$\mathrm{GL}^+(2,\mathbb{R})$-ORBITS IN PRYM EIGENFORM LOCI

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ABSTRACT. This paper is devoted to the classification of $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures of surfaces in the intersection of the Prym eigenform locus with various strata of Abelian differentials. We show that the following dichotomy holds: an orbit is either closed or dense in a connected component of the Prym eigenform locus.

The proof uses several topological properties of Prym eigenforms, in particular the tools and the proof are independent of the recent results of Eskin-Mirzakhani-Mohammadi.

As an application we obtain a finiteness result for the number of closed $\mathrm{GL}^+(2,\mathbb{R})$-orbits (not necessarily primitive) in the Prym eigenform locus $\Omega E_D(2,2)$ for any fixed D that is not a square.

1. INTRODUCTION

For any $g \geq 1$ and any integer partition $\kappa = (\kappa_1, \dots, \kappa_r)$ of $2g-2$ we denote by $\mathcal{H}(\kappa)$ a stratum of the moduli space of pairs (X,ω), where X is a Riemann surface of genus g and ω is a holomorphic 1-form having r zeros with prescribed multiplicities $\kappa_1, \dots, \kappa_r$. Analogously, one defines the strata of the moduli space of *quadratic differentials* $\mathcal{Q}(\kappa')$ having zeros and simple poles of multiplicities $\kappa'_1, \dots, \kappa'_s$ with $\sum_{i=1}^{s} \kappa'_s = 4g-4$ (simple poles correspond to zeros of multiplicity -1).

The 1-form ω defines a canonical flat metric on X with conical singularities at the zeros of ω. Therefore we will refer to points of $\mathcal{H}(\kappa)$ as flat surfaces or *translation surfaces*. The strata admit a natural action of the group $\mathrm{GL}^+(2,\mathbb{R})$ that can be viewed as a generalization of the $\mathrm{GL}^+(2,\mathbb{R})$ action on the space $\mathrm{GL}^+(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of flat tori. For an introduction to this subject, we refer to the excellent surveys [MT02, Zor06].

It has been discovered that many topological and dynamical properties of a translation surface can be revealed by its $\mathrm{GL}^+(2,\mathbb{R})-$orbit closure. The most spectacular example of this phenomenon is the case of *Veech surfaces*, or *lattice surfaces*, that is surfaces whose $\mathrm{GL}^+(2,\mathbb{R})$-orbit is a closed subset in its stratum; for such surfaces, the famous *Veech dichotomy* holds: the linear flow in any direction is either periodic or uniquely ergodic.

It follows from the foundation results of Masur and Veech that most of $\mathrm{GL}^+(2,\mathbb{R})$ orbits are dense in their stratum. However, in any stratum there always exist surfaces whose orbits are closed: *e.g.* coverings of the standard flat torus and are commonly known as *square-tiled surfaces*.

During the past three decades, much effort has been made in order to obtain the list of possible $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures and to understand their structure as subsets of strata. So far, such a list is only known in genus two by the work of McMullen [McM07], but the problem is wide open in higher genus, even though some breakthroughs have been achieved recently (see below).

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In genus two the complex dimensions of the connected strata $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ are, respectively, 4 and 5. In this situation, McMullen proved that if a $\mathrm{GL}^+(2,\mathbb{R})$-orbit is not dense, then it belongs to a *Prym eigenform locus*, which is a submanifold of complex dimension 3. In this case, the orbit is either closed or dense in the whole Prym eigenform locus. These (closed) invariant submanifolds, that we denote by ΩE_D, where D is a *discriminant* (that is $D \in \mathbb{N},\ D \equiv 0, 1 \mod 4$), are characterized by the following properties:

(1) Every surface $(X,\omega) \in \Omega E_D$ has a holomorphic involution $\tau : X \to X$, and
(2) The *Prym variety* $\mathrm{Prym}(X,\tau) = (\Omega^-(X,\tau))^*/H_1(X,\mathbb{Z})^-$ admits a real multiplication by some quadratic order $\mathcal{O}_D := \mathbb{Z}[x]/(x^2+bx+c),\ b,c \in \mathbb{Z},\ b^2-4c = D$.

(where $\Omega^-(X,\tau) = \{\eta \in \Omega(X) :\ \tau^*\eta = -\eta\}$).

Later, these properties were extended to higher genera (up to genus five) by McMullen (see [McM03a, McM06, LN13] for more details).

Recently, Eskin-Mirzakhani-Mohammadi [EMi13, EMiMo13] have announced a proof of the conjecture that any $\mathrm{GL}^+(2,\mathbb{R})$-orbit closure is an affine invariant submanifold of $\mathcal{H}(\kappa)$. This result is of great importance in view of the classification of orbit closures as it provides some very important characterizations of such subsets. However *a priori* this result does not allow us to construct explicitly such invariant submanifolds.

So far, most of $\mathrm{GL}^+(2,\mathbb{R})$-invariant submanifolds of a stratum are obtained from coverings of translation surfaces of lower genera. The only known examples of invariant submanifolds *not* arising from this construction belong to one of the following families:

(1) Primitive Teichmüller curves (closed orbits), and
(2) Prym eigenforms.

This paper is concerned with the classification of $\mathrm{GL}^+(2,\mathbb{R})-$orbit closures in the space of Prym eigenforms. To be more precise, for any non empty stratum $\mathcal{Q}(\kappa')$, there is a (local) affine map $\phi : \mathcal{Q}_{g'}(\kappa') \to \mathcal{H}_g(\kappa)$ given by the orientating double covering (the indices g and g' are the genus of the corresponding Riemann surfaces). When $g - g' = 2$, following McMullen [McM06] we call the image of ϕ a Prym locus and denote it by $\mathrm{Prym}(\kappa)$. Those Prym loci contain $\mathrm{GL}^+(2,\mathbb{R})$-invariant suborbifolds denoted by $\Omega E_D(\kappa)$ (see Section 2 for more precise definitions). We will investigate the $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures in $\Omega E_D(\kappa)$. The first main theorem of this paper is the following.

Theorem 1.1. *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform, where $\Omega E_D(\kappa)$ has complex dimension* 3 *(*i.e. *$\Omega E_D(\kappa)$ is contained in one of the Prym loci in Table 1). We denote by $\mathcal{O}$ its orbit under* $\mathrm{GL}^+(2,\mathbb{R})$*. Then*

(1) *Either $\mathcal{O}$ is closed (i.e. (X,ω) is a Veech surface), or*
(2) *$\overline{\mathcal{O}}$ is a connected component of $\Omega E_D(\kappa)$.*

Observe that the case $\kappa = (1,1)$ is part of McMullen's classification in genus two, which is obtained via decompositions of translation surfaces of genus two into connected sums of two tori (see [McM07]).

Remark 1.2. *The classification of connected components of $\Omega E_D(2,2)$ and $\Omega E_D(1,1,2)$ will be addressed in a forthcoming paper* [LN13c] *(see also* [LN13] *for related work). The statement is the following: for any discriminant $D \geq 8$ and $\kappa \in \{(2,2),(1,1,2)\}$, the locus $\Omega E_D(\kappa)$ is non-empty if and*

$Q(\kappa')$	$\mathrm{Prym}(\kappa)$	$g(X)$
$Q_0(-1^6,2)$	$\mathrm{Prym}(1,1)\simeq\mathcal{H}(1,1)$	2
$Q_1(-1^3,1,2)$	$\mathrm{Prym}(1,1,2)$	3
$Q_1(-1^4,4)$	$\mathrm{Prym}(2,2)^{\mathrm{odd}}$	3
$Q_2(-1^2,6)$	$\mathrm{Prym}(3,3)\simeq\mathcal{H}(1,1)$	4

$Q(\kappa')$	$\mathrm{Prym}(\kappa)$	$g(X)$
$Q_2(1^2,2)$	$\mathrm{Prym}(1^2,2^2)\simeq\mathcal{H}(0^2,2)$	4
$Q_2(-1,2,3)$	$\mathrm{Prym}(1,1,4)$	4
$Q_2(-1,1,4)$	$\mathrm{Prym}(2,2,2)^{\mathrm{even}}$	4
$Q_3(8)$	$\mathrm{Prym}(4,4)^{\mathrm{even}}$	5

TABLE 1. Prym loci for which the corresponding stratum of quadratic differentials has (complex) dimension 5. The Prym eigenform locus $\Omega E_D(\kappa)$ has complex dimension 3. Observe that the stratum $\mathcal{H}(1,1)$ in genus 2 is a particular case of Prym locus.

only if $D\equiv 0,1,4 \mod 8$, and it is connected if $D\equiv 0,4 \mod 8$, and has two connected components otherwise.

