RUELLE SPECTRUM OF LINEAR PSEUDO-ANOSOV MAPS

FRÉDÉRIC FAURE, SÉBASTIEN GOUËZEL AND ERWAN LANNEAU

Abstract. The Ruelle resonances of a dynamical system are spectral data describing the precise asymptotics of correlations. We classify them completely for a class of chaotic two-dimensional maps, the linear pseudo-Anosov maps, in terms of the action of the map on cohomology. As applications, we obtain a full description of the distributions which are invariant under the linear flow in the stable direction of such a linear pseudo-Anosov map, and we solve the cohomological equation for this flow.

Contents

Date: August 30, 2018.
We thank Corinna Ulcigrai, Mauro Artigiani and Giovanni Forni for their enlightening comments.

1. Introduction, statements of results

Ruelle resonances. Consider a map T on a smooth manifold X, preserving a probability measure μ. One feature that encapsulates a lot of information on its probabilistic behavior is the speed of decay of correlations. Consider two smooth functions f and g. Then one expects that $\int f \cdot g \circ T^n \,\mathrm{d}\mu$ converges to $(\int f \,\mathrm{d}\mu) \cdot (\int g \,\mathrm{d}\mu)$ if iterating the dynamics creates more and more independence – if this is the case, T is said to be mixing for the measure μ. Often, one can say more than just the mere convergence to 0 of the correlations $\int f \cdot g \circ T^n \,\mathrm{d}\mu - (\int f \,\mathrm{d}\mu) \cdot (\int g \,\mathrm{d}\mu)$, and this is important for applications. For instance, the central limit theorem for the Birkhoff sums $S_n f = \sum_{k=0}^{n-1} f \circ T^k$ of a function f with 0 average often follows from the summability of the correlations between f and $f \circ T^n$.

When T is very chaotic, the correlations tend exponentially fast to 0. It is sometimes possible to obtain the next few terms in their asymptotic expansion, in terms of the *Ruelle spectrum* (or *Ruelle resonances*) of the map.

Definition 1.1. *Let T be a map on a space X, preserving a probability measure μ. Consider a space of bounded functions $\mathcal{C}$ on X. Let I be a finite or countable set, let $\Lambda = (\lambda_i)_{i \in I}$ be a set of complex numbers with $|\lambda_i| \in (0, 1]$ such that for any $\varepsilon > 0$ there are only finitely many i with $|\lambda_i| \geqslant \varepsilon$, and let $(N_i)_{i \in I}$ be nonnegative integers. We say that T has the Ruelle spectrum $(\lambda_i)_{i \in I}$ with Jordan blocks dimension $(N_i)_{i \in I}$ on the space of functions $\mathcal{C}$ if, for any $f, g \in \mathcal{C}$ and for any $\varepsilon > 0$, there is an asymptotic expansion*

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \varepsilon} \sum_{j \leqslant N_i} \lambda_i^n n^j c_{i,j}(f, g) + o(\varepsilon^n),$$

where $c_{i,j}(f, g)$ are bilinear functions of f and g, that we suppose finite rank but non zero.

In other words, there is an asymptotic expansion for the correlations of functions in $\mathcal{C}$, up to an arbitrarily small exponential error. With this definition, it is clear that the Ruelle spectrum is an intrinsic object, only depending on T, μ and the space of functions $\mathcal{C}$. In general, one takes for $\mathcal{C}$ the space of C^∞ functions on a manifold.

As an example, assume that T is a C^∞ uniformly expanding map on a manifold and μ is its unique invariant probability measure in the Lebesgue measure class. Then the correlations of C^r functions admit an asymptotic expansion up to an exponential term ε_r^n, where ε_r tends to 0 when r tends to infinity. Hence, Definition 1.1 is not satisfied for $\mathcal{C} = C^r$, but it is satisfied for $\mathcal{C} = C^\infty(M)$. The same holds for Anosov maps, when μ is a Gibbs measure.

The first question one may ask is if it makes sense to talk about the Ruelle spectrum, i.e., if Definition 1.1 holds for some $\Lambda = (\lambda_i)_{i \in I}$. Virtually all proofs of such an abstract existence result follow from spectral considerations, exhibiting the λ_i as the spectrum of an operator associated to T, acting on a Banach space or a scale of Banach spaces. General spectral theorems taking advantage of compactness or quasi-compactness properties of this operator then imply that there is some set Λ for which Definition 1.1 holds (and moreover all elements of Λ have finite multiplicity), but without giving any information whatsoever on Λ in addition to the fact that it is discrete and at most countable – in particular, it is not guaranteed that Λ is not reduced to the eigenvalue 1, which is always a Ruelle resonance as one can see by taking $f = g = 1$. Indeed, if T is the doubling map $x \mapsto 2x \mod 1$ on the circle and $\mathcal{C} = C^\infty(\mathbb{S}^1)$, then there is no other resonance. In the same way, there is no other resonance for linear Anosov map of the torus (these facts are easy to check by computing the correlations using Fourier series). That Definition 1.1 holds is notably known for uniformly expanding and uniformly hyperbolic smooth maps, see [Rue90, BT07, GL08].

Once the answer to this first question is positive, there is a whole range of questions one may ask about Λ: is it reduced to $\{1\}$? is it infinite? are there asymptotics for $\mathrm{Card}(\Lambda \cap \{|z| \geqslant \varepsilon\})$ (possibly counted with multiplicities) when ε tends to 0? is it possible to describe explicitly Λ? The answers to these questions depend on the map under consideration. Let us only mention the results of Naud [Nau12] (for generic analytic expanding maps, there is nontrivial Ruelle spectrum, with density at 0 bounded below explicitly), Adam [Ada17] (the spectrum is generically non-empty for hyperbolic maps), Bandtlow-Jenkinson [BJ08] (upper bound for the density of Ruelle resonances at 0 in analytic expanding maps, extending previous results of Fried), Bandtlow-Just-Slipantschuk [BJS13, BJS17] (construction of expanding or hyperbolic maps for which the Ruelle spectrum is completely explicit), Dyatlov-Faure-Guillarmou [DFG15] (classification of the Ruelle resonances for the geodesic flow on compact hyperbolic manifolds in any dimension).

Our goal in this article is to investigate these questions for a class of maps of geometric origin, namely linear pseudo-Anosov maps. They are analogues of linear Anosov maps of the two-dimensional torus, but on higher genus surfaces. The difference with the torus case is that the expanding and contracting foliations have singularities. Apart from these singularities, the local picture is exactly the same as for linear Anosov maps of the torus (in particular, it is the same everywhere in the manifold). We will obtain a complete description of the Ruelle spectrum of linear pseudo-Anosov map. Then, using the philosophy of Giulietti-Liverani [GL14] that Ruelle resonances contain information on the translation flow along the stable manifold on the map, we will discuss consequences of these results on the vertical translation flow in translation surfaces supporting a pseudo-Anosov map. We will in particular obtain complete results on the set of distributions which are invariant under the vertical flow, and on smooth solutions to the cohomological equation, recovering in this case results due to Forni on generic translation surfaces [For97, For02, For07].

Linear pseudo-Anosov maps. There are several equivalent definitions of pseudo-Anosov maps (especially in terms of foliations carrying a transverse measure). We will use the following one in which the foliations have already been straightened (i.e., we use coordinates where the foliations are horizontal and vertical), in terms of half-translation surfaces (see e.g. [Zor06] for a nice survey on half-translation surfaces).

Definition 1.2. *Let M be a compact connected surface and let Σ be a finite subset of M. A half-translation structure on (M, Σ) is an atlas on $M - \Sigma$ for which the coordinate changes have the form $x \mapsto x + v$ or $x \mapsto -x + v$. Moreover, we require that around each point of Σ the half translation surface is isomorphic to a finite ramified cover of $\mathbb{R}^2/\pm \mathrm{Id}$ around 0.*

A half-translation surface carries a canonical complex structure: it is just the canonical complex structure in the charts away from Σ, which extends to the singularities. In particular, it also has a C^∞ structure, and it is orientable.

In a half-translation structure, the horizontal and vertical lines in the charts define two foliations of $M - \Sigma$, called the horizontal and vertical foliations. Of particular importance to us will be the case where the coordinate changes are of the form $x \mapsto x + v$. In this case, we say that M is a translation surface. Singularities are then finite ramified cover of $\mathbb{R}^2$ around 0. Moreover, the horizontal and vertical foliations carry a canonical orientation.

Definition 1.3. *Consider a half-translation structure on (M, Σ). A homeomorphism $T : M \to M$ is a linear pseudo-Anosov map for this structure if $T(\Sigma) = \Sigma$ and there exists $\lambda > 1$ such that, for any $x \in M - \Sigma$, one has in half-translation charts around x and Tx the equality $Ty = \begin{pmatrix} \pm\lambda & 0 \\ 0 & \pm\lambda^{-1} \end{pmatrix} y$, where the choice of signs depends on the choice of coordinate charts. We say that λ is the* expansion factor *of T.*

In other words, T sends horizontal segments to horizontal segments and vertical segments to vertical segments, expanding by λ in the horizontal direction and contracting by λ in the vertical direction. In particular, Lebesgue measure is invariant under T.

When M is a translation surface, there are two global signs ε_h and ε_v saying if T preserves or reverses the orientation of the horizontal and vertical foliations. The simplest case is when $\varepsilon_h = \varepsilon_v = 1$. In this case, T preserves the orientation of both foliations, and can be written in local charts as $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

While we obtain a complete description of the Ruelle spectrum in all situations (orientable foliations or not, ε_v and ε_h equal to 1 or -1), it is easier to explain in the simplest case of translation surfaces with $\varepsilon_v = \varepsilon_h = 1$. We will refer to this case as linear pseudo-Anosov maps preserving orientations. We will focus on this case in this introduction and most of the paper, and refer to Section 6 for the general situation (that we will deduce from the case of linear pseudo-Anosov maps preserving orientations).

In the definition of Ruelle resonances, there is a subtlety related to the choice of the space of functions $\mathcal{C}$ for which we want asymptotic expansions of the correlations. While it is clear that we want C^∞ functions away from the singularities, the requirements at the singularities are less obvious. Denote by $C_c^\infty(M - \Sigma)$ the space of C^∞ functions that vanish on a neighborhood of the singularities. This is the space we will use for definiteness.

Let T be a linear pseudo-Anosov map, preserving orientations, on a genus g translation surface M. Let λ be its expansion factor. As the local picture for T is the same everywhere, it should not be surprising that the only data influencing the Ruelle spectrum are of global nature, related to the action of T on the first cohomology group $H^1(M)$ (a vector space of dimension $2g$). By Thurston [Thu88], λ and λ^{-1} are two simple eigenvalues of $T^* : H^1(M) \to H^1(M)$ (the corresponding eigenvectors are the cohomology classes of the horizontal and the vertical foliations). The orthogonal subspace to these two cohomology classes has dimension $2g-2$, it is invariant under T^*, and the spectrum $\Xi = \{\mu_1, \dots, \mu_{2g-2}\}$ of T^* on this subspace is made of eigenvalues satisfying $\lambda^{-1} < |\mu_i| < \lambda$ for all i.

Here is our main theorem when T preserves orientations.

Theorem 1.4. *Let T be a linear pseudo-Anosov map preserving orientations on a genus g compact surface M, with expansion factor λ and singularity set Σ. Then T has a Ruelle spectrum on $\mathcal{C} = C_c^\infty(M - \Sigma)$ given as follows. First, there is a simple eigenvalue at 1. Denote by $\Xi = \{\mu_1, \dots, \mu_{2g-2}\}$ the spectrum of T^* on the orthogonal subspace to the classes of the horizontal and vertical foliations in $H^1(M)$. Then, for any i and for any integer $n \geqslant 1$, there is a Ruelle resonance at $\lambda^{-n}\mu_i$ of multiplicity n.*

Note that a complex number z may sometimes be written in different ways as $\lambda^{-n}\mu_i$ (for instance if the spectrum of T^* is not simple, i.e., if there is $i \neq j$ with $\mu_i = \mu_j$ – but it can also happen that there is $i \neq j$ with $\mu_i = \lambda^{-1}\mu_j$, which will lead to more superpositions). In this case, to get the multiplicity of z, one should add all the multiplicities from the theorem corresponding to the different possible decompositions.

Let us note that some nonzero functions can be orthogonal to all Ruelle resonances. For instance, if T lifts a linear Anosov map of the torus to a higher genus surface covering the torus, then the correlations of any two smooth functions lifted from the torus tend to 0 faster than any exponential, as this is the case in the torus.

A quick sketch of the proof. Before we discuss further results, we should explain briefly the strategy to prove Theorem 1.4. First, we want to show that Ruelle resonances make sense as in Definition 1.1. This part is classical. We introduce a scale of Banach spaces of distributions, denoted by $\mathcal{B}^{-k_h,k_v}$, which behaves well under the composition operator $\mathcal{T} : f \mapsto f \circ T$. The elements of $\mathcal{B}^{-k_h,k_v}$ are objects that can be integrated along horizontal segments against C^{k_h}-functions, and moreover have k_v vertical derivatives: this is an anisotropic Banach space, taking advantage of the contraction of T in the vertical direction and of its expansion in the horizontal direction, as is customary in the study of hyperbolic dynamics. On the technical level, the definition of $\mathcal{B}^{-k_h,k_v}$ is less involved than in many articles on hyperbolic dynamics (see for instance [GL08, BT07]), as we may take advantage of the fact that the stable and unstable directions are smooth – in this respect, it is closer to [Bal05, AG13]. The only additional difficulty compared to the literature is the singularities, but it turns out that they do not play any role in this part. Hence, we can prove that the essential spectral radius of $\mathcal{T}$ on $\mathcal{B}^{-k_h,k_v}$ is at most $\lambda^{-\min(k_h,k_v)}$. The existence of Ruelle resonances in the sense of Definition 1.1 readily follows. One important point we want to stress here is that, since we are interested in Ruelle resonances for functions in $C_c^\infty(M - \Sigma)$, we take for $\mathcal{B}^{-k_h,k_v}$ the closure of $C_c^\infty(M - \Sigma)$ for an anisotropic norm as described above. In particular, smooth functions are dense in $\mathcal{B}^{-k_h,k_v}$.

The second step in the proof is to show that the elements described in Theorem 1.4 belong to the set of Ruelle resonances or, equivalently, to the spectrum of $\mathcal{T}$ on $\mathcal{B}^{-k_h,k_v}$ when k_h and k_v are large enough. It is rather easy to show that 1 and $\lambda^{-1}\mu_i$ belong to the spectrum, by considering a smooth 1-form $\omega = \omega_x \,\mathrm{d}x + \omega_y \,\mathrm{d}y$ whose cohomology class is an eigenfunction for the iteration of T^*, and looking at the asymptotics of $\mathcal{T}^n\omega_x$ to obtain an element $f \in \mathcal{B}^{-k_h,k_v}$ with $\mathcal{T}f = \lambda^{-1}\mu_i f$. Then, one deduces that $\lambda^{-n}\mu_i$ also belongs to the spectrum, as $L_h^{n-1}f$ is an eigenfunction for this eigenvalue, where L_h denotes the derivative in the horizontal direction.

The most interesting part of the proof is to show that there is no other eigenvalue, and that the multiplicities are as stated in the theorem. For this, start from an eigenfunction $f \in \mathcal{B}^{-k_h,k_v}$ for an eigenvalue ρ. Denote by L_v the derivative in the vertical direction. Then $L_v^n f$ is an eigenfunction for the eigenvalue $\lambda^n\rho$. Since all eigenvalues have modulus at most 1, we deduce that $L_v^n f = 0$ for large enough n. Consider the last index n where $L_v^n f \neq 0$, and write $g = L_v^n f$. It is an eigenfunction, and $L_v g = 0$. If we can prove that the corresponding eigenvalue has the form $\lambda^{-k}\mu_i$ for some k and i, then we get $\rho = \lambda^{-(n+k)}\mu_i$, as desired. To summarize, it is enough to understand eigenfunctions that, additionally, satisfy $L_v g = 0$. For this, we introduce a cohomological interpretation of elements of $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$.

Heuristically, elements of $\mathcal{B}^{-k_h,k_v}$ can be integrated along horizontal segments by definition, so what really matters is not the distribution g, but the 1-current $g\,\mathrm{d}x$. (In the language of Forni [For02], elements g of $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$ are the vertically invariant distributions, see his Definition 6.4, while $g\,\mathrm{d}x$ is the corresponding basic current on M.) Formally, its differential is

$$\mathrm{d}(g\,\mathrm{d}x) = (\partial_x g\,\mathrm{d}x + \partial_y g\,\mathrm{d}y) \wedge \mathrm{d}x = -L_v g\,\mathrm{d}x \wedge \mathrm{d}y.$$

Hence, elements of $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$ give rise to closed currents, and have an associated cohomology class in $H^1(M)$ by de Rham Theorem (in fact, we do not use de Rham theorem directly, but a custom version suited for our needs that deals more carefully with the singularities). From the equality $\mathcal{T} g = \rho_g g$ one deduces that this class is an eigenfunction for T^* acting on $H^1(M)$, for the eigenvalue $\lambda\rho_g$. If the class is nonzero, we get that $\lambda\rho_g$ is one of the μ_i, and $\rho_g = \lambda^{-1}\mu_i$ as desired. If the class is zero, this means that $g\,\mathrm{d}x$ is itself the differential of a 0-current $\tilde{g}$. It turns out that $\tilde{g}$ belongs to our scale of Banach spaces, and is an eigenfunction for the eigenvalue $\lambda\rho_g$. One can then argue in this way by induction to show that all eigenvalues are of the form claimed in Theorem 1.4. There are additional difficulties related to the eigenvalue λ^{-1} of $T^* : H^1(M) \to H^1(M)$: it does not show up in the statement of Theorem 1.4, but this does not follow from the sketch we have just given. Moreover, getting the precise multiplicities requires further arguments, based on duality arguments and beyond this introduction.

Here is the precise description we get in the end, illustrated on Figure 1, assuming to simplify that μ_i is simple for $T^* : H^1(M) \to H^1(M)$ and that $\lambda^{-1}\mu_i$ is not an eigenvalue of T^*. Then the eigenvalue $\lambda^{-1}\mu_i$ for $\mathcal{T}$ is simple, and realized by a distribution f_i which is annihilated by L_v (i.e., it is invariant under vertical translation) and such that the cohomology class of $f_i\,\mathrm{d}x$ is the eigenfunction in $H^1(M)$ under T^*, for the eigenvalue μ_i. Denoting by E_α the generalized eigenspace associated to the eigenvalue α, then L_v is onto from $E_{\lambda^{-n-1}\mu_i}$ to $E_{\lambda^{-n}\mu_i}$, and its kernel is one-dimensional, equal to $L_h^n E_{\lambda^{-1}\mu_i}$. Therefore, there is a flag decomposition

$$\{0\} \subset L_h^n E_{\lambda^{-1}\mu_i} \subset L_h^{n-1} E_{\lambda^{-2}\mu_i} \subset \cdots \subset L_h^2 E_{\lambda^{-n+1}\mu_i} \subset L_h E_{\lambda^{-n}\mu_i} \subset E_{\lambda^{-n-1}\mu_i}, \tag{1.1}$$

in which the k-th term $L_h^{n+1-k} E_{\lambda^{-k}\mu_i}$ has dimension k, and is equal to $E_{\lambda^{-n-1}\mu_i} \cap \ker L_v^k$. This decomposition shows that the elements of $E_{\lambda^{-n-1}\mu_i}$ behave like polynomials of degree n when one moves along the vertical direction. Moreover, the decomposition (1.1) is invariant under the transfer operator $\mathcal{T}$, which is thus in upper triangular form with $\lambda^{-n-1}\mu_i$ on the diagonal. We do not know if there are genuine Jordan blocks, or a choice of basis for which $\mathcal{T}$ is diagonal. In particular, we do not identify in Theorem 1.4 the Jordan blocks dimension of the Ruelle resonances, in the sense of Definition 1.1. The decomposition (1.1) can also be interpreted in terms of the operator $N = L_h L_v$, which is nilpotent of order $n+1$ on the $n+1$-dimension space $E_{\lambda^{-n-1}\mu_i}$: the k-th term is the kernel of N^k, and also the image of N^{n+1-k}.

Invariant distributions for the vertical flow. The above description is a first step into the direction of classifying all distributions on $M - \Sigma$ which are invariant under the vertical flow. We will call such distributions vertically invariant, or L_v-annihilated, or sometimes L_v-invariant. It turns out that there is another family of such L_v-annihilated distributions, which do not show up in the Ruelle resonances and correspond to relative homology. They

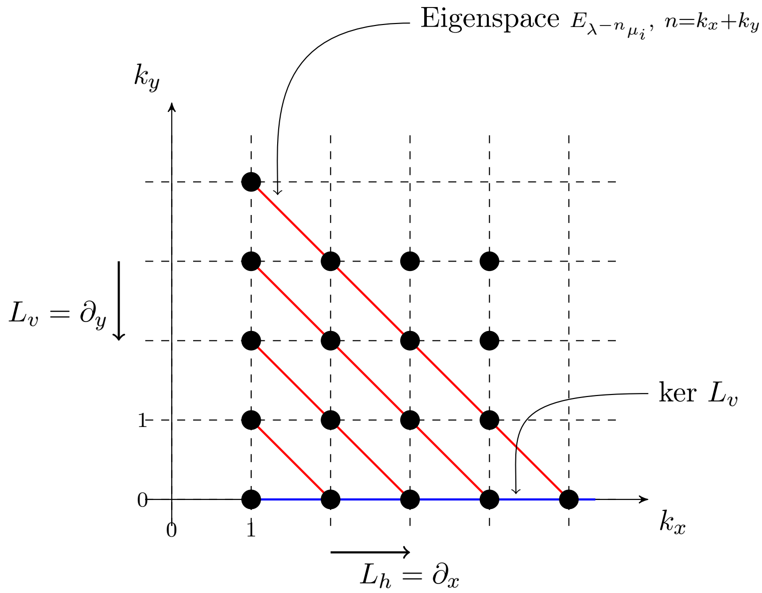

FIGURE 1. For a given eigenvalue μ_i of T^* ($\mu_i \in (\lambda^{-1}, \lambda)$), each black point of the lattice $(k_x, k_y)_{k_x \geqslant 1, k_y \geqslant 0}$ represents an independent Ruelle distribution $u_{(k_x,k_y)}$. In particular $f_i = u_{(1,0)}$. The eigenvalues of the transfer operator $\mathcal{T}$ are $\lambda^{-n}\mu_i$ with $n \geqslant 1$ and the associated eigenspace is $E_{\lambda^{-n}\mu_i} = \operatorname{Span}\left\{u_{k_x,k_y}, k_x + k_y = n\right\}$ with dimension n and represented by a diagonal red line. The operator $L_h \equiv \partial_x$ maps $u_{(k_x,k_y)}$ to $u_{(k_x+1,k_y)}$ and $L_v \equiv \partial_y$ maps $u_{(k_x,k_y)}$ to $u_{(k_x,k_y-1)}$. In particular the space $\ker L_v$ is represented by the first horizontal blue line $k_y = 0$.

belong to an extended space $\mathcal{B}^{-k_h,k_v}_{ext}$ defined like $\mathcal{B}^{-k_h,k_v}$ above, except that we do not restrict to the closure of the set of smooth functions. (In the language of Forni [For02], elements g of $\mathcal{B}^{-k_h,k_v}_{ext} \cap \ker L_v$ are the vertically quasi-invariant distributions, see his Definition 6.4, while $g\,\mathrm{d}x$ is the corresponding basic current on $M - \Sigma$). An example of an element of $\mathcal{B}^{-k_h,k_v}_{ext} \setminus \mathcal{B}^{-k_h,k_v}$ is as follows: consider a vertical segment Γ_σ ending at a singularity σ, a function ρ on this segment which is equal to 1 on a neighborhood of the singularity and to 0 on a neighborhood of the other endpoint of the segment, and define a distribution $\xi^{(0)}_\sigma$ by $\langle \xi^{(0)}_\sigma, f\rangle = \int_{\Gamma_\sigma} \rho(y) f(y)\,\mathrm{d}y$. In other words, the corresponding distribution on a horizontal segment I is equal to $\rho(x_I)\delta_{x_I}$ if I intersects Γ_σ at a point x_I, and 0 otherwise. It turns out that these are essentially the only elements of $\mathcal{B}^{-k_h,k_v}_{ext} \setminus \mathcal{B}^{-k_h,k_v}$: the latter has (almost) finite codimension in the former (see Proposition 4.4 for a precise statement). Note that if one chooses another vertical segment Γ'_σ ending on the same singularity, then the difference of the two distributions associated to Γ_σ and Γ'_σ belongs to $\mathcal{B}^{-k_h,k_v}$ when $k_h \geqslant 1$. The same happens if one replaces ρ by another function ρ'. Hence, modulo $\mathcal{B}^{-k_h,k_v}$, the distribution $\xi^{(0)}_\sigma$ is canonically defined and depends only on σ.

Proposition 1.5. *Let $k_h, k_v \geqslant 3$. For $\sigma \in \Sigma$, there exists a distribution $\xi_\sigma \in \mathcal{B}^{-k_h,k_v}_{ext}$ such that $\xi_\sigma - \xi_\sigma^{(0)} \in \mathcal{B}^{-k_h,k_v}$ and $L_v \xi_\sigma$ is the constant distribution equal to $1/\operatorname{Leb}(M)$. Therefore, the distributions $\xi_\sigma - \xi_{\sigma'}$ span a subspace of dimension $\operatorname{Card}\Sigma - 1$ of L_v-annihilated distributions.*

The full description of L_v-annihilated distributions is given in the next theorem. It states that all such distributions come from the distributions associated to Ruelle resonances described in Theorem 1.4, and additional spurious distributions coming from the singularities as in Proposition 1.5.

To give a precise statement, we have to deal carefully with the exceptional situation when there is an eigenvalue μ' of T^* such that $\mu = \lambda^{-1}\mu'$ is also an eigenvalue of T^*: then $L_h E_{\lambda^{-1}\mu'}$ is contained in $E_{\lambda^{-1}\mu}$, and there are some formal difficulties.

For each eigenvalue $\mu \in \Xi = \{\mu_1, \dots, \mu_{2g-2}\}$, there is a map $f \mapsto [f]$ from $E_{\lambda^{-1}\mu} \cap \ker L_v$ to $H^1(M)$, whose image is the generalized eigenspace associated to the eigenvalue μ of T^*. It is an isomorphism except in the exceptional situation above where it is onto, with a kernel equal to $L_h E_{\lambda^{-1}\mu'}$. Denote by $E^H_{\lambda^{-1}\mu}$ a subspace of $E_{\lambda^{-1}\mu} \cap \ker L_v$ which is sent isomorphically to the generalized eigenspace of T^* for the eigenvalue μ, i.e., $E^H_{\lambda^{-1}\mu} = E_{\lambda^{-1}\mu}$, except in the exceptional case above where $E^H_{\lambda^{-1}\mu}$ is a vector complement to $L_h E_{\lambda^{-1}\mu'}$ in $E_{\lambda^{-1}\mu} \cap \ker L_v$.

Theorem 1.6. *Let T be a linear pseudo-Anosov map preserving orientations on a genus g compact surface M, with expansion factor λ and singularity set Σ. Let L_v denote the differentiation in the vertical direction. Then the space of distributions in the kernel of L_v is exactly given by the direct sum of the constant functions, of the spaces $L_h^n E^H_{\lambda^{-1}\mu_i}$ for $n \geqslant 0$ and $i = 1, \dots, 2g-2$, of the multiples of the distributions $\xi_\sigma - \xi_{\sigma'}$ for $\sigma, \sigma' \in \Sigma$, and of the multiples of $L_h^n \xi_\sigma$ for $n \geqslant 1$ and $\sigma \in \Sigma$, where ξ_σ is defined in Proposition 1.5.*

In particular, the space of L_v-annihilated distributions of order $\geqslant -N$ is finite-dimensional for any N, and its dimension grows like $(2g - 2 + \operatorname{Card}\Sigma)N$ when $N \to \infty$. This is an analogue of [For02, Theorem 7.7(i)] in our context (see Remark 4.8 for a further cohomological description). If one restricts to L_v-annihilated distributions coming from $\mathcal{B}^{-k_h,k_v}$, one should remove the distributions $\xi_\sigma - \xi_{\sigma'}$ and $L_h^n \xi_\sigma$. Their dimension grows like $(2g-2)N$, corresponding to [For02, Theorem 7.7(ii)].

Bufetov has also studied vertically invariant distributions of the vertical foliation of a linear pseudo-Anosov map in [Buf14a]. In this article, the author is only interested in distributions of small order, which can be integrated against characteristic functions of intervals. He obtains a full description of such distributions, by more combinatorial means, and gets further properties such as their local Hölder behavior. These distributions correspond exactly to the elements of $\bigcup_{|\alpha| > \lambda^{-1}} E_\alpha$.

Solving the cohomological equation for the vertical flow. One of the main motivations to study L_v-annihilated distributions is that they are related to the cohomological equation for the vertical flow. Indeed, if one wants to write a function f as $L_v F$ for some function F with some smoothness, then one should have for any distribution ω in the kernel of L_v the equality

$$\langle \omega, f \rangle = \langle \omega, L_v F \rangle = -\langle L_v \omega, F \rangle = 0, \tag{1.2}$$

at least if F is more smooth than the order of ω and if L_v is antiselfadjoint on the relevant distributions (note that, in general, F will not be supported away from the singularities, so the fact the $\langle \omega, F\rangle$ or $\langle L_v\omega, F\rangle$ are well defined is not obvious, and neither is the formal equality $\langle \omega, L_v F\rangle = -\langle L_v \omega, F\rangle$). Such necessary conditions to have a coboundary are also often sufficient. In this direction, we obtain the following statement. The philosophy that results on the coboundary equation should follow from results on Ruelle resonances comes from Giulietti-Liverani [GL14]. Note that the converse is also true: in a resent work, Forni [For18] studied Ruelle resonances and obstructions to the existence of solution to the cohomological equation. In particular his work independently reproves some of the results of our paper (with very different methods). The cohomological equation was first solved for a large class of interval exchange maps (including the ones corresponding to pseudo-Anosov maps) in [?]. The proof we give of the next theorem also owes a lot to the techniques of [GL14] (although the local affine structure makes many arguments simpler compared to their article, but the presence of singularities creates new difficulties, as usual).

