However, a different path can be chosen, which simplifies the computation and which also works for odd n: from 0 to R to $R\exp(2\pi i/n)$ to 0.]

Answer: $(\pi/n)/\sin(\pi/n)$.

14 Suppose Ω_1 and Ω_2 are plane regions, f and g are nonconstant complex functions defined in Ω_1 and Ω_2, respectively, and $f(\Omega_1) \subset \Omega_2$. Put $h = g \circ f$. If f and g are holomorphic, we know that h is holomorphic. Suppose we know that f and h are holomorphic. Can we conclude anything about g? What if we know that g and h are holomorphic?

15 Suppose Ω is a region, $\varphi \in H(\Omega)$, φ' has no zero in Ω, $f \in H(\varphi(\Omega))$, $g = f \circ \varphi$, $z_0 \in \Omega$, and $w_0 = \varphi(z_0)$. Prove that if f has a zero of order m at w_0, then g also has a zero of order m at z_0. How is this modified if φ' has a zero of order k at z_0?

16 Suppose μ is a complex measure on a measure space X, Ω is an open set in the plane, φ is a bounded function on $\Omega \times X$ such that $\varphi(z, t)$ is a measurable function of t, for each $z \in \Omega$, and $\varphi(z, t)$ is holomorphic in Ω, for each $t \in X$. Define

$$f(z) = \int_X \varphi(z, t)\, d\mu(t)$$

for $z \in \Omega$. Prove that $f \in H(\Omega)$. *Hint*: Show that to every compact $K \subset \Omega$ there corresponds a constant $M < \infty$ such that

$$\left| \frac{\varphi(z, t) - \varphi(z_0, t)}{z - z_0} \right| < M \qquad (z \text{ and } z_0 \in K, t \in X).$$

17 Determine the regions in which the following functions are defined and holomorphic:

$$f(z) = \int_0^1 \frac{dt}{1 + tz}, \qquad g(z) = \int_0^\infty \frac{e^{tz}}{1 + t^2} dt, \qquad h(z) = \int_{-1}^1 \frac{e^{tz}}{1 + t^2} dt.$$

Hint: Either use Exercise 16, or combine Morera's theorem with Fubini's.

18 Suppose $f \in H(\Omega)$, $\overline{D}(a; r) \subset \Omega$, γ is the positively oriented circle with center at a and radius r, and f has no zero on γ^*. For $p = 0$, the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} z^p \, dz$$

is equal to the number of zeros of f in $D(a; r)$. What is the value of this integral (in terms of the zeros of f) for $p = 1, 2, 3, \ldots$? What is the answer if z^p is replaced by any $\varphi \in H(\Omega)$?

19 Suppose $f \in H(U)$, $g \in H(U)$, and neither f nor g has a zero in U. If

$$\frac{f'}{f}\left(\frac{1}{n}\right) = \frac{g'}{g}\left(\frac{1}{n}\right) \qquad (n = 1, 2, 3, \ldots)$$

find another simple relation between f and g.

20 Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, none of the functions f_n has a zero in Ω, and $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Prove that either f has no zero in Ω or $f(z) = 0$ for all $z \in \Omega$.

If Ω' is a region that contains every $f_n(\Omega)$, and if f is not constant, prove that $f(\Omega) \subset \Omega'$.

21 Suppose $f \in H(\Omega)$, Ω contains the closed unit disc, and $|f(z)| < 1$ if $|z| = 1$. How many fixed points must f have in the disc? That is, how many solutions does the equation $f(z) = z$ have there?

22 Suppose $f \in H(\Omega)$, Ω contains the closed unit disc, $|f(z)| > 2$ if $|z| = 1$, and $f(0) = 1$. Must f have a zero in the unit disc?

23 Suppose $P_n(z) = 1 + z/1! + \cdots + z^n/n!$, $Q_n(z) = P_n(z) - 1$, where $n = 1, 2, 3, \ldots$. What can you say about the location of the zeros of P_n and Q_n for large n? Be as specific as you can.

24 Prove the following general form of Rouché's theorem: Let Ω be the interior of a compact set K in the plane. Suppose f and g are continuous on K and holomorphic in Ω, and $|f(z) - g(z)| < |f(z)|$ for all $z \in K - \Omega$. Then f and g have the same number of zeros in Ω.

25 Let A be the annulus $\{z : r_1 < |z| < r_2\}$, where r_1 and r_2 are given positive numbers.

(*a*) Show that the Cauchy formula

$$f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_1} + \int_{\gamma_2}\right)\frac{f(\zeta)}{\zeta - z}\,d\zeta$$

is valid under the following conditions: $f \in H(A)$,

$$r_1 + \epsilon < |z| < r_2 - \epsilon,$$

and

$$\gamma_1(t) = (r_1 + \epsilon)e^{-it}, \qquad \gamma_2(t) = (r_2 - \epsilon)e^{it} \qquad (0 \le t \le 2\pi).$$

(*b*) Show by means of (*a*) that every $f \in H(A)$ can be decomposed into a sum $f = f_1 + f_2$, where f_1 is holomorphic outside $\overline{D}(0; r_1)$ and $f_2 \in H(D(0; r_2))$; the decomposition is unique if we require that $f_1(z) \to 0$ as $|z| \to \infty$.

(*c*) Use this decomposition to associate with each $f \in H(A)$ its so-called "Laurent series"

$$\sum_{-\infty}^{\infty} c_n z^n$$

which converges to f in A. Show that there is only one such series for each f. Show that it converges to f uniformly on compact subsets of A.

(*d*) If $f \in H(A)$ and f is bounded in A, show that the components f_1 and f_2 are also bounded.

(*e*) How much of the foregoing can you extend to the case $r_1 = 0$ (or $r_2 = \infty$, or both)?

(*f*) How much of the foregoing can you extend to regions bounded by finitely many (more than two) circles?

26 It is required to expand the function

$$\frac{1}{1 - z^2} + \frac{1}{3 - z}$$

in a series of the form $\sum_{-\infty}^{\infty} c_n z^n$.

How many such expansions are there? In which region is each of them valid? Find the coefficients c_n explicitly for each of these expansions.

27 Suppose Ω is a horizontal strip, determined by the inequalities $a < y < b$, say. Suppose $f \in H(\Omega)$, and $f(z) = f(z+1)$ for all $z \in \Omega$. Prove that f has a Fourier expansion in Ω,

$$f(z) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n z},$$

which converges uniformly in $\{z: a + \epsilon \le y \le b - \epsilon\}$, for every $\epsilon > 0$. *Hint*: The map $z \to e^{2\pi i z}$ converts f to a function in an annulus.

Find the integral formulas by means of which the coefficients c_n can be computed from f.

28 Suppose Γ is a closed curve in the plane, with parameter interval $[0, 2\pi]$. Take $\alpha \notin \Gamma^*$. Approximate Γ uniformly by trigonometric polynomials Γ_n. Show that $\text{Ind}_{\Gamma_n}(\alpha) = \text{Ind}_{\Gamma_m}(\alpha)$ if m and n are sufficiently large. Define this common value to be $\text{Ind}_\Gamma(\alpha)$. Prove that the result does not depend on the choice of $\{\Gamma_n\}$; prove that Lemma 10.39 is now true for closed curves, and use this to give a different proof of Theorem 10.40.