Even though Theorem 1.1 is a particular case of the recent results of Eskin-Mirzakhani and Eskin-Mirzakhani-Mohammadi [EMi13, EMiMo13], our proof is independent from these works. It is based on the geometry of the kernel foliation on the space of Prym eigenforms. It is also likely to us that the method introduced here can be generalized to yield Eskin-Mirzakhani-Mohammadi's result in invariant submanifolds which possess the *complete periodic* property (see Section 2.3).

We will also prove a finiteness result for Teichmüller curves in the locus $\Omega E_D(2,2)^{\mathrm{odd}}$; this is our second main result:

Theorem 1.3. *If D is not a square then there exist only finitely many closed* $\mathrm{GL}^+(2,\mathbb{R})$*-orbits in* $\Omega E_D(2,2)^{\mathrm{odd}}$.

We end with a few remarks on Theorem 1.3.

Remark 1.4.

- *To the authors' knowledge, such finiteness result is not a direct consequence of the work by Eskin-Mirzakhani-Mohammadi.*
- *In* $\mathrm{Prym}(1,1)$ *a stronger statement holds: there exist only finitely many* $\mathrm{GL}^+(2,\mathbb{R})$*-closed orbits in* $\bigsqcup_{D \text{ not a square}} \Omega E_D(1,1)$ *(see* [McM05b, McM06a]*). The same result holds for* $\mathrm{Prym}(1,1,2)$*: this is proved in a forthcoming paper by the first author and M. Möller* [LMöl13]. *However, this is no longer true in* $\mathrm{Prym}(2,2)^{\mathrm{odd}}$ *as we will see in Theorem A.1.*
- *Other finiteness results on Teichmüller curves have been obtained in other situations by different methods, see for instance* [Möl08, BaMöl12, MaWri13].

Outline of the paper. We end this section with a sketch of the proofs of Theorem 1.1 and Theorem 1.3. Before going into the details, we single out the relevant properties of $\Omega E_D(\kappa)$ for our purpose. In what follows (X,ω) will denote a surface in $\Omega E_D(\kappa)$ (sometimes we will simply use X when there is no confusion).

(1) Each locus is preserved by the *kernel foliation*, that is, $(X,\omega)+v$ is well defined for any sufficiently small vector $v\in\mathbb{R}^2$ (see Section 3). Up to action of $\mathrm{GL}^+(2,\mathbb{R})$, there exists $\varepsilon>0$ such that a neighborhood of (X,ω) in $\Omega E_D(\kappa)$ can be identified with the set
$$\{(X,\omega)+v,\ |v|<\varepsilon\}.$$

(2) Every surface in $\Omega E_D(\kappa)$ is completely periodic in the sense of Calta: any direction of a simple closed geodesic is actually completely periodic, which means that the surface is decomposed into cylinders in this direction. The number of cylinders is bounded from above by $g + |\kappa| - 1$, where $|\kappa|$ is the number of zeros of ω (see Section 2).
(3) Assume that (X, ω) decomposes into cylinders in the horizontal direction, then the moduli of those cylinders are related by some equations with rational coefficients (see Proposition 4.12).
(4) The cylinder decomposition in a completely periodic direction is said to be *stable* if there is no saddle connection connecting two different zeros in this direction. The stable periodic directions are *generic* for the kernel foliation in the following sense: if the horizontal direction is stable for (X, ω), then there exists $\varepsilon > 0$ such that for any $v \in \mathbb{R}^2$ with $|v| < \varepsilon$, the horizontal direction is also periodic and stable on $X + v$. If the horizontal direction is *unstable* then there exists $\varepsilon > 0$ such that for any $v = (x, y)$ with $|v| < \varepsilon$ and $y \neq 0$ the horizontal direction is periodic and stable on $X + v$.

The properties (1)-(2)-(3) are explained in [LN13a] (see Section 3.1 and Corollary 3.2, Theorem 1.5, Theorem 7.2, respectively). We will give more details on Property (4) in Section 4.

We now give a sketch of the proof of our results. The first part of the paper (Sections 3-6) is devoted to the proof of Theorem 1.1, while the second part (Sections 7-11) is concerned with Theorem 1.3.

Sketch of proof of Theorem 1.1. Let $(X, \omega) \in \Omega E_D(\kappa)$ be a Prym eigenform and let $\mathcal{O} := \mathrm{GL}^+(2, \mathbb{R}) \cdot (X, \omega)$ be the corresponding $\mathrm{GL}^+(2, \mathbb{R})$−orbit. We will show that if $\mathcal{O}$ is not a closed subset in $\Omega E_D(\kappa)$ then it is dense in a connected component of $\Omega E_D(\kappa)$.

We first prove a weaker version of Theorem 1.1 (see Section 5) under the additional condition that there exists a completely periodic direction θ on (X, ω) that is not parabolic. We start by applying the horocycle flow in that periodic direction, and use the classical Kronecker's theorem to show that the orbit closure contains the set $(X, \omega) + x\vec{v}$, where $\vec{v}$ is the unit vector in direction θ, and $x \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small enough. Then we apply the same argument to the surfaces $(X, \omega) + x\vec{v}$ in another periodic direction that is transverse to θ. It follows that $\overline{\mathcal{O}}$ contains a neighborhood of (X, ω), and hence for any $g \in \mathrm{GL}^+(2, \mathbb{R})$, $\overline{\mathcal{O}}$ contains a neighborhood of $g \cdot (X, \omega)$. Using this fact, we show that for any $(Y, \eta) \in \overline{\mathcal{O}} \setminus \mathcal{O}$, the closure $\overline{\mathcal{O}}$ also contains a neighborhood of (Y, η), from which we deduce that $\overline{\mathcal{O}}$ is an open subset of $\Omega E_D(\kappa)$. Hence $\overline{\mathcal{O}}$ must be a connected component of $\Omega E_D(\kappa)$.

In full generality, (see Section 6) we show that if the orbit is not closed and all the periodic directions are parabolic, then it is also dense in a component of $\Omega E_D(\kappa)$. For this, we consider a surface $(Y, \eta) \in \overline{\mathcal{O}} \setminus \mathcal{O}$ for which the horizontal direction is periodic. From Property (1), we see that there is a sequence $((X_n, \omega_n))_{n \in \mathbb{N}}$ of surfaces in $\mathcal{O}$ converging to (Y, η) such that we can write $(X_n, \omega_n) = (Y, \eta) + (x_n, y_n)$, where $(x_n, y_n) \longrightarrow (0, 0)$. Property (4) then implies that the horizontal direction is periodic for (X_n, ω_n). Moreover, we can assume that the corresponding cylinder decomposition in (X_n, ω_n) is stable (for n large enough).

For any $x \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is small enough, we show that (up to taking a subsequence) the orbit of the horocycle flow though (X_n, ω_n) contains a surface $(X_n, \omega_n) + (x_n, 0)$ such that the sequence (x_n) converges to x. As a consequence, we see that $\overline{\mathcal{O}}$ contains $(Y, \eta) + (x, 0)$ for every $x \in (-\varepsilon, \varepsilon)$. We can now conclude that $\overline{\mathcal{O}}$ is a component of $\Omega E_D(\kappa)$ by the weaker version of Theorem 1.1.

Sketch of proof of Theorem 1.3. We first show a finiteness result up to the (real) kernel foliation for surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ (see Theorem 11.2): If D is not a square then there exists a finite family $\mathbb{P}_D \subset \Omega E_D(2,2)^{\mathrm{odd}}$ such that for any $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ with an unstable cylinder decomposition, up to rescaling by $\mathrm{GL}^+(2,\mathbb{R})$, we have the following

$$(X,\omega) = (X_k,\omega_k) + (x,0) \qquad \text{for some } (X_k,\omega_k) \in \mathbb{P}_D.$$

Compare to [McM05a, LN13] where a similar result is established.

Now let us assume that there exists an infinite family, say $\mathcal{Y} = \bigcup_{i\in I} \mathrm{GL}^+(2,\mathbb{R}) \cdot (X_i,\omega_i)$, of closed $\mathrm{GL}^+(2,\mathbb{R})$-orbits, generated by Veech surfaces (X_i,ω_i), $i \in I$.