Theorem 1.7. *In the setting of Theorem 1.6, consider a C^∞ function f with compact support in $M - \Sigma$. Assume that $\langle \omega, f\rangle = 0$ for all $\omega \in \bigcup_{|\alpha| \geqslant \lambda^{-k-1}} E_\alpha \cap \ker L_v$. Then there exists a function F on M which is C^k whose k derivatives are bounded and continuous on M, such that $f = L_v F$ on $M - \Sigma$.*

The fact that f is C^∞ and compactly supported in $M - \Sigma$ is for the simplicity of the statement. Indeed, the theorem also holds if f is continuous on $M - \Sigma$ and C^{k+2} along horizontal lines, with $L_h^j f$ uniformly bounded for any $j \leqslant k+2$, see the more precise Theorem 5.9 below (in this case, the primitive F is C^k along horizontal lines). Even more, $C^{k+1+\varepsilon}$ along horizontal lines would suffice, for any $\varepsilon > 0$. So, the loss of derivatives in the above theorem is really $1+\varepsilon$ (which is optimal). Moreover, the k-th derivative of the solution of the coboundary equation is automatically Hölder continuous. This corresponds in our context respectively to the results of [For07] and [MY16].

It is not surprising that distributions in $E_\alpha \cap \ker L_v$ show up as conditions to solve the cohomological equation, as explained before the theorem. The main outcome of Theorem 1.7 is that there are finitely many obstructions to be a C^k coboundary. The number of such obstructions grows like $(2g-2)k$ when $k \to \infty$, by the classification of the Ruelle spectrum given in Theorem 1.4 and the following discussion. This answers the problem raised by Forni at the end of [For97], where a similar theorem is proved for the vertical flow on generic translation surfaces, using different methods based on the Laplacian.

Note that the distributions that appear in Theorem 1.7 only come from the Ruelle spectrum. The other L_v-annihilated distributions from Theorem 1.6 do not play a role. The reason is that the formal computation in (1.2) does not work for these distributions, as F is not compactly supported away from Σ. These distributions would appear if one were trying to find a vertical primitive of f which, additionally, vanishes at all singularities.

Trace formula. In finite dimension, the trace of an operator is the sum of its eigenvalues. This does not hold in general in infinite dimension (sometimes for lack of a good notion of trace, or for lack of summability of the eigenvalues), but it sometimes does for well behaved operators. In the dynamical world, this often holds for analytic maps (for which the transfer operator can be interpreted as a nuclear operator on a suitable space), but it fails most of the time outside of this class, see [Jéz17] and references therein.

In our case, it is easy to investigate this question, as we have a full description of the Ruelle spectrum. One should also define a suitable trace of the composition operator $\mathcal{T}$. On smooth manifolds, one can define the flat trace of a composition operator as the limit of the integral along the diagonal of the Schwartz kernel of a smoothed version of $\mathcal{T}$, when the smoothing parameter tends to 0. When T is a diffeomorphism with isolated fixed points, this reduces to a sum over the fixed points of $1/|\det(\mathrm{Id} - DT(x))|$, as follows from an easy computation involving the change of variables $y = x - Tx$.

In our case, the determinant is $(1-\lambda)(1-\lambda^{-1})$ everywhere, but one should also deal with the singularities, where the smoothing procedure is not clear (one can not convolve with a kernel because of the singularity). We recall the notion of *Lefschetz index* of an isolated fixed point x of a homeomorphism T in two dimensions (see for instance [HK95, Section 8.4]): it is the number

$$\mathrm{ind}_T(x) = \deg(p \mapsto (p - Tp)/\|p - Tp\|),$$

where the degree is computed on a small curve around x, identified with $\mathbb{S}^1$. If one could make sense of a smoothing at the singularity σ, then its contribution to the flat trace would be $\mathrm{ind}_T \sigma/((1-\lambda)(1-\lambda^{-1}))$, as follows from the same formal computation with the change of variables $y = x - Tx$ (the index comes from the number of branches of this map, giving a multiplicity when one computes the integral). Thus, to have a sound definition independent of an unclear smoothing procedure, we *define* the flat trace of $\mathcal{T}^n$ as

$$\mathrm{tr}^\flat(\mathcal{T}^n) = \sum_{T^n x = x} \frac{\mathrm{ind}_{T^n} x}{(1-\lambda^n)(1-\lambda^{-n})}.$$

If T^n is smooth at a fixed point x, then its index is -1 and we recover the usual contribution of x to the flat trace. More generally, if T is such that $\det(I - DT)$ has a limit at all fixed points of T (regular or singular) then one defines its flat trace as the sum over all fixed points x of $\mathrm{ind}_T x/(\lim_x \det(I - DT))$.

Theorem 1.8. *Let T be a linear pseudo-Anosov map preserving orientations on a compact surface M. Then, for all n,*

$$\mathrm{tr}^\flat(\mathcal{T}^n) = \sum_\alpha d_\alpha \alpha^n, \tag{1.3}$$

where the sum is over all Ruelle resonances α of T, and d_α denotes the multiplicity of α.

Proof. The Lefschetz fixed-point formula (see [HK95, Theorem 8.6.2]) gives

$$\begin{aligned}\sum_{T^n x = x} \mathrm{ind}_{T^n} x &= \mathrm{tr}((T^n)^*_{|H^0(M)}) - \mathrm{tr}((T^n)^*_{|H^1(M)}) + \mathrm{tr}((T^n)^*_{|H^2(M)}) \\ &= 1 - \left(\lambda^n + \lambda^{-n} + \sum_{i=1}^{2g-2} \mu_i^n\right) + 1,\end{aligned}$$

where $\{\mu_1, \dots, \mu_{2g-2}\}$ denote the eigenvalues of T^* on the subspace of $H^1(M)$ orthogonal to $[\mathrm{d}x]$ and $[\mathrm{d}y]$, as in the statement of Theorem 1.4. We can also compute the right hand side of (1.3), using the description of Ruelle resonances: 1 has multiplicity one, and $\lambda^{-k}\mu_i$ has multiplicity k for $k \geqslant 1$. As $\sum kx^k = x/(1-x)^2 = -1/((1-x)(1-x^{-1}))$, we get

$$\sum_\alpha d_\alpha \alpha^n = 1 + \sum_{i=1}^{2g-2}\sum_{k=1}^{\infty} k\lambda^{-nk}\mu_i^n = 1 - \sum_{i=1}^{2g-2} \frac{\mu_i^n}{(1-\lambda^{-n})(1-\lambda^n)}$$

$$= \frac{(1-\lambda^{-n})(1-\lambda^n) - \sum_{i=1}^{2g-2}\mu_i^n}{(1-\lambda^{-n})(1-\lambda^n)} = \frac{2 - \left(\lambda^n + \lambda^{-n} + \sum_{i=1}^{2g-2}\mu_i^n\right)}{(1-\lambda^{-n})(1-\lambda^n)}.$$

Combining the two formulas with the definition of the flat trace, we get the conclusion of the theorem. □

Organization of the paper. In Section 2, we define the anisotropic Banach spaces $\mathcal{B}^{-k_h,k_v}$ we will use to understand the spectrum of the composition operator $\mathcal{T}$. The construction works in any translation surface. We prove the basic properties of these Banach spaces, including notably compact inclusion statements, a duality result, and a cohomological interpretation of elements of the space which are vertically invariant. All these tools are put to good use in Section 3, where we describe the Ruelle spectrum of a linear pseudo-Anosov map preserving orientations, proving Theorem 1.4. Then, we use (and extend) this theorem in Section 4 to classify all vertically invariant distributions (proving Theorem 1.6), and in Section 5 to find smooth solutions to the cohomological equation (proving Theorem 1.7). Finally, Section 6 is devoted to the discussion of the Ruelle spectrum for linear pseudo-Anosov maps which do not preserve orientations.

2. Functional spaces on translation surfaces

2.1. **Anisotropic Banach spaces on translation surfaces.** In this section, we consider a translation surface (M, Σ). We wish to define anisotropic Banach spaces of distributions on such a surface, i.e., spaces of distributions which are smooth along the vertical direction, and dual of smooth along the horizontal direction. Indeed, this is the kind of space on which the transfer operator associated to a pseudo-Anosov map will be well behaved, leading ultimately to the existence of Ruelle spectrum for such a map, and to its explicit description. The definition we use below is of geometric nature: we will require that the objects in our space can be integrated along horizontal segments when multiplied by smooth functions, and that they have vertical derivatives with the same property. This simple-minded definition in the spirit of [GL08, AG13] is very well suited for the constructions we have in mind below (especially for the cohomological interpretation in Paragraph 2.4 below) and makes it possible to deal transparently with the singularities. However, it is probably possible to use other approaches as explained in [Bal17] and references therein.

Let V^h be the unit norm positively oriented horizontal vector field, i.e., the vector field equal to $1 \in \mathbb{C}$ in the translation charts. It is C^∞ on $M - \Sigma$, but singular at Σ. In particular, the derivation L_h given by this vector field acts on $C^\infty(M-\Sigma)$. In the same way, the vertical vector field V^v (equal to $\mathbf{i}$ in the complex translation charts) is C^∞ on $M - \Sigma$, and the corresponding derivation L_v acts on $C^\infty(M-\Sigma)$. On this space, the two derivations L_v and L_h commute, as this is the case in $\mathbb{C}$.

Choose two real numbers $k \geqslant 0$ and $\beta > 0$. Denote by $\mathcal{I}^h_\beta$ the set of horizontal segments of length β in $M - \Sigma$. For $I \in \mathcal{I}^h_\beta$, denote by $C^k_c(I)$ the set of C^k functions on I which vanish on a neighborhood of the boundary of I, endowed with the C^k norm (when k is not an integer, this is the set of functions of class $C^{\lfloor k \rfloor}$ whose $\lfloor k \rfloor$-th derivative is Hölder continuous with exponent $k - \lfloor k \rfloor$).

When $k_h \geqslant 0$ is a nonnegative real number, and $k_v \geqslant 0$ is an integer, we define a seminorm on $C^\infty_c(M - \Sigma)$ by

$$\|f\|'_{-k_h,k_v,\beta} = \sup_{I \in \mathcal{I}^h_\beta} \sup_{\varphi \in C^{k_h}_c(I), \|\varphi\|_{C^{k_h}} \leqslant 1} \left| \int_I \varphi \cdot (L_v)^{k_v} f \,\mathrm{d}x \right|.$$

Essentially, this seminorm measures k_v derivatives in the vertical direction, and $-k_h$ derivatives in the horizontal direction (as one is integrating against a function with k_h derivatives). Hence, it is indeed a norm of anisotropic type. One could define many such norms, but this one is arguably the simplest one: it takes advantage of the fact that the horizontal and vertical foliations are smooth, and even affine.

Proposition 2.1. *If β is smaller than the length of the shortest horizontal saddle connection, then this seminorm does not really depend on β: if β_1 is another such number, then there exists a constant $C = C(\beta, \beta_1, k_h, k_v)$ such that , for any $f \in C^\infty_c(M - \Sigma)$,*

$$C^{-1}\|f\|'_{-k_h,k_v,\beta_1} \leqslant \|f\|'_{-k_h,k_v,\beta} \leqslant C\|f\|'_{-k_h,k_v,\beta_1}.$$

We recall that a horizontal saddle connection is a horizontal segment connecting two singularities. There is no horizontal saddle connection in a surface carrying a pseudo-Anosov map: otherwise, iterating the inverse of the map (which contracts uniformly the horizontal segments), we would deduce the existence of arbitrarily short horizontal saddle connections, a contradiction.

Proof. Assume for instance $\beta_1 > \beta$. The inequality $\|f\|'_{-k_h,k_v,\beta} \leqslant \|f\|'_{-k_h,k_v,\beta_1}$ is clear: an interval $I \in \mathcal{I}^h_\beta$ is contained in an interval I_1 in $\mathcal{I}^h_{\beta_1}$ as β_1 is smaller than the length of any horizontal saddle connection. Moreover, a compactly supported test function φ on I can be extended by 0 to outside of I to get a test function on I_1. The result follows readily.

Conversely, consider a smooth partition of unity $(\rho_j)_{j \in J}$ on $[0, \beta_1]$ by C^∞ functions whose support has length at most β (we do not require that the functions vanish at 0 or β_1. Using this partition of unity, for $I_1 \in \mathcal{I}^h_{\beta_1}$, one may decompose a test function $\varphi \in C^{k_h}_c(I_1)$ as the sum of the functions $\varphi \cdot \rho_j$, which are all compactly supported on intervals belonging to $\mathcal{I}^h_\beta$. Moreover, their C^{k_h} norms are controlled by the C^{k_h} norm of φ. It follows that the integrals defining $\|f\|'_{-k_h,k_v,\beta_1}$ are controlled by finitely many integrals that appear in the definition of $\|f\|'_{-k_h,k_v,\beta}$, giving the inequality $\|f\|'_{-k_h,k_v,\beta_1} \leqslant C\|f\|'_{-k_h,k_v,\beta}$. □

By the above proposition, we may use any small enough β. For definiteness, let us choose once and for all $\beta = \beta_0$ much smaller than the distance between any two singularities. This implies that, in all the local discussions, we will have to consider at most one singularity. From this point on, we will keep β_0 implicit, unless there is an ambiguity.

The seminorms $\|\cdot\|'_{-k_h,k_v}$ are not norms in general on $C^\infty_c(M - \Sigma)$. For instance, if there is a cylinder made of closed vertical leaves, then one may find a function which is constant on each vertical leaf, vanishes close to the singularities, and is nevertheless not everywhere zero. Then $L_v f = 0$, so that $\|f\|'_{-k_h,k_v} = 0$ if $k_v > 0$, but still $f \neq 0$. This is not the case when there is no vertical connection: in this case, all vertical leaves are dense, hence a function which is constant along vertical leaves and vanishes on a neighborhood of the singularities has to vanish everywhere. In general, this remark indicates that the above seminorms do not behave very well by themselves. On the other hand, the following norm is much nicer:

$$\|f\|_{-k_h,k_v} = \sup_{j\leqslant k_v} \|f\|'_{-k_h,j} = \sup_{j\leqslant k_v} \sup_{I\in\mathcal{I}^h} \sup_{\varphi\in C_c^{k_h}(I), \|\varphi\|_{C^{k_h}}\leqslant 1} \left| \int_I \varphi \cdot L_v^j f \,\mathrm{d}x \right|. \tag{2.1}$$

This is obviously a norm on $C_c^\infty(M-\Sigma)$. Indeed, if a function f is not identically zero, then it is nonzero at some point x. Taking a horizontal interval I around x and a test function φ on I supported on a small neighborhood of x, one gets $\int_I \varphi f \,\mathrm{d}x \neq 0$, and therefore $\|f\|_{-k_h,k_v} > 0$.

Then, let us define the space $\mathcal{B}^{-k_h,k_v}$ as the (abstract) completion of $C_c^\infty(M-\Sigma)$ for this norm. Note that all the linear forms $\ell_{I,\varphi,j} : f \mapsto \int_I \varphi \cdot L_v^j f \,\mathrm{d}x$, initially defined on $C_c^\infty(M-\Sigma)$, extend by continuity to $\mathcal{B}^{-k_h,k_v}$ (for $I \in \mathcal{I}^h$ and $\varphi \in C_c^{k_h}(I)$ and $j \leqslant k_v$). Heuristically, an element in $\mathcal{B}^{-k_h,k_v}$ can be differentiated in the vertical direction, and integrated in the horizontal direction. Moreover, the norm of an element in $\mathcal{B}^{-k_h,k_v}$ is

$$\|f\|_{-k_h,k_v} = \sup_{j\leqslant k_v} \sup_{I\in\mathcal{I}^h} \sup_{\varphi\in C_c^{k_h}(I), \|\varphi\|_{C^{k_h}}\leqslant 1} |\ell_{I,\varphi,j}(f)|. \tag{2.2}$$

This follows directly from the definition of the norm on $C_c^\infty(M-\Sigma)$ and from the construction of $\mathcal{B}^{-k_h,k_v}$ as its completion.

Remark 2.2. In the spaces $\mathcal{B}^{-k_h,k_v}$ we have just defined, the parameter k_h of horizontal regularity can be any nonnegative real, but the parameter k_v of vertical regularity has to be an integer, as it counts a number of derivatives. One could also use a non-integer vertical parameter k_v, requiring additionally the following control: if $k_v = k + r$ where k is an integer and $r \in (0,1)$, then we require the boundedness of

$$\varepsilon^{-r} \left| \int_{I_0} \varphi_0 L_v^k f \,\mathrm{d}x - \int_{I_\varepsilon} \varphi_\varepsilon L_v^k f \,\mathrm{d}x \right|$$

when I_0 is a horizontal interval of length β_0, φ_0 is a compactly supported C^{k_h} function on I_0 with norm at most 1, $\varepsilon \in [0,\beta_0]$ is such that one can translate vertically the interval I_0 into an interval I_ε without hitting any singularity, and φ_ε is the push-forward of φ_0 on I_ε using the vertical translation. In other words, we are requiring that $L_v^k f$ is Hölder continuous of order r vertically, in the distributional sense. All the results that follow are true for such a norm, but the proofs become more cumbersome while the results are not essentially stronger, so we will only consider integer k_v for the sake of simplicity.

Let φ be a C^∞ function on M, and denote by dLeb the flat Lebesgue measure on M. Then $\ell_\varphi : f \mapsto \int f\varphi \,\mathrm{dLeb}$ is a linear form on $C_c^\infty(M-\Sigma)$. Contrary to the previous linear forms, ℓ_φ does *not* extend to a linear form on $\mathcal{B}^{-k_h,k_v}$, because of the singularities: from the point of view of the C^∞ structure, horizontals and verticals close to the singularity have a lot of curvature, so that the restriction of φ to $I \in \mathcal{I}^h$ is C^k, but with a large C^k

norm (larger when I is closer to the singularity). This prevents the extension of ℓ_φ to $\mathcal{B}^{-k_h,k_v}$. On the other hand, if φ is supported by $M - B(\Sigma, \delta)$, then one has a control of the form $|\ell_\varphi(f)| \leqslant C(\delta) \|\varphi\|_{C^{k_h}} \|f\|_{-k_h,k_v}$, so that ℓ_φ extends continuously to $\mathcal{B}^{-k_h,k_v}$. More precisely, denote by $\mathcal{D}^\infty(M - \Sigma)$ the set of distributions on $M - \Sigma$, i.e., the dual space of $C_c^\infty(M - \Sigma)$ with its natural topology. Then the above argument shows that there is a map $i : \mathcal{B}^{-k_h,k_v} \to \mathcal{D}^\infty(M - \Sigma)$, extending the canonical inclusion $C_c^\infty(M - \Sigma) \to \mathcal{D}^\infty(M - \Sigma)$ given by $\langle i(f), \varphi \rangle = \int f\varphi \,\mathrm{dLeb}$. Locally, if φ is supported by a small rectangle foliated by horizontal segments $I_t \in \mathcal{I}^h$ (where t is an arc-length parametrization along the vertical direction), one has the explicit description

$$\langle i(f), \varphi \rangle = \int \ell_{I_t, \varphi_{|I_t}, 0}(f) \,\mathrm{d}t. \tag{2.3}$$

Indeed, this formula holds when f is C^∞, and extends by uniform limit to all elements of $\mathcal{B}^{-k_h,k_v}$.

Proposition 2.3. *The map $i : \mathcal{B}^{-k_h,k_v} \to \mathcal{D}^\infty(M - \Sigma)$ is injective. Therefore, one can identify $\mathcal{B}^{-k_h,k_v}$ with a space of distributions on $M - \Sigma$.*

Proof. Consider $I \in \mathcal{I}^h$ and $\varphi \in C_c^{k_h}(I)$. For small enough t, one can shift vertically I by t, and obtain a new interval $I_t \in \mathcal{I}^h$, as well as a function $\varphi_t : I_t \to \mathbb{R}$ (equal to the composition of the vertical projection from I_t to I, and of φ). For any $f \in C_c^\infty(M - \Sigma)$, the function $t \mapsto \ell_{I_t,\varphi_t,0}(f)$ is C^{k_v}, with successive derivatives $t \mapsto \ell_{I_t,\varphi_t,j}(f)$. An element $f \in \mathcal{B}^{-k_h,k_v}$ can be written as a limit of a Cauchy sequence of smooth functions. Then $\ell_{I_t,\varphi_t,j}(f_n)$ converges uniformly to $\ell_{I_t,\varphi_t,j}(f)$. Passing to the limit in n, we deduce that $t \mapsto \ell_{I_t,\varphi_t,0}(f)$ is C^{k_v}, with successive derivatives $t \mapsto \ell_{I_t,\varphi_t,j}(f)$.

Consider a nonzero $f \in \mathcal{B}^{-k_h,k_v}$, with norm $c > 0$. By (2.2), there exist I, φ and j such that $|\ell_{I,\varphi,j}(f)| \geqslant c/2$. Let us shift I vertically as above. The function $t \mapsto \ell_{I_t,\varphi_t,0}(f)$ has a j-th derivative which is nonzero at 0, hence it is not locally constant. In particular, it does not vanish at some parameter t_0. Consider δ such that it is almost constant on the interval $[t_0 - \delta, t_0 + \delta]$ by continuity. Let ψ be a smooth function with positive integral, supported by $[t_0 - \delta, t_0 + \delta]$. In local coordinates, let us finally write $\zeta(x, y) = \varphi(x)\psi(y)$. It satisfies $\langle i(f), \zeta \rangle \neq 0$ thanks to the explicit description (2.3) for $i(f)$. □

It follows that one can think of elements of $\mathcal{B}^{-k_h,k_v}$ as objects that can be integrated along horizontal segments, or after an additional vertical integration as distributions. Even better, since the elements of $\mathcal{B}^{-k_h,k_v}$ are designed to be integrated horizontally, the natural object to consider is rather $f \,\mathrm{d}x$. This is a current, i.e., a differential form with distributional coefficients, but it is nicer than general currents as it can really be integrated along horizontal segments (i.e., it is regular in the vertical direction). The process that associates to such an object a global distribution is simply the exterior product with $\mathrm{d}y$. Going back and forth like that between 0-currents and 1-currents will be an essential feature of the forthcoming arguments.

The next lemma makes it possible to use partitions of unity, to decompose an element of $\mathcal{B}^{-k_h,k_v}$ into a sum of elements supported in arbitrarily small balls.

Lemma 2.4. *Let $\psi \in C^\infty(M)$ be constant in the neighborhood of each singularity. Then the map $f \mapsto \psi f$, initially defined on $C_c^\infty(M-\Sigma)$, extends continuously to a linear map on $\mathcal{B}^{-k_h,k_v}$.*

Proof. We have to bound $\int_I \varphi \cdot L_v^j(\psi f)\,\mathrm{d}x$ when I is a horizontal interval, φ a compactly supported C^{k_h} function on I, and $j \leqslant k_v$. We have $L_v^j(\psi f) = \sum_{k\leqslant j}\binom{j}{k} L_v^{j-k}\psi \cdot L_v^k f$, hence this integral can be decomposed as a sum of integrals of $L_v^k f$ against the functions $\varphi \cdot L_v^{j-k}\psi$ which are C^{k_h} and compactly supported on I. This concludes the proof, by definition of $\mathcal{B}^{-k_h,k_v}$. □

One may wonder how rich the space $\mathcal{B}^{-k_h,k_v}$ is, and if the choice to take the closure of the set of functions vanishing on a neighborhood of the singularities really matters. Other functions are natural, for instance the constants, or more generally the smooth functions that factorize through the covering projection $\pi : z \mapsto z^p$ around each singularity of angle $2\pi p$. The largest natural class is the space of functions f which are C^∞ on $M-\Sigma$ and such that, for all indices a_h and a_v, the function $L_v^{a_v}L_h^{a_h} f$ is bounded. The next lemma asserts that starting from any of these classes of functions would not make any difference, as our space $\mathcal{B}^{-k_h,k_v}$ is already rich enough to contain all of them.

Lemma 2.5. *Consider a function f on M which is C^{k_v} on every vertical segment and such that $L_v^k f$ is bounded and continuous on $M-\Sigma$ for any $k \leqslant k_v$. Then the function f (or rather the corresponding distribution $i(f)$) belongs to $\mathcal{B}^{-k_h,k_v}$ for any $k_h \geqslant 0$. This is in particular the case of the constant function $f = 1$.*

Proof. First, if f is supported away from the singularities, one shows that $f \in \mathcal{B}^{-k_h,k_v}$ by convolving it with a smooth kernel ρ_ε: the sequence $f_\varepsilon = f * \rho_\varepsilon$ thus constructed is C^∞ and forms a Cauchy sequence in $\mathcal{B}^{-k_h,k_v}$, hence it converges in this space to a limit. As it converges to f in the distributional sense, this shows $f \in \mathcal{B}^{-k_h,k_v}$.

To handle the general case, by taking a partition of unity, it suffices to treat the case of a function f supported in a small neighborhood of a singularity, such that $L_v^k f$ is continuous and bounded for any $k \leqslant k_v$. Let π denote the covering projection, defined on a neighborhood of this singularity. Let u be a real function, equal to 1 on a neighborhood of 0, supported in $[-1,1]$. Let $N > 0$ be large enough. For $\delta > 0$, we define a function $\rho_\delta(x+\mathbf{i}y) = u(x/\delta^N)u(y/\delta)$, supported on the neighborhood $[-\delta^N,\delta^N]+\mathbf{i}[-\delta,\delta]$ of 0 in $\mathbb{C}$.

We claim that, if $N > k_v$, then in $\mathbb{C}$ one has $\|\rho_\delta\|_{-k_h,k_v} \to 0$ when $\delta \to 0$, where by $\|\cdot\|_{-k_h,k_v}$ we mean the formal expression (2.1), which makes sense for any function but could be infinite. To prove this, consider a horizontal interval I of length β_0, a function $\varphi \in C_c^{k_h}(I)$ with norm at most 1, and a differentiation order $j \leqslant k_v$. Then

$$\begin{aligned}\left|\int_I \varphi \cdot L_v^j \rho_\delta\,\mathrm{d}x\right| &= \delta^{-j}\left|\int_I \varphi \cdot u(x/\delta^N)u^{(j)}(y/\delta)\,\mathrm{d}x\right| \\ &\leqslant \delta^{-j}\|\varphi\|_{C^0}\|u\|_{C^0}\|u^{(j)}\|_{C^0}\,\mathrm{Leb}([-\delta^N,\delta^N]).\end{aligned}$$

This quantity tends to 0 if $N > j$, as claimed.

The same computation, taking moreover into account the fact that the vertical derivatives of f are bounded, shows that $\|f\cdot\rho_\delta\circ\pi\|_{-k_h,k_v} \to 0$ when $\delta \to 0$. It follows that the sequence $f_n = f(1-\rho_{1/n}\circ\pi)$ is a Cauchy sequence in $\mathcal{B}^{-k_h,k_v}$, made of functions in $C_c^{k_v}(M-\Sigma)$

(which is indeed included in $\mathcal{B}^{-k_h,k_v}$ by the first step). It converges (in L^1, and therefore in the sense of distributions) to f, which has therefore to coincide with its limit in $\mathcal{B}^{-k_h,k_v}$. □

In particular, if Σ contains an artificial singularity σ (i.e., around which the angle is equal to 2π), then one gets the same space $\mathcal{B}^{-k_h,k_v}$ by using the singularity sets Σ or $\Sigma - \{\sigma\}$.

The horizontal and vertical derivations L_h and L_v act on $C_c^\infty(M-\Sigma)$. By duality, they also act on $\mathcal{D}^\infty(M-\Sigma)$. In view of Proposition 2.3 asserting that $\mathcal{B}^{-k_h,k_v}$ is a space of distributions, it makes sense to ask if they stabilize these spaces, or if they send one into the other.

Proposition 2.6. *The derivation L_h maps continuously $\mathcal{B}^{-k_h,k_v}$ to $\mathcal{B}^{-k_h-1,k_v}$, and it satisfies $\ell_{I,\varphi,j}(L_h f) = -\ell_{I,\varphi',j}(f)$ for every $I \in \mathcal{I}^h$, $\varphi \in C_c^{k_h+1}(I)$, $j \leqslant k_v$ and $f \in \mathcal{B}^{-k_h,k_v}$.*

The derivation L_v maps continuously $\mathcal{B}^{-k_h,k_v}$ to $\mathcal{B}^{-k_h,k_v-1}$ if $k_v > 0$, and it satisfies $\ell_{I,\varphi,j}(L_v f) = \ell_{I,\varphi,j+1}(f)$ for every $I \in \mathcal{I}^h$, $\varphi \in C_c^{k_h}(I)$, $j \leqslant k_v - 1$ and $f \in \mathcal{B}^{-k_h,k_v}$.

Proof. The formulas $\ell_{I,\varphi,j}(L_h f) = -\ell_{I,\varphi',j}(f)$ and $\ell_{I,\varphi,j}(L_v f) = \ell_{I,\varphi,j+1}(f)$ are obvious when f is a smooth function. The general result follows by density. □

Lemma 2.7. *Assume that there is no horizontal saddle connection in M. Let $f \in \mathcal{B}^{-k_h,k_v}$ satisfy $L_h f = 0$. Then f is a constant function.*

Proof. As $L_h f = 0$, one has $\ell_{I,\varphi',0}(f) = 0$ for any smooth function φ on a horizontal interval I. Denoting by τ_h the translation by h, one gets $\ell_{I,\varphi,0}(f) = \ell_{I,\varphi\circ\tau_h,0}(f)$ if φ and $\varphi\circ\tau_h$ both have their support in I. It follows that the distribution induced by f on a bi-infinite horizontal leaf is invariant by translation. Therefore, it is a multiple $c\,\mathrm{dLeb}$ of Lebesgue measure. Since there is no horizontal saddle connection by assumption, the horizontal flow is minimal by Keane's Criterion. In particular, the above bi-infinite horizontal leaf is dense. At the quantities $\ell_{I,\varphi,0}(f)$ vary continuously when one moves I vertically, it follows that f is equal to $c\,\mathrm{dLeb}$ on all horizontal intervals. □

We want to stress that Lemma 2.7 is wrong for L_v. A measure μ which is invariant for the vertical flow can locally be written as $\nu \otimes \mathrm{d}y$, where ν is a measure along horizontal leaves, invariant under vertical holonomy. Writing ν as a limit of measures which are equivalent to Lebesgue and with smooth densities, one checks that μ belongs to $\mathcal{B}^{-k_h,k_v}$, and moreover it satisfies $L_v\mu = 0$. In a translation surface in which the vertical flow is minimal but not uniquely ergodic, one can find such examples where μ is not Lebesgue measure.

In the case of surfaces associated to pseudo-Anosov maps, the vertical flow is uniquely ergodic, so this argument does not apply. However, we will see later that there are still many nonconstant distributions f in $\mathcal{B}^{-k_h,k_v}$ which satisfy $L_v f = 0$.