By previous finiteness result, up to taking a subsequence, we assume that $(X_i,\omega_i) = (X,\omega) + (x_i,0)$ for some $(X,\omega) \in \mathbb{P}_D$, where x_i belongs to a finite open interval (a,b) which is independent of i (see Theorem 8.1). Up to taking a subsequence, one can assume that the sequence (x_i) converges to some $x \in [a,b]$. Hence the sequence $(X_i,\omega_i) = (X,\omega) + (x_i,0)$ converges to $(Y,\eta) := (X,\omega) + (x,0)$.

If $x \in (a,b)$ then (Y,η) belongs to $\Omega E_D(2,2)^{\mathrm{odd}}$, otherwise, that is $x \in \{a,b\}$, (Y,η) belongs to one of the following loci $\Omega E_D(0,0,0), \Omega E_D(4)$, or $\Omega E_{D'}(2)^*$, with $D' \in \{D, D/4\}$ (see Section 8). Then by using a by-product of the proof of Theorem 1.1, replacing $\mathcal{O}$ by $\mathcal{Y}$ (see Theorem 6.2 and Theorem 9.4) we obtain that $\mathcal{Y}$ is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$. We conclude with Theorem 10.1 which asserts that the set of closed $\mathrm{GL}^+(2,\mathbb{R})-$orbits is not dense in any component of $\Omega E_D(2,2)^{\mathrm{odd}}$ when D is not a square.

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2. Background

For an introduction to translation surfaces, and a nice survey on this topic, see *e.g.* [MT02, Zor06]. In this section we recall necessary background and relevant properties of $\Omega E_D(\kappa)$ for our purpose. For a general reference on Prym eigenforms, see [McM06].

We will use the following notations along the paper:
$\mathbf{B}(\varepsilon) = \{v \in \mathbb{R}^2,\ |v| < \varepsilon\}$, $\mathbf{B}(M,\varepsilon) = \{A \in \mathrm{GL}^+(2,\mathbb{R}),\ \|A - M\| < \varepsilon\}$ and
$\omega(\gamma) := \int_\gamma \omega$, for any $\gamma \in H_1(X,\mathbb{Z})$,
where $|.|$ is the Euclidean norm on $\mathbb{R}^2$, and $\|.\|$ is some norm on $\mathbf{M}(2,\mathbb{R})$.

2.1. Prym loci and Prym eigenforms.

Let X be a compact Riemann surface, and $\tau : X \to X$ be a holomorphic involution of X. We define the *Prym variety* of X:

$$\mathrm{Prym}(X,\tau) = (\Omega^-(X,\tau))^*/H_1(X,\mathbb{Z})^-,$$

where $\Omega^-(X,\tau) = \{\eta \in \Omega(X) : \tau^*\eta = -\eta\}$. It is a sub-Abelian variety of the Jacobian variety $\mathrm{Jac}(X) := \Omega(X)^*/H_1(X,\mathbb{Z})$.

For any integer vector $\kappa = (k_1,\dots,k_n)$ with nonnegative entries, we denote by $\mathrm{Prym}(\kappa) \subset \mathcal{H}(\kappa)$ the subset of pairs (X,ω) such that there exists an involution $\tau : X \to X$ satisfying $\tau^*\omega = -\omega$, and

$\dim_{\mathbb{C}} \Omega^-(X,\tau) = 2$. Following McMullen [McM06], we will call an element of $\mathrm{Prym}(\kappa)$ a *Prym form*. For instance, in genus two, one has $\mathrm{Prym}(2) \simeq \mathcal{H}(2)$ and $\mathrm{Prym}(1,1) \simeq \mathcal{H}(1,1)$ (the Prym involution being the hyperelliptic involution).

Let Y be the quotient of X by the Prym involution (here $g(Y) = g(X) - 2$) and π the corresponding (possibly ramified) double covering from X to Y. By push forward, there exists a meromorphic quadratic differential q on Y (with at most simple poles) so that $\pi^* q = \omega^2$. Let κ' be the integer vector that records the orders of the zeros and poles of q. Then there is a $\mathrm{GL}^+(2,\mathbb{R})$-equivariant bijection between $Q(\kappa')$ and $\mathrm{Prym}(\kappa)$ [L04, p. 6].

All the strata of quadratic differentials of dimension 5 are recorded in Table 1. It turns out that the corresponding Prym varieties have complex dimension two (*i.e* if (X,ω) is the orientating double covering of (Y,q) then $g(X) - g(Y) = 2$).

We now give the definition of Prym eigenforms. Recall that a quadratic order is a ring isomorphic to $O_D = \mathbb{Z}[X]/(X^2 + bX + c)$, where $D = b^2 - 4c > 0$ (quadratic orders being classified by their discriminant D).

Definition 2.1 (Real multiplication)**.** *Let A be an Abelian variety of dimension* 2*. We say that A admits a real multiplication by O_D if there exists an injective homomorphism* $\mathfrak{i} : O_D \to \mathrm{End}(A)$*, such that* $\mathfrak{i}(O_D)$ *is a self-adjoint, proper subring of* $\mathrm{End}(A)$ *(*i.e. *for any* $f \in \mathrm{End}(A)$*, if there exists* $n \in \mathbb{Z}\backslash\{0\}$ *such that* $nf \in \mathfrak{i}(O_D)$ *then* $f \in \mathfrak{i}(O_D)$*).*

Definition 2.2 (Prym eigenform)**.** *For any quadratic discriminant* $D > 0$*, we denote by* $\Omega E_D(\kappa)$ *the set of* $(X,\omega) \in \mathrm{Prym}(\kappa)$ *such that* $\dim_{\mathbb{C}} \mathrm{Prym}(X,\tau) = 2$*,* $\mathrm{Prym}(X,\tau)$ *admits a multiplication by* O_D*, and* ω *is an eigenvector of* O_D*. Surfaces in* $\Omega E_D(\kappa)$ *are called* Prym eigenforms*.*

Prym eigenforms do exist in each Prym locus described in Table 1, as real multiplications arise naturally with pseudo-Anosov homeomorphisms commuting with τ (see [McM06]).

2.2. **Periodic directions and Cylinder decompositions.** We collect here several results concerning surfaces having a decomposition into periodic cylinders.

Let (X,ω) be a translation surface. A *cylinder* is a topological annulus embedded in X, isometric to a flat cylinder $\mathbb{R}/w\mathbb{Z} \times (0,h)$. In what follows all cylinders are supposed to be *maximal*, that is, they are not properly contained in a larger one. If $g \geq 2$, the boundary of a maximal cylinder is a finite union of saddle connections. If C is a cylinder, we will denote by $w(C), h(C), \mu(C)$ the width, height, and modulus of C respectively ($\mu(C) = h(C)/w(C)$).

Another important parameter of a cylinder is its twist $t(C)$. Note that we only define $t(C)$ when C is a horizontal cylinder. For that, we first mark a pair of oriented saddle connections on the bottom and the top boundaries of C. This allows us to define a saddle connection contained in C joining the origins of the marked saddle connections. This gives us a twist vector, its vertical component equals $h(C)$ and its horizontal component is $t(C)$. We emphasis that $t(C)$ depends on the marking (see [HLM06, Section 3]). However, the choice of the marking is irrelevant for our arguments throughout this paper. Therefore, we will refer to $t(C)$ as the twist associated to any marking.

A direction θ is *completely periodic* or simply *periodic* on X if all regular geodesics in this direction are closed. This means that X is the closure of a finite number of cylinders in direction θ, we will say that X admits a *cylinder decomposition* in this direction.

We can associate to any cylinder decomposition a separatrix diagram which encodes the way the cylinders are glued together, see [KZ03]). Given such a diagram, one can reconstruct the surface (X,ω) (up to a rotation) from the widths, heights, and twists of the cylinders (see Section 4).

2.3. **Complete periodicity.** A translation surface (X,ω) is said to be *completely periodic* if it satisfies the following property: let $\theta \in \mathbb{RP}^1$ be a direction, if the linear flow $\mathcal{F}_\theta$ in the direction θ has a regular closed orbit on X, then θ is a periodic direction. Flat tori and their ramified coverings are completely periodic, as well as Veech surfaces.