It is enlightening to try to prove that $f \in \mathcal{B}^{-k_h,k_v}$ with $L_v f = 0$ has to be constant, and see where the argument fails. The problem stems from the fact that f is a distribution on horizontal segments. Let F be a dense vertical leaf, let I_t be a small horizontal interval around the point at height t on F, and let φ be a function on I_0 that we push vertically to a function on I_t (still denoted φ) while this is possible. Then we get $\int_{I_t} \varphi f \,\mathrm{d}x = \int_{I_0} \varphi f \,\mathrm{d}x$ as $L_v f = 0$. If this were true for all real t, then we would deduce that f is constant. However, the support of φ has positive length. Hence, when we push it vertically, we will encounter a singularity in finite time, and the argument is void afterwards. We could say something on a longer time interval if we used a function $\tilde{\varphi}$ with smaller support, but the same problem will happen again. The key point is a competition between the speed at which F fills the surface, and how close to singularities it passes. The existence of non-constant distributions f with $L_v f = 0$ is a manifestation of the fact that F is often too close to singularities.

A related but more detailed discussion is made before the proof of Theorem 3.11, where we study the existence of primitives under L_v of some eigendistributions, not only 0.

2.2. **Compact inclusions.** In this paragraph, we prove the following proposition, ensuring that there is inclusion (resp. compact inclusion) in the family of spaces $\mathcal{B}^{-k_h,k_v}$ if one requires less (resp. strictly less) regularity in all directions. This corresponds to the usual intuitions.

Proposition 2.8. *Consider k'_h with $-k'_h \leqslant -k_h$ (i.e., $k'_h \geqslant k_h$) and k'_v with $k'_v \leqslant k_v$. Then there is a continuous inclusion $\mathcal{B}^{-k_h,k_v} \subseteq \mathcal{B}^{-k'_h,k'_v}$. If the two inequalities are strict, this inclusion is compact.*

Proof. The inclusion $\mathcal{B}^{-k_h,k_v} \subseteq \mathcal{B}^{-k'_u,k'_v}$ when $k'_h \geqslant k_h$ and $k'_v \leqslant k_v$ is obvious, as one uses less linear forms in the second space than in the first space to define the norm.

For the compact inclusion, we will use the following criterion. Let $\mathcal{B} \subseteq \mathcal{C}$ be two Banach spaces. Assume that, for every $\varepsilon > 0$, there exist finitely many continuous linear forms $\ell_1, \dots, \ell_P$ on $\mathcal{B}$ such that, for any $x \in \mathcal{B}$,

$$\|x\|_{\mathcal{C}} \leqslant \varepsilon \|x\|_{\mathcal{B}} + \sum_{p \leqslant P} |\ell_p(x)|. \tag{2.4}$$

Then the inclusion of $\mathcal{B}$ in $\mathcal{C}$ is compact.

To prove the criterion, suppose its assumptions are satisfied, and consider a sequence $x_n \in \mathcal{B}$ of elements with norm at most 1. Extracting a subsequence, one can ensure that all the sequences $\ell_i(x_n)$ converge, for $i \leqslant P$. We deduce from the above inequality that $\limsup_{m,n\to\infty} \|x_m - x_n\|_{\mathcal{C}} \leqslant 2\varepsilon$. By a diagonal argument, one can then extract a subsequence of x_n which is a Cauchy sequence in $\mathcal{C}$, and therefore converges.

Let us now apply the criterion to $\mathcal{B} = \mathcal{B}_{5\beta_0}^{-k_h,k_v}$ and $\mathcal{C} = \mathcal{B}_{\beta_0}^{-k'_h,k'_v}$ with $k'_h > k_h$ and $k'_v < k_v$. We take larger intervals in the first space than in the second space for technical convenience, but this is irrelevant for the result as the spaces do not depend on β, see Proposition 2.1.

Let us first fix a finite family of intervals $(J_n)_{n \leqslant N}$ in $\mathcal{I}^h_{5\beta_0}$ such that any interval in $\mathcal{I}^h_{\beta_0}$ can be translated vertically by at most $\varepsilon/2$, without hitting a singularity, and end up in one of the J_n, or even better in its central part denoted by $J_n[\beta_0, 4\beta_0]$. Such a family exists by compactness, and the singularities do not create any problem there. Then, on each J_n, let us fix finitely many functions $(\varphi_{n,k})_{k \leqslant K}$ in $C_c^{k'_h}(J_n)$ with norm at most 1 such that, for any function $\varphi \in C^{k'_h}(J_n)$ with $C^{k'_h}$ norm at most 1 and with support included in $J_n[\beta_0, 4\beta_0]$, there exists k such that $\|\varphi - \varphi_{n,k}\|_{C^{k_h}} \leqslant \varepsilon/2$. Their existence follows from the compactness of the inclusion of $C^{k'_h}$ in C^{k_h}. We will use the linear forms $\ell_{n,k,j} = \ell_{J_n,\varphi_{n,k},j}$ for $n \leqslant N$, $k \leqslant K$ and $j \leqslant k_v$ to apply the criterion (2.4).

Let us fix $f \in \mathcal{B}_{5\beta_0}^{-k_h,k_v}$. We want to bound its norm in $\mathcal{B}_{\beta_0}^{-k'_h,k'_v}$. By density, it is enough to do it for $f \in C_c^\infty(M - \Sigma)$ – this does not change anything to the following argument, but it is comforting. Consider thus $I \in \mathcal{I}^h_{\beta_0}$, and $\varphi \in C_c^{k'_h}(I)$ with norm at most 1, and $j \leqslant k'_v < k_v$. Let $(I_t)_{0 \leqslant t \leqslant \delta}$ be vertical shifts of I, parameterized by the vertical length t, with I_δ included in an interval $J_n[\beta_0, 4\beta_0]$ and $\delta \leqslant \varepsilon/2$. Denote by φ_t the push-forward of φ on I_t. Integrating by parts, one gets

$$\int_{I_0} \varphi \cdot L_v^j f \,\mathrm{d}x = \int_{I_\delta} \varphi_\delta \cdot L_v^j f \,\mathrm{d}x - \int_0^\delta \left(\int_{I_t} \varphi_t L_v^{j+1} f \,\mathrm{d}x \right) \mathrm{d}t.$$

The integrals on each I_t are bounded by $\|f\|_{-k_h,k_v}$ as $j \leqslant k'_v < k_v$. Hence, the last term is at most $\delta \|f\|_{-k_h,k_v} \leqslant (\varepsilon/2)\|f\|_{-k_h,k_v}$. In the first term, choose k such that $\|\varphi_\delta - \varphi_{n,k}\|_{C^{k_h}} \leqslant \varepsilon/2$. Then this integral is bounded by $(\varepsilon/2)\|f\|_{-k_h,k_v} + |\ell_{n,k,j}(f)|$. We have proved that

$$\|f\|_{-k'_h,k'_v} \leqslant \varepsilon \|f\|_{-k_h,k_v} + \max_{n,k,j} |\ell_{n,k,j}(f)|.$$

This shows that the compactness criterion (2.4) applies, and concludes the proof. □

2.3. **Duality.** Let us define the spaces $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ just like the spaces $\mathcal{B}^{-k_h,k_v}$ but exchanging horizontals and verticals. Hence, $\check{k}_v$ quantifies the regularity of a test function in the vertical direction, and $\check{k}_h$ the number of permitted derivatives in the horizontal direction. The derivations L_v and L_h still act on $\check{\mathcal{B}}$, as in Proposition 2.6, but their roles are swapped compared to $\mathcal{B}$.

Some of the arguments later to identify the spectrum and the multiplicities of a pseudo-Anosov map rely on a duality argument, exchanging the roles of the horizontal and vertical directions. To carry out this argument, we need to show that there is a duality between the spaces $\mathcal{B}^{-k_h,k_v}$ and $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ when the global regularity is positive enough in every direction, i.e., when $-k_h+\check{k}_h \geqslant 2$ and $k_v - \check{k}_v \geqslant 0$ (or conversely, as one can exchange the two directions – it is possible that the duality holds if $\check{k}_h - k_h \geqslant 0$ and $k_v - \check{k}_v \geqslant 0$, but our proof requires a little bit more). This is not surprising: $g \in \check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ has essentially $\check{k}_h$ derivatives along horizontals, and $f \in \mathcal{B}^{-k_h,k_v}$ can be integrated along horizontals against C^{k_h} functions, so if $\check{k}_h \geqslant k_h$ one expects that one can integrate the product fg along horizontals, and therefore globally. This argument is wrong since the horizontal regularity of g is only in the distributional sense, so we will also have to take advantage of the vertical smoothness of f. Using a computation based on suitable integrations by parts, it is easy to make this argument rigorous away from singularities. However, as it is often the case, the proof is much more delicate close to singularities, as integrations by parts can not cross the singularity, giving rise to additional boundary terms that can a priori not be controlled, unless one proceeds in a roundabout way as in the following proof. The technical difficulty of this proof is probably related to our choice of Banach spaces: it is possible that another choice of Banach space makes this proposition essentially trivial. This proof can be skipped on first reading.

Proposition 2.9. *Assume $-k_h + \check{k}_h \geqslant 2$ and $k_v - \check{k}_v \geqslant 0$. Then there exists $C > 0$ such that, for any $f, g \in C_c^\infty(M - \Sigma)$, one has*

$$\left| \int fg \,\mathrm{dLeb} \right| \leqslant C \|f\|_{\mathcal{B}^{-k_h,k_v}} \cdot \|g\|_{\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}}.$$

Therefore, the map $(f,g) \mapsto \int fg \,\mathrm{dLeb}$ extends by continuity to a bilinear map on $\mathcal{B}^{-k_h,k_v} \times \check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ that we denote by $\langle f, g \rangle$.

The proof will rely on a decomposition of f into basic pieces for which all the above integrals can be controlled. We will denote by $\mathcal{H}$ the set of local half-planes around all singularities, bounded by horizontal or vertical lines. Specifically, if σ is a singularity of angle $2\boldsymbol{\pi}\kappa$ with covering projection π, these sets are the κ components of $\pi^{-1}\{z : \Re z \geqslant 0\}$ in a neighborhood of σ, intersected with a small disk around σ, and similarly for the upper half-planes, lower half-planes and left half-planes, giving rise to 4κ half-planes around σ.

Lemma 2.10. *Fix k_h and k_v. There exist N, C, and rectangles $(R_i)_{i\leqslant N}$ away from the singularities with the following property. For any $f \in C_c^\infty(M - \Sigma)$, there is a decomposition*

$$f = \sum_{i=1}^{N} f_i + \sum_{\sigma\in\Sigma} f_\sigma + \sum_{H\in\mathcal{H}} f_H \tag{2.5}$$

where all the f_i and f_σ and f_H are C^{k_v} functions with compact support in $M - \Sigma$. They belong to $\mathcal{B}^{-k_h,k_v}$ and have norm at most $C\|f\|_{-k_h,k_v}$. Moreover, each f_i is supported in R_i, each f_H is supported in H, and each f_σ is supported in a small disk D_σ around σ and is constant on the fibers of the covering projection π around σ.

Proof. Multiplying f by a partition of unity, we can assume that f is supported in a small disk around a singularity σ with angle $2\boldsymbol{\pi}\kappa$ (the terms away from the singularities will give rise to the terms f_i in the decomposition (2.5)). We have to construct a decomposition

$$f = f_\sigma + \sum_{H\in\mathcal{H}_\sigma} f_H \tag{2.6}$$

as in the statement of the lemma, where $\mathcal{H}_\sigma$ denotes the set of half-planes around σ. We assume $\|f\|_{-k_h,k_v} \leqslant 1$ for definiteness.

Let $\pi = \pi_\sigma$ be the covering projection, sending σ to 0. We may assume that $\pi^{-1}([-a,a]^2)$ only contains σ as a singularity, and that f is supported in $\pi^{-1}([-a/2,a/2]^2)$. Denote by $\omega = e^{2\mathbf{i}\boldsymbol{\pi}/\kappa}$ the fundamental κ-th root of unity. Let R be the rotation by $2\boldsymbol{\pi}$ around σ. For $q \in \mathbb{Z}/\kappa\mathbb{Z}$, let $f_q(z) = \kappa^{-1}\sum_{j=0}^{\kappa-1}\omega^{qj} f(R^j z)$. This is the component of f that is multiplied by ω^q when one turns by $2\boldsymbol{\pi}$ around σ. We have $f = \sum f_q$ by construction, and each f_q is C^∞, compactly supported, and satisfies $\|f_q\|_{\mathcal{B}^{-k_h,k_v}} \leqslant 1$ since this is the case for f.

The function f_0 is constant along the fibers of π. It will be the function f_σ in the decomposition of f. Consider now $q \neq 0$. We will first work in a chart U sent by π on $[-a,a]^2 - [0,\infty)$, i.e., a chart cut along the positive real axis. When one crosses this axis from top to bottom, the function f_q is multiplied by ω^q. We will use the canonical complex coordinates on U.

Let us first show the following: for $\varphi \in C_c^{k_h}([-a,a])$ and $j \leqslant k_v$, one has

$$\left|\int_{-a}^{0} \varphi \cdot L_v^j f_q \,\mathrm{d}x\right| \leqslant C\|\varphi\|_{C^{k_h}}. \tag{2.7}$$

The interest of this estimate is that φ is a priori not compactly supported in $[-a,0]$, so that this integral can not be controlled directly using $\|f_q\|_{-k_h,k_v}$.

For small $y > 0$ and $\varepsilon \in \{-1,1\}$, the interval $[-a,a] + \varepsilon\mathbf{i}y$ is included in U. Therefore,

$$\left|\int_{-a}^{a} \varphi(x) f_q(x + \varepsilon\mathbf{i}y)\,\mathrm{d}x\right| \leqslant \|\varphi\|_{C^{k_h}}. \tag{2.8}$$

Let y tend to 0. For $x \leqslant 0$, $f_q(x+\varepsilon\mathbf{i}y)$ tends to $f_q(x)$. On the other hand, for $x>0$, the limit depends on ε: one gets $f_q(x^+)$ for $\varepsilon=1$ and $f_q(x^-)=\omega^q f_q(x^+)$ for $\varepsilon=-1$. Hence,

$$\int_{-a}^{a}\varphi(x)f_q(x+\mathbf{i}y)\,\mathrm{d}x-\omega^{-q}\int_{-a}^{a}\varphi(x)f_q(x-\mathbf{i}y)\,\mathrm{d}x\to(1-\omega^{-q})\int_{-a}^{0}\varphi(x)f_q(x)\,\mathrm{d}x.$$

Combined with the control (2.8), this proves (2.7) for $j=0$ (for $C=2/|1-\omega^{-q}|$). The argument is the same for $j>0$.

Consider a C^∞ function ρ_2 which is equal to 1 on $[-a/2,a/2]^2$ and vanishes outside of $[-a,a]^2$. We define a function f_U on U by $f_U(x+\mathbf{i}y)=1_{x\leqslant 0}\rho_2(x+\mathbf{i}y)\sum_{j\leqslant k_v}y^jL_v^jf_q(x)$. This is a C^∞ function, compactly supported in $M-\Sigma$ (we recall that f, and therefore f_q, vanishes in a neighborhood of σ, so that $f_q(x)=0$ for x close to 0 in the chart U). This function is supported by U. Its interest is that its germ along $[-a,0]$ is the same as that of f_q. Moreover, it follows from (2.7) that the norm of f_U in $\mathcal{B}^{-k_h,k_v}$ is uniformly bounded. This function is supported in the left half-plane $H\in\mathcal{H}$ contained in U. Let us denote it by $f_{q,H}$. It will be part of the term f_H in the decomposition (2.6).

For each horizontal segment τ coming out of the singularity σ, one can consider a chart U as above cut along τ (with the difference that $[-a,a]^2$ can be cut along either the positive real axis, or the negative real axis, depending on τ), and then the associated function f_U. Let $\tilde f_q=f_q-\sum_U f_U$. This function is bounded by a constant in $\mathcal{B}^{-k_h,k_v}$. Its interest is that it vanishes along every horizontal segment coming out of σ, and moreover all its vertical derivatives up to order k_v also vanish there. In particular, the restriction of $\tilde f_q$ to any upper half-plane or lower half-plane $H\in\mathcal{H}$ is still C^{k_v} and it can be extended to the rest of the manifold by zero. Denote this extended function by $f_{q,H}$. It belongs to $\mathcal{B}^{-k_h,k_v}$ and has a bounded norm in this space, and it is supported in H.

Finally, the decomposition (2.6) of f is obtained by letting $f_\sigma=f_0$ and $f_H=\sum_{q\neq 0}f_{q,H}$. □

Proof of Proposition 2.9. Decomposing f as in Lemma 2.9, it suffices to show the inequality $\int fg\,\mathrm{dLeb}\leqslant C\leqslant C\|f\|_{\mathcal{B}^{-k_h,k_v}}\cdot\|g\|_{\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}}$ when f is:

(1) supported away from the singularities,
(2) or supported on a small neighborhood of a singularity, and constant on the fibers of the covering projection,
(3) or supported in a half-plane close to a singularity.

For definiteness, we will also assume $\|f\|_{\mathcal{B}^{-k_h,k_v}}\leqslant 1$ and $\|g\|_{\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}}\leqslant 1$.

Let us first handle the case where f is supported in a small rectangle $[-a,a]^2$ away from the singularities. We can even assume that f is supported in $[-a/4,a/4]^2$. Multiplying g by a cutoff function, we can assume that it is also supported in $[-a/2,a/2]^2$.

Using a local chart, we may work in $\mathbb{C}$. Along the horizontal interval $[-a,a]+\mathbf{i}y$, the successive primitives of $F_0=f$ vanishing at $-a+\mathbf{i}y$ are given by

$$F_k(x+\mathbf{i}y)=\int_{-a}^{x}f(t+\mathbf{i}y)(x-t)^{k-1}/(k-1)!\,\mathrm{d}t, \tag{2.9}$$

as one checks easily by induction over k. Let us take $k = k_h + 2$. With k integrations by parts, one gets

$$\int_{[-a,a]+\mathbf{i}y} fg \,\mathrm{d}x = (-1)^k \int_{[-a,a]+\mathbf{i}y} F_k \cdot L_h^k g \,\mathrm{d}x. \tag{2.10}$$

Let us consider a function $\rho(x)$ equal to 1 for $x \geqslant -a/2$ and vanishing on a neighborhood of $-a$. As f is supported by $[-a/2, a/2]^2$, one has

$$F_k(x + \mathbf{i}y) = \int_{-a}^{a} f(t + \mathbf{i}y) \cdot \rho(t) 1_{t \leqslant x} (x - t)^{k-1}/(k-1)! \,\mathrm{d}t.$$

The function

$$t \mapsto \rho(t) 1_{t \leqslant x} (x-t)^{k-1}/(k-1)! \tag{2.11}$$

is of class C^{k-2} on $[-a, a]$, with a bounded C^{k-2} norm: Its singularity at x is a zero of order $k-1$ to the left of x, and of infinite order to the right of x, so that everything matches in C^{k-2} topology. Therefore, by the definition of $\mathcal{B}^{-k_h, k_v}$ and the choice $k = k_h + 2$, one has $|F_k(x + \mathbf{i}y)| \leqslant C$ as $\|f\|_{\mathcal{B}^{-k_h,k_v}} \leqslant 1$. In the same way, the vertical derivatives of F_k involve vertical derivatives of f, which can be integrated against C^{k_h} functions along horizontals. We get, for all $j \leqslant k_v$ and all $x + \mathbf{i}y \in [-a, a]^2$, the inequality $\left| L_v^j F_k(x + \mathbf{i}y) \right| \leqslant C$. Therefore, along any vertical segment of the form $x + \mathbf{i}[-a, a]$, the function F_k is C^{k_v} with bounded norm, and it is compactly supported as it vanishes for $|y| \geqslant a/2$ (as f is supported by $[-a/2, a/2]^2$).

Let us integrate the equality (2.10) with respect to y. We get

$$\int fg \,\mathrm{dLeb} = (-1)^k \int_{x \in [-a,a]} \left(\int_{x + \mathbf{i}[-a,a]} F_k \cdot L_h^k g \,\mathrm{d}y \right) \mathrm{d}x. \tag{2.12}$$

When x is fixed, every integral $\int_{x+\mathbf{i}[-a,a]} F_k \cdot L_h^k g \,\mathrm{d}y$ is the integral against a C^{k_v} function with bounded norm of the function $L_h^k g$, with $k \leqslant \check{k}_h$ and $k_v \geqslant \check{k}_v$ by assumption. By definition, this integral is bounded by $\|F_k\|_{C^{k_v}} \|g\|_{\check{\mathcal{B}}^{\check{k}_h, -\check{k}_v}} \leqslant C$. Integrating in x, we obtain the desired inequality $\left| \int fg \,\mathrm{dLeb} \right| \leqslant C$.

We still have to consider the case where f is supported in the neighborhood of a singularity σ with angle $2\boldsymbol{\pi}\kappa$. Multiplying g by a cutoff function, we can assume that g is also supported there. Write π for the corresponding covering projection, sending σ to 0. We may assume that $\pi^{-1}([-a, a]^2)$ only contains σ as a singularity, and that f and g are supported by $\pi^{-1}([-a/2, a/2]^2)$. We would like to carry out the same argument as before, but the function F_k one obtains by integrating along a horizontal line is smooth along vertical lines to the left of the singularity, but it is discontinuous on vertical lines on the right of the singularity, breaking the argument.

Assume first that f is invariant under the covering projection π. Denote by $\omega = e^{2\mathbf{i}\boldsymbol{\pi}/\kappa}$ the fundamental κ-th root of unity. Let R be the rotation by $2\boldsymbol{\pi}$ around σ. For $q \in \mathbb{Z}/\kappa\mathbb{Z}$, let $g_q(z) = \kappa^{-1} \sum_{j=0}^{\kappa-1} \omega^{qj} g(R^j z)$. This is the component of g that is multiplied by ω^q when one turns by $2\boldsymbol{\pi}$ around σ. For $q \neq 0$, the function fg_q is multiplied by ω^q when one turns around the singularity. Therefore, $\int fg_q \,\mathrm{dLeb} = \omega^q \int fg_q \,\mathrm{dLeb}$, which implies $\int fg_q \,\mathrm{dLeb} = 0$ (this is just the classical fact that two functions living in different irreducible representations are orthogonal). Let us now handle g_0. The functions f and g_0 are both R-invariant. They can be written as $\tilde f\circ\pi$ and $\tilde g\circ\pi$ where $\tilde f$ and $\tilde g$ are functions on $\mathbb{C}$ supported by $[-a/2,a/2]^2$. The norms of these functions (in $\mathcal{B}^{-k_h,k_v}$ and $\check{\mathcal{B}}^{\check k_h,-\check k_v}$ respectively) are bounded by 1. The case of functions away from singularities, that we have already treated, shows that $\left|\int \tilde f\tilde g \,\mathrm{dLeb}\right| \leqslant C$. This gives the same estimate for $\int f g_0 \,\mathrm{dLeb}$.

Assume now that f is supported in a vertical half-plane H, to the left of σ for instance. Let us show that

$$\left|\int f g \,\mathrm{dLeb}\right| \leqslant C. \tag{2.13}$$

We proceed like in the proof away from singularities, making integrations by parts along horizontals. Let F_j be the j-th primitive of f along horizontals, vanishing at $-a+\mathbf{i}y$. It is given by the formula (2.9). Then, we do $k=k_h+2$ integrations by parts along each horizontal line, to get

$$\int_{[-a,0]+\mathbf{i}y} f g \,\mathrm{d}x = (-1)^k \int_{[-a,0]+\mathbf{i}y} F_k\cdot L_h^k g\,\mathrm{d}x + \sum_{j<k} (-1)^j F_{j+1}(\mathbf{i}y) L_h^j g(\mathbf{i}y).$$

The difference with (2.10) is the boundary terms, due to the fact that g does not vanish on the line $x=0$. Integrating in y, we obtain

$$\int f g\,\mathrm{dLeb} = (-1)^k \int_{x\in[-a,0]} \left(\int_{x+\mathbf{i}[-a,a]} F_k\cdot L_h^k g\,\mathrm{d}y\right)\mathrm{d}x + \sum_{j<k}(-1)^j \left(\int_{\mathbf{i}[-a,a]} F_{j+1}\cdot L_h^j g\,\mathrm{d}y\right). \tag{2.14}$$

The first term is controlled as in the case away from singularities, as the function F_k is bounded and C^{k_v} along vertical segments since $k=k_h+2$. On the other hand, the boundary terms are more delicate. The difficulty is that, a priori, $F_{j+1}(\mathbf{i}y)$ is not bounded just in terms of $\|f\|_{\mathcal{B}^{-k_h,k_v}}$: The function (2.11) (with k replaced by j and $x=0$) is not C^{k_h} for $j<k$ because of its singularity at 0. Nevertheless, as the distribution f is supported in H, we may replace the function in (2.11) by another function which coincides with it on $[-a,0]$ and is C^{k_h} with bounded norm on $[-a,a]$, without changing the value of the integral. It follows that in fact $F_j(\mathbf{i}y)$ is bounded in terms of $\|f\|_{\mathcal{B}^{-k_h,k_v}}$. In the same way, its vertical derivatives are also bounded. As $g\in\check{\mathcal{B}}^{\check k_h,-\check k_v}$ has norm at most 1, we obtain (integrating on a segment with horizontal coordinate $-x$ with x small to avoid the singularity)

$$\left(\int_{-a}^{a} F_{j+1}(\mathbf{i}y)\cdot L_h^j g(-x+\mathbf{i}y)\,\mathrm{d}y\right) \leqslant C.$$

Letting x tend to 0, we obtain that the second term in (2.14) is uniformly bounded. This proves (2.13).

Finally, assume that f is supported in a horizontal half-plane H, for instance an upper half plane above σ. We proceed exactly as in the case without singularities, integrating by parts along horizontal segments. Let F_j be the j-th primitive of f that vanishes on $-a+\mathbf{i}(0,a]$. The only difference is at the end of the argument: the analog of (2.12) in our case is

$$\int_H f \cdot g \,\mathrm{dLeb} = (-1)^k \int_{x\in[-a,a]} \left(\int_{x+\mathbf{i}(0,a]} F_k \cdot L_h^k g \,\mathrm{d}y \right) \mathrm{d}x.$$

The function F_k is still smooth along vertical segments, with uniformly bounded derivatives. However, it is not compactly supported in $x + \mathbf{i}[0, a]$, which prevents us from writing.

$$\left| \int_{x+\mathbf{i}(0,a]} F_k \cdot L_h^k g \,\mathrm{d}y \right| \leqslant C \|g\|_{\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}}. \tag{2.15}$$

On the other hand, F_k vanishes on $[-a, a]$, as well as its successive derivatives. Indeed, f is supported in H and smooth vertically, so by approximating the left and half parts of the boundary of H from below one obtains this vanishing property. Therefore, we may extend F_k by 0 for points with negative imaginary part. This extension is still C^{k_v} along vertical lines. This justifies the inequality (2.15). Integrating in x, we obtain the desired inequality $\left|\int f \cdot g \,\mathrm{dLeb}\right| \leqslant C$. □

Lemma 2.11. *We have the following duality formulas for $f \in \mathcal{B}^{-k_h,k_v}$ and $g \in \check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$:*

$$\langle L_h f, g\rangle = -\langle f, L_h g\rangle, \quad \langle L_v f, g\rangle = -\langle f, L_v g\rangle. \tag{2.16}$$

Proof. It is enough to check these formulas for functions in $C_c^\infty(M - \Sigma)$, as they extend by density to the whole spaces thanks to Proposition 2.9. The function fg vanishes on a neighborhood of the singularities. Denote by Ω the complement of a union of small disks around the singularities such that $fg = 0$ outside of Ω. We have

$$\int_M L_h(fg) \,\mathrm{d}x \wedge \mathrm{d}y = \int_\Omega \mathrm{d}(fg \,\mathrm{d}y) = -\int_{\partial\Omega} fg \,\mathrm{d}y = 0.$$

Hence, $\int L_h f \cdot g \,\mathrm{dLeb} + \int f \cdot L_h g \,\mathrm{dLeb} = 0$. This proves the first identity in (2.16). The second one is identical, upon exchanging the roles of x and y. □

2.4. **Cohomological interpretation.** In the study of the Ruelle spectrum of pseudo-Anosov maps, a special role will be played by the elements of $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$. Heuristically, the relevant object associated to $f \in \mathcal{B}^{-k_h,k_v}$ is the current $f \,\mathrm{d}x$. When f satisfies additionally $L_v f = 0$, then the formal derivative of this current is $\mathrm{d}(f \,\mathrm{d}x) = (\partial_x f \,\mathrm{d}x + \partial_y f \,\mathrm{d}y) \wedge \mathrm{d}x$. The term $\mathrm{d}x \wedge \mathrm{d}x$ vanishes. When $L_v f = 0$, one has $\partial_y f = 0$, and one gets $\mathrm{d}(f \,\mathrm{d}x) = 0$. Therefore, the current $f \,\mathrm{d}x$ is closed. It defines a cohomology class in $H^1(M - \Sigma)$. We will give a more explicit description of this cohomology class, and show that it even belongs to $H^1(M)$ (i.e., it vanishes if one integrates it along a small path around a singularity).

Let γ be a continuous closed loop in $M - \Sigma$ and let $f \in \mathcal{B}^{-k_h,k_v} \cap \ker L_v$. We define the integral of f along γ, denoted by $\int_\gamma f \,\mathrm{d}x$, as follows. Deforming γ slightly, we can first transform it into a loop made of finitely many horizontal and vertical segments. In $\int_\gamma f \,\mathrm{d}x$, the vertical components of γ do not appear. For a horizontal component I, we would like it to contribute by $\int_I f \,\mathrm{d}x$, but this does not make sense since f can only be integrated against smooth functions, which is not the case for the characteristic function of I. Let us smoothen this function by adding to the end of I a smooth function going from 1 to 0. In the next horizontal interval J, that follows I in γ, on the contrary, we subtract φ (pushed forward by the vertical translation from I to J) to the characteristic function χ_J of J – this process changes it to the function $\chi_J - \varphi$, which is smooth. In this way, we obtain integrals that are well defined. As f is invariant under vertical holonomy by the assumption $L_v f = 0$, it follows that the result is independent of the choice of φ, and of the choice of the initial deformation of γ in $M - \Sigma$. This concludes the definition of $\int_\gamma f \,\mathrm{d}x$. This construction is reminiscent of [Buf14b, Paragraph 1.3], although the fact that our distributions can not be integrated against characteristic functions enforces an additional smoothing step in the definition above.

Proposition 2.12. *Let $f \in \mathcal{B}^{-k_h,k_v} \cap \ker L_v$. Then the integral $\int_\gamma f \,\mathrm{d}x$ only depends on the homology class of γ in $H_1(M)$. Therefore, the map $\gamma \mapsto \int_\gamma f \,\mathrm{d}x$ defines a linear map from $H_1(M)$ to $\mathbb{R}$, i.e., a cohomology class in $H^1(M)$ which we denote by $[f]$ or $[f \,\mathrm{d}x]$.*

Proof. The fact that $\int_\gamma f \,\mathrm{d}x$ only depends on the homology class of γ in $M - \Sigma$ follows directly from the definitions. The only assertion that remains to be checked is that this integral is not modified when one crosses a singularity. Equivalently, we have to show that $\int_\gamma f \,\mathrm{d}x = 0$ when γ is a positive path around a singularity σ.

Let π be the covering projection around σ, well defined on a neighborhood of size $\delta \in (0, \beta_0/10)$. Let us fix a function φ on $\mathbb{R}$ equal to 1 around 0, with support included in $[-\delta, \delta]$. For $y > 0$, we may construct a path γ around σ by considering $I_y^+ = \pi^{-1}([-\delta, \delta] + \mathbf{i}y)$ (a union of κ horizontal segments, where κ is the degree of σ), crossed negatively, and $I_y^- = \pi^{-1}([-\delta, \delta] - \mathbf{i}y)$ (a union of κ horizontal segments), crossed positively, as well as the corresponding vertical segments. Then

$$\int_\gamma f \,\mathrm{d}x = \int_{I_y^-} \varphi(x) f \,\mathrm{d}x - \int_{I_y^+} \varphi(x) f \,\mathrm{d}x \tag{2.17}$$

for any $y > 0$, by definition.

Let $\varepsilon > 0$. By definition of $\mathcal{B}^{-k_h,k_v}$, we may choose $g \in C_c^\infty(M - \Sigma)$ with $\|f - g\|_{-k_h,k_v} < \varepsilon$. When y tends to 0, we have $\int_{I_y^-} \varphi g \,\mathrm{d}x - \int_{I_y^+} \varphi g \,\mathrm{d}x \to 0$ as the horizontal segments compensate each other, and the singularity does not contribute as g vanishes close to σ. We can in particular choose y for which this quantity is less than ε. We have

$$\left| \int_{I_y^-} \varphi g \,\mathrm{d}x - \int_{I_y^-} \varphi f \,\mathrm{d}x \right| \leqslant \kappa \|\varphi\|_{C^{k_h}} \|g - f\|_{-k_h,k_v} \leqslant C\varepsilon,$$

as the integral along each of the κ horizontal segments composing I_y^- is bounded by $\|\varphi\|_{C^{k_h}} \|g - f\|_{-k_h,k_v}$. The same holds on I_y^+. Finally, we get $\left| \int_{I_y^-} \varphi f \,\mathrm{d}x - \int_{I_y^+} \varphi f \,\mathrm{d}x \right| \leqslant (2C + 1)\varepsilon$. This concludes the proof thanks to (2.17). □

By definition of cohomology, a closed current of degree 1 vanishes in cohomology if and only if it is the differential of a current of degree 0. In the case of currents in $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$, we will see that this primitive is of the same type in the next proposition. The primitive of the current $f \,\mathrm{d}x$ is obtained by integrating f along horizontal leaves. We will have to see that this makes sense, and that the primitive thus defined has all the required regularity properties. Equivalently, the primitive g has to satisfy $L_h g = f$.

Proposition 2.13. *Assume that there is no horizontal saddle connection. Consider* $f \in \mathcal{B}^{-k_h,k_v} \cap \ker L_v$ *such that* $[f] = 0 \in H^1(M)$*, with* $k_h > 0$*. Then there exists* $g \in \mathcal{B}^{-k_h+1,k_v} \cap \ker L_v$ *such that* $f = L_h g$*.*

Proof. Let x_0 be a basepoint, and F a horizontal half-line starting at x_0, positively oriented, which does not end at a singularity. Since we assume there is no horizontal saddle connection, it is dense. We identify it with $[0,\infty)$. We will denote by x_t the point of F at horizontal distance t of x_0. Choose on F a function ρ_0 equal to 1 in a neighborhood of x_0, and to 0 on $[\delta/2, +\infty)$, where δ is small enough that there is no singularity in the ball of radius 10δ around x_0.

Let φ be a C^{k_h-1} function on F with compact support. Let Φ be its unique primitive that vanishes at x_0. It is constant after some time T, equal to $\int \varphi$. Choose a time $t > T$ such that x_t belongs to the vertical segment of size δ through x_0 (it exists as the half-line F is dense). Consider then the function Φ_t equal to Φ on $[0,t]$, to $(\int \varphi)\cdot \rho_0$ on $[t, t+\delta]$ (where ρ_0 is pushed vertically to $[x_t, x_{t+\delta}]$), and to 0 further on. This is a function of class C^{k_h} with compact support in F, so that $\int_F \Phi_t f \,\mathrm{d}x$ is well defined. Then we define formally an object g by the formula

$$\int \varphi \cdot g \,\mathrm{d}x = -\int \Phi_t \cdot f \,\mathrm{d}x. \tag{2.18}$$

Let us first notice that this quantity does not depend on t. Indeed, if we choose another time $s > t$ such that x_s also belongs to the vertical segment of size δ through x_0, then the difference between these two quantities is given by $(\int \varphi) \int_\gamma f \,\mathrm{d}x$, where γ is the union of the piece of F between x_t and x_s, and a subsegment of the vertical segment through x_0. As $[f] = 0$, this integral vanishes. Note that, for now, g is only a distribution along F.

The interest of this definition is the following. If we prove that g defines a genuine element of $\mathcal{B}^{-k_h+1,k_v}$, we will have by definition of L_h that, for any function φ with compact support on a segment $I \subseteq F$,

$$\int_I \varphi \cdot L_h g \,\mathrm{d}x = -\int_I \varphi' \cdot g \,\mathrm{d}x = \int_I \Psi_t \cdot f \,\mathrm{d}x,$$

where Ψ_t is the primitive of φ' vanishing at x_0, extended to the right by $(\int \varphi')\rho_0 = 0$. Hence, $\Psi_t = \varphi$. This formula shows that $L_h g = f$, at least along subintervals of F. As we will see later that g is invariant under vertical holonomy, we will obtain $L_h g = f$ everywhere, as desired.

The same argument using $[f] = 0$ shows that, if two segments I and J of F are obtained one from the other by a vertical translation in a small chart without singularity, and if φ_I is a function on I, then $\int_I \varphi_I g \,\mathrm{d}x = \int_J \varphi_J g \,\mathrm{d}x$, where φ_J is the push-forward to J of φ_I by vertical translation. This makes it possible to define $\int_I \varphi g \,\mathrm{d}x$ for any horizontal segment I and any $\varphi \in C_c^{k_h-1}(I)$, by using the integral on a small vertical translate of I included in F. By the above, it does not depend on the choice of the translate.

Let $\delta > 0$ be such that any horizontal segment of length β_0 can be translated vertically, in the positive or negative direction, by at least δ. If T is large enough, then $F[0,T]$ is δ-dense in M. This implies that, to compute $\int_I \varphi g \,\mathrm{d}x$ for any interval I of length β_0, one can first translate it vertically to reduce the computation to an interval included in $F[0, T+\beta_0]$, and then use a time t independent of I. The function Φ_t obtained in this way has a C^{k_h} norm which is bounded by $C\|\varphi\|_{C^{k_h-1}}$. This shows that, uniformly in $I \in \mathcal{I}^h$,

$$\left|\int_I \varphi \cdot g \, \mathrm{d}x\right| \leqslant C\|\varphi\|_{C^{k_h-1}}.$$

Moreover, as g is locally invariant under vertical translations, we have $\int_I \varphi \cdot L_v^j g \, \mathrm{d}x = 0$ for all $j > 0$. Therefore, g satisfies all the inequalities that are satisfied by the elements of $\mathcal{B}^{-k_h+1,k_v}$.

However, this is not enough to conclude that g is indeed an element of $\mathcal{B}^{-k_h+1,k_v}$. We should come back to the definition of this space as the closure of $C_c^\infty(M-\Sigma)$, and show that g is a limit of smooth functions with compact support. This is the hardest part of the proof, as one may not regularize g blindly by convolving it with a smooth kernel along horizontal segments: this fails for segments that hit the singularity. We prove the statement locally, as one can then extend it using a partition of unity. We treat the harder case of the neighborhood of a singularity σ, the case away from singularities is easier. Let $\pi : U \to \mathbb{C}$ be the covering projection of a neighborhood U of σ in $\mathbb{C}$, sending σ to 0. We write $U_r = \pi^{-1}([-r,r] + \mathbf{i}[-r,r])$. Let $a > 0$ be small enough. We fix a smooth function ρ that is equal to 1 on U_{4a} and vanishes outside of U_{5a}.

By assumption, f itself is the limit in $\mathcal{B}^{-k_h,k_v}$ of a sequence of functions $f_n \in C_c^\infty(M-\Sigma)$. Let us consider around σ the function g_n^0 which is a primitive of f_n along every horizontal segment, and vanishes on the vertical segments going through σ. Then $\rho g_n^0 \in C_c^\infty(M-\Sigma)$. However, g_n^0 will not converge in general to g, as one has to adjust integration constants. The difficulty is that, if we adjust the integration constant by considering what happens to the left of σ in complex charts (i.e., on the set of points whose image under π has negative real part), then this integration constant will behave nicely along vertical segments to the left of σ, but it will be discontinuous along vertical segments to the right of σ. The converse problem shows up if we fix the integration constant by using what happens to the right of σ. The idea will be to have two integration constants, coming from the left and from the right, and to show that they are necessarily close.

Let η be a nonnegative C^∞ function on $\mathbb{R}$ with support in $[0,a]$ and with integral 1. We will write η_t for $\eta(\cdot - t)$, whose support is contained in $[t, t+a]$. Given a point y on a vertical segment through σ, we write

$$\begin{aligned} c_n^+(y) &= \int_{[6a,7a]+\mathbf{i}y} \eta_{6a} g \, \mathrm{d}x - \int_{[6a,7a]+\mathbf{i}y} \eta_{6a} g_n^0 \, \mathrm{d}x, \\ c_n^-(y) &= \int_{[-7a,-6a]+\mathbf{i}y} \eta_{-7a} g \, \mathrm{d}x - \int_{[-7a,-6a]+\mathbf{i}y} \eta_{-7a} g_n^0 \, \mathrm{d}x \end{aligned}$$

(where we used the local complex coordinates given by π). These functions are uniformly bounded. As g is invariant under vertical shift and as g_n^0 is C^∞, they are smooth along vertical segments. More precisely, c_n^+ is C^∞ along vertical segments on the right of the singularity (in the chart π), while c_n^- is C^∞ along vertical segments to the left of the singularity.

We claim that, for y as above, for any function $\varphi \in C_c^{k_h-1}([-3a,3a]+\mathbf{i}y)$ with norm at most 1, and for any sign $s=\pm$,

$$\left| \int \rho\varphi g \,\mathrm{d}x - \int \rho\varphi g_n^0 \,\mathrm{d}x - \left(\int \rho\varphi\right) c_n^s(y) \right| \leqslant C\|f-f_n\|_{-k_h,k_v}, \tag{2.19}$$

where C does not depend on n. Let us prove this for $s=+$ for instance. By density of F and by continuity of all the objects under consideration, it suffices to prove it if $y\in F$. The function $\rho\varphi - (\int \rho\varphi)\eta_{6a}$ has a vanishing integral on $[-3a,7a]+\mathbf{i}y$. Its primitive Φ vanishing at $-3a+\mathbf{i}y$ also vanishes at $7a+\mathbf{i}y$. The definition of g entails

$$\int \left(\rho\varphi - \left(\int \rho\varphi\right)\eta_{6a}\right) g = -\int \Phi f.$$

Moreover,

$$\int \left(\rho\varphi - \left(\int \rho\varphi\right)\eta_{6a}\right) g_n^0 = \int \Phi' g_n^0 = -\int \Phi (g_n^0)' = -\int \Phi f_n.$$

Taking the difference between these two equations and using the definition of $c_n^+(y)$ yields

$$\int \rho\varphi g - \int \rho\varphi g_n^0 - \left(\int \rho\varphi\right) c_n^+(y) = \int \Phi f_n - \int \Phi f.$$

Thanks to the definition of the norm, this proves (2.19) since Φ is C^{k_h} with norm and support uniformly bounded.

Let us now consider a function φ supported by $[-3a,3a]$ with integral 1. We have $\int \rho\varphi = 1$ if $|y|\leqslant 3a$ by definition of ρ. Using the inequalities (2.19) with the signs $+$ and $-$ and taking their differences, we get in particular

$$|c_n^+(y) - c_n^-(y)| \leqslant C\|f-f_n\|_{-k_h,k_v}. \tag{2.20}$$

Let h_n be a smooth function on $\mathbb{R}$ equal to 0 in a neighborhood of 0 and to 1 for $|x|\geqslant 1/n$. We define g_n by $g_n(x+\mathbf{i}y) = g_n^0(x+\mathbf{i}y) + c_n^{\operatorname{sgn} x}(y) h_n(x)$. This is a C^∞ function on U_{3a}, vanishing in a neighborhood of σ. Let $\bar\rho$ be a smooth function equal to 1 on U_a, vanishing outside of U_{2a}. Let us show that $\bar\rho g_n$ converges to $\bar\rho g$ in $\mathcal{B}^{-k_h+1,k_v}$, to conclude the proof.

We first control what happens without vertical derivatives. Let I be a horizontal interval. We may assume that it is close to σ, at height y with $|y|<2a$, otherwise $\bar\rho$ vanishes on I and everything is trivial. Consider also $\varphi\in C_c^{k_h-1}(I)$. Then

$$\begin{aligned}\int_I \varphi\cdot\bar\rho g \,\mathrm{d}x - \int_I \varphi\cdot\bar\rho g_n \,\mathrm{d}x &= \int_I \rho\cdot\bar\rho\varphi\cdot g\,\mathrm{d}x - \int_I \rho\cdot\bar\rho\varphi\cdot g_n^0\,\mathrm{d}x - \int_I \rho\cdot\bar\rho\varphi c_n^{\operatorname{sgn} x}(y) h_n(x)\,\mathrm{d}x \\ &= \left(\int_I \rho\cdot\bar\rho\varphi\right) c_n^+(y) - \int_I \rho\cdot\bar\rho\varphi\cdot c_n^{\operatorname{sgn} x}(y) h_n(x)\,\mathrm{d}x + O(\|f-f_n\|_{-k_h,k_v}),\end{aligned}$$

where the first equality comes from the definition of g_n, and the second one from (2.19). In the last integral, if one replaces $c_n^-(y)$ by $c_n^+(y)$, one makes a mistake which is bounded by $C\|f-f_n\|_{-k_h,k_v}$, thanks to (2.20). We are left with

$$c_n^+(y)\cdot \int_I \rho\cdot\bar\rho\varphi\cdot(1-h_n(x))\,\mathrm{d}x + O(\|f-f_n\|_{-k_h,k_v}).$$

Since $1-h_n$ is supported in an interval of length $2/n$ and since the function $\rho\cdot\bar\rho\varphi$ is uniformly bounded, as well as c_n^+, this quantity is bounded by $C/n + C\|f-f_n\|_{-k_h,k_v}$, which tends to 0 with n. We have therefore proved that $\|\bar\rho g_n - \bar\rho g\|_{-k_h+1,0} \to 0$.

Let us then consider what happens with successive derivatives in the vertical direction. In $L_v^j(\bar\rho g)$, if one differentiates $\bar\rho$, then the number of derivatives of g is less than j, and one concludes by induction. We are left with proving the convergence to 0 of

$$\int_I \varphi\cdot\bar\rho L_v^j g \,\mathrm{d}x - \int_I \varphi\cdot\bar\rho L_v^j g_n \,\mathrm{d}x.$$

As the vertical derivative of g vanishes, the first term is 0. For the second term, the vertical derivatives of f_n, integrated against a smooth function, are small since they are close to the corresponding term for f, which vanishes as $L_v f = 0$. Integrating horizontally, we deduce that the vertical derivatives of g_n^0 are small in the distributional sense. As a consequence, the vertical derivatives of c_n^+ and c_n^- are also small. The same is true for the vertical derivatives of g_n. This concludes the proof. □

The following lemma will be very important for us, to show that the eigenvalue λ^{-1} of a pseudo-Anosov map acting on $H^1(M)$ does not show up in its Ruelle spectrum.

Lemma 2.14. *There is no $f\in\mathcal{B}^{-k_h,k_v}\cap\ker L_v$ with $[f]=[\mathrm{d}y]$.*

Proof. We argue by contradiction, assuming that $f\in\mathcal{B}^{-k_h,k_v}\cap\ker L_v$ satisfies $[f]=[\mathrm{d}y]$. Increasing k_h (which only makes the space larger), we can assume $k_h\geqslant 1$. Since f is in the kernel of L_v, its vertical smoothness is infinite, so we can also assume $k_v\geqslant 3$. We claim that, in this case, there exists $g\in\mathcal{B}^{-k_h+1,k_v}$ with $L_h g = f$ and $L_v g = 1$.

We follow the construction in Proposition 2.13 to construct the primitive g of f. Let us use all the notations of the corresponding proof. In particular, let F be a half-infinite horizontal leaf starting at a point x_0, and let x_t be the point at distance t of x_0 in F, and let ρ_0 be a function on F which is equal to 1 on a neighborhood of x_0 and to 0 on $[\delta/2,+\infty[$, where δ is small enough.

Let φ be a C^{k_h-1} function on F, with compact support. Denote by Φ its primitive that vanishes at 0. It is eventually constant and equal to $\int\varphi$ after some time T. Choose $t>T$ such that x_t belongs to the vertical segment of size δ through x_0 (such a time exists as the half-leaf F is dense), at a vertical distance $y(x_t)$. Let us consider the function Φ_t equal to Φ on $[0,t]$, to $(\int\varphi)\cdot\rho_0$ on $[t,t+\delta]$ (where ρ_0 is pushed vertically to $[x_t,x_{t+\delta}]$), and to 0 afterwards. This is a compactly support C^{k_h} function on F. Therefore, $\int_F \Phi_t f\,\mathrm{d}x$ is well defined. Let us define formally

$$\int\varphi\cdot g\,\mathrm{d}x = -\int\Phi_t\cdot f\,\mathrm{d}x - y(x_t)\cdot\int\varphi. \tag{2.21}$$

The last term is the only difference with (2.18).

This quantity does not depend on t. Indeed, choose $s > t$ such that x_s is also on the vertical leaf of size δ through x_0. Then

$$\left(-\int \Phi_s \cdot f \,\mathrm{d}x - y(x_s)\cdot \int \varphi\right) - \left(-\int \Phi_t \cdot f \,\mathrm{d}x - y(x_t)\cdot \int \varphi\right) = -\left(\int \varphi\right)\left(\int_\gamma f \,\mathrm{d}x + y(x_s) - y(x_t)\right),$$

where γ is the union of the piece of F between x_t and x_s, and of the small vertical segment between x_s and x_t. As $[f] = [\mathrm{d}y]$, we have $\int_\gamma f \,\mathrm{d}x = y(x_t) - y(x_s)$. Therefore, the above difference vanishes.

Let I_0 be a subsegment of F, let φ be a compactly supported function on I_0, let I_ε be a vertical translate of I_0 by a small parameter ε so that there is no singularity in between and so that I_ε is also included in F. Then we have

$$\int_{I_\varepsilon} \varphi \cdot g \,\mathrm{d}x - \int_{I_0} \varphi \cdot g \,\mathrm{d}x = \left(\int \varphi\right)\varepsilon. \tag{2.22}$$

Indeed, let us use in Definition (2.21) a time t which is large enough to work as well for I_0 and I_ε. The difference between the primitives of φ on I_0 and I_ε is then supported on the subsegment of F between I_0 and I_ε, and is equal to $\int \varphi$ except in the boundaries I_0 and I_ε. We obtain

$$\int_{I_\varepsilon} \varphi \cdot g \,\mathrm{d}x - \int_{I_0} \varphi \cdot g \,\mathrm{d}x = -\left(\int \varphi\right) \int_\gamma f \,\mathrm{d}x,$$

where γ is made of a horizontal piece of F and of the vertical segment between the left endpoints of I_ε and I_0, with length ε. As $[f] = [\mathrm{d}y]$, we have $\int_\gamma f \,\mathrm{d}x = \int_\gamma \mathrm{d}y = -\varepsilon$. This proves (2.22).

We can then extend by continuity g to all horizontal segments, ensuring that (2.22) is always satisfied. Then, by definition, $L_v g = 1$ in the distributional sense. It remains to check that g belongs to $\mathcal{B}^{-k_h+1,k_v}$. The argument is completely identical to the corresponding argument in the proof of Proposition 2.13.

We have obtained $g \in \mathcal{B}^{-k_h+1,k_v}$ with $L_v g = 1$. With the duality from Lemma 2.11, we get

$$\operatorname{Leb} M = \langle 1, 1\rangle = \langle L_v g, 1\rangle = -\langle g, L_v 1\rangle = 0.$$

This is a contradiction, concluding the proof of the lemma. □

3. The Ruelle spectrum of pseudo-Anosov maps with orientable foliations

Let T be a pseudo-Anosov map preserving orientations, on a translation surface (M, Σ). This section is devoted to the description of its Ruelle spectrum, culminating with the proof of Theorem 1.4.

3.1. Quasi-compactness of the transfer operator.

In this paragraph, we show that the operator $\mathcal{T}$ of composition with T acts on $\mathcal{B}^{-k_h,k_v}$, and is quasi-compact with a small essential spectral radius. Namely:

Theorem 3.1. *The operator $\mathcal{T}$ acting on $\mathcal{B}^{-k_h,k_v}$ has a spectral radius bounded by 1, and an essential spectral radius bounded by $\lambda^{-\min(k_h,k_v)}$.*

The proof will use a Lasota-Yorke inequality given in the next proposition.

Proof of Theorem 3.1 assuming Proposition 3.2. This follows readily from Hennion's Theorem [Hen93], from the compact embedding proposition 2.8 and from the Lasota-Yorke inequality given in Proposition 3.2. □

Proposition 3.2. *Let $k_h, l_v \geqslant 0$. The operator $\mathcal{T} : f \mapsto f \circ T$, initially defined for $f \in C_c^\infty(M - \Sigma)$, extends to a continuous linear operator on $\mathcal{B}^{-k_h,k_v}$, whose iterates are uniformly bounded. Moreover, it satisfies the inequality*

$$\|\mathcal{T}^n f\|_{-k_h,k_v} \leqslant C\lambda^{-\min(k_h,k_v)n}\|f\|_{-k_h,k_v} + C_n\|f\|_{-k_h-1,k_v-1}, \tag{3.1}$$

where C and C_n are constants that do not depend on f. (When $k_v = 0$, the last term should be omitted).

Proof. Assume that we can prove the inequality (3.1) for $f \in C_c^\infty(M - \Sigma)$. Then, it extends to $\mathcal{B}^{-k_h,k_v}$ by density, and proves that $\mathcal{T}$ acts continuously on this space thanks to the inclusion $\mathcal{B}^{-k_h,k_v} \subseteq \mathcal{B}^{-k_h-1,k_v-1}$.

Let us now prove (3.1) for smooth f. In the course of the proof, we will also establish the boundedness of the iterates of $\mathcal{T}$ on $\mathcal{B}^{-k_h,k_v}$. First, we estimate the contribution of $\|\mathcal{T}^n f\|'_{-k_h,k_v}$ to $\|\mathcal{T}^n f\|_{-k_h,k_v}$. Consider $I \in \mathcal{I}^h$ and $\varphi \in C_c^{k_h}(I)$ with norm at most 1, and compute

$$\int_I \varphi \cdot L_v^{k_v}(f \circ T^n)\,\mathrm{d}x = \lambda^{-k_v n}\int_I \varphi \cdot (L_v^{k_v} f) \circ T^n\,\mathrm{d}x = \lambda^{-k_v n}\cdot\lambda^{-n}\int_{T^n I} \varphi \circ T^{-n} \cdot L_v^{k_v} f\,\mathrm{d}x.$$

Let us then introduce a partition of unity ρ_p on $T^n I$ into smooth functions with supports of size $\leqslant \beta_0$ and bounded intersection multiplicity. Thus, we decompose $T^n I$ as a union of at most $C\lambda^n$ intervals in $\mathcal{I}^h$. On each of these intervals, the integral is bounded by $C\|f\|'_{-k_h,k_v}$ as the function $\varphi \circ T^{-n} \cdot \rho_p$ has a C^{k_h}-norm which is uniformly bounded (this is the case for φ and ρ_p, and the map T^{-n} only makes things better as it is a uniform contraction by λ^{-n}). Summing over p, we get a bound $C\lambda^{-k_v n}\|f\|'_{-k_h,k_v}$. Hence,

$$\|\mathcal{T}^n f\|'_{-k_h,k_v} \leqslant C\lambda^{-k_v n}\|f\|_{-k_h,k_v}. \tag{3.2}$$

If we use the same argument with a norm involving $j < k_v$ stable derivatives, we get a weaker gain λ^{-jn}. Summing over j, this shows that the iterates of $\mathcal{T}$ are uniformly bounded on $\mathcal{B}^{-k_h,k_v}$, but this is not enough to prove (3.1). To prove it, we will take advantage of the expansion in the horizontal direction, which we have not used yet. We can extend I in one of the two horizontal directions without meeting a singularity, for instance to its right, to an interval $I' \in \mathcal{I}^h_{2\beta_0}$. Let $\varphi_\varepsilon = \varphi \star \theta_\varepsilon$ where θ_ε is a kernel supported on $[0, \varepsilon]$, and $\varepsilon < \beta_0$ is a small parameter that will be chosen later on, depending on n. (If the interval I had been extended to its left, we would have taken the support of θ_ε in $[-\varepsilon, 0]$). Then φ_ε is compactly supported in I' if $\varepsilon < \beta_0$, and it satisfies

$$\|\varphi - \varphi_\varepsilon\|_{C^{k_h-1}} \leqslant C\varepsilon, \quad \|\varphi_\varepsilon\|_{C^{k_h}} \leqslant C, \quad \|\varphi_\varepsilon\|_{C^{k_h+1}} \leqslant C/\varepsilon. \tag{3.3}$$

Let us compute as above, introducing a partition of unity ρ_p on $T^n I'$. We get

$$\int_I \varphi \cdot L_v^j(f\circ T^n)\,\mathrm{d}x = \lambda^{-jn}\cdot\lambda^{-n}\sum_p \int_{I_p}(\varphi-\varphi_\varepsilon)\circ T^{-n}\cdot\rho_p\cdot L_v^j f\,\mathrm{d}x + \lambda^{-jn}\cdot\lambda^{-n}\sum_p\int_{I_p}\varphi_\varepsilon\circ T^{-n}\cdot\rho_p\cdot L_v^j f\,\mathrm{d}x.$$

In the second sum, the test function $\varphi_\varepsilon\circ T^{-n}\cdot\rho_p$ has a C^{k_h+1} norm which is bounded by C/ε. As the number j of derivatives we consider is $< k_v$, we deduce that this term is bounded by $C\lambda^{-jn}\varepsilon^{-1}\|f\|_{-k_h-1,k_v-1} \leqslant C(\varepsilon,n)\|f\|_{-k_h-1,k_v-1}$. In the first sum, the first k_h-1 derivatives of $(\varphi-\varphi_\varepsilon)\circ T^{-n}$ are bounded by $C\varepsilon$, as this already holds for $\varphi-\varphi_\varepsilon$ by (3.3). The k_h-th derivative of $\varphi-\varphi_\varepsilon$ is only bounded by a constant. As T^{-n} contracts by λ^{-n}, the k_h-th derivative of $(\varphi-\varphi_\varepsilon)\circ T^{-n}$ is therefore bounded by $C\lambda^{-k_h n}$. Hence, taking $\varepsilon=\lambda^{-k_h n}$, we get $\|(\varphi-\varphi_\varepsilon)\circ T^{-n}\|_{C^{k_h}}\leqslant C\lambda^{-k_h n}$. Multiplying by ρ_p (whose derivatives are all bounded) and then integrating and summing, we find that the first sum is bounded by $C\lambda^{-(j+k_h)n}\|f\|_{-k_h,k_v}\leqslant C\lambda^{-k_h n}\|f\|_{-k_h,k_v}$.

Finally, we have proved that, for $j<k_v$,

$$\|\mathcal{T}^n f\|'_{-k_h,j}\leqslant C\lambda^{-k_h n}\|f\|_{-k_h,k_v}+C_n\|f\|_{-k_h-1,k_v-1}.$$

Together with the inequality (3.2), we get the conclusion of the proposition. □

Theorem 3.1 shows that the spectrum of $\mathcal{T}$ acting on $\mathcal{B}^{-k_h,k_v}$ is discrete in $\{z : |z| > \lambda^{-\min(k_h,k_v)}\}$, made of at most countably many eigenvalues which are all discrete and of finite multiplicity. A priori, the spectrum could depend on the space $\mathcal{B}^{-k_h,k_v}$ we consider. However, all these spaces contain the dense subspace $C_c^\infty(M-\Sigma)$ and they are all continuously embedded in the distribution space $\mathcal{D}^\infty(M-\Sigma)$. A theorem of Baladi-Tsujii [BT08, Lemma A.1] then ensures that the spectrum (and even the eigenspaces, considered as subspaces of the space of distributions) do not depend on the space one considers, if one is beyond the essential spectral radius. Hence, it makes sense to talk about the *spectrum* of $\mathcal{T}$, independently of the space $\mathcal{B}^{-k_h,k_v}$. We have proved the existence of a Ruelle spectrum for T in the sense of Definition 1.1. To complete the proof of Theorem 1.4, we still have to identify this spectrum.

For $\alpha\neq 0$, let us denote by $E_\alpha^{(1)}$ the eigenspace corresponding to the eigenvalue α, and by E_α the corresponding generalized eigenspace (containing the eigenvectors and more generally the generalized eigenvectors, i.e., such that $(\mathcal{T}-\alpha I)^k f=0$ for some $k>0$). They are included in $\mathcal{B}^{-k_h,k_v}$ when $|\alpha|>\lambda^{-\min(k_h,k_v)}$.

3.2. **Description of the spectrum.** To describe the spectrum, we will rely crucially on the action of the operators L_h and L_v.

Proposition 3.3. *We have $\mathcal{T}\circ L_v=\lambda L_v\circ\mathcal{T}$ on $C_c^\infty(M-\Sigma)$. This equality still holds on all spaces to which these operators extend continuously, in particular as operators from $\mathcal{B}^{-k_h,k_v}$ to $\mathcal{B}^{-k_h,k_v-1}$ when $k_v>0$.*

In the same way, $\mathcal{T}\circ L_h=\lambda^{-1}L_h\circ\mathcal{T}$ on $C_c^\infty(M-\Sigma)$. This equality still holds on all spaces to which these operators extend continuously, in particular as operators from $\mathcal{B}^{-k_h,k_v}$ to $\mathcal{B}^{-k_h-1,k_v}$.