It turns out that, if the genus is at least two, the set of surfaces having this property has measure zero. Indeed complete periodicity is locally expressed via proportionality of a non-empty set of relative periods, and thus is defined by some quadratic equations in the period coordinates. This property has been initiated by Calta [C04] (see also [CS07]) where she proved that *any* surface in $\Omega E_D(2)$ and $\Omega E_D(1,1)$ is completely periodic. Later the authors extended this property to any Prym eigenform given by Table 1. This property is also proved by A. Wright [Wri13] by a different argument.

Theorem 2.3 ([C04, LN13a, Wri13]). *Any Prym eigenform in the loci $\Omega E_D(\kappa) \subset \mathrm{Prym}(\kappa)$ of Table 1 is completely periodic.*

3. KERNEL FOLIATION ON PRYM LOCI

We briefly recall the kernel foliation for Prym loci (see [EMZ03, MZ08, C04, MW08] and [Zor06, §9.6] for related constructions). We refer to [LN13a, Section 3.1] for details. This notion was introduced by Eskin-Masur-Zorich, and was certainly known to Kontsevich.

Let $(X,\omega) \in \mathcal{H}(\kappa)$ be a translation surface with *several distinct* zeros. Using the period mapping, we can identify a neighborhood of (X,ω) in $\mathcal{H}(\kappa)$ with an open subset $U \subset \mathbb{C}^d$, where $d = \dim \mathcal{H}(\kappa)$. We have a foliation of U by subsets consisting of surfaces having the same absolute periods. The set of surfaces in this neighborhood that have the same absolute coordinates as X corresponds to the intersection of U with an affine subspace of dimension $|\kappa| - 1$. Therefore the leaves of this foliation have dimension $|\kappa| - 1$. It is not difficult to see that this foliation is invariant by the coordinate changes of the period mappings. Thus we have a foliation defined globally in $\mathcal{H}(\kappa)$, this is the *kernel foliation*.

It turns out that the kernel foliation also exists in $\mathrm{Prym}(\kappa)$ and $\Omega E_D(\kappa)$, for all κ in Table 1. In particular, the leaves of the kernel foliation in $\Omega E_D(\kappa)$ have dimension one. We refer to [LN13a, Section 3.1] for a description of this foliation in $\Omega E_D(\kappa)$ with more details.

Since the leaves of the kernel foliation in $\Omega E_D(\kappa)$ have dimension one, we have a local action of $\mathbb{C}$ on $\Omega E_D(\kappa)$ as follows: for any Prym eigenform (X,ω) and $w \in \mathbb{C}$ with $|w|$ small enough, $(X',\omega') := (X,\omega) + w$ is the unique surface in the neighborhood of (X,ω) (in $\Omega E_D(\kappa)$) such that ω' has the same absolute periods as ω, and for a chosen relative relative cycle $c \in H_1(X,\Sigma,\mathbb{Z})$, we have $\omega'(c) = \omega(c) + w$ (Σ is the set of zeros of ω). An explicit construction for $(X,\omega) + w$ will be given in Section 4.3.

have

It is worth noticing that we do not have a global action of $\mathbb{C}$ on each leaf of the kernel foliation, *i.e* even $(X,\omega) + w_1$ and $(X,\omega) + w_2$ exist, $(X,\omega) + w_1 + w_2$ may not be well defined. Nevertheless, there still exists a local action of $\mathbb{C}$ in a neighborhood of (X,ω) on which a local chart (by period mappings) can be defined. In particular, if $|w_1|$ and $|w_2|$ are small enough then $(X,\omega) + (w_1 + w_2) = ((X,\omega) + w_1) + w_2 = ((X,\omega) + w_2) + w_1$.

Convention : Throughout this paper, we only consider the intersection of kernel foliation leaves with a neighborhood of (X,ω) on which this local action of $\mathbb{C}$ is well-defined, and by $(X,\omega)+w$ we will mean the surface obtained from (X,ω) by the construction described above.

The relative periods of $(X',\omega') := (X,\omega)+w$ are characterized by the following lemma (see Figure 1 for an example in $\mathrm{Prym}(1,1,2)$).

Lemma 3.1. *Let c be a path on X joining two zeros of ω, and c' be the corresponding path on X'. Then*

(1) *If the two endpoints of c are exchanged by τ then $\omega'(c')-\omega(c)=\pm w$.*
(2) *If one endpoint of c is fixed by τ, but the other is not, then $\omega'(c')-\omega(c)=\pm w/2$.*

The sign of the difference is determined by the orientation of c.

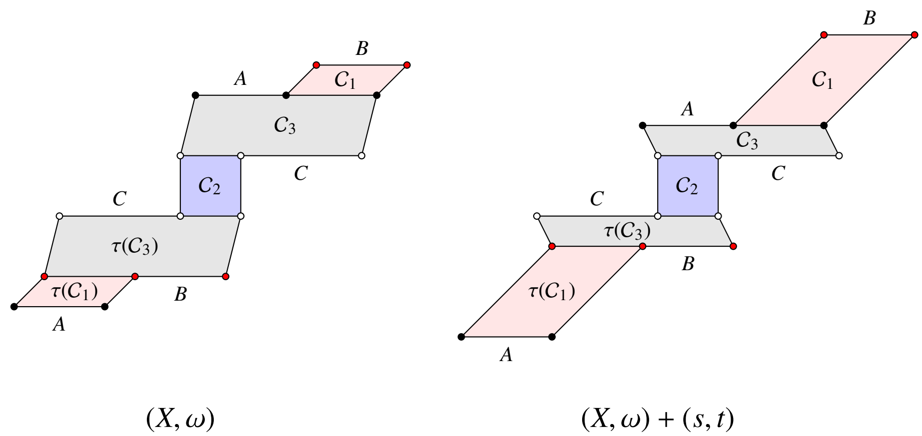

FIGURE 1. Decomposition of a surface $(X,\omega)\in\mathrm{Prym}(1,1,2)$. The cylinder C_2 is fixed by the Prym involution τ, while the cylinders C_i and $\tau(C_i)$ are exchanged for $i=1,3$. Along a kernel foliation leaf $(X,\omega)+(s,t)$ the twists and heights change as follows: $t_1(s)=t_1-s$, $t_2(s)=t_2$, $t_3(s)=t_3+s/2$ and $h_1(t)=h_1-t$, $h_2(t)=h_2$, $h_3(t)=h_3+t/2$. We emphasis that the formula for the twists does not depend on the choice of the marking.

We end this section by a description of a neighborhood of a Prym eigenform: up to the action of $\mathrm{GL}^+(2,\mathbb{R})$ a neighborhood of a point (X,ω) in $\Omega E_D(\kappa)$ can be identified with the ball $\{(X,\omega)+w, |w|<\varepsilon\}$.

Proposition 3.2 ([LN13a])**.** *For any $(X,\omega)\in\Omega E_D(\kappa)$, if (X',ω') is a Prym eigenform in $\Omega E_D(\kappa)$ close enough to (X,ω), then there exists a unique pair (g,w), where $g\in\mathrm{GL}^+(2,\mathbb{R})$ close to Id, and $w\in\mathbb{R}^2$ with $|w|$ small, such that $(X',\omega')=g\cdot((X,\omega)+w)$.*

Proof. For completeness we include the proof here (see [LN13a, Section 3.2]).
Let $(Y,\eta)=(X,\omega)+w$, with $|w|$ small, be a surface in the leaf of the kernel foliation through (X,ω). We denote by $[\omega]$ and $[\eta]$ the classes of ω and η in $H^1(X,\Sigma;\mathbb{C})^-$. Let $\rho: H^1(X,\Sigma;\mathbb{C})^-\to H^1(X,\mathbb{C})^-$ be the natural projection. We then have $[\eta]-[\omega]\in\ker\rho$. On the other hand, the action of $g\in\mathrm{GL}^+(2,\mathbb{R})$ on

$H^1(X,\Sigma;\mathbb{C})^-$ satisfies $\rho(g\cdot[\omega]) = g\cdot\rho([\omega])$. Therefore the leaves of the kernel foliation and the orbits of $\mathrm{GL}^+(2,\mathbb{R})$ are transversal. Since their dimensions are complementary, the proposition follows. □

4. Stable and Unstable cylinder decompositions

4.1. **Cylinder decompositions.** A *separatrix* is a geodesic ray emanating from a zero of ω. It is a well-known fact that a direction is periodic if and only if all the separatrices in this direction are saddle connections. In this case the surface decomposes into finitely many cylinders in this direction. Since the Prym involution τ preserves the set of cylinders, it naturally induces an equivalence relation on this set. We will often use the term "number of cylinders up to Prym involution" for the number of τ-equivalence classes of cylinders.

Definition 4.1. *A cylinder decomposition of* (X,ω) *is said to be* stable *if every separatrix joins a zero of* ω *to itself. The decomposition is said to be* unstable *otherwise.*

Lemma 4.2. *Let* θ *be a periodic direction for* $(X,\omega)\in\mathcal{H}(\kappa)$ *and* g *be the genus of* g*. If* X *has* $g+|\kappa|-1$ *cylinders in the direction* θ*, then the cylinder decomposition in this direction is stable (*$|\kappa|$ *is the number of zeros of* ω*).*

Proof. Let $C_1,\dots,C_n$ be the cylinders in the direction θ of X. For $i=1,\dots,n$, let c_i be a core curve of C_i. Cutting X along c_i we obtain r compact surfaces with boundary denoted by $X_1,\dots,X_r$. Note that each of X_i must contain at least a zero of ω. Therefore we have $r\leq|\kappa|$. Let n_i be the number of boundary components of X_i. Remark that we have $\sum_{1\leq i\leq r} n_i = 2n$, and $\chi(X_i)\leq 2-n_i$, where $\chi(.)$ is the Euler characteristic. By construction, we have

$$2-2g=\chi(X)=\sum_{i=1}^{r}\chi(X_i)\leq\sum_{i=1}^{r}(2-n_i)=2r-\sum_{i=1}^{r}n_i=2r-2n.$$

It follows immediately that

$$n\leq g+r-1\leq g+|\kappa|-1.$$

From the previous inequalities, we see that the equality $n=g+|\kappa|-1$ is realized if and only if $r=|\kappa|$ and each X_i has genus zero. In particular, if $n=g+|\kappa|-1$, then each component X_i contains a unique zero of ω. If there is a saddle connection joining two distinct zeros of ω, then these two zeros must belong to the same X_i, and we have a contradiction. Therefore, the cylinder decomposition must be stable. □

Remark 4.3. *In* $\mathcal{H}(1,1)$ *the maximal number of cylinders in a cylinder decomposition is three, and a cylinder decomposition is stable if and only if this maximal number is attained. In higher genus, there are stable cylinder decompositions with less than* $n+|\kappa|-1$ *cylinders.*

Lemma 4.4. *Let* $(X,\omega)\in\mathrm{Prym}(\kappa)$ *be a surface in one of the strata given by Table 1. If the horizontal direction is periodic for* (X,ω) *then the number* n *of horizontal cylinders, counted up to the Prym involution, satisfies* $n\leq 3$*. Moreover, if* $\kappa\neq(1,1,2,2)$ *and* $n=3$ *then the cylinder decomposition in the horizontal direction is stable.*

Remark 4.5. *Observe that Lemma 4.4 is false for the stratum* $\mathrm{Prym}(1,1,2,2)$*. However, using the identification* $\mathrm{Prym}(1,1,2,2)\simeq\mathcal{H}(0,0,2)$ *the statement becomes true with the convention that a cylinder decomposition of* $(X,\omega)\in\mathrm{Prym}(1,1,2,2)$ *is stable if and only if the decomposition of the corresponding surface in* $\mathcal{H}(0,0,2)$ *is.*

Proof. Let us assume that the horizontal direction is completely periodic. We first show that the number n of horizontal cylinders, counted up to the Prym involution, satisfies $n \leq 3$. Let n_f be the number of fixed cylinders (by the Prym involution) and let $2 \cdot n_p$ be the number of non-invariant cylinders. Obviously $n = n_f + n_p$.

The next observation is that each fixed cylinder contains exactly two *regular* fixed points of the Prym involution, which project to simple poles of the corresponding quadratic differential. Hence if $\mathrm{Prym}(\kappa)$ is the covering of $\mathcal{Q}(-1^p, k_1, \dots, k_n)$ where $k_i \geq 0$ then $n_f \leq \lfloor p/2 \rfloor$. Now since the number of cylinders is at most $g + |\kappa| - 1$, we get $n_p \leq \lfloor (g + |\kappa| - 1 - n_f)/2 \rfloor$. Hence

$$n = n_f + n_p \leq \lfloor (g + |\kappa| - 1 + n_f)/2 \rfloor.$$

The values of $g + |\kappa| - 1$ for the different cases of Table 1 are the following:

$\mathcal{Q}(\kappa')$	$\mathrm{Prym}(\kappa)$	$g+\lvert\kappa\rvert-1$
$\mathcal{Q}_0(-1^6, 2)$	$\mathrm{Prym}(1,1)$	3
$\mathcal{Q}_1(-1^3, 1, 2)$	$\mathrm{Prym}(1,1,2)$	5
$\mathcal{Q}_1(-1^4, 4)$	$\mathrm{Prym}(2,2)^{\mathrm{odd}}$	4
$\mathcal{Q}_2(-1^2, 6)$	$\mathrm{Prym}(3,3)$	5

$\mathcal{Q}(\kappa')$	$\mathrm{Prym}(\kappa)$	$g+\lvert\kappa\rvert-1$
$\mathcal{Q}_2(1^2, 2)$	$\mathrm{Prym}(1^2, 2^2)$	7
$\mathcal{Q}_2(-1, 2, 3)$	$\mathrm{Prym}(1,1,4)$	6
$\mathcal{Q}_2(-1, 1, 4)$	$\mathrm{Prym}(2,2,2)^{\mathrm{even}}$	6
$\mathcal{Q}_3(8)$	$\mathrm{Prym}(4,4)^{\mathrm{even}}$	6

On the right table, the inequality $p \leq 1$ holds for all cases, thus $n_f = 0$. Therefore $n \leq \lfloor 7/2 \rfloor = 3$.

For all the other cases on the left table, one has, respectively:

(1) If $\kappa = (1,1)$ then $n_f \leq 3$ and $n \leq \lfloor (3 + n_f)/2 \rfloor \leq 3$.
(2) If $\kappa = (1,1,2)$ then $n_f \leq 1$ and $n \leq \lfloor (5 + n_f)/2 \rfloor \leq 3$.
(3) If $\kappa = (2,2)$ then $n_f \leq 2$ and $n \leq \lfloor (4 + n_f)/2 \rfloor \leq 3$.
(4) If $\kappa = (3,3)$ then $n_f \leq 1$ and $n \leq \lfloor (5 + n_f)/2 \rfloor \leq 3$.

The first statement of the lemma is proved. Now we notice that if $n = 3$ then in every case, but $\kappa = (1,1,2,2)$, one has $n_f + 2 \cdot n_p = g + |\kappa| - 1$. Hence by Lemma 4.2 the horizontal direction is stable. $\square$

4.2. **Combinatorial data.** Let (X, ω) be a surface for which the horizontal direction is completely periodic. Since each saddle connection is contained in the upper (respectively, lower) boundary of a unique cylinder, we can associate to the cylinder decomposition the following data:

- two partitions of the set of saddle connections into k subsets, where k is the number of cylinders, each subset in these partitions is equipped with a cyclic ordering, and
- a pairing of subsets in these two partitions.

We will call these data the *combinatorial data* or *topological model* of the cylinder decomposition. Note that while there exists only one topological model for cylinder decompositions with maximal number of cylinders in $\mathrm{Prym}(1,1)$, in general, there are several topological models for such decompositions in other Prym loci in Table 1.

4.3. **Kernel foliation and stable decomposition.** We will now carefully investigate the kernel foliation leaf nearby a surface (X, ω) for which the horizontal direction is periodic. In what follows, we only consider the intersection of the kernel foliation leaves with a neighborhood of (X, ω) on which a local chart by the period mapping is defined. This restriction means that the surfaces in the same leaf as (X, ω) can be written as $(X, \omega) + v$, with $|v|$ small enough. Remark that for all Prym loci in Table 1,

ω has either 2 or 3 zeros, two of them are permuted by the Prym involution, the third one (if exist) is fixed. To keep the exposition easy to follow, let us assume that ω has two zeros, denoted by P and Q.