Proof. We compute: $(\mathcal{T} \circ L_v)(f) = (L_v f) \circ T$, and $(L_v \circ \mathcal{T})(f) = L_v(f \circ T) = \lambda^{-1}(L_v f) \circ T$ as T contracts by λ^{-1} in the vertical direction. This proves the desired equality for L_v. The argument is the same for L_h. □

Corollary 3.4. *The operator L_v sends E_α to $E_{\lambda\alpha}$. The operator L_h sends E_α to $E_{\lambda^{-1}\alpha}$.*

Proof. A generalized eigendistribution f for α satisfies $(\mathcal{T} - \alpha I)^k f = 0$ for large enough k. Moreover, we have $(\mathcal{T} - \lambda\alpha I) \circ L_v = \lambda L_v \circ (\mathcal{T} - \alpha I)$ by Proposition 3.3. By induction, $(\mathcal{T} - \lambda\alpha I)^k \circ L_v = \lambda^k L_v \circ (\mathcal{T} - \alpha I)^k$. Therefore, $(\mathcal{T} - \lambda\alpha I)^k (L_v f) = \lambda^k L_v((\mathcal{T} - \alpha I)^k f) = 0$. This shows that L_v maps E_α to $E_{\lambda\alpha}$. The argument is the same for L_h. □

Corollary 3.5. *For $f \in E_\alpha$, we have $L_v^k f = 0$ when k is large enough, more specifically when $\lambda^k|\alpha| > 1$.*

Proof. We have $L_v^k f \in E_{\lambda^k\alpha}$. This space is trivial if $|\lambda^k\alpha| > 1$ as the iterates of $\mathcal{T}$ are bounded on $\mathcal{B}^{-k_h,k_v}$ by Proposition 3.2. □

If we start from a nonzero generalized eigendistribution, we can consider the smallest k such that $L_v^k f = 0$. Then $L_v^{k-1} f$ is a generalized eigendistribution for $\mathcal{T}$, and it satisfies $L_v f = 0$. Such elements are the main building blocks to describe the spectrum of $\mathcal{T}$. We will take advantage of the cohomological description of such objects we have given in Paragraph 2.4 to go further in the description of the spectrum.

Let us now try to see if any cohomology class can be realized by elements in $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$ – and if the class is a (generalized) eigenfunction for the action of T on cohomology we will try to realize it by a (generalized) eigendistribution for $\mathcal{T}$, for the same eigenvalue. This is not always possible: if one considers the action of a linear Anosov matrix on the torus, then the cohomology has dimension 2, but the spectrum of $\mathcal{T}$ is reduced to $\{1\}$: it is not possible to realize in this way the cohomology class corresponding to the stable foliation. We will see that this is the only obstruction: all the other eigenvectors in cohomology (which correspond to eigenvalues in $(\lambda^{-1}, \lambda]$) can be realized.

Theorem 3.6. *Let $h \in H^1(M)$ be a cohomology class which is a generalized eigenfunction for the linear action of T on cohomology: we have $(T^* - \mu)^J h = 0$ for some $J \geqslant 1$ and some μ with $|\mu| \in [\lambda^{-1}, \lambda]$ (where $\mu = \lambda$ if and only if h is a multiple of the class of the horizontal foliation $\mathrm{d}x$, and $\mu = \lambda^{-1}$ if and only if h is a multiple of the class of the vertical foliation $\mathrm{d}y$). We assume $\mu \neq \lambda^{-1}$, i.e., we exclude multiples of $\mathrm{d}y$.*

Then, for $\min(k_h, k_v) \geqslant 3$, there exists $f \in \mathcal{B}^{-k_h,k_v} \cap \ker L_v$ in the generalized eigenspace $E_{\lambda^{-1}\mu}$ whose cohomology class $[f]$ is equal to h. In particular, if $h \neq 0$, the eigenspace is nontrivial.

Proof. Let ω be a closed 1-form with compact support in $M - \Sigma$ such that $[\omega] = h$, i.e., $\int_\gamma \omega = \langle h, \gamma \rangle$ for any closed curve γ. It is possible to choose such an ω which vanishes on a neighborhood of Σ as part of the long exact sequence in cohomology reads $H_c^1(M - \Sigma) \to H_c^1(M) \to H_c^1(\Sigma)$. As the last term is 0, the previous arrow is onto.

Let us write $\omega = \omega_x \,\mathrm{d}x + \omega_y \,\mathrm{d}y$ where ω_x and ω_y belong to $C_c^\infty(M - \Sigma)$. Then we have

$$(T^n)^*\omega = \lambda^n(\mathcal{T}^n\omega_x)\,\mathrm{d}x + \lambda^{-n}(\mathcal{T}^n\omega_y)\,\mathrm{d}y,$$

as T expands horizontally by λ and contracts vertically by λ.

Consider a closed path γ made of horizontal and vertical segments, away from the singularities. Denote by γ_t the same path but shifted horizontally by t. If t is small enough, it does not meet any singularity either. Let $\bar{\gamma} = \int_t \eta(t)\gamma_t$ where η is a smooth function whose support is small enough to ensure that this is well defined. This integral should be understood in the weak sense, i.e., for any form ω the integral of ω on $\bar{\gamma}$ is by definition $\int_t \eta(t)(\int_{\gamma_t} \omega)$. Then $\bar{\gamma}$ is made of horizontal segments weighted by a C^∞ compactly supported function – we denote this part by $\bar{\gamma}_h$ – and of vertical parts that we denote by $\bar{\gamma}_v$. Then

$$\int_{\bar{\gamma}_h} (\mathcal{T}^n \omega_x)\,\mathrm{d}x = \lambda^{-n} \int_{\bar{\gamma}} (T^n)^* \omega - \lambda^{-2n} \int_{\bar{\gamma}_v} (\mathcal{T}^n \omega_y)\,\mathrm{d}y.$$

The last integral is uniformly bounded as ω_y is a bounded function. Hence, its contribution is $O(\lambda^{-2n})$. In the first term, as $(T^n)^*\omega$ is closed, it is equivalent to integrate just on γ. This only depends on the homology class h of ω, which is a generalized eigenvector for T^*. By Jordan's decomposition, we may write

$$(T^n)^* h = \mu^n \sum_{j<J} n^j h_j,$$

with $h_0 = h$. We get

$$\int_{\bar{\gamma}_h} (\mathcal{T}^n \omega_x)\,\mathrm{d}x = \left(\int \eta\right) \cdot (\lambda^{-1}\mu)^n \sum_{j<J} n^j \langle h_j, \gamma\rangle + O(\lambda^{-2n}). \tag{3.4}$$

In $\mathcal{B}^{-k_h,k_v}$, we can write

$$\mathcal{T}^n \omega_x = \sum_{|r|\geqslant \lambda^{-2}} \sum_{j\leqslant C} r^n n^j f_{r,j} + R_n,$$

where r runs along the eigenvalues of modulus $\geqslant \lambda^{-2}$ of $\mathcal{T}$, the $f_{r,j}$ belong to E_r and R_n is a remainder term which decays faster than λ^{-2n}. Identifying the terms in the asymptotic (3.4) thanks to the assumption $|\mu| > \lambda^{-1}$ and using $h_0 = h$, we obtain for $f = f_{\lambda^{-1}\mu,0}$ the equality

$$\int_{\bar{\gamma}_h} f\,\mathrm{d}x = \left(\int \eta\right)\langle h, \gamma\rangle. \tag{3.5}$$

Let us show that f satisfies $L_v f = 0$. Consider a horizontal interval $I_0 = [0, q]$, a small vertical translate $I_\varepsilon = I_0 + \mathbf{i}\varepsilon$ of this interval (in a chart away from singularities), and a compactly supported test function φ_0 on I_0. We want to show that $\int_{I_0} \varphi_0 f\,\mathrm{d}x = \int_{I_\varepsilon} \varphi_\varepsilon f\,\mathrm{d}x$ where φ_ε is the vertical push-forward of φ_0 on I_ε. To do this, denote by γ_t the path from 0 to t then to $\mathbf{i}\varepsilon + t$ then to $\mathbf{i}\varepsilon$ then to 0. 0. Let also $\eta(t) = -\varphi_0'(t)$. In $\bar{\gamma} = \int \eta(t)\gamma_t\,\mathrm{d}t$, a point $x \in [0, q]$ is counted with a weight $\int_{t\in[x,q]} \eta(t)\,\mathrm{d}t = -\varphi_0(q) + \varphi_0(x) = \varphi_0(x)$. One can argue similarly along I_ε. Therefore, by definition, $\int_{I_0} \varphi_0 f\,\mathrm{d}x - \int_{I_\varepsilon} \varphi_\varepsilon f\,\mathrm{d}x = \int_{\bar{\gamma}_h} f\,\mathrm{d}x$. This integral vanishes by (3.5) as $\int \eta = 0$. This shows that f is invariant under vertical translation, i.e., $L_v f = 0$.

The cohomology class $[f]$ is then well defined by Proposition 2.12, as well as $\int_\gamma f\,\mathrm{d}x$ for any closed path. By definition of this integral, it coincides with $\int_{\bar{\gamma}_h} f\,\mathrm{d}x$ when $\bar{\gamma}$ is a smoothing of γ as above and η has integral 1. We deduce from (3.5) that $\int_\gamma f\,\mathrm{d}x = \langle h, \gamma\rangle$ for any closed path γ. By definition, this shows that $[f] = h$. □

We can use this statement to show that the spectrum of T contains the set mentioned in Theorem 1.4:

Corollary 3.7. *The Ruelle spectrum of T contains all the $\lambda^{-n}\mu$ for $n \geqslant 1$ and $\mu \in \Xi$, where Ξ is the spectrum of T^* on the subspace of $H^1(M)$ made of 1-forms which are orthogonal to $\mathrm{d}x$ and $\mathrm{d}y$, as in the statement of Theorem 1.4.*

Proof. Theorem 3.6 ensures that $\lambda^{-1}\mu$ belongs to the Ruelle spectrum of T. The map L_h is injective on the generalized eigenspace $E_{\lambda^{-1}\mu}$ by Lemma 2.7, as the kernel of L_h is included in E_1. It sends it to $E_{\lambda^{-2}\mu}$ by Corollary 3.4, hence this space is nontrivial. By induction, one proves in the same way that all the spaces $E_{\lambda^{-n}\mu}$ are nontrivial. □

Proposition 3.8. *For any $\alpha \neq 0$, the operator L_h is onto from $E_\alpha \cap \ker L_v$ to $E_{\lambda^{-1}\alpha} \cap \ker L_v \cap \ker[\cdot]$. It is bijective for $\alpha \neq 1$.*

Proof. First, L_h sends E_α to $E_{\lambda^{-1}\alpha}$ by Corollary 3.4. As it commutes with L_v, it even sends $E_\alpha \cap \ker L_v$ to $E_{\lambda^{-1}\alpha} \cap \ker L_v$. Let us show that its image is contained in $\ker[\cdot]$. Let $f \in \ker L_v$, we have to see that $[L_h f] = 0$. Consider a path γ made of horizontal and vertical segments. We compute $\int_\gamma L_h f \,\mathrm{d}x$ by coming back to its definition. Informally, we have $\int_\gamma L_h f \,\mathrm{d}x = \sum_I \int_I L_h f \,\mathrm{d}x$ where the sum is over horizontal parts of γ. With an integration by parts, $\int_\gamma L_h f \,\mathrm{d}x = \sum_I (f(y_I) - f(x_I))$ where y_I and x_I are the endpoints of I. As γ is a closed path and f is invariant vertically each $f(y_I)$ cancels out with $-f(x_J)$ where J is the horizontal interval following I in γ. We are left with $\int_\gamma L_h f \,\mathrm{d}x = 0$.

This computation is not rigorous as f can not be integrated against characteristic functions, and $f(y_I)$ makes no sense (f is only a distribution). This is why $\int_\gamma L_h f \,\mathrm{d}x$ is defined in Paragraph 2.4 by using a regularization of the characteristic function of I. The above argument works with the regularization. As f is vertically invariant, the contribution of the end of the interval I to $\int_\gamma L_h f \,\mathrm{d}x$ compensates exactly with the contribution of the beginning of the next interval, and we are left with $\int_\gamma L_h f = 0$ as desired.

It remains to show that $L_h : E_\alpha \cap \ker L_v \to E_{\lambda^{-1}\alpha} \cap \ker L_v \cap \ker[\cdot]$ is surjective (its bijectivity for $\alpha \neq 1$ follows directly as L_h is injective away from constants by Lemma 2.7). Fix $f \in E_{\lambda^{-1}\alpha} \cap \ker L_v \cap \ker[\cdot]$. By Proposition 2.13, if k_h and k_v are large enough, there exists $g \in \mathcal{B}^{-k_h+1,k_v}$ such that $L_v g = 0$ and $L_h g = f$. The question is whether one can take $g \in E_\alpha$.

Consider j such that $(\mathcal{T} - \lambda^{-1}\alpha)^j f = 0$. We have $(\mathcal{T} - \lambda^{-1}\alpha)^j \circ L_h = \lambda^{-j} L_h \circ (\mathcal{T} - \alpha)^j$ by Proposition 3.3. Therefore, $L_h((\mathcal{T} - \alpha)^j g) = 0$, i.e., there exists a constant c such that $(\mathcal{T} - \alpha)^j g = c$ by Lemma 2.7. If $\alpha \neq 1$, we have then $(\mathcal{T} - \alpha)^j (g - c/(1-\alpha)^j) = 0$. Therefore, $\tilde{g} = g - c/(1-\alpha)^j$ satisfies $\tilde{g} \in E_\alpha \cap \ker L_v$ and $L_h \tilde{g} = f$, as announced. If $\alpha = 1$, then $(\mathcal{T} - \alpha)^{j+1} g = (\mathcal{T} - 1)c = 0$, so g itself already belongs to E_α. □

There are two possible spectral values, corresponding to the eigenvalues λ and λ^{-1} of $T^* : H^1(M) \to H^1(M)$, i.e., to $\mathrm{d}x$ and $\mathrm{d}y$. They have a special status in Theorem 1.4: the first one is simple and does not interact with the rest of the spectrum, while the second one does not belong to the Ruelle spectrum. Let us now give the specific results about these values that we will need to classify the Ruelle spectrum.

Lemma 3.9. *The generalized eigenspace E_1 is one-dimensional, made of constants.*

Proof. The generalized eigenspace E_1 contains the constants as the function 1 belongs to $\mathcal{B}^{-k_h,k_v}$ by Lemma 2.5. Moreover, any element f of E_1 satisfies $L_v f = 0$ (as $L_v f$ belongs to E_λ by Corollary 3.4, and this space is trivial by Theorem 3.1). Therefore, there is a linear map $f \mapsto [f]$ from E_1 to $H^1(M)$, taking its values in the generalized eigenspace for the eigenvalue λ of T^*. This space has dimension 1. To conclude, it suffices to show that this map is injective, i.e., if $f \in E_1$ satisfies $[f] = 0$ then f vanishes. When $[f] = 0$, Proposition 3.8 shows that f can be written as $L_h g$ with $g \in E_\lambda$. As this space is trivial, we get $g = 0$ and then $f = 0$. □

We have almost all the tools to show that the Ruelle spectrum of T is given exactly by the set described in Theorem 1.4. More precisely, we can already show the following partial result.

Proposition 3.10. *The Ruelle spectrum of T is given exactly by the set described in Theorem 1.4, i.e., it is made of 1 and of the numbers $\lambda^{-n}\mu$ with $n \geqslant 1$ and $\mu \in \Xi$.*

Proof. On the one hand, 1 belongs to the spectrum by Lemma 3.9. On the other hand, for $\mu \in \Xi$ and $n \geqslant 1$, then $E_{\lambda^{-n}\mu}$ is nontrivial by Corollary 3.7. This shows one inclusion in the proposition.

For the converse, consider $\alpha \neq 0$ such that E_α is nontrivial, and take a nonzero $f \in E_\alpha$. Let $k \geqslant 0$ be the integer such that $L_v^k f \neq 0$ and $L_v^{k+1} f = 0$. It exists by Corollary 3.5. The function $f_k = L_v^k f$ belongs to $E_{\lambda^k \alpha}$ by Corollary 3.5, and to $\ker L_v$ by construction. If $[f_k] = 0$, Proposition 3.8 shows that there exists $f_{k+1} \in E_{\lambda^{k+1}\alpha} \cap \ker L_v$ with $L_h f_{k+1} = f_k$. If $[f_{k+1}] = 0$, we can iterate the same process. It has to stop at some point as $E_{\lambda^{k+n}\alpha}$ is trivial for n large. Therefore, we get an integer n and a distribution $f_{k+n} \in E_{\lambda^{k+n}\alpha} \cap \ker L_v$ with $L_h^n f_{k+n} = f_k$ and $[f_{k+n}] \neq 0$. The cohomology class $[f_{k+n}]$ belongs to the generalized eigenspace for $T^* : H^1(M) \to H^1(M)$ for the eigenvalue $\alpha' = \lambda^{k+n+1}\alpha$. We have $\alpha' \neq \lambda^{-1}$, since otherwise the corresponding cohomology class would be a nonzero multiple of $[\mathrm{d}y]$, contradicting Lemma 2.14. Hence, $\alpha' \in \Xi$ or $\alpha' = \lambda$. If $\alpha' \in \Xi$, we have written α as $\lambda^{-p}\alpha'$ with $p \geqslant 1$, in accordance with the claim of the proposition. If $\alpha' = \lambda$, then $f_{k+n} \in E_1$. By Lemma 3.9, f_{k+n} is constant. As $L_h^n f_{k+n} = f_k \neq 0$, we deduce $n = 0$. Then $L_v^k f = f_k$ is a nonzero constant c. Using the duality formula from Lemma 2.11, we get

$$c \operatorname{Leb} M = \langle f_k, 1\rangle = \langle L_v^k f, 1\rangle = -\langle f, L_v^k 1\rangle.$$

If k were nonzero, then $L_v^k 1$ would vanish and we would get a contradiction. Therefore, $k = 0$. Finally, $\alpha = 1$, again in accordance with the claim. □

The conclusion of the proof of Theorem 1.4 relies on the following statement.

Theorem 3.11. *Let $\alpha \notin \{0, 1\}$. Then $L_v : E_{\lambda^{-1}\alpha} \to E_\alpha$ is onto.*

Before proving the theorem, let us show how we can conclude the proof of Theorem 1.4.

Proof of Theorem 1.4 using Theorem 3.11. To simplify the notations, we will assume that for $\mu \in \Xi$ then $\lambda^{-1}\mu \notin \Xi$ (otherwise, there is a superposition phenomenon as explained after the statement of Theorem 1.4, which makes things more complicated to write but does not change anything to the proof).

In Proposition 3.10, we have described exactly the spectrum of T, and moreover we have shown how the generalized eigenspaces were constructed. On the one hand, there is the space E_1, which is one-dimensional by Lemma 3.9. On the other hand, for $\mu \in \Xi$, the space $E_{\lambda^{-1}\mu}$ is in bijection with the generalized eigenspace for the action of T^* on $H^1(M)$ and the eigenvalue μ, with dimension d_μ.

Finally, $E_{\lambda^{-(n+1)}\mu}$ is made of elements sent by L_v to $E_{\lambda^{-n}\mu}$, and of elements in $E_{\lambda^{-(n+1)}\mu} \cap \ker L_v$. Proposition 3.8 shows that L_h is a bijection between $E_{\lambda^{-n}\mu} \cap \ker L_v$ and $E_{\lambda^{-(n+1)}\mu} \cap \ker L_v$ (as, on the second space, the condition $[f] = 0$ is always satisfied thanks to our non-superposition assumption). Therefore, by induction, all these spaces have dimension d_μ. As $L_v : E_{\lambda^{-(n+1)}\mu} \to E_{\lambda^{-n}\mu}$ is onto by Theorem 3.11, we get

$$\dim E_{\lambda^{-(n+1)}\mu} = \dim E_{\lambda^{-n}\mu} + \dim E_{\lambda^{-(n+1)}\mu} \cap \ker L_v = \dim E_{\lambda^{-n}\mu} + d_\mu.$$

By induction, we obtain $\dim E_{\lambda^{-n}\mu} = n d_\mu$. In fact, we have even proved the flag decomposition expressed in (1.1). □

We recall that L_v sends $E_{\lambda^{-1}\alpha}$ to E_α by Corollary 3.4. To prove Theorem 3.11, the most natural approach would be to start from an element of E_α with $\alpha \notin \{0,1\}$ and to construct a preimage under L_v, by integrating along vertical lines as we did in the proof of Proposition 2.13. But we have no cohomological condition to use, and moreover we only have a distributional object for which the meaning of vertical integration is not clear. If one thinks about it, the result of the theorem is even counterintuitive.

Let us try to prove the opposite of Theorem 3.11, to see the subtlety. Assume for instance that $f \in E_\alpha$ is nonzero and satisfies $L_v f = 0$, and that we can find a vertical primitive g of f, i.e., one has $L_v g = f$. Let us try to prove that $f = 0$. We should not succeed (this would be a contradiction with Theorem 3.11), but we will see that there is a strong nonrigorous argument in favor of the equality $f = 0$. Consider an embedded rectangle with horizontal sides I_0 and I_R and very long vertical sides of length R. Fix a smooth compactly supported function φ on I_0, and push it vertically to I_R. We should have $\int_{I_R} \varphi g \,\mathrm{d}x - \int_{I_0} \varphi g \,\mathrm{d}x = R \int_{I_0} \varphi f \,\mathrm{d}x$. As the left hand side is bounded, we obtain

$$\int_{I_0} \varphi f \,\mathrm{d}x = O(\|\varphi\|_{C^{k_h}}/R).$$

Letting R tend to infinity, we can almost deduce that f vanishes, except that this argument is not correct as one can not take R arbitrarily large because of the singularities. If one tries to cut I_0 into smaller pieces for which one can increase R, then we will use a partition of unity with a large C^{k_h} norm, so that we will improve the bound at the level of $1/R$, but lose at the level of $\|\varphi\|_{C^{k_h}}$. Therefore, we can not prove in this way that f vanishes, so there is hope that Theorem 3.11 is true. But this shows that this theorem is non-trivial, and follows from a subtle balance.

The proof we will give of Theorem 3.11 will not follow the constructive approach we sketched above. Instead, it will follow from an indirect duality argument: we will show that the adjoint of L_v is injective. To do this, let us define the operator $\check{\mathcal{T}}$ which extends to $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ the operator $f \mapsto f \circ T^{-1}$ initially defined on $C_c^\infty(M-\Sigma)$. As T^{-1} is a pseudo-Anosov map, all the results of the previous paragraphs apply to $\check{\mathcal{T}}$. In particular, one can talk about its Ruelle spectrum. We will write $\check{E}_\alpha$ for the generalized eigenspace of $\check{\mathcal{T}}$ associated to the eigenvalue α, on any space $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ with $|\alpha| > \lambda^{-\min(\check{k}_h,\check{k}_v)}$.

From this point on, we will only consider non-negative integers k_h, k_v, $\check{k}_h$ and $\check{k}_v$ that satisfy the conditions of the duality Proposition 2.9, i.e., $-k_h + \check{k}_h \geqslant 2$ and $k_v - \check{k}_v \geqslant 0$ (or conversely). If we are dealing with an eigenvalue α, we will moreover choose them with $|\alpha| > \lambda^{-\min(k_h,k_v)}$ and $|\alpha| > \lambda^{-\min(\check{k}_h,\check{k}_v)}$ to ensure that the corresponding generalized eigenspaces for $\mathcal{T}$ and $\check{\mathcal{T}}$ are included respectively in $\mathcal{B}^{-k_h,k_v}$ and $\mathcal{B}^{\check{k}_h,-\check{k}_v}$. This implies in particular that the duality is well defined on $E_\alpha \times \check{E}_{\alpha'}$ for all $\alpha, \alpha' \neq 0$.

In addition to the duality formulas for L_h and L_v given in Lemma 2.11, we will also use the following one: For $f \in \mathcal{B}^{-k_h,k_v}$ and $g \in \check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$,

$$\langle \mathcal{T} f, g\rangle = \langle f, \check{\mathcal{T}} g\rangle. \tag{3.6}$$

It follows readily from the definitions and the fact that T preserves Lebesgue measure.

Lemma 3.12. *We have $\langle f, g\rangle = 0$ for $f \in E_\alpha$ and $g \in \check{E}_{\alpha'}$ with $\alpha \neq \alpha'$. Moreover, $(f, g) \mapsto \langle f, g\rangle$ is a perfect duality on $E_\alpha \times \check{E}_\alpha$, i.e., it identifies E_α with the dual of $\check{E}_\alpha$, and conversely.*

Proof. Take $f \in E_\alpha$. Then $\mathcal{T}^n f = \sum_{j\leqslant J} \alpha^n n^j f_j$ for some $f_j \in E_\alpha$, with $f_0 = f$. In the same way, for $g \in \check{E}_{\alpha'}$, we have $\check{\mathcal{T}}^n g = \sum_{j\leqslant J} (\alpha')^n n^j g_j$ for some $g_j \in \check{E}_{\alpha'}$ with $g_0 = g$. Using the duality (3.6), we obtain for all n

$$\sum \alpha^n n^j \langle f_j, g\rangle = \langle \mathcal{T}^n f, g\rangle = \langle f, \check{\mathcal{T}}^n g\rangle = \sum (\alpha')^n n^j \langle f, g_j\rangle.$$

When $\alpha \neq \alpha'$, one gets by identifying the asymptotics that $\langle f_j, g\rangle = 0$ for all j. In particular, for $j = 0$, this gives $\langle f, g\rangle = 0$ and shows that E_α and $\check{E}_{\alpha'}$ are orthogonal.

To prove that there is a perfect duality between E_α and $\check{E}_\alpha$, we have to show that the duality is nondegenerate: for any $f \in E_\alpha$, we have to find $g \in \check{E}_\alpha$ with $\langle f, g\rangle \neq 0$ (and conversely, but the argument is the same). As f is a distribution, there exists a function $h \in C_c^\infty(M - \Sigma)$ with $\langle f, h\rangle \neq 0$. We think of h as an element of $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$, and we write its spectral decomposition for $\check{\mathcal{T}}$: we have $\check{\mathcal{T}}^n h = \sum_{i,j} \alpha_i^n n^j h_{i,j} + O(\varepsilon^n)$ where $\varepsilon < |\alpha|$ and $h_{i,j} \in \check{E}_{\alpha_i}$. As above, using (3.6), we find

$$\sum \alpha^n n^j \langle f_j, h\rangle = \langle \mathcal{T}^n f, h\rangle = \langle f, \check{\mathcal{T}}^n h\rangle = \sum_{i,j} \alpha_i^n n^j \langle f, h_{i,j}\rangle + O(\varepsilon^n).$$

In the sum on the left, there is the term $\alpha^n\langle f_0, h\rangle$ with $\langle f_0, h\rangle = \langle f, h\rangle \neq 0$. Therefore, there also has to be a term in α^n on the right hand side. This entails that one of the α_i equals α, and the corresponding function $g = h_{i,0}$ belongs to $\check{E}_\alpha$ and satisfies $\langle f, g\rangle \neq 0$, as desired. □

Proof of Theorem 3.11. Let $\alpha \notin \{0, 1\}$. We want to show that $L_v : E_{\lambda^{-1}\alpha} \to E_\alpha$ is onto. Equivalently, we want to show that its adjoint, from E_α^* to $E_{\lambda^{-1}\alpha}^*$, is injective. These spaces are identified respectively with $\check{E}_\alpha$ and $\check{E}_{\lambda^{-1}\alpha}$ by the duality of Lemma 3.12, and the adjoint of L_v is $-L_v$ by (2.16). Hence, it is enough to show that $L_v : \check{E}_\alpha \to \check{E}_{\lambda^{-1}\alpha}$ is injective. This follows from Lemma 2.7 (we recall that L_v plays in $\check{\mathcal{B}}$ the same role as L_h in $\mathcal{B}$). □

4. Vertically invariant distributions

Let (M, Σ) be a translation surface, and T a linear pseudo-Anosov map on (M, Σ), preserving orientations. Theorem 1.4 and its proof give a whole set of distributions which are annihilated by L_v. Indeed, this is the case of the constant distribution, of the distributions in $E_{\lambda^{-1}\mu_i} \cap \ker L_v$, and of their images under L_h^n. These are the only distributions in $\mathcal{B}^{-k_h,k_v}$ which are vertically invariant:

Lemma 4.1. *Any distribution in $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$ belongs to the linear span of the constant distributions and of the spaces $L_h^n(E_{\lambda^{-1}\mu_i} \cap \ker L_v)$ for $i = 1, \dots, 2g-2$ and $n \geqslant 0$.*

Proof. This follows from the same inductive strategy used to classify Ruelle resonances. We show that any $\omega \in \mathcal{B}^{-k_h,k_v} \cap \ker L_v$ belongs to the space F spanned by the constant distributions and the spaces $L_h^n(E_{\lambda^{-1}\mu_i} \cap \ker L_v)$ for $i = 1, \dots, 2g-2$ and $n \geqslant 0$, by induction on the order of ω.

The constant distributions and the distributions in $E_{\lambda^{-1}\mu_i} \cap \ker L_v$ have cohomology classes which span all the classes without any $[\mathrm{d}y]$ components, i.e., the orthogonal to $[\mathrm{d}x]$. Therefore, there exists $\tilde{\omega}$ in F such that $[\omega - \tilde{\omega}]$ is a multiple of $[\mathrm{d}y]$. By Lemma 2.14, we have in fact $[\omega - \tilde{\omega}] = 0$. Therefore, by Proposition 2.13, there exists $\eta \in \mathcal{B}^{-k_h+1,k_v} \cap \ker L_v$ (and therefore in $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$) such that $\omega - \tilde{\omega} = L_h \eta$. The order of η being strictly smaller than the order of ω, the induction assumption ensures that $\eta \in F$. As F is stable under L_h, we get $\omega = \tilde{\omega} + L_h \eta \in F$.

We should also check the initial step of the induction, when ω is of order 0. With the same construction as above, η is a continuous function. As it is vertically invariant, we deduce that it is constant by minimality of the vertical flow. In particular, it belongs to F, and so does ω. □

However, there are some distributions that are not seen with this point of view, as they are not in the closure of $C_c^\infty(M - \Sigma)$. To describe them, we will follow the same route as above, but replacing our Banach space $\mathcal{B}^{-k_h,k_v}$ by an extended space $\mathcal{B}_{ext}^{-k_h,k_v}$.

We define an element ω of $\mathcal{B}_{ext}^{-k_h,k_v}$ to be a family of distributions ω_I of order at most k_h on all horizontal segments I in $\mathcal{I}^h$, with the following conditions:

1. Compatibility: if two segments $I, I' \in \mathcal{I}^h$ intersect, then the corresponding distributions coincide on functions supported in $I \cap I'$.
2. Smoothness in the vertical direction: for any interval $I \in \mathcal{I}$, and any test function $\varphi \in C_c^{k_h}(I)$ with norm at most 1, denote by I_t the vertical translation by t of I for small enough t, and by φ_t the vertical push-forward of φ on I_t. Then we require that $t \mapsto \int_{I_t} \varphi_t \omega_{I_t}$ is C^{k_v}, with all derivatives bounded by a constant C independent of I or φ. The best such C is by definition the norm of ω in $\mathcal{B}_{ext}^{-k_h,k_v}$.
3. Extension to the singularity: if $(I_t)_{t \in (0,\varepsilon]}$ is a family of vertical translates of a horizontal segment, parameterized by height, such that the limit I_0 contains a singularity, then we require that ω_{I_t} and all its k_v vertical derivatives extend continuously up to I_0.

The first two conditions are very natural, and reproduce directly what we have imposed in the construction of $\mathcal{B}^{-k_h,k_v}$ in Paragraph 2.1. The third condition is to exclude pathological behaviour such as in the following example. Consider a vertical segment $\Gamma = (0, \varepsilon]$ ending on a singularity at 0, a function ρ on Γ with support in $[0, \varepsilon/2]$ that oscillates like $\sin(1/t)$ at 0, and define ω_I to be equal to $\rho(x_I)\delta_{x_I}$ if I intersects Γ_σ at a point x_I, and 0 otherwise. Then this would be an element of our extended space without the third condition. Recall that $\mathcal{B}_{ext}^{-k_h,k_v} \neq \mathcal{B}^{-k_h,k_v}$ (see the example on Page 7).

With this definition, many of the results of the previous sections extend readily. We indicate in the next proposition all the results for which the statements and the proofs do not need any modification.

Proposition 4.2. *The spaces $\mathcal{B}_{ext}^{-k_h,k_v}$ have the following properties:*

1. *The space $\mathcal{B}^{-k_h,k_v}$ is a closed subspace of $\mathcal{B}_{ext}^{-k_h,k_v}$.*
2. *The space $\mathcal{B}_{ext}^{-k_h,k_v}$ is canonically a space of distributions, as in Proposition 2.3.*
3. *Multiplication by C^∞ functions which are constant on a neighborhood of the singularities, or more generally by $C^{k_h+k_v}$-functions on $M-\Sigma$ with $L_h^a L_v^b \psi$ uniformly bounded for $a \leqslant k_h$ and $b \leqslant k_v$, maps $\mathcal{B}_{ext}^{-k_h,k_v}$ into itself continuously, as in Lemma 2.4.*
4. *The derivation L_h maps continuously $\mathcal{B}_{ext}^{-k_h,k_v}$ to $\mathcal{B}_{ext}^{-k_h-1,k_v}$. The derivation L_v maps continuously $\mathcal{B}_{ext}^{-k_h,k_v}$ to $\mathcal{B}_{ext}^{-k_h,k_v-1}$ if $k_v \geqslant 1$, as in Proposition 2.6.*
5. *As there is no horizontal saddle connection, an element in $\mathcal{B}_{ext}^{-k_h,k_v}$ satisfying $L_h f = 0$ is constant, as in Lemma 2.7.*
6. *The space $\mathcal{B}_{ext}^{-k_h,k_v}$ is continuously included in $\mathcal{B}^{-k'_h,k'_v}$ if $k'_h \geqslant k_h$ and $k'_v \leqslant k_v$. This inclusion is compact if both inequalities are strict, as in Proposition 2.8.*
7. *The composition operator $\mathcal{T}$ acts continuously on $\mathcal{B}_{ext}^{-k_h,k_v}$, and it satisfies a Lasota-Yorke inequality (3.1). Therefore, its spectral radius is bounded by 1, and its essential spectral radius is at most $\lambda^{-\min(k_h,k_v)}$, as in Theorem 3.1.*
8. *We have $\mathcal{T} \circ L_v = \lambda L_v \circ \mathcal{T}$ and $\mathcal{T} \circ L_h = \lambda^{-1} L_h \circ \mathcal{T}$, as in Proposition 3.3.*

The space $\mathcal{B}_{ext}^{-k_h,k_v}$ is relevant to study vertically invariant distributions, as all such distributions belong to these spaces:

Lemma 4.3. *Assume that ω is an L_v-annihilated distribution. Then for large enough k_h and for any k_v one has $\omega \in \mathcal{B}_{ext}^{-k_h,k_v}$.*

Proof. Let ω be an L_v-annihilated distribution. For an interval $I \in \mathcal{I}^h$, define a distribution η_I on I by the equality $\int_I \varphi(x)\eta_I(x) = \int \varphi(x)\rho(y)\omega(x,y)$, where ρ is a smooth function supported in $[-\delta, \delta]$ (where δ is small enough so that $I \times [-\delta, \delta]$ does not contain any singularity) with $\int \rho = 1$. We claim that this quantity does not depend on ρ. Indeed, if $\tilde{\rho}$ is another such function, then $(x, y) \mapsto \varphi(x)(\rho(y) - \tilde{\rho}(y))$ has zero average along every vertical segment through $I \times [-\delta, \delta]$, hence it can be written as $L_v f$ for some function f supported in $I \times [-\delta, \delta]$. Then

$$0 = \langle L_v \omega, f\rangle = -\langle \omega, L_v f\rangle = \langle \omega, \varphi(x)\tilde{\rho}(y)\rangle - \langle \omega, \varphi(x)\rho(y)\rangle.$$

This shows that η_I is well defined. It is a finite order distribution on any interval I. Moreover, as ω is vertically invariant, one has $\eta_{I_t} = \eta_I$ if I_t is a vertical family of horizontal segments through I.

By compactness of the manifold, there is a finite family of horizontal segments such that any horizontal segment can be obtained as a subinterval of a vertical translate of one interval in the finite family. If follows that the order of all the distributions η_I is uniformly bounded, independently of $I \in \mathcal{I}^h$. By vertical invariance, it follows that the family η_I defines an element $\eta \in \mathcal{B}_{ext}^{-k_h,k_v}$ if k_h is large enough.

Let us finally prove that $\omega = \eta$ as distributions. Consider a smooth function φ supported by a rectangle $I \times [-\delta, \delta]$ away from singularities. Then

$$\begin{aligned}\langle \eta, \varphi \rangle &= \int_{t=-\delta}^{\delta} \int_{I_t} \varphi(x,t)\eta_{I_t} = \int_{t=-\delta}^{\delta} \int_{I_t} \varphi(x,t)\eta_I = \int_I \left(\int_{t=-\delta}^{\delta} \varphi(x,t)\,\mathrm{d}t \right) \eta_I \\ &= \int \left(\int_{t=-\delta}^{\delta} \varphi(x,t)\,\mathrm{d}t \right) \rho(y)\omega(x,y),\end{aligned}$$

where the last equality is the definition of η_I. Since the integrals of $\left(\int_{t=-\delta}^{\delta} \varphi(x,t)\,\mathrm{d}t\right)\rho(y)$ and φ are the same along all vertical segments, this is equal to $\int \varphi\omega$ thanks to the vertical invariance of ω as we have explained above.

We have proved that $\langle \eta, \varphi \rangle = \langle \omega, \varphi \rangle$ for any smooth function φ with compact support in a rectangle away from the singularities. As any $\varphi \in C_c^\infty(M - \Sigma)$ can be decomposed as a finite sum of such functions, we obtain $\eta = \omega$ as desired. □

Since the space $C_c^\infty(M - \Sigma)$ is *not* dense in $\mathcal{B}_{ext}^{-k_h,k_v}$, we can not use the theorem of Baladi-Tsujii to claim that the eigenspaces beyond the essential spectral radius do not depend on k_h or k_v. Nevertheless, we will show that this is the case, by describing explicitly the new eigenvalues compared to $\mathcal{B}^{-k_h,k_v}$.

For $\sigma \in \Sigma$ and $i_h, i_v \geqslant 0$, we define a distribution $\xi_{\sigma,i_h,i_v}^{(0)}$ as follows. Choose a vertical segment Γ_σ ending on σ and whose image under the covering projection is in the negative half-plane, choose a function ρ on this segment which is equal to 1 on a neighborhood of the singularity and to 0 on a neighborhood of the other endpoint of the segment, and define a distribution $\xi_{\sigma,i_h,i_v}^{(0)} \in \mathcal{B}_{ext}^{-k_h,k_v}$ by $\langle \xi_{\sigma,i_h,i_v}^{(0)} f \rangle = \int_{\Gamma_\sigma} \rho(y) y^{i_v} L_v^{i_h} f(y)\,\mathrm{d}y$. In other words, the corresponding distribution on a horizontal segment I is equal to $\rho(y_I) y_I^{i_v} \delta_{x_I}^{(i_h)}$ if I intersects Γ_σ at a point $z_I = (x_I, y_I)$, and 0 otherwise. This is clearly an element of $\mathcal{B}_{ext}^{-k_h,k_v}$ if $i_h \leqslant k_h$.

Proposition 4.4. *An element ω of $\mathcal{B}_{ext}^{-k_h,k_v}$ can be written uniquely as*

$$\omega = \tilde{\omega} + \sum_{\sigma \in \Sigma} \sum_{i_h \leqslant k_h, i_v \leqslant k_v} c_{\sigma,i_h,i_v} \xi_{\sigma,i_h,i_v}^{(0)}, \tag{4.1}$$

with $\tilde{\omega} \in \mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}$. Moreover, this decomposition depends continuously on ω.

The reason we have $\tilde{\omega} \in \mathcal{B}^{-k_h-1,k_v}$ and not $\tilde{\omega} \in \mathcal{B}^{-k_h,k_v}$ in the statement is that a distribution of order k_h is not well approximated in $(C^{k_h})^*$ by a regularization by convolution: one needs to use smoother test functions, in C^{k_h+1}, to get uniform norm controls.

Proof. Let us first prove the uniqueness in the decomposition (4.1). Consider a singularity σ, of angle $2\pi\kappa$. There are κ half-planes above σ, and κ half-planes below σ. Along any of these half-planes U, consider horizontal intervals I_t which are all vertical translates of an interval $I_0 = I_0(U)$ through the singularity σ, identified with $[-\delta, \delta] \subset \mathbb{C}$ by the covering projection sending σ to 0. By Condition (3) in the definition of $\mathcal{B}_{ext}^{-k_h,k_v}$, the corresponding distributions ω_{I_t} converge to $\omega_{I_0(U)}$. Consider now the distribution on $[-\delta, \delta]$ defined by

$$\omega_\sigma := \sum \omega_{I_0(U^+)} - \sum \omega_{I_0(U^-)}$$

where the first sum is over all half-planes above σ, and the second sum is over all half-planes below σ. By vertical continuity to the left and to the right of the singularity, there are many cancellations in the definition of ω_σ, so that this distribution on $[-\delta, \delta]$ is in fact supported at 0. Therefore, it is a linear combination of derivatives of Dirac masses [Hör03, Theorem 2.3.4], of the form $\sum_{i\leqslant k_h} c_i \delta_0^{(i)}$. Let us do the same construction with the term on the right of (4.1). For functions $f \in C_c^\infty(M - \Sigma)$, the distribution f_σ is obviously 0. By density, this extends to $\mathcal{B}^{-k_h-1,k_v}$, hence $\tilde{\omega}_\sigma = 0$. In the same way, the singularities different from σ do not contribute, and the functions $\xi^{(0)}_{\sigma,i_h,i_v}$ contribute only when $i_v = 0$, with a distribution $\delta^{(i_h)}$. Identifying the coefficients, we get that $c_{\sigma,i_h,0} = c_i$ is uniquely defined by ω. In the same way, we can identify c_{σ,i_h,i_v} from ω by the same process after i_v vertical differentiations. This shows that the decomposition (4.1) is unique. Moreover, the continuity of the decomposition follows from the continuity of all the coefficients c_{σ,i_h,i_v}, which is obvious from the construction.

For the existence, let us decompose ω as

$$\omega = \sum_{i=1}^{N} \omega_i + \sum_{\sigma\in\Sigma} \omega_\sigma + \sum_{H\in\mathcal{H}} \omega_H$$

as in Lemma 2.10, where ω_i is supported in a rectangle R_i away from the singularities, and ω_σ is supported in a small disk around the singularity σ and is constant along fibers of the covering projection π_σ, and ω_H is supported in a local half-plane H based at a singularity. Indeed, the proof of Lemma 2.10 goes through in $\mathcal{B}^{-k_h,k_v}_{ext}$. We will show that each term in this decomposition can be written as in (4.1).

We start with ω_i. Let $\rho_\varepsilon(x)$ be a real C^∞ approximation of the identity. For $z = (x, y)$ in a chart, define

$$f_\varepsilon(z) = \omega_i * \rho_\varepsilon(z) = \int \omega_i(x - h, y)\rho_\varepsilon(h)\,\mathrm{d}h.$$

This is an integral of ω_i along a small horizontal interval against a C_c^∞ function, hence it is well defined. Moreover, f_ε is C^∞ along the horizontal direction, C^{k_v} along the vertical direction, and compactly supported away from the singularities. By Lemma 2.5, $f_\varepsilon \in \mathcal{B}^{-k_h-1,k_v}$. Moreover, f_ε converges in $\mathcal{B}^{-k_h-1,k_v}$ to ω_i thanks to the fact that ω_i is of order k_h and to the fact that we are using C^{k_h+1} test functions: standard properties of convolutions ensure that their difference is bounded by $O(\varepsilon)$ in norm. It follows that $\omega_i \in \mathcal{B}^{-k_h-1,k_v}$. This gives the decomposition (4.1) for ω_i, just taking $\tilde{\omega} = \omega_i$ and the other terms equal to 0.

Let us now consider ω_σ. Its push-forward $\eta = \pi_*\omega_\sigma$ under the covering projection π is almost in $\mathcal{B}^{-k_h,k_v}_{ext}(\mathbb{C})$, except for the fact that the horizontal distributions do not have to match when one reaches 0 from above and from below. The difference is exactly given by a sum of the form $\sum_{i_h,i_v} c_{i_h,i_v}\xi^{(0)}_{0,i_h,i_v}$ as constructed above. In other words, we have

$$\eta = \tilde{\eta} + \sum_{i_h,i_v} c_{i_h,i_v}\xi^{(0)}_{0,i_h,i_v},$$

with $\tilde{\eta} \in \mathcal{B}_{ext}^{-k_h,k_v}(\mathbb{C})$. The case away from singularities shows that $\tilde{\eta} \in \mathcal{B}^{-k_h-1,k_v}(\mathbb{C})$. Lifting everything with π, we get

$$\omega_\sigma = \eta \circ \pi = \tilde{\eta} \circ \pi + \sum_{i_h,i_v} c_{i_h,i_v} \xi^{(0)}_{0,i_h,i_v} \circ \pi.$$

The first term $\tilde{\eta} \circ \pi$ belongs to $\mathcal{B}^{-k_h-1,k_v}$. For the other terms, $\xi^{(0)}_{0,i_h,i_v} \circ \pi$ is not equal to $\xi^{(0)}_{\sigma,i_h,i_v}$ as the latter is supported on one single vertical segment ending on σ while the former is supported on all κ such segments. We claim that the difference belongs to $\mathcal{B}^{-k_h-1,k_v}$, which will conclude the proof.

To prove this, consider a vertical half-plane H with σ in its boundary, and denote by Γ_+ and Γ_- the two components of its boundary, above and below σ. Define a distribution $\alpha_H = \int_{\Gamma_-} y^{i_v} \delta^{(i_h)} \rho(y) \,\mathrm{d}y + \int_{\Gamma_+} y^{i_v} \delta^{(i_h)} \rho(y) \,\mathrm{d}y$ where ρ is smooth and equal to 1 on a neighborhood of 0. This distribution belongs to $\mathcal{B}^{-k_h-1,k_v}$, as it is the limit of a smooth function supported in the interior of H, constructed by approximating inside H the derivative of the Dirac mass with a smooth function. Consider now two consecutive half-planes H and H' sharing the same Γ_+. Taking the difference between α_H and $\alpha_{H'}$, we deduce that

$$\int_{\Gamma_-} y^{i_v} \delta^{(i_h)} \rho(y) \,\mathrm{d}y - \int_{\Gamma'_-} y^{i_v} \delta^{(i_h)} \rho(y) \,\mathrm{d}y \in \mathcal{B}^{-k_h-1,k_v}.$$

Iterating the argument using a sequence of half-planes, we deduce that the same holds for any vertical segments Γ_- and Γ'_- ending at σ. This concludes the proof of the decomposition for ω_σ.

Let us now consider ω_H where H is a local vertical half-plane with a singularity σ in its boundary. This case is easy: as in the case away from singularities, one can smoothen ω_i by convolving it with a kernel ρ_ε, with the additional condition that ρ_ε is supported in $[\varepsilon, 2\varepsilon]$ if H is to the right of σ, and in $[-2\varepsilon, -\varepsilon]$ if H is to the left of σ: this ensures that $\omega_i * \rho_\varepsilon$ is supported in H and everything matches vertically. In fact, the resulting distribution will not be smooth vertically if there is a discrepancy between what happens on the boundaries Γ_+ and Γ_- of H above and below σ. This discrepancy is handled as in the case of ω_σ, by first subtracting a distribution supported on Γ_- to make sure there is no discrepancy, and then arguing that this distribution supported on Γ_- can be written in the form (4.1).

Finally, let us consider ω_H where H is a local horizontal half-plane with a singularity σ in its boundary. Subtracting if necessary a distribution η supported in the vertical segment inside H ending on σ, we can assume that the distribution induced by ω_H on the boundary of H vanishes, as well as all its vertical derivatives up to order k_v. The distribution η is handled as in the two previous cases. Let us then smoothen ω_H by convolving with a kernel ρ_ε in the horizontal direction. Inside H, we get a smooth function. On the boundary of H, this function vanishes, as well as its vertical derivatives up to order k_v. Hence, if one extends this function by 0 outside of H, we get a C^{k_v} function, which belongs to $\mathcal{B}^{-k_h-1,k_v}$ by Lemma 2.5. It approximates ω_H in the $\mathcal{B}_{ext}^{-k_h,k_v}$ norm, showing that $\omega_H \in \mathcal{B}^{-k_h-1,k_v}$. This concludes the proof. □

Corollary 4.5. *The spectrum of $\mathcal{T}$ on $\mathcal{B}_{ext}^{-k_h,k_v}$ in $\{z : |z| > \lambda^{-\min(k_h,k_v)}\}$ is given by the spectrum of $\mathcal{T}$ on $\mathcal{B}^{-k_h,k_v}$ in this region as described in Theorem 1.4, and additionally $j \operatorname{Card}\Sigma$ eigenvalues of modulus λ^{-j} for any $j \geqslant 1$ with $j < \min(k_h, k_v)$.*

One can be more specific about the additional eigenvalues. If T stabilizes pointwise each singularity, then λ^{-j} itself is an eigenvalue of multiplicity $j \operatorname{Card}\Sigma$. Otherwise, there are cycles of singularities, and each cycle of length p gives rise to eigenvalues $e^{2\mathrm{i}k\pi/p}\lambda^{-j}$ with multiplicity j for $k = 0, \dots, p-1$.

We can also formulate the results in terms of the action of T^* on relative cohomology group $H^1(M, \Sigma, \mathbb{C})$ (the eigenvalues of T^* are then $\lambda, \lambda^{-1}, \mu_i$ for $i = 1, \dots, 2g-2$ and roots of unity $e^{2\mathrm{i}k\pi/p}$ for some p corresponding to cycles of singularities of length p).

Proof. Define $E = \mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}$ and $F = \mathcal{B}_{ext}^{-k_h,k_v}/(\mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v})$. The space E is closed and the space F is finite-dimensional, isomorphic to the span of $\xi^{(0)}_{\sigma,i_h,i_v}$ for $i_h \leqslant k_h$ and $i_v \leqslant k_v$, by Proposition 4.4.

The space E is stable under $\mathcal{T}$, and the essential spectral radius of $\mathcal{T}$ on this space is $\leqslant \lambda^{-\min(k_h,k_v)}$ as this is the case on the whole space $\mathcal{B}_{ext}^{-k_h,k_v}$ by Proposition 4.2(7). Since $C_c^\infty(M - \Sigma)$ is dense in E, it follows from the theorem of Baladi-Tsujii that the spectrum of $\mathcal{T}$ on E beyond $\lambda^{-\min(k_h,k_v)}$ is the same as on $\mathcal{B}^{-k_h,k_v}$. Moreover, since $\mathcal{T}$ stabilizes E, its spectrum on the whole space is the union of its spectrum on E and on F. To conclude, we should thus describe the spectrum of $\mathcal{T}$ on F.

The image under $\mathcal{T}$ of $\xi^{(0)}_{\sigma,i_h,i_v}$ is equal to the sum of $\lambda^{-1-i_h-i_v}\xi^{(0)}_{T^{-1}\sigma,i_h,i_v}$ and of a distribution in $\mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}$. Indeed, this follows readily from the definition if the vertical segment $\Gamma_{T^{-1}\sigma}$ is sent by T to Γ_σ. In general, it is sent to another vertical segment ending on σ, but Proposition 4.4 shows that changing the choice of the vertical segment results in a difference in $\mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}$. This shows that the matrix of $\mathcal{T}$ on the finite-dimensional space F is a union of permutation matrices multiplied by λ^{-j} for $j = 1 + i_h + i_v$. The spectrum of such a permutation matrix, along a cycle of length p, is made of the eigenvalues $e^{2\mathrm{i}k\pi/p}$ for $k = 0, \dots, p-1$. Hence, the spectrum of $\mathcal{T}$ on F is made of eigenvalues of modulus λ^{-j}, and the number of such eigenvalues is

$$\operatorname{Card}\{(i_h, i_v) : i_h \leqslant k_h, i_v \leqslant k_v,\ j = i_h + iv + 1\} \cdot \operatorname{Card}\Sigma.$$

For $j < \min(k_h, k_v)$, this is equal to $j \operatorname{Card}\Sigma$. □

The description of the spectrum of $\mathcal{T}$ on F in this proof is reminiscent of the description of the spectrum of $\mathcal{T}$ on $\mathcal{B}^{-k_h,k_v}$, but in a simpler situation. Assume to simplify the discussion that T acts as the identity on Σ. Then there are some basic eigenfunctions for the eigenvalue λ^{-1}, which are the classes of the functions $\xi^{(0)}_\sigma = \xi^{(0)}_{\sigma,0,0} = \int_{\Gamma_\sigma} \delta \cdot \rho(y) \,\mathrm{d}y$ modulo $\mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}$. The other eigenfunctions are given by $\xi^{(0)}_{\sigma,i_h,i_v} = \int_{\Gamma_\sigma} y^{i_v} \delta^{(i_h)} \cdot \rho(y) \,\mathrm{d}y$. They are obtained by differentiating the original function i_h times in the horizontal direction, and integrating it i_v times in the vertical direction. To obtain the eigenvalue λ^{-j}, the total number of such operations $i_h + i_v$ should be equal to $j - 1$, giving j choices.

It follows from the above corollary that one can define the generalized eigenspace $E_{\alpha,ext}$ associated to the eigenvalue α of $\mathcal{T}$ acting on $\mathcal{B}^{-k_h,k_v}$ for large enough k_h and k_v. This space of distributions does not depend on k_h and k_v if they are large enough. Moreover, L_v maps $E_{\alpha,ext}$ to $E_{\lambda\alpha,ext}$ and L_h maps $E_{\alpha,ext}$ to $E_{\lambda^{-1}\alpha,ext}$ as in Corollary 3.4.

To proceed, we will need some ingredients of duality. In general, there is no canonical way to define a pairing between $\mathcal{B}_{ext}^{-k_h,k_v}$ and $\check{\mathcal{B}}_{ext}^{\check{k}_h,-\check{k}_v}$. Indeed, consider a distribution φ on $[-1,1]$ for which $\int_{-1}^{1} 1_{y\leqslant 0}\varphi(y)$ does not make sense, and define a distribution $\omega \in \check{\mathcal{B}}_{ext}^{-\check{k}_h,\check{k}_v}$ which is equal to φ on each vertical leaf around a singularity σ, multiplied by a cutoff function to extend it by 0 elsewhere. Then one can not make sense of $\langle \xi^{(0)}_{\sigma,0,0}, \varphi\rangle$. However, there is no difficulty to define $\langle \omega, 1\rangle$ by integrating a partition of unity along horizontal segments, and then summing over the partition of unity. When ω belongs to $\mathcal{B}^{-k_h,k_v}$, this coincides with the duality between $\mathcal{B}^{-k_h,k_v}$ and $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$ defined in Proposition 2.9 if one considers the distribution 1 as an element of $\check{\mathcal{B}}^{\check{k}_h,-\check{k}_v}$. The main property of this linear form we will use is the following.

Lemma 4.6. *Let $\omega \in \mathcal{B}_{ext}^{-k_h,k_v}$. Consider its decomposition given by Proposition 4.4. Then*

$$\langle L_v\omega, 1\rangle = \sum_\sigma c_{\sigma,0,0}.$$

Proof. We should show that $\langle L_v\tilde{\omega}, 1\rangle = 0$, and that $\langle L_v \xi^{(0)}_{\sigma,i_h,i_v}, 1\rangle = 1$ if $i_h = i_v = 0$ and 0 otherwise. First, $\langle L_v\tilde{\omega}, 1\rangle = -\langle \tilde{\omega}, L_v 1\rangle = 0$ by Lemma 2.11. The fact that L_v is antiselfadjoint does not apply to $\xi^{(0)}_{\sigma,i_h,i_v}$ as additional boundary terms show up when one does integrations by parts (contrary to the case of elements of $\mathcal{B}^{-k_h,k_v}$, which are in the closure of compactly supported functions and for which there is therefore no boundary term). These boundary terms are responsible for the formula in the lemma, as we will see in the following computation.

We show that $\langle L_v \xi^{(0)}_{\sigma,0,0}, 1\rangle = 1$, the other case is similar. Write $\xi^{(0)}_{\sigma,0,0} = \int_{y=-\delta}^{0} \rho(y)\delta_{(x,y)}\,\mathrm{d}y$ as in its definition, where we are integrating on a vertical segment ending at a singularity and ρ vanishes on a neighborhood of $-\delta$ and is equal to 1 on a neighborhood of 0. Then $L_v\xi^{(0)}_{\sigma,0,0} = \int_{y=-\delta}^{0} \rho'(y)\delta_{(x,y)}\,\mathrm{d}y$. Therefore,

$$\langle L_v \xi^{(0)}_{\sigma,0,0}, 1\rangle = \int_{y=-\delta}^{0} \rho'(y)\,\mathrm{d}y = \rho(0) - \rho(-\delta) = 1. \qquad \square$$

We can now prove Proposition 1.5, asserting that $\xi^{(0)}_\sigma = \xi^{(0)}_{\sigma,0,0}$ can be modified by adding an element of $\mathcal{B}^{-k_h,k_v}$ to obtain a distribution which is mapped by L_v to the constant distribution $1/\operatorname{Leb} M$. As in the statement of the proposition, we will denote this modified distribution by ξ_σ or $\xi_{\sigma,0,0}$.

Proof of Proposition 1.5. We work in $\mathcal{B}_{ext}^{-2,k_v}$. On this space, the essential spectral radius of $\mathcal{T}$ is $\leqslant \lambda^{-2} < \lambda^{-1}$. Replacing T by a power of T if necessary, we can assume without loss of generality that σ is fixed by T. Then $\mathcal{T}\xi^{(0)}_\sigma = \lambda^{-1}\xi^{(0)}_\sigma + \eta$ where $\eta \in E = \mathcal{B}^{-3,k_v} \cap \mathcal{B}_{ext}^{-2,k_v}$ as explained in the proof of Corollary 4.5. Since the essential spectral radius of $\mathcal{T}$ on E is $\leqslant \lambda^{-2}$ (see again the proof of Corollary 4.5), we can decompose $\eta = \eta_1 + \eta_2$ where η_1 is in the generalized eigenspace associated to λ^{-1}, and η_2 belongs to its spectral complement, on which $\mathcal{T} - \lambda^{-1}$ is invertible. Therefore, we can write $\eta_2 = -(\mathcal{T} - \lambda^{-1})\omega$ for some $\omega \in E$. Finally, we have

$$(\mathcal{T} - \lambda^{-1})(\xi_\sigma^{(0)} + \omega) = \eta - \eta_2 = \eta_1.$$

Since η_1 is a generalized eigenvector for the eigenvalue λ^{-1}, we have $(\mathcal{T} - \lambda^{-1})^N \eta_1 = 0$ for large enough N. Hence, $(\mathcal{T} - \lambda^{-1})^{N+1}(\xi_\sigma^{(0)} + \omega) = 0$. This shows that $\xi_\sigma^{(0)} + \omega$ belongs to the generalized eigenspace $E_{\lambda^{-1},ext}$ associated to the eigenvalue λ^{-1} of $\mathcal{T}$ acting on $\mathcal{B}_{ext}^{-2,2}$. Moreover, as $\min(k_h, k_v) \geqslant 3$, we have $\omega \in \mathcal{B}^{-k_h,k_v}$.

To conclude the proof, it remains to show that $L_v(\xi_\sigma^{(0)} + \omega) = 1/\operatorname{Leb} M$. Since $\xi_\sigma^{(0)} + \omega \in E_{\lambda^{-1},ext}$, we have $L_v(\xi_\sigma^{(0)} + \omega) \in E_{1,ext}$. The description of the spectrum in Corollary 4.5 shows that this space is just E_1. By Lemma 3.9, it is made of constants. We get the existence of a constant c such that $L_v(\xi_\sigma^{(0)} + \omega) = c$.