By assumption, X is decomposed into cylinders in the horizontal direction. Let h be the minimal height among the heights of the cylinders, and ℓ be the length of the shortest horizontal saddle connection. For any $\varepsilon > 0$ such that $\varepsilon < 1/2\min\{h,\ell\}$, the sets $D(P,\varepsilon) := \{x \in X,\ \mathbf{d}(x,P) < \varepsilon\}$ and $D(Q,\varepsilon) := \{x \in X,\ \mathbf{d}(x,Q) < \varepsilon\}$ are two disjoint embedded disks in X. In what follows, we fix an $\varepsilon < 1/2\min\{h,\ell\}$ such that a neighborhood of (X,ω) in its stratum can be identified (via the period mapping) with an open subset of $\mathbb{C}^d$ which contains the ball of radius ε centered at (X,ω). For any vector $v \in \mathbb{R}^2, |v| < \varepsilon$, there is a unique surface (X',ω') in this neighborhood such that ω' has the same absolute periods as ω, and $\omega'(c) - \omega(c) = v$, where c is fixed a relative cycle. We denote (X',ω') by $(X,\omega)+v$. Remark that we have $(X,\omega)+(v_1+v_2) = ((X,\omega)+v_1)+v_2$ as long as $|v_1| < \varepsilon$, $|v_2| < \varepsilon$, and $|v_1|+|v_2| < \varepsilon$.

We now construct a surface $(X,\omega)+v$ from surgeries inside the discs $D(P,\varepsilon)$ and $D(Q,\varepsilon)$. Let $D(\varepsilon)$ denote the disc of radius ε centered at $(0,0)$ in $\mathbb{R}^2$. We will denote the center of $D(\varepsilon)$ by c, and let a,b denote respectively the bottom and top endpoints of its vertical diameter. Let $D_-(\varepsilon)$ and $D_+(\varepsilon)$ denote respectively the left and right half-disks of $D(\varepsilon)$ that are cut out by the diameter $\overline{ab}$.

Since P and Q are permuted by the Prym involution, their cone angles are the same and equal to $2\pi m$, $m \in \mathbb{N}$. The disk $D(P,\varepsilon)$ (resp. $D(Q,\varepsilon)$) can be constructed from m copies of $D_+(\varepsilon)$ and m copies of $D_-(\varepsilon)$ glued together following a circular fashion. Denote the copies of $D_+(\varepsilon)$ and $D_-(\varepsilon)$ in $D(P,\varepsilon)$ (resp. in $D(Q,\varepsilon)$) respectively by D^P_{i+} and D^P_{i-} (resp. D^Q_{i+} and D^Q_{i-}) with $i = 1,\dots,m$. For each copy $D^\alpha_{i\pm}$, we denote by $a^\alpha_{i\pm}, b^\alpha_{i\pm}, c^\alpha_{i\pm}$, the points corresponding to a,b,c respectively. The labeling of the half-discs is chosen so that D^α_{i+} is glued to D^α_{i-} along the segment $\overline{bc}$, and D^α_{i-} is glued to $D^\alpha_{(i+1)+}$ along the segment $\overline{ca}$, for $\alpha \in \{P,Q\}$ (here we identify D^P_{i+} and D^P_{i-} with $D_+(\varepsilon)$ and $D_-(\varepsilon)$, and use the convention $m+1 = 1$). Note that by construction, all the points $c^P_{i\pm}$ are identified with P, all the points $c^Q_{i\pm}$ are identified with Q, a^α_{i-} is identified with $a^\alpha_{(i+1)+}$ and b^α_{i-} is identified with b^α_{i+}.

We call a ray in direction $(1,0)$ a *positive horizontal ray*, and a ray in direction $(-1,0)$ a *negative horizontal ray*. Since the horizontal direction is periodic, any horizontal ray from a zero of ω must end in another zero. Observe that a positive horizontal saddle connection must start in a copy of D_+ and end in a copy of D_-, hence it joins a point c^α_{i+} to a point c^β_{j-}. Therefore, the horizontal saddle connections of X provide us with a bijection

$$\pi : \{c^\alpha_{k+},\ \alpha \in \{P,Q\},\ k = 1,\dots,m\} \to \{c^\alpha_{k-},\ \alpha \in \{P,Q\},\ k = 1,\dots,m\}.$$

Consider a horizontal saddle connection starting in the half-disk D^α_{i+} and ending in the half-disk D^β_{j-}. Using the developing map, we see that the horizontal positive ray starting from a^α_{i+} passes through a^β_{j-}. By the gluing rules, we know that a^β_{j-} is identified with $a^\beta_{(j+1)+}$, thus we have a horizontal segment from a^α_{i+} to $a^\beta_{(j+1)+}$. We can now encode this information in a permutation σ_a of the set $\{a^\alpha_{k+}, k = 1,\dots,m, \alpha \in \{P,Q\}\}$ by defining $\sigma_a(a^\alpha_{i+}) = a^\beta_{(j+1)+}$.

Similarly, we also have a horizontal segment joining b^α_{i+} to b^β_{j+}, which induces a permutation σ_b on the set $\{b^\alpha_{k+}, k = 1,\dots,m,\ \alpha \in \{P,Q\}\}$. Remark that we have a bijection between the set of horizontal cylinders in X and the cycles of σ_a (resp. σ_b). Thus the map π and the pair of permutations (σ_a, σ_b) completely determine the combinatorial data of the cylinder decomposition of X.

We first observe

Lemma 4.6. *Let v be a vector in $\mathbb{R}^2$ such that $|v| < \varepsilon$. If v is horizontal then the cylinder decompositions of $(X, \omega) + v$ and (X, ω) have the same combinatorial data. Moreover, the corresponding cylinders have the same width.*

Proof. If v is horizontal then clearly all the horizontal saddle connections in (X, ω) persist in $(X, \omega) + v$ (the lengths of some of them may be changed), hence $(X, \omega) + v$ also has a cylinder decomposition in the horizontal direction with the same combinatorial as (X, ω). Remark that the widths of the corresponding cylinders must be the same since they are absolute periods of ω. □

We now describe the surgery inside $D(P, \varepsilon)$ and $D(Q, \varepsilon)$ to obtain $(X, \omega) + (0, t)$, with $|t| < \varepsilon$. Let us assume that $t > 0$ (the case $t < 0$ is completely similar). Let a' and b' denote respectively the points $(0, -t/2)$ and $(0, t/2)$ in $D(\varepsilon)$. Let $a'^{\alpha}_{i\pm}$ and $b'^{\alpha}_{i\pm}$ be the corresponding points in the boundary of $D^{\alpha}_{i\pm}$. To obtain $(X, \omega) + (0, t)$ we glue the copies of $D_+(\varepsilon)$ and $D_-(\varepsilon)$ as follows:

- In $D(P, \varepsilon)$, D^P_{i+} is glued to D^P_{i-} along the segment corresponding to $\overline{a'b}$, and D^P_{i-} is glued to $D^P_{(i+1)+}$ along the segment corresponding to $\overline{aa'}$.
- In $D(Q, \varepsilon)$, D^Q_{i+} is glued to D^Q_{i-} along the segment corresponding to $\overline{b'b}$, and D^Q_{i-} is glued to $D^Q_{(i+1)+}$ along the segment corresponding to $\overline{ab'}$.

From this construction, observe that we have

(i) All the points $a'^{P}_{i\pm}$ are identified to give a point P' with cone angle $2\pi m$.
(ii) All the points $b'^{Q}_{i\pm}$ are identified to give a point Q' with cone angle $2\pi m$.
(iii) c^P_{i-} is identified with c^P_{i+}, but c^Q_{i-} is identified with $c^Q_{(i+1)+}$, and those identifications give regular points in the new surface.
(iv) If $\pi(c^{\alpha}_{i+}) = c^{\beta}_{j-}$, then there is a (positive) horizontal segment from a'^{α}_{i+} to a'^{β}_{j-}, and a horizontal segment from b'^{α}_{i+} to b'^{β}_{j-}.
(v) b'^{P}_{i-} is identified with b'^{P}_{i+}, and a'^{Q}_{i-} is identified with $a'^{Q}_{(i+1)+}$.

Since this surgery does not change the flat metric outside of $D(P, \varepsilon)$ and $D(Q, \varepsilon)$, it is not difficult to see that the new surface is $(X, \omega) + (0, t)$ (see [LN13a]).

We are now ready to prove the following two propositions which will play an important role in the sequel.