To identify c, we compute

$$c \operatorname{Leb} M = \langle c, 1\rangle = \langle L_v(\xi_\sigma^{(0)} + \omega), 1\rangle = 1,$$

thanks to Lemma 4.6. This proves that $c = 1/\operatorname{Leb} M$. □

Lemma 4.7. *Let $k_h, k_v \geqslant 3$. Then all L_v-annihilated distributions in $\mathcal{B}_{ext}^{-k_h,k_v}$ are of the form described in Theorem 1.6, i.e., they are linear combinations of distributions $\xi_\sigma - \xi_{\sigma'}$ for $\sigma, \sigma' \in \Sigma$, of $L_h^n \xi_\sigma$ with $n \geqslant 1$ and $\sigma \in \Sigma$, of 1, and of $L_h^n E^H_{\lambda^{-1}\mu_i}$ with $n \geqslant 0$ and $i = 1, \dots, 2g-2$.*

Proof. Define a distribution $\xi_{\sigma,i_h,i_v} = L_h^{i_h}\xi_{\sigma,0,0}$ if $i_v = 0$ and $\xi_{\sigma,i_h,i_v} = \xi^{(0)}_{\sigma,i_h,i_v}$ otherwise. Then we have

$$\mathcal{B}_{ext}^{-k_h,k_v} = (\mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}) \oplus \bigoplus_{i_h \leqslant k_h, i_v \leqslant k_v} \mathbb{R}\xi_{\sigma,i_h,i_v},$$

by Proposition 4.4 and the fact that $\xi_{\sigma,i_h,i_v} - \xi^{(0)}_{\sigma,i_h,i_v} \in \mathcal{B}^{-k_h-1,k_v} \cap \mathcal{B}_{ext}^{-k_h,k_v}$. Write this decomposition as $\mathcal{B}_{ext}^{-k_h,k_v} = E \oplus F$. On $\mathcal{B}_{ext}^{-k_h,k_v}/E$, the operator L_v maps ξ_{σ,i_h,i_v} to ξ_{σ,i_h,i_v-1} if $i_v > 0$, and to 0 if $i_v = 0$. Therefore, a distribution ω with $L_v\omega = 0$ must have zero components on ξ_{σ,i_h,i_v} for $i_v > 0$: it can be written as $\tilde\omega + \sum_{i_h \leqslant k_h} c_{\sigma,i_h}\xi_{\sigma,i_h,0}$. Moreover, $L_v\tilde\omega = 0$.

By Lemma 4.6, we have

$$0 = \langle L_v\omega, 1\rangle = \sum_\sigma c_{\sigma,0}.$$

This shows that $\omega - \tilde\omega$ belongs to the vector space generated by the $\xi_\sigma - \xi_{\sigma'}$ over σ, σ', and by all the $L_h^n\xi_\sigma$ for $n > 0$. Moreover, Lemma 4.1 shows that $\tilde\omega$ belongs to the span of the constant distribution and of $L_h^n E^H_{\lambda^{-1}\mu_i}$ with $n \geqslant 0$ and $i = 1, \dots, 2g-2$. This concludes the proof. □

Since all L_v-invariant distributions belong to some space $\mathcal{B}_{ext}^{-k_h,k_v}$ by Lemma 4.3, Theorem 1.6 giving the classification of all vertically invariant distributions follows directly from Lemma 4.7.

Remark 4.8. Although it is not needed for the above proof, it is enlightening to describe a cohomological interpretation for all the elements of $\mathcal{B}_{ext}^{-k_h,k_v} \ker L_v$, i.e., for all vertically invariant distributions.

If γ is a continuous closed loop in $M-\Sigma$ and $\omega \in \mathcal{B}_{ext}^{-k_h,k_v} \cap \ker L_v$, one can define the integral $\int_\gamma \omega$ just like for elements in $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$ (see the discussion before Proposition 2.12). This integral only depends on γ up to deformation in $M-\Sigma$. Therefore, it defines an element of $H^1(M-\Sigma)$, that we denote by $[\omega]_{ext}$. Contrary to the case of $\mathcal{B}^{-k_h,k_v} \cap \ker L_v$, however, the integral $\int_{\gamma_\sigma} \omega$ along a small loop γ_σ around a singularity σ does not have to vanish, so that $[\omega]_{ext}$ is *not* an element of $H^1(M)$ in general. Indeed, if one considers two different singularities σ and σ', then $\xi_\sigma - \xi'_\sigma$ is annihilated by L_v, but the corresponding cohomology class integrates to 1 along a small positive loop around σ, and to -1 along a small positive loop around σ'. This is a direct consequence of the definition of $\xi_\sigma^{(0)}$, with a Dirac mass along a vertical segment ending at σ, that will be intersected once by a small loop around σ. In general, for $\omega \in \mathcal{B}_{ext}^{-k_h,k_v} \cap \ker L_v$, one has

$$\int_{\gamma_\sigma} [\omega]_{ext} = c_{\sigma,0,0}(\omega), \tag{4.2}$$

where $c_{\sigma,0,0}$ is defined in the decomposition of Proposition 4.4. Indeed, $\xi_{\sigma,0,0}^{(0)}$ contributes by 1 to the integral along a small loop around σ, while the contribution of all the other terms tends to 0 when the loop tends to σ. In fact, the map $\omega \mapsto c_{\sigma,0,0}(\omega)$ corresponds to the *boundary operator* of [MY16] (it does not appear in the case of Ruelle resonances as all our functions are continuous in this setting).

If a distribution $\omega \in \mathcal{B}_{ext}^{-k_h,k_v} \cap \ker L_v$ satisfies $[f]_{ext} = 0$, then one proves as in Proposition 2.13 that it can be written as $\omega = L_h \eta$ for some $\eta \in \mathcal{B}_{ext}^{-k_h+1,k_v} \cap \ker L_v$. Indeed, the proof of this proposition goes through, and it is in fact easier as one does not need to show that the resulting object one constructs by horizontal integration belongs to the closure of $C_c^\infty(M-\Sigma)$, which is the hard part in Proposition 2.13.

With (4.2) and Lemma 4.6, one has

$$\sum_\sigma \int_{\gamma_\sigma} [\omega]_{ext} = \sum_\sigma c_{\sigma,0,0}(\omega) = \langle L_v \omega, 1\rangle = 0.$$

This corresponds to the fact that, in the homology of $M-\Sigma$, one has $\sum [\gamma_\sigma] = 0$.

The cohomology classes one can get in this way are all cohomology classes without any $[\mathrm{d}y]$ component, i.e., orthogonal to $[\mathrm{d}x]$, as one can realize all such classes in $H^1(M)$ using $\mathcal{B}^{-k_h,k_v}$, and one can account for the additional $\operatorname{Card}\Sigma - 1$ dimensions in $H^1(M-\Sigma)$ by using the $\xi_\sigma - \xi_{\sigma'}$. It turns out that one can also recover the class $[\mathrm{d}y]$. Indeed, start from $\xi_{\sigma,0,0}$ and consider a path γ made of horizontal and vertical segments. As $\mathrm{d}\xi_{\sigma,0,0}$ is exact, one may compute formally

$$0 = \int_\gamma \mathrm{d}\xi_{\sigma,0,0} = \int_\gamma L_h \xi_{\sigma,0,0} \,\mathrm{d}x + \int_\gamma L_v \xi_{\sigma,0,0} \,\mathrm{d}y = \int_\gamma L_h \xi_{\sigma,0,0} \,\mathrm{d}x + \frac{1}{\operatorname{Leb} M} \int_\gamma \mathrm{d}y,$$

where the last equality follows from Proposition 1.5. It follows that the element $-\operatorname{Leb} M \cdot L_h \xi_{\sigma,0,0}$, which belongs to $\mathcal{B}_{ext}^{-k_h,k_v} \cap \ker L_v$, has a cohomology class whose integral along any path coincides with the integral of $\mathrm{d}y$ along this path, i.e., $[-\operatorname{Leb} M \cdot L_h \xi_{\sigma,0,0}]_{ext} = [\mathrm{d}y]$. The above formal computation can be made rigorous by smoothing the path γ horizontally, as we did to define the cohomology classes. This shows that, for $k_h, k_v \geqslant 3$, the map from

$\mathcal{B}_{ext}^{-k_h,k_v} \cap \ker L_v$ to $H^1(M-\Sigma)$ is onto. This is the analogue of [For02, Theorem 7.1(ii)] in our setting.

5. Solving the cohomological equation

Consider a C^∞ function f which is compactly supported away from the singularity set Σ on a translation surface M. Solving the cohomological equation for the vertical flow on M amounts to finding a function F, which is smooth along vertical lines, and satisfies the equality $L_v F = f$. In general, the function F will not be compactly supported on $M-\Sigma$, but it will hopefully be continuous on M. More generally, one may ask how smooth the solution F can be chosen.

A direct obstruction to solve the cohomological equation with a smooth solution is given by distributions in the kernel of L_v: if $L_v\omega = 0$, then

$$\langle \omega, f\rangle = \langle \omega, L_v F\rangle = -\langle L_v \omega, F\rangle = 0,$$

where the last equalities make sense if F belongs to the space on which ω acts. Indeed, in general, a distribution $\omega \in \mathcal{D}^\infty(M-\Sigma)$ is in the dual of $C_c^\infty(M-\Sigma)$, so that $\langle\omega, F\rangle$ does not make sense if F is not C^∞ or not compactly supported away from Σ. However, many distributions act on larger classes of functions, so an important question in the discussion below will be to see if $\langle \omega, F\rangle$ is meaningful.

The Gottschalk-Hedlund theorem states that, for a minimal continuous flow on a compact manifold, a continuous function is a continuous coboundary if and only if its Birkhoff integrals $\int_0^T f(g_t x)\,\mathrm{d}t$ are bounded independently of x and T. We will use a variation around this result due to Giulietti-Liverani [GL14]. Its interest is that it gives an explicit formula for the coboundary, which we will use to study its smoothness.

In this section, we fix once and for all a C^∞ function $\chi : \mathbb{R} \to [0,1]$ which is equal to 1 on a neighborhood of $(-\infty, 0]$ and to 0 on a neighborhood of $[1,\infty)$.

Lemma 5.1. *Consider a semiflow g_t on a space X, and a function $f : X \to \mathbb{R}$ for which there exist $C > 0$ and $\varepsilon > 0$ and $r \in \mathbb{N}$ with the following property: for any $x \in X$, for any $\tau \geqslant 1$, for any function φ which is compactly supported on $(0,1)$,*

$$\left| \int_{t=0}^{\tau} \varphi(t/\tau) f(g_t x)\,\mathrm{d}t \right| \leqslant C \|\varphi\|_{C^r} / \tau^\varepsilon. \tag{5.1}$$

Then f is a coboundary: there exists a function F such that $\int_0^\tau f(g_t x)\,\mathrm{d}t = F(x) - F(g_\tau x)$ for all $x \in X$ and all $\tau \geqslant 0$.

More specifically, F can be constructed as follows. Fix $\lambda > 1$. Define a function $F_n(x) = \int_{t=0}^{\lambda^n} \chi(t/\lambda^n) f(g_t x)\,\mathrm{d}t$. Then F_n converges uniformly to a function F as above. Moreover, $|F_n(x) - F(x)| \leqslant C\lambda^{-\varepsilon n}$ where C does not depend on x or n.

In fact, one can even prove that $\int_{t=0}^{\tau} \chi(t/\tau) f(g_t x)\,\mathrm{d}t$ converges to $F(x)$ at a uniform rate $O(1/\tau^\varepsilon)$ when $\tau \to \infty$, without having to restrict to the subsequence λ^n, with a small modification of the following proof. We will not need this more precise version of the lemma.

Proof. This is essentially a reformulation of [GL14, Lemmas 1.4 and 3.1].

Define $\varphi(t) = \chi(t) - \chi(\lambda t)$. This is a C^∞ function with compact support on $(0,1)$. Moreover,

$$F_{n+1}(x) - F_n(x) = \int_{t=0}^{\lambda^{n+1}} (\chi(t/\lambda^{n+1}) - \chi(t/\lambda^n)) f(g_t x) \,\mathrm{d}t = \int_{t=0}^{\lambda^{n+1}} \varphi(t/\lambda^{n+1}) f(g_t x) \,\mathrm{d}t,$$

Under the assumptions of the lemma, this is bounded by $C(\varphi)/\lambda^{(n+1)\varepsilon}$. This shows that $F_n(x)$ is a Cauchy sequence, converging uniformly to a limit $F(x)$ with $|F_n(x) - F(x)| \leqslant C\lambda^{-\varepsilon n}$.

To conclude, we should show that F solves the cohomological equation. Let us fix x and τ. We have

$$\begin{aligned} F_n(g_\tau x) &+ \int_0^\tau f(g_t x) \,\mathrm{d}t - F_n(x) \\ &= \int_\tau^{\lambda^n+\tau} \chi((t-\tau)/\lambda^n) f(g_t x) + \int_0^\tau \chi((t-\tau)/\lambda^n) f(g_t x) \,\mathrm{d}t - \int_0^{\lambda^n} \chi(t/\lambda^n) f(g_t x) \,\mathrm{d}t \\ &= \int_0^{\lambda^n+\tau} \varphi_{n,\tau}(t/(\lambda^n+\tau)) f(g_t x) \,\mathrm{d}t, \end{aligned}$$

where

$$\varphi_{n,\tau}(s) = \chi(((\lambda^n+\tau)s - \tau)/\lambda^n) - \chi((\lambda^n+\tau)s/\lambda^n).$$

The function $\varphi_{n,\tau}$ has compact support in $(0,1)$ and uniformly bounded C^r norm when n tends to infinity. By (5.1) applied to $\varphi_{n,\tau}$, we deduce that $F_n(g_\tau x) + \int_0^\tau f(g_t x)\,\mathrm{d}t - F_n(x)$ tends to 0. Passing to the limit, we get $F(g_\tau x) + \int_0^\tau f(g_t x) - F(x) = 0$. □

We will denote by C_h^k the space of functions $M \to \mathbb{R}$ which are C^k along the horizontal direction and such that $L_h^i f$ is continuous and bounded on $M - \Sigma$ for $i \leqslant k$. Elements of C_h^k belong to $\check{\mathcal{B}}^{k,0}$ by Lemma 2.5. To formulate the assumptions of our theorems, we will use the following fact:

$$\langle \omega, f\rangle \text{ makes sense for } f \in C_h^{k+2} \text{ and } \omega \in E_\alpha \text{ with } |\alpha| \geqslant \lambda^{-k-1}. \tag{5.2}$$

Indeed, elements of E_α for $|\alpha| \geqslant \lambda^{-k-1}$ belong to $\mathcal{B}^{-k-2,k+2}$ as the essential spectral radius of $\mathcal{T}$ on this space is $\leqslant \lambda^{-k-2} < \lambda^{-k-1}$. Therefore, since $f \in \check{\mathcal{B}}^{k+2,0}$, the coupling $\langle \omega, f\rangle$ is well defined by Proposition 2.9 (exchanging the roles of the horizontal and the vertical direction to make sure that the inequalities on the exponents are satisfied). One could even weaken slightly more the conditions, by requiring only $f \in C_h^{k+1+\varepsilon}$ for $\varepsilon > 0$, by exploring the route alluded to in Remark 2.2 if one were striving for minimal assumptions.

We will apply the previous lemma in the setting of the vertical flow on a translation surface endowed with a pseudo-Anosov map preserving orientations, with expansion factorλ. We obtain the following criterion to have a continuous coboundary.

Theorem 5.2. *Let T be a linear pseudo-Anosov map preserving orientations on a translation surface (M, Σ). Denote by g_t the vertical flow on this surface. Consider a function f on M in C_h^2. Assume that, for any $\omega \in \bigcup_{|\alpha|\geqslant\lambda^{-1}} E_\alpha$, one has $\langle \omega, f\rangle = 0$. Then f is a continuous coboundary: there exists a continuous function F on M such that, for any x and any τ such that $g_t x$ is well defined for $t \in [0,\tau]$, holds*

$$\int_0^\tau f(g_t x)\,\mathrm{d}t = F(x) - F(g_\tau x). \tag{5.3}$$

The assumptions of the theorem make sense by (5.2). The distributions appearing in the statement of the theorem have been completely classified in Theorem 1.4 and its proof. In particular, they are all vertically invariant.

To prove this theorem, let us first check that the assumptions of the Giulietti-Liverani criterion of Lemma 5.1 are satisfied.

Lemma 5.3. *Under the assumptions of Theorem 5.2, there exists $\varepsilon > 0$ such that the inequality $\left|\int_{t=0}^{\tau} \varphi(t/\tau) f(g_t x)\,\mathrm{d}t\right| \leqslant C\|\varphi\|_{C^2}/\tau^{\varepsilon}$ in (5.1) holds, with $r = 2$.*

Proof. It suffices to prove the estimate for τ of the form λ^n, as the case of a general τ follows by using n such that $\tau \in [\lambda^{n-1}, \lambda^n]$. Fix x and φ. We have

$$\int_0^{\lambda^n} \varphi(t/\lambda^n) f(g_t x)\,\mathrm{d}t = \lambda^n \int_0^1 \varphi(s) f(T^{-n}(g_s(T^n x)))\,\mathrm{d}s = \lambda^n \int_0^1 \varphi(s) \check{\mathcal{T}}^n f(g_s y)\,\mathrm{d}s, \tag{5.4}$$

for $y = T^n x$. The integral is the integral of $\check{\mathcal{T}}^n f \in \check{\mathcal{B}}^{2,-2}$ along a vertical manifold against a C^2 smooth function. Therefore, this is bounded by $\lambda^n \|\varphi\|_{C^2} \|\check{\mathcal{T}}^n f\|_{\check{\mathcal{B}}^{2,-2}}$.

On this space, the essential spectral radius of $\check{\mathcal{T}}$ is $\leqslant \lambda^{-2} < \lambda^{-1}$, by Theorem 3.1. Let us decompose f as $\sum_\alpha f_\alpha + \tilde{f}$, where α runs among the (finitely many) eigenvalues of $\check{\mathcal{T}}$ of modulus $> \lambda^{-2}$, and f_α is the component of f on the corresponding generalized eigenspace $\check{E}_\alpha$. By assumption, $\langle \omega, f\rangle = 0$ for any $\omega \in E_\alpha$ with $|\alpha| \geqslant \lambda^{-1}$. Thanks to the perfect duality statement given in Lemma 3.12, this gives $f_\alpha = 0$ for all such α. Let $\gamma < \lambda^{-1}$ be such that all eigenvalues of modulus $< \lambda^{-1}$ have in fact modulus $< \gamma$. We deduce that $\|\check{\mathcal{T}}^n f\|_{\check{\mathcal{B}}^{2,-2}}$ grows at most like $C\gamma^n$. Together with (5.4), this gives

$$\left|\int_0^{\lambda^n} \varphi(t/\lambda^n) f(g_t x)\,\mathrm{d}t\right| \leqslant C\|\varphi\|_{C^2} (\lambda\gamma)^n.$$

As $\lambda\gamma < 1$, one may write $\lambda\gamma = \lambda^{-\varepsilon}$ for some $\varepsilon > 0$. Then this bound is of the form $C\|\varphi\|_{C^2}/(\lambda^n)^{\varepsilon}$, as requested. □

There is a difficulty to apply Lemma 5.1 due to the singularities, which imply that the flow is not defined everywhere for all times. One can circumvent the difficulty by going to a bigger space in which trajectories ending on a singularity are split into two trajectories going on both sides of the singularity. This results in a compact space with a Cantor transverse structure and a minimal flow, to which Lemma 5.1 applies. This classical strategy works well for continuous coboundary results, but there are difficulties in higher smoothness. Instead, we will use a strategy which avoids the use of such an extension, and works also for higher smoothness. The idea is to iterate the flow in forward time or backward time depending on the point one considers.

Proof of Theorem 5.2. Let $M_n^+ \subseteq M$ be the set of points for which the vertical flow is defined for all times in $[0, \lambda^n]$, and let $M^+ = \bigcap_n M_n^+$, i.e., the set of points that do not reach a singularity in finite positive time. In the same way, but using backward time, we define M_n^- and M^-. Then $M - \Sigma = M^+ \cup M^-$ as there is no vertical saddle connection.

Let us define functions $F_n^+(x)$ on M_n^+ and F_n^- on M_n^- by

$$F_n^+(x) = \int_0^{\lambda^n} \chi(t/\lambda^n) f(g_t x)\,\mathrm{d}t, \quad F_n^-(x) = -\int_0^{\lambda^n} \chi(t/\lambda^n) f(g_{-t} x)\,\mathrm{d}t.$$

For $x \in M_n^+ \cap M_n^-$, the difference $F_n^+(x) - F_n^-(x)$ can be written as

$$F_n^+(x) - F_n^-(x) = \int_{-\lambda^n}^{\lambda^n} \tilde{\chi}(t/\lambda^n) f(g_t x)\, \mathrm{d}t,$$

where $\tilde{\chi}(t) = \chi(|t|)$. By Lemma 5.3, this tends to 0 like $C(\tilde{\chi})/(2\lambda^n)^\varepsilon$.

Lemma 5.1 applied to the semiflow g_t on M^+, and to the semiflow g_{-t} on M^-, shows that $F_n^+(x)$ converges uniformly to a function $F^+(x)$ on M^+, and that $F_n^-(x)$ converges uniformly to a function $F^-(x)$ on M^-. From the fact that the difference between F_n^+ and F_n^- is small where defined, we deduce that $F^+ = F^-$ on $M^+ \cap M^-$. Let us define a function F on $M - \Sigma$, equal to F^+ on M^+ and to F^- on M^-. By the above, we have

$$|F_n^+(x) - F(x)| \leqslant C/\lambda^{\varepsilon n} \text{ for } x \in M_n^+, \quad |F_n^-(x) - F(x)| \leqslant C/\lambda^{\varepsilon n} \text{ for } x \in M_n^-. \tag{5.5}$$

Moreover, the function F satisfies the coboundary equation (5.3), as F^+ and F^- satisfy it respectively on M^+ and M^- by Lemma 5.1.

Let us show that F is continuous on $M - \Sigma$. Take $x \in M - \Sigma$, for instance in M^+. Let $\delta > 0$. Let n be large. The function F_n^+ is well defined and continuous on a neighborhood of x. In particular, it oscillates by at most δ on a neighborhood of x. As F differs from F_n^+ by C/λ^n, we deduce that F oscillates by at most $\delta + C/\lambda^n$ on a neighborhood of x. This proves the continuity of F at x.

Finally, let us show that F extends continuously to Σ. It suffices to show that it is uniformly continuous on $M - \Sigma$. For this, it suffices to show that it is uniformly continuous on small horizontal segments close to a singularity, as uniform continuity along vertical segments follows from the coboundary equation. Let $(I_t)_{t \in (0,\delta]}$ be a family of vertical translates of horizontal segments such that I_0 contains a singularity. For $x, y \in I_0$, we have $F(x) - F(y) = F(g_t x) - F(g_t y) + \int_0^t (f(g_s x) - f(g_s y))\, \mathrm{d}s$. Thanks to the boundedness of $L_h f$, the last integral is small if x and y are close and t is small, while the first difference is small if x and y are close enough thanks to the continuity of F on I_t. Hence, $F(x) - F(y)$ itself is small. This concludes the proof. □

To get further smoothness results, one needs to assume more cancellations for f. The next theorem gives such conditions ensuring that F is C^1.

Theorem 5.4. *Under the assumptions of Theorem 5.2, assume additionally that $f \in C_h^3$. Assume moreover that, for any $\omega \in \bigcup_{|\alpha| \geqslant \lambda^{-2}} E_\alpha \cap \ker L_v$, one has $\langle \omega, f \rangle = 0$. Then the function F solving the cohomological equation* (5.3) *is C^1 along the horizontal direction, and $L_h F$ extends continuously to M.*

The assumptions of the theorem make sense by (5.2). The distributions appearing in the statement of the theorem have been completely classified in Theorem 1.4 and its proof.

Let us start with a preliminary reduction.

Lemma 5.5. *To prove Theorem 5.4, it is sufficient to prove it assuming the stronger condition that $\langle \omega, f \rangle = 0$ for all $\omega \in \bigcup_{|\alpha| \geqslant \lambda^{-2}} E_\alpha$.*

The difference with the assumptions in Theorem 5.4 is that our new assumption is not restricted only to the vertically invariant distributions.

Proof. Consider a function $f \in C^3_h$ such that $\langle \omega, f\rangle = 0$ for all $\omega \in \bigcup_{|\alpha|\geqslant \lambda^{-2}} E_\alpha \cap \ker L_v$. We can not deduce from the assumptions of the lemma that f is a smooth coboundary, as there might exist distributions $\omega \in E_\alpha - \ker L_v$ with $\langle \omega, f\rangle \neq 0$. We will bring these quantities back to 0 by subtracting from f a suitable coboundary. The additional distributions we have to handle belong to $E_{\lambda^{-2}\mu_i}$ for some μ_i with $|\mu_i| \in [1, \lambda)$. Denote by F_i a subspace of $E_{\lambda^{-2}\mu_i}$ sent isomorphically by L_v to $E_{\lambda^{-1}\mu_i}$. Then $E_{\lambda^{-2}\mu_i} = F_i \oplus (E_{\lambda^{-2}\mu_i} \cap \ker L_v)$, see (1.1).

Consider on $\bigoplus_{|\mu_i|\in[1,\lambda)} E_{\lambda^{-1}\mu_i}$ the linear form $\omega \mapsto \langle L_v^{-1}\omega, f\rangle$, where by $L_v^{-1}\omega$ we mean the unique $\tilde\omega \in \bigoplus F_i$ with $L_v\tilde\omega = \omega$. As $\mathcal{B}^{-k_h,k_v}$ is a space of distributions, any linear form on a finite-dimensional subspace can be realized by a smooth function. Hence, there exists $g_0 \in C^\infty_c(M - \Sigma)$ such that, for any $\omega \in \bigoplus_{|\mu_i|\in[1,\lambda)} E_{\lambda^{-1}\mu_i}$, then $\langle L_v^{-1}\omega, f\rangle = \langle \omega, g_0\rangle$. Hence, for $\tilde\omega \in \bigoplus F_i$, applying the previous equality to $\omega = L_v\tilde\omega$, we have

$$\langle \tilde\omega, f\rangle = \langle L_v\tilde\omega, g_0\rangle = -\langle \tilde\omega, L_v g_0\rangle.$$

This shows that the function $\tilde f = f + L_v g_0$ vanishes against any distribution in $\bigoplus F_i$. It also vanishes against any distribution on $\bigcup_{|\alpha|\geqslant\lambda^{-2}} E_\alpha \cap \ker L_v$, as this is the case of f by assumption, and of $L_v g_0$. Hence, it vanishes against all distributions in $\bigcup_{|\alpha|\geqslant\lambda^{-2}} E_\alpha$. Under the assumptions of the lemma, it follows that $f + L_v g_0$ can be written as $L_v F$ for some function $F \in C^1_h$. Then $f = L_v(F - g_0)$, concluding the proof. □

From this point on, we will assume that f satisfies the strengthened assumptions of Lemma 5.5. To prove the theorem, we start with a stronger version of Lemma 5.3.

Lemma 5.6. *Under the assumptions of Lemma 5.5, there exists $\varepsilon > 0$ such that the inequality $\left|\int_{t=0}^{\tau} \varphi(t/\tau) f(g_t x)\,\mathrm{d}t\right| \leqslant C\|\varphi\|_{C^3}/\tau^{1+\varepsilon}$ in (5.1) holds, with $r = 3$.*

Proof. The proof is the same as for Lemma 5.6, with the difference that the additional vanishing conditions in Lemma 5.5 give more vanishing terms in the spectral decomposition of f, and thus a faster decay of $\check{\mathcal{T}}^n f$. □

Let us now prove that the function F given by Theorem 5.2 is Lipschitz along horizontal segments. This is the main step of the proof.

Lemma 5.7. *Under the assumptions of Lemma 5.5, there exists C such that, for any points x, y on the same horizontal segment, one has $|F(x) - F(y)| \leqslant Cd(x, y)$.*

Proof. It suffices to prove the result for nearby points. Let $\delta > 0$ be such that any horizontal segment of size $\leqslant \delta$ can be completed above or below to form a rectangle of vertical size 1, not containing any singularity. We will show the statement when $d = d(x, y)$ belongs to $(0, \delta/\lambda)$.

Let $n \geqslant 1$ be the integer such that $\lambda^n d \in (\delta/\lambda, \delta]$. Let I be the horizontal interval between x and y. Assume for instance that $T^n I$ (which is of length $\leqslant \delta$) can be completed above by a rectangle of height 1 (otherwise, it can be completed below, and the argument is the same but using F_n^- instead of F_n^+). In particular, there is no singularity in the rectangle of height λ^n above I. Note first that

$$\left|F_0^+(x) - F_0^+(y)\right| = \left|\int_{t=0}^{1} \chi(t) f(g_t x) - f(g_t y)\,\mathrm{d}t\right|.$$

As $L_h f$ is bounded by assumption and $g_t x$ and $g_t y$ are at distance d along a horizontal segment, we get

$$\left|F_0^+(x) - F_0^+(y)\right| \leqslant Cd. \tag{5.6}$$

Next, for $0 < k \leqslant n$, we have $F_k^+(x) - F_{k-1}^+(x) = \int_{t=0}^{\lambda^k} \varphi(t/\lambda^k) f(g_t x) \, \mathrm{d}t$ where $\varphi(t) = \chi(t) - \chi(\lambda t)$. Taking the difference, we get

$$\begin{aligned}(F_k^+(x) - F_{k-1}^+(x)) - (F_k^+(y) - F_{k-1}^+(y)) &= \int_{t=0}^{\lambda^k} \varphi(t/\lambda^k)(f(g_t x) - f(g_t y)) \, \mathrm{d}t \\ &= \lambda^k \int_{s=0}^{1} \varphi(s)(\check{\mathcal{T}}^k f(g_s x_k) - \check{\mathcal{T}}^k f(g_s y_k)) \, \mathrm{d}s,\end{aligned}$$

for $x_k = T^k x$ and $y_k = T^k y$, as in (5.4). Since the points $g_s x_k$ and $g_s y_k$ are on the same horizontal segment of length $\lambda^k d$, we can integrate by parts and get

$$(F_k^+(x) - F_{k-1}^+(x)) - (F_k^+(y) - F_{k-1}^+(y)) = \lambda^k \int_{u=y_k}^{x_k} \left(\int_{s=0}^{1} \varphi(s) L_h \check{\mathcal{T}}^k f(g_s u) \, \mathrm{d}s \right) \mathrm{d}u.$$

Each integral over s is an integral over a vertical segment, against a smooth function φ. By the definition of $\check{\mathcal{B}}$, it is bounded by $C\|\varphi\|_{C^3} \|\check{\mathcal{T}}^k f\|_{\check{\mathcal{B}}^{3,-3}}$. Moreover, the vanishing conditions on f in the assumptions of Theorem 5.4 ensure that $\|\check{\mathcal{T}}^k f\|_{\check{\mathcal{B}}^{3,-3}}$ decays like $C\lambda^{-(2+\varepsilon)k}$ for some $\varepsilon > 0$. We get

$$\begin{aligned}\left|(F_k^+(x) - F_{k-1}^+(x)) - (F_k^+(y) - F_{k-1}^+(y))\right| &\leqslant C\lambda^k |x_k - y_k| \lambda^{-(2+\varepsilon)k} = C\lambda^k \cdot \lambda^k d \cdot \lambda^{-(2+\varepsilon)k} \\ &= Cd\lambda^{-\varepsilon k}.\end{aligned}$$

As the geometric series $\lambda^{-\varepsilon k}$ is summable, we get starting from (5.6) and summing over k from 1 to n the inequality

$$\left|F_n^+(x) - F_n^+(y)\right| \leqslant Cd. \tag{5.7}$$

Moreover, by (5.5) (but with ε replaced by $1 + \varepsilon$ thanks to Lemma 5.6), we have

$$\left|F_n^+(x) - F(x)\right| \leqslant C/\lambda^{(1+\varepsilon)n} \leqslant C\lambda^{-n} \leqslant C(\lambda d/\delta),$$

thanks to the inequality $\lambda^n d \geqslant \delta/\lambda$. This is bounded by Cd. In the same way, $|F_n^+(y) - F(y)| \leqslant Cd$. Together with (5.7), this gives $|F(x) - F(y)| \leqslant Cd$. □

Remark 5.8. Under the weaker assumptions of Theorem 5.2, then the same proof goes through to prove that $|F(x) - F(y)| \leqslant Cd(x, y)^{\varepsilon}$, where ε comes from Lemma 5.3. Hence, the solution F to the cohomological equation is automatically Hölder continuous, without any further assumption. This corresponds in a different setting to the main result of [MY16].

Proof of Theorem 5.4. Consider a function f satisfying the assumptions of Lemma 5.5. We have to show that it is a C^1 coboundary. Let F be the solution to the coboundary equation given by Theorem 5.2. By Lemma 5.7, along any horizontal segment, it is differentiable almost everywhere, and equal to the primitive of its derivative. We get a bounded measurable function F_h such that, for every horizontal interval I, for every $x, y \in I$, one has

$$F(y) - F(x) = \int_x^y F_h(u) \, \mathrm{d}u. \tag{5.8}$$

The difficulty is that we do not know if F_h is continuous and well defined everywhere.

The function $L_h f$ belongs to C^2_h. Moreover, it satisfies $\langle \omega, L_h f\rangle = 0$ for $\omega \in \bigcup_{|\alpha|\geqslant \lambda^{-1}} E_\alpha$, as this is equal to $-\langle L_h\omega, f\rangle$, which vanishes under the assumptions of Lemma 5.5 as $L_h\omega \in \bigcup_{|\alpha|\geqslant\lambda^{-2}} E_\alpha$. It follows that $L_h f$ satisfies all the assumptions of Theorem 5.2. Hence, there exists a continuous function G on M such that $\int_0^\tau L_h f(g_t x) = G(x) - G(g\tau x)$ for all x and τ.

Consider two points x and y on a small horizontal interval, and $\tau > 0$ so that there is no singularity between the orbits $(g_s x)_{s\leqslant\tau}$ and $(g_s y)_{s\leqslant\tau}$. Then one can compute

$$\begin{aligned}\int_{u=x}^{y} (G-F_h)(u) - &(G-F_h)(g_\tau u)\,\mathrm{d}u \\ &= \int_{u=x}^{y}\int_0^\tau L_h f(g_t u)\,\mathrm{d}t\,\mathrm{d}u - (F(y)-F(x)) + (F(g_\tau y) - F(g_\tau x)) \\ &= \int_0^\tau f(g_t y) - f(g_t x)\,\mathrm{d}t - (F(y)-F(x)) + (F(g_\tau y)-F(g_\tau x)) = 0.\end{aligned}$$

Since this also holds along any subsegment $[x', y']$ of $[x, y]$, it follows that $(G-F_h)(u) - (G-F_h)(g_\tau u)$ vanishes almost everywhere on the segment $[x, y]$. One deduces that, for almost every $\tau \geqslant 0$ and almost every $u \in M$, one has $(G-F_h)(g_\tau u) = (G-F_h)(u)$. By ergodicity of the vertical flow, it follows that $G - F_h$ is almost everywhere constant, and we can even assume that this constant vanishes by subtracting it from G if necessary.

By Fubini, for almost every horizontal interval I one has $F_h = G$ almost everywhere on I. On such an interval, we deduce from (5.8) the equality $F(y) - F(x) = \int_x^y G(u)\,\mathrm{d}u$. By continuity of F and G, this equality extends to all horizontal intervals. It follows from this formula that F is differentiable in the horizontal direction, with derivative G. As G is continuous on M, this concludes the proof of the theorem. □

The following theorem is the precise version of Theorem 1.7 on C^k solutions to the cohomological equation.

Theorem 5.9. *Under the assumptions of Theorem 5.2, assume additionally that $f \in C_h^{k+2}$. Assume moreover that, for any $\omega \in \bigcup_{|\alpha|\geqslant\lambda^{-k-1}} E_\alpha \cap \ker L_v$, one has $\langle\omega, f\rangle = 0$. Then the function F solving the cohomological equation* (5.3) *is C^k along the horizontal direction, and $L_h^j F$ extends continuously to M for all $j \leqslant k$.*

The assumptions of the theorem make sense by (5.2). As explained after that equation, the assumptions of the theorem could even be weakened to $f \in C_h^{k+1+\varepsilon}$. The loss of $1+\varepsilon$ derivatives corresponds in this setting to the result of Forni on the regularity loss in the cohomological equation on almost every translation surface [For07]. The conclusion can also be strengthened as the k-th derivative is also Hölder continuous for some small exponent, see Remark 5.8.

Proof. We argue by induction on k, the cases $k = 0$ and $k = 1$ being true thanks to Theorems 5.2 and 5.4. Assume $k \geqslant 2$. By Theorem 5.4, there exists a function F solving the cohomological equation for f, such that $L_h F$ is well defined and continuous. Differentiating horizontally, one gets that $L_h F$ is a continuous function, solving the cohomological equation for $L_h f$.

Moreover, the function $L_h f$ satisfies all the assumptions of the theorem for the smoothness degree $k-1$. By the inductive assumption, there exists a function G solving the cohomological equation for $L_h f$, such that $L_h^i G$ is well defined for $i \leqslant k-1$. The functions G and $L_h F$ solve the same cohomological equation. Hence, $G - L_h F$ is constant along orbits of the vertical flow. As this flow is minimal, it follows that $G - L_h F$ is constant. Therefore, $L_h F$ has $k-1$ continuous horizontal derivatives. This concludes the proof. □

6. When orientations are not preserved

6.1. **Orientable foliations whose orientations are not preserved.** Consider a translation surface (M, Σ), and a linear pseudo-Anosov map T on M which does not necessarily preserve the orientations of the horizontal and vertical foliations. There are two global signs ε_h and ε_v indicating respectively if T preserves the orientations of the horizontal and the vertical foliations. Then the spectrum of T^* on $H^1(M)$ is given by $\varepsilon_h \lambda$, by $\varepsilon_v \lambda^{-1}$, and by $\Xi = \{\mu_1, \dots, \mu_{2g-2}\}$ with $|\mu_i| \in (\lambda^{-1}, \lambda)$ (where this last property follows from the same result for the map T^2, which preserves orientations). One can describe the Ruelle spectrum exactly as we did in the orientations preserving case, with the only difference that the commutation relations between the composition operator $\mathcal{T}$ and the horizontal and vertical derivatives are not the same: Proposition 3.3 should be replaced by the equalities

$$\mathcal{T} \circ L_v = \varepsilon_v \lambda L_v \circ \mathcal{T}, \quad \mathcal{T} \circ L_h = \varepsilon_h \lambda^{-1} L_h \circ \mathcal{T}$$

on appropriate spaces. On the other hand, the definition of the Banach spaces $\mathcal{B}^{-k_h, k_v}$ need not be changed (their very definition in Section 2 is independent of the existence of a pseudo-Anosov map on the surface).

The largest eigenvalues of $\mathcal{T}$, in addition to 1, are given by $\varepsilon_h \lambda^{-1} \mu_i$. Then, to build new eigenfunctions from such an eigenfunction, one can either differentiate in the horizontal direction, or integrate in the vertical direction. When $\varepsilon_h \neq \varepsilon_v$, this gives rise to two different eigenvalues, while when they coincide one obtains the same eigenvalue again. In general, choosing to apply $k-1$ horizontal derivatives and ℓ vertical integrations (with $k \geqslant 1$ and $\ell \geqslant 0$) gives an eigenfunction for the eigenvalue $\varepsilon_h^k \varepsilon_v^\ell \lambda^{-k-\ell} \mu_i$. Hence, one obtains the following description of the spectrum:

Theorem 6.1. *Let T be a linear pseudo-Anosov map on a translation surface of genus g, with orientable horizontal and vertical foliations. Denoting by $\lambda > 1$ its expansion factor, then the spectrum of T^* on $H^1(M)$ has the form $\{\varepsilon_h \lambda, \varepsilon_v \lambda^{-1}, \mu_1, \dots, \mu_{2g-2}\}$ with $|\mu_i| \in (\lambda^{-1}, \lambda)$ for all $i = 1, \dots, 2g-2$. Then T has a Ruelle spectrum on $\mathcal{C} = C_c^\infty(M - \Sigma)$, given (with multiplicities) by*

$$\{1\} \cup \bigcup_{i=1}^{2g-2} \bigcup_{k \geqslant 1} \bigcup_{\ell \geqslant 0} \{\varepsilon_h^k \varepsilon_v^\ell \lambda^{-k-\ell} \mu_i\}.$$

For $\varepsilon_h = \varepsilon_v = 1$, one recovers Theorem 1.4.

One can also obtain a full description of the vertically invariant distributions, and solve the cohomological equation for the vertical flow. However, the simplest way to do this is certainly to apply the results of the previous sections to the map T^2, which preserves orientations, so we will not discuss these results any further.

It is more interesting to check that the trace formula of Theorem 1.8 still holds in this more general context.

Theorem 6.2. *Let T be a linear pseudo-Anosov map on a compact surface with orientable horizontal and vertical foliations. Then, for all n,*

$$\operatorname{tr}^\flat(\mathcal{T}^n) = \sum_\alpha d_\alpha \alpha^n, \tag{6.1}$$

where the sum is over all Ruelle resonances α of T, and d_α denotes the multiplicity of α.

Proof. We follow the proof of Theorem 1.8, with appropriate modifications. The Lefschetz fixed-point formula gives

$$\begin{aligned}\sum_{T^n x = x} \operatorname{ind}_{T^n} x &= \operatorname{tr}((T^n)^*_{|H^0(M)}) - \operatorname{tr}((T^n)^*_{|H^1(M)}) + \operatorname{tr}((T^n)^*_{|H^2(M)}) \\ &= 1 - \left(\varepsilon_h^n \lambda^n + \varepsilon_v^n \lambda^{-n} + \sum_{i=1}^{2g-2} \mu_i^n \right) + \varepsilon_h^n \varepsilon_v^n,\end{aligned}$$

where $\{\mu_1, \dots, \mu_{2g-2}\}$ denote the eigenvalues of T^* on the subspace of $H^1(M)$ orthogonal to $[\mathrm{d}x]$ and $[\mathrm{d}y]$, as in the statement of Theorem 1.4. The last term $\varepsilon_h^n \varepsilon_v^n$ is equal to 1 if T^n preserves orientation, -1 if it reverses orientation.

We can also compute the right hand side of (6.1), using the description of Ruelle resonances: By Theorem 6.1, $\sum d_\alpha \alpha^n$ is given by

$$\begin{aligned}1 + \sum_{i=1}^{2g-2} \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} (\varepsilon_h^k \lambda^{-k})^n (\varepsilon_v^\ell \lambda^{-\ell})^n \mu_i^n &= 1 + \sum_{i=1}^{2g-2} \frac{\varepsilon_h^n \lambda^{-n}}{1 - \varepsilon_h^n \lambda^{-n}} \cdot \frac{1}{1 - \varepsilon_v^n \lambda^{-n}} \cdot \mu_i^n \\ = 1 - \sum_{i=1}^{2g-2} \frac{\mu_i^n}{(1 - \varepsilon_h^n \lambda^n) \cdot (1 - \varepsilon_v^n \lambda^{-n})} &= \frac{(1 - \varepsilon_h^n \lambda^n) \cdot (1 - \varepsilon_v^n \lambda^{-n}) - \sum_{i=1}^{2g-2} \mu_i^n}{(1 - \varepsilon_h^n \lambda^n) \cdot (1 - \varepsilon_v^n \lambda^{-n})} \\ = \frac{1 - \left(\varepsilon_h^n \lambda^n + \varepsilon_v^n \lambda^{-n} + \sum_{i=1}^{2g-2} \mu_i^n \right) + \varepsilon_h^n \varepsilon_v^n}{(1 - \varepsilon_h^n \lambda^n) \cdot (1 - \varepsilon_v^n \lambda^{-n})}.\end{aligned}$$

Combining the two formulas with the definition of the flat trace, we get the conclusion of the theorem. □

6.2. **Non-orientable foliations.** Consider a pseudo-Anosov map T on a half-translation surface M, but such that the horizontal and vertical foliations are not orientable. Note that, with our Definition 1.2, a half-translation surface is always orientable as $x \mapsto -x$ preserves orientation in $\mathbb{R}^2$. Hence, if the horizontal foliation is not orientable, then neither is the vertical foliation, and conversely. In this case, one can not argue directly in M as the differentiation operators L_h and L_v do not make sense anymore: there is a sign ambiguity regarding the direction of differentiation. (On the other hand, the squares L_h^2 and L_v^2 of these operators are well defined.)

Let $\bar{M}$ be the two fold orientation (ramified) covering of M: away from singularities, an element of $\bar{M}$ is a pair (x, v) where $x \in M - \Sigma$ and v is an orientation of the horizontal foliation at x (equivalently, it is a horizontal unit-norm vector). Let $\bar{\pi} : \bar{M} \to M$ be the covering projection, and write $\bar{\Sigma} = \bar{\pi}^{-1}(\Sigma)$. Then $(\bar{M}, \bar{\Sigma})$ is a translation surface. Let $i : \bar{M} \to \bar{M}$ be the involution $i(x, v) = (x, -v)$. It is a homeomorphism of $\bar{M}$.

T lifts to two pseudo-Anosov maps $\bar{T}$ and $i \circ \bar{T}$ of $\bar{M}$ and the homeomorphism i commutes with $\bar{T}$. Let us consider $\varepsilon_h,\ \varepsilon_v$ where $\varepsilon_h, \varepsilon_v \in \{\pm 1\}$ indicate whether $\bar{T}$ fixes or reverses the orientation in the horizontal (resp. vertical) direction, as in Paragraph 6.1. Obviously the corresponding pair associated to the other lift $i \circ \bar{T}$ is $(-\varepsilon_h, -\varepsilon_v)$.

The action of i^* gives rise to a splitting of $H^1(\bar{M})$ as the direct sum of the two subspaces $H^1_\pm(\bar{M}) = \{h \in H^1(\bar{M}) : i^* h = \pm h\}$. The invariant part $H^1_+(\bar{M})$ corresponds to classes that are lifts of classes in $H^1(M)$. On the other hand, $[\mathrm{d}x]$ and $[\mathrm{d}y]$ belong to the anti-invariant part. If f is a function on M, then $f \circ \pi \cdot \mathrm{d}x$ if also anti-invariant.

The spectrum of $\bar{T}^*$ on $H^1_+(\bar{M})$ is equal to the spectrum of T on $H^1(M)$, given by $2g$ eigenvalues that we denote by $\mu_1^+, \dots, \mu_{2g}^+$. Let us denote the spectrum of $\bar{T}^*$ on $H^1_-(\bar{M})$ by $\varepsilon_h \lambda$, $\varepsilon_v \lambda^{-1}$ and $\mu_1^-, \dots, \mu_{2g_- -2}^-$. The Ruelle spectrum of $\bar{T}$ is expressed in terms of all these data as in Theorem 6.1, but the Ruelle spectrum of T is a strict subset of the Ruelle spectrum of $\bar{T}$ as one should only consider those distributions in the spectrum that do not vanish on functions coming from the basis.

Theorem 6.3. *In this setting, T has a Ruelle spectrum on $\mathcal{C} = C_c^\infty(M - \Sigma)$, given (with multiplicities) by*

$$\{1\} \cup \bigcup_{i=1}^{2g} \bigcup_{\substack{k \geqslant 1, \ell \geqslant 0 \\ k+\ell \text{ even}}} \{\varepsilon_h^k \varepsilon_v^\ell \lambda^{-k-\ell} \mu_i^+\} \cup \bigcup_{i=1}^{2g_- -2} \bigcup_{\substack{k \geqslant 1, \ell \geqslant 0 \\ k+\ell \text{ odd}}} \{\varepsilon_h^k \varepsilon_v^\ell \lambda^{-k-\ell} \mu_i^-\}.$$

It is remarkable that, in this theorem only mentioning the correlations of functions in M, all the eigenvalues of $\bar{T}^*$ appear: both the invariant and anti-invariant parts of the cohomology can be read off the correlations of functions in M.

This statement does not depend on the choice of the lift of T. Indeed, if one chooses the other lift $i \circ \bar{T}$ of T, then the μ_i^+ do not change, but ε_h, ε_v and μ_i^- are replaced by their opposites, so that the above spectrum is not modified.

Proof. Among the distributions constructed in the proof of Theorem 6.1, one should understand which are orthogonal to functions from the basis, and which come from the basis. First, for the cohomology classes, one writes them as $h = f\,\mathrm{d}x$ for some f in the Banach space $\mathcal{B}^{-k_h, k_v}$. As $\mathrm{d}x$ is anti-invariant, it follows that f is invariant if and only if h is anti-invariant. Hence, the eigenvalues μ_i^- give rise to distributions coming from the base, for the eigenvalue $\varepsilon_h \lambda^{-1} \mu_i^-$. On the other hand, the eigendistributions for $\varepsilon_h \lambda^{-1} \mu_i^+$ are anti-invariant, and do not appear in the Ruelle spectrum of T. Then, in $\bar{M}$, differentiating with respect to L_h or integrating with respect to L_v exchanges the invariant and anti-invariant subspaces. The full description of the spectrum follows. □

In this context, the trace formula of Theorem 1.8 still holds.

Theorem 6.4. *Let T be a linear pseudo-Anosov map. Then, for all n,*

$$\operatorname{tr}^\flat(\mathcal{T}^n) = \sum_\alpha d_\alpha \alpha^n, \tag{6.2}$$

where the sum is over all Ruelle resonances α of T, and d_α denotes the multiplicity of α.

Proof. We have already proved this result when the foliations are orientable, in Theorem 6.2. Hence, we can assume that the foliations are not orientable. In this case, the Ruelle spectrum is given in Theorem 6.3.

Let x be a fixed point of T^n. Denote by x_1 and x_2 its two lifts. They are either fixed or exchanged by $\bar{T}^n$. We say that x is positively fixed if its lifts are fixed by $\bar{T}^n$, and negatively fixed if they are exchanged by $\bar{T}^n$, i.e., fixed by $i \circ \bar{T}^n$. Let $\mathrm{Fix}^+(T^n)$ and $\mathrm{Fix}^-(T^n)$ denote respectively the set of positively and negatively fixed points of T^n. Around a positively fixed point, the local picture of T^n is the same as the local picture of $\bar{T}^n$ around the lifts. In particular, $\det(I - DT^n)$ is equal to $(1-\varepsilon_h^n\lambda^n)(1-\varepsilon_v^n\lambda^n)$. If x is negatively fixed, on the other hand, the local picture of T^n is the same as that of $i \circ \bar{T}^n$, hence locally $\det(I - DT^n) = (1+\varepsilon_h^n\lambda^n)(1+\varepsilon_v^n\lambda^n)$. With the definition of the flat trace, we get

$$\mathrm{tr}^\flat(\mathcal{T}^n) = \sum_{x\in \mathrm{Fix}^+(T^n)} \frac{\mathrm{ind}_{T^n} x}{(1-\varepsilon_h^n\lambda^n)(1-\varepsilon_v^n\lambda^{-n})} + \sum_{x\in \mathrm{Fix}^-(T^n)} \frac{\mathrm{ind}_{T^n} x}{(1+\varepsilon_h^n\lambda^n)(1+\varepsilon_v^n\lambda^{-n})} \tag{6.3}$$

To proceed, we note that to one point in $\mathrm{Fix}^+(T^n)$ correspond two fixed points of $\bar{T}^n$, with the same Lefschetz index. Therefore,

$$2\sum_{x\in \mathrm{Fix}^+(T^n)} \mathrm{ind}_{T^n}(x) = \sum_{\bar{T}^n y = y} \mathrm{ind}_{\bar{T}^n}(y).$$

We can apply Lefschetz index formula for $\bar{T}^n$ to the last sum, yielding

$$\begin{aligned}
2\sum_{x\in \mathrm{Fix}^+(T^n)} \mathrm{ind}_{T^n}(x) &= \mathrm{tr}((\bar{T}^n)^*_{|H^0(\bar{M})}) - \mathrm{tr}((\bar{T}^n)^*_{|H^1(\bar{M})}) + \mathrm{tr}((\bar{T}^n)^*_{|H^2(\bar{M})}) \\
&= 1 - \left(\varepsilon_h^n\lambda^n + \varepsilon_v^n\lambda^{-n} + \sum_{i=1}^{2g}(\mu_i^+)^n + \sum_{i=1}^{2g_- -2}(\mu_i^-)^n\right) + \varepsilon_h^n\varepsilon_v^n \\
&= (1-\varepsilon_h^n\lambda^n)(1-\varepsilon_v^n\lambda^{-n}) - \sum_{i=1}^{2g}(\mu_i^+)^n - \sum_{i=1}^{2g_- -2}(\mu_i^-)^n.
\end{aligned}$$

A point in $\mathrm{Fix}^-(T^n)$ corresponds to two fixed points of $i\circ\bar{T}^n$. Applying the Lefschetz formula to $i\circ\bar{T}^n$, we get in the same way

$$2\sum_{x\in \mathrm{Fix}^-(T^n)} \mathrm{ind}_{T^n}(x) = (1+\varepsilon_h^n\lambda^n)(1+\varepsilon_v^n\lambda^{-n}) - \sum_{i=1}^{2g}(\mu_i^+)^n + \sum_{i=1}^{2g_- -2}(\mu_i^-)^n,$$

as the eigenvalues of $i\circ\bar{T}^n$ in cohomology are $-\varepsilon_h^n\lambda^n$, $-\varepsilon_v^n\lambda^n$, $(\mu_i^+)^n$ and $-(\mu_i^-)^n$. Combining these two formulas with (6.3), we obtain

$$\begin{aligned}
\mathrm{tr}^\flat(\mathcal{T}^n) = 1 &- \frac{1}{2}\sum(\mu_i^+)^n\left(\frac{1}{(1-\varepsilon_h^n\lambda^n)(1-\varepsilon_v^n\lambda^{-n})} + \frac{1}{(1+\varepsilon_h^n\lambda^n)(1+\varepsilon_v^n\lambda^{-n})}\right) \\
&- \frac{1}{2}\sum(\mu_i^-)^n\left(\frac{1}{(1-\varepsilon_h^n\lambda^n)(1-\varepsilon_v^n\lambda^{-n})} - \frac{1}{(1+\varepsilon_h^n\lambda^n)(1+\varepsilon_v^n\lambda^{-n})}\right).
\end{aligned} \tag{6.4}$$

Let us expand

$$\begin{aligned}\frac{1}{(1-\varepsilon_h^n\lambda^n)(1-\varepsilon_v^n\lambda^{-n})} &= -\varepsilon_h^n\lambda^{-n}\frac{1}{1-\varepsilon_h^n\lambda^{-n}}\cdot\frac{1}{1-\varepsilon_v^n\lambda^{-n}}\\ &= -\varepsilon_h^n\lambda^{-n}\left(\sum_{k\geqslant 0}(\varepsilon_h^n\lambda^{-n})^k\right)\left(\sum_{k\geqslant 0}(\varepsilon_v^n\lambda^{-n})^\ell\right)\\ &= -\sum_{k\geqslant 1,\ell\geqslant 0}(\varepsilon_h^k\varepsilon_v^\ell\lambda^{-k-\ell})^n\end{aligned}$$

and analogously

$$\frac{1}{(1+\varepsilon_h^n\lambda^n)(1+\varepsilon_v^n\lambda^{-n})} = -\sum_{k\geqslant 1,\ell\geqslant 0}(-1)^{k+\ell}(\varepsilon_h^k\varepsilon_v^\ell\lambda^{-k-\ell})^n$$

Therefore, when one computes the terms in (6.4), there comes out a factor $(1+(-1)^{k+\ell})/2$ on the first line, which is 1 when $k+\ell$ is even and 0 otherwise, and a factor $(1-(-1)^{k+\ell})/2$ on the second line, which is 1 when $k+\ell$ is odd and 0 otherwise. We finally get

$$\operatorname{tr}^\flat(\mathcal{T}^n) = 1+\sum_{i=1}^{2g}\sum_{\substack{k\geqslant 1,\ell\geqslant 0\\ k+\ell \text{ even}}}(\varepsilon_h^k\varepsilon_v^\ell\lambda^{-k-\ell}\mu_i^+)^n+\sum_{i=1}^{2g_--2}\sum_{\substack{k\geqslant 1,\ell\geqslant 0\\ k+\ell \text{ odd}}}(\varepsilon_h^k\varepsilon_v^\ell\lambda^{-k-\ell}\mu_i^+)^n.$$

In view of the expression for the Ruelle spectrum given in Theorem 6.3, this is the desired result. □

References

[Ada17] Alexander Adam, *Generic non-trivial resonances for Anosov diffeomorphisms*, Nonlinearity **30** (2017), no. 3, 1146–1164. 3622282. Cited page 3.

[AG13] Artur Avila and Sébastien Gouëzel, *Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow*, Ann. of Math. (2) **178** (2013), 385–442. MR3071503. Cited pages 6 and 13.

[Bal05] Viviane Baladi, *Anisotropic Sobolev spaces and dynamical transfer operators: C^∞ foliations*, Algebraic and topological dynamics, Contemp. Math., vol. 385, Amer. Math. Soc., Providence, RI, 2005, pp. 123–135. MR2180233. Cited page 6.

[Bal17] ———, *The quest for the ultimate anisotropic Banach space*, J. Stat. Phys. **166** (2017), no. 3-4, 525–557. 3607580. Cited page 13.

[BJ08] Oscar F. Bandtlow and Oliver Jenkinson, *Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions*, Adv. Math. **218** (2008), no. 3, 902–925. 2414325. Cited page 3.

[BJS13] Oscar F. Bandtlow, Wolfram Just, and Julia Slipantschuk, *Analytic expanding circle maps with explicit spectra*, Nonlinearity **26** (2013), no. 12, 3231–3245. 3141853. Cited page 3.

[BJS17] ———, *Complete spectral data for analytic Anosov maps of the torus*, Nonlinearity **30** (2017), no. 7, 2667–2686. 3670002. Cited page 3.

[BT07] Viviane Baladi and Masato Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*, Ann. Inst. Fourier (Grenoble) **57** (2007), 127–154. MR2313087. Cited pages 3 and 5.

[BT08] ———, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms*, Geometric and probabilistic structures in dynamics, Contemp. Math., vol. 469, Amer. Math. Soc., Providence, RI, 2008, pp. 29–68. MR2478465. Cited page 34.

[Buf14a] Alexander I. Bufetov, *Finitely-additive measures on the asymptotic foliations of a Markov compactum*, Mosc. Math. J. **14** (2014), no. 2, 205–224, 426. 3236492. Cited page 9.

[Buf14b] ______, *Limit theorems for translation flows*, Ann. of Math. (2) **179** (2014), no. 2, 431–499. 3152940. Cited page 26.

[DFG15] Semyon Dyatlov, Frédéric Faure, and Colin Guillarmou, *Power spectrum of the geodesic flow on hyperbolic manifolds*, Anal. PDE **8** (2015), no. 4, 923–1000. 3366007. Cited page 3.

[For97] Giovanni Forni, *Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus*, Ann. of Math. (2) **146** (1997), no. 2, 295–344. 1477760. Cited pages 3 and 10.

[For02] ______, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math. (2) **155** (2002), 1–103. MR1888794. Cited pages 3, 6, 7, 9, and 52.

[For07] Giovanni Forni, *Sobolev regularity of solutions of the cohomological equation*, Preprint, 2007. Cited pages 3, 10, and 59.

[For18] ______, *Ruelle resonances dnd cohomological equations*, Preprint, 2018. Cited page 10.

[GL08] Sébastien Gouëzel and Carlangelo Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, J. Differential Geom. **79** (2008), 433–477. MR2433929. Cited pages 3, 5, and 13.

[GL14] Paolo Giulietti and Carlangelo Liverani, *Parabolic dynamics and anisotropic Banach spaces*, preprint, 2014. Cited pages 3, 10, 52, and 53.

[Hen93] Hubert Hennion, *Sur un théorème spectral et son application aux noyaux lipchitziens*, Proc. Amer. Math. Soc. **118** (1993), 627–634. MR1129880. Cited page 33.

[HK95] Boris Hasselblatt and Anatole Katok, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza. MR1326374. Cited pages 11 and 12.

[Hör03] Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003, Distribution theory and Fourier analysis, Reprint of the second (1990) edition. MR1996773. Cited page 45.

[Jéz17] Malo Jézéquel, *Local and global trace formulae for $\mathcal{C}^\infty$ hyperbolic diffeomorphisms*, Preprint, 2017. Cited page 11.

[MY16] Stefano Marmi and Jean-Christophe Yoccoz, *Hölder regularity of the solutions of the cohomological equation for Roth type interval exchange maps*, Comm. Math. Phys. **344** (2016), no. 1, 117–139. 3493139. Cited pages 10, 51, and 58.

[Nau12] Frédéric Naud, *The Ruelle spectrum of generic transfer operators*, Discrete Contin. Dyn. Syst. **32** (2012), no. 7, 2521–2531. 2900558. Cited page 3.

[Rue90] David Ruelle, *An extension of the theory of Fredholm determinants*, Inst. Hautes Études Sci. Publ. Math. **72** (1990), 175–193. MR1087395. Cited page 3.

[Thu88] William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431. 956596. Cited page 5.

[Zor06] Anton Zorich, *Flat surfaces*, Frontiers in number theory, physics, and geometry (2006), 437–583. Cited page 4.

Univ. Grenoble Alpes, CNRS UMR 5582, Institut Fourier, F-38000 Grenoble, France
E-mail address: `frederic.faure@univ-grenoble-alpes.fr`

Laboratoire Jean Leray, CNRS UMR 6629, Université de Nantes, 2 rue de la Houssinière, 44322 Nantes, France
E-mail address: `sebastien.gouezel@univ-nantes.fr`

Univ. Grenoble Alpes, CNRS UMR 5582, Institut Fourier, F-38000 Grenoble, France
E-mail address: `erwan.lanneau@univ-grenoble-alpes.fr`