Proposition 4.7. *Let $(X, \omega) \in \Omega E_D(\kappa)$, where κ is one of the strata in Table 1. If (X, ω) admits a stable cylinder decomposition in the horizontal direction then there exists $\varepsilon' > 0$ such that for every $v \in (-\varepsilon', \varepsilon') \times (-\varepsilon', \varepsilon') \subset \mathbb{R}^2$, $(X, \omega) + v$ admits a stable cylinder decomposition (in the horizontal direction) with the same combinatorial data and the same widths of cylinders.*

Remark 4.8. *Observe that if the horizontal direction on (X, ω) is stable then the horizontal kernel foliation is well defined for all time $s \in \mathbb{R}$ (see Section 4.4).*

Proof. We only give the proof for the case where ω has two zeros P and Q permuted by the Prym involution since the arguments for the other cases are completely similar. We choose $\varepsilon' > 0$ small enough so that for any point (s,t) in the square $(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$, the disk of radius $\varepsilon/2$ centered at (s,t) is included in the disk $D(\varepsilon)$.

Suppose that v is vertical, that is $v=(0,t)$ with $|t|<\varepsilon'$. The assumption that the cylinder decomposition is stable means that the bijection π maps $\{c^P_{i+},\ i=1,\dots,m\}$ to $\{c^P_{i-},\ i=1,\dots,m\}$, and $\{c^Q_{i+},\ i=1,\dots,m\}$ to $\{c^Q_{i-},\ i=1,\dots,m\}$. It follows from the property (iv) above that any positive horizontal ray emanating from P' ends in P', and any horizontal ray from Q' also ends in Q'. Thus $(X,\omega)+v$ also admit a stable cylinder decomposition in the horizontal direction. Moreover, we have a bijection between the sets of horizontal saddle connections of (X,ω) and $(X,\omega)+v$. Since the sets of cylinders of (X,ω) and $(X,\omega)+v$ are in bijection with the cycles of σ_a and σ_b, we conclude that the cylinder decomposition of $(X,\omega)+v$ has the same combinatorial data as the one of (X,ω).

For the general case $v=(s,t)\in(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$, we write $(X,\omega)+(s,t)=((X,\omega)+(0,t))+(s,0)$, the proposition then follows from the case v is vertical, and Lemma 4.6. □

Proposition 4.9. *Let $(X,\omega)\in\Omega E_D(\kappa)$, where κ is one of the strata in Table 1. If (X,ω) admits an* unstable *cylinder decomposition in the horizontal direction, then there exists $\varepsilon'>0$ such that for every $v=(s,t)\in(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$, with $t\neq 0$, $(X,\omega)+v$ admits a stable cylinder decomposition in the horizontal direction. Moreover, the combinatorial data of the decomposition and the widths of the cylinders depend only on the sign of t.*

Proof. Again, we only give the proof for the case ω only has two zeros permuted by the Prym involution. We also choose $\varepsilon'>0$ such that for every $(s,t)\in(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$, the disk of radius $\varepsilon/2$ centered at (s,t) is included in $D(\varepsilon)$, and keep the same notations as in the proof of Proposition 4.7. By Lemma 4.6, we only need to consider the case $s=0$. Let us assume that $t>0$. The condition that the cylinder decomposition is unstable means that there exist $i,j\in\{1,\dots,m\}$ such that $\pi(c^P_{i+})=c^Q_{j-}$.

We first claim that any positive horizontal ray emanating from P' ends in P'. By construction, such a ray starts at a point ${a'}^P_{i+}$, from property (iv) this ray passes through a point ${a'}^\beta_{i'-}$, where β and i' satisfy $\pi(c^P_{i+})=c^\beta_{i'-}$. If $\beta=P$, then we have a saddle connection joining P' to itself. Otherwise, we have $\beta=Q$, in this case ${a'}^Q_{i'-}$ is identified with ${a'}^Q_{(i'+1)+}$ (property (v)), thus the ray can be continued and passes through another point ${a'}^\gamma_{i''-}$ where γ and i'' are determined by π.

Continuing this procedure, we see that this ray must end at a point ${a'}^P_{j-}$, and we get a saddle connection joining P' to itself (see Figure 2). Note that the index j can be determined from σ_a by the following rule: there is a unique sequence $(\alpha_0,i_0),\dots,(\alpha_k,i_k)$, where $\alpha_r\in\{P,Q\}$ and $i_r\in\{1,\dots,m\}$, such that

- $\alpha_0=\alpha_k=P$, $\alpha_1=\dots=\alpha_{k-1}=Q$,
- $i_0=i$, $i_k=j$, and $\sigma_a(a^{\alpha_r}_{i_r+})=a^{\alpha_{r+1}}_{i_{r+1}+}$.

It follows from the same arguments that any horizontal ray emanating from Q' ends in Q', and those saddle connections are encoded in σ_b. We can then conclude that $(X,\omega)+(0,t)$ admits a stable cylinder decomposition in the horizontal direction.

We will now show that the combinatorial data of this decomposition are encoded in π,σ_a,σ_b, which implies that those data only depend on the sign of t. There are two kinds of horizontal cylinders in $(X,\omega)+(0,t)$, the ones that already exist in (X,ω) (recall that the central core curve of any cylinder in X does not intersect $D(P,\varepsilon)\sqcup D(Q,\varepsilon)$ thus remains in $(X,\omega)+(0,t)$), and the new ones that contain

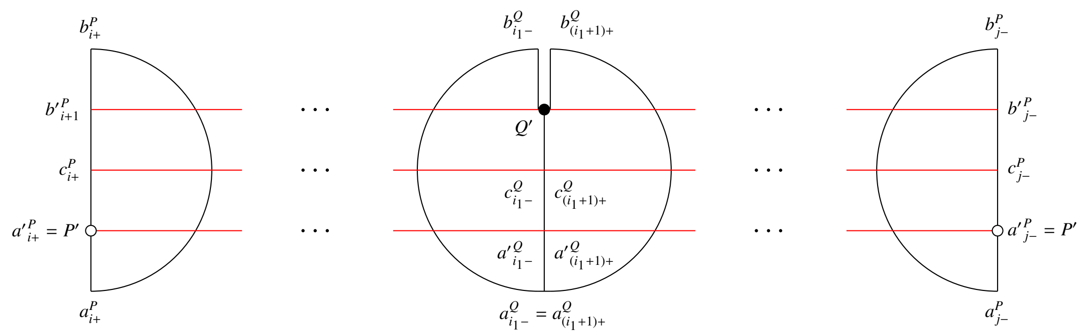

FIGURE 2. Horizontal saddle connection from P' to itself

some of the points $c^\alpha_{i\pm}$. Cylinders in the first family are encoded by cycles of σ_a and σ_b. Cylinders in the second family are encoded by cycles in a permutation σ_c of $\{c^\alpha_{i+},\ \alpha \in \{P,Q\},\ i=1,\dots,m\}$ which is defined as follows: for any c^α_{i+} there is a horizontal segment joining c^α_{i+} to c^β_{j-}, where $c^\beta_{j-} = \pi(c^\alpha_{i+})$. From property (iii), if $\beta = P$, then c^P_{j-} is identified with c^P_{j+}, and we define $\sigma_c(c^\alpha_{i+}) = c^P_{j+}$. Otherwise, if $\beta = Q$ then c^Q_{j-} is identified with $c^Q_{(j+1)+}$, and we define $\sigma_c(c^\alpha_{i+}) = c^Q_{(j+1)+}$. It is clear from the definition that the set of cycles of σ_c is in bijection with the set of cylinders that contain some of the points $c^\alpha_{i\pm}$. Since the definition of σ_c only depends on the sign of t, we derive that the combinatorial data of the cylinder decomposition of $(X,\omega)+(0,t)$ only depends on the sign of t. □

4.4. **Action of the kernel foliation on cylinders.**

4.4.1. *Horizontal kernel foliation.* Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform with κ in Table 1. The kernel foliation in those Prym egeinform loci implies that there exist a maximal interval $0 \in I \subset \mathbb{R}$, and a continuous map $\varphi : I \to \Omega E_D(\kappa),\ s \mapsto (X_s,\omega_s)$ such that

- $(X_0,\omega_0) = (X,\omega)$,
- for all $s \in I$, ω_s has the same absolute periods as ω,
- for a fixed relative homology class $c \in H(X,\Sigma,\mathbb{Z})$, where Σ is the set of zeros of ω, and for all $s_1, s_2 \in I$, $\omega_{s_1}(c) - \omega_{s_2}(c) = s_1 - s_2$.

We will call the set $\{(X_s,\omega_s),\ s \in I\}$ the leaf of the *horizontal (or real) kernel foliation* through (X,ω), and write $(X,\omega)+(s,0) = (X_s,\omega_s)$. If $I = \mathbb{R}$, we say that $(X,\omega)+(s,0)$ is defined for all s.

Assume that (X,ω) admits a stable horizontal cylinder decomposition, then $(X,\omega)+(s,0)$ is defined for all $s \in \mathbb{R}$. Moreover, $(X,\omega)+(s,0)$ also admits a stable cylinder decomposition in the horizontal direction with the same combinatorial data. Let $C_i,\ i=1,\dots,k$, be the horizontal cylinders in X, and w_i, h_i, t_i be respectively the width, height, and twist of C_i. The cylinder in $(X,\omega)+(s,0)$ corresponding to C_i will be denoted by $C_i(s,0)$. Let $w(C_i(s,0)), h(C_i(s,0)), t(C_i(s,0))$ denote the width, height, and twist of $C_i(s,0)$. Since the cylinder decomposition is stable, the upper (resp. lower) boundary of C_i contains only one zero of ω. By construction, we have

$$\begin{cases} w(C_i(s,0)) = w(C_i) = w_i, \\ h(C_i(s,0)) = h(C_i) = h_i, \end{cases}$$

for any s. However, in general $t(C_i(s,0))$ is a non-constant function of s.

Lemma 4.10. *We have $t(C_i(s,0)) = t_i + \alpha_i s$, where*

$$\alpha_i = \begin{cases} 0 & \text{if the zeros in the upper and lower boundaries of } C_i \text{ are the same,} \\ \pm 1 & \text{if the zeros are exchanged by the Prym involution,} \\ \pm 1/2 & \text{if one zero is fixed, the other is mapped to the third one by the Prym involution.} \end{cases}$$

Again we emphasis that the formula does not depend on the marking (even if the twists depend on the marking).

4.4.2. *Vertical kernel foliation.* Again, assume that (X,ω) admits a stable cylinder decomposition in the horizontal directions with cylinders denoted by $C_1,\dots,C_k$. If $v = (0,t)$, then by Proposition 4.7, $(X,\omega)+(0,t)$ is well defined whenever $|t|$ is small enough. Let $C_i(0,t)$ denote the cylinder in $(X,\omega)+(0,t)$ that corresponds to C_i. The widths (as they are absolute periods) and the twists of the cylinders $C_i(0,t)$ are unchanged, only their heights vary. Namely,

Lemma 4.11. *We have $h(C_i(0,t)) = h_i + \alpha_i t$, where*

$$\alpha_i = \begin{cases} 0 & \text{if the zeros in the upper and lower boundaries are the same,} \\ \pm 1 & \text{if the zeros are exchanged by the Prym involution,} \\ \pm 1/2 & \text{if one zero is fixed, the other is mapped to the third one by the Prym involution.} \end{cases}$$

The proofs of Lemma 4.10 and Lemma 4.11 are elementary and left to the reader.

4.5. **Action of the horizontal horocycle flow on cylinders.** The (horizontal) horocycle flow is defined as the action of the one parameter subgroup $U = \{u_s, \quad s \in \mathbb{R}\}$ of $\mathrm{GL}^+(2,\mathbb{R})$, where $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. If the horizontal direction on (X,ω) is completely periodic, then obviously the action of u_s on (X,ω) preserves the cylinder decomposition topologically. Moreover each cylinder C_i with parameters $(w_i, h_i, t_i \mod w_i)$ is mapped to a cylinder $C_i(s) := u_s(C_i)$ of $u_s\cdot(X,\omega)$ with the same width and height, while the twist is given by

$$t(C_i(s)) = t_i + sh_i \mod w_i. \tag{1}$$

4.6. **Cylinders decomposition: relation of moduli.** The aim of this section is to establish the following result:

Proposition 4.12. *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform with κ in Table 1 such that the horizontal direction is periodic. Let n be the number of τ-equivalence classes of horizontal cylinders (recall that $n \le 3$), and $C_1,\dots,C_n$ be a family of cylinders representing the n equivalence classes. Then we have*

(a) *If $n=3$ then there exists $(r_1,r_2,r_3) \in \mathbb{Q}^3 \setminus \{0\}$ such that*

$$r_1\mu_1 + r\mu_2 + r_3\mu_3 = 0. \tag{2}$$

Moreover, let $\alpha_i \in \{0,\pm1/2,\pm1\}$ be the coefficient given by Lemmas 4.10 and 4.11 associated to C_i, then (r_1,r_2,r_3) satisfies

$$r_1\frac{\alpha_1}{w_1} + r_2\frac{\alpha_2}{w_2} + r_3\frac{\alpha_3}{w_3} = 0. \tag{3}$$

(b) *If the cylinder decomposition is unstable then the horizontal direction is parabolic.*

We first recall the following result when D is not a square.

Theorem 4.13 (McMullen [McM03b]). *Let $K = \mathbb{Q}(\sqrt{D}) \subset \mathbb{R}$ be a real quadratic field and let $(X, \omega) \in \Omega E_D(\kappa)$ be a Prym eigenform such that all the absolute periods of ω belong to $K(\imath)$. Assume that the horizontal direction is periodic with k cylinders, then we have*

$$\sum_{i=1}^{k} w'_i h_i = 0,$$

where w_i, h_i are respectively the width and the height of the i-th cylinder, and w'_i is the Galois conjugate of w_i in K.

Sketch of proof. A remarkable property of Prym eigenform is that the *complex flux* vanishes. Namely (see [McM03b, Theorem 9.7])

$$\int_X \omega \wedge \omega' = \int_X \omega \wedge \overline{\omega}' = 0.$$

Here $\overline{\omega}$ and ω' are respectively the complex conjugate and the Galois conjugate of ω. The argument is as follows: let T be a generator of the order $\mathcal{O}_D$, we have a pair of 2-dimensional eigenspaces $S \oplus S' = H^1(X, \mathbb{R})^-$ on which T acts by multiplication by a scalar, where S is spanned by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$, and S' is spanned by $\mathrm{Re}(\omega')$ and $\mathrm{Im}(\omega')$. Since T is self-adjoint, S and S' are orthogonal with respect to the cup product. This shows the equalities above. Now we have

$$\int_{C_i} \mathrm{Im}(\omega) \wedge \mathrm{Re}(\omega') = w'_i h_i,$$

where $C_1, \dots, C_k$ are the horizontal cylinders in X. Since the surface X is covered by those cylinders, it follows

$$\sum_{i=1}^{k} w'_i h_i = \sum_{i=1}^{k} \int_{C_i} \mathrm{Im}(\omega) \wedge \mathrm{Re}(\omega') = \int_X \mathrm{Im}(\omega) \wedge \mathrm{Re}(\omega') = \frac{1}{4\imath} \int_X (\omega - \overline{\omega}) \wedge (\omega' + \overline{\omega}') = 0.$$

Theorem 4.13 is proved. □

4.6.1. *Proof of Proposition 4.12: case D is not a square.*

Proof. Let $\beta_i \in \{1, 2\}$ be the number of cylinders in the τ-equivalence class of C_i ($\beta_i = 1$ if C_i is fixed by τ, $\beta_i = 2$ if C_i is exchanged with another cylinder). Set $r_i = \beta_i w_i w'_i \in \mathbb{Q}$.

For the case $n = 3$, the first equality follows directly from Theorem 4.13. Namely,

$$0 = \sum_{i=1}^{k} w'_i h_i = \sum_{i=1}^{3} \beta_i (w_i w'_i) \mu_i = \sum_{i=1}^{3} r_i \mu_i.$$

When $n = 3$, Lemma 4.4 implies that the cylinder decomposition is stable. Thus we can associate to each cylinder C_i a coefficient $\alpha_i \in \{0, \pm 1/2, \pm 1\}$ (by Lemmas 4.10 and 4.11). Observe that moving in the leaves of the kernel foliation does not change the area of the surface, therefore

$$\mathrm{Area}(X, \omega) = \mathrm{Area}((X, \omega) + (0, s)) \quad \Rightarrow \quad \sum_{i=1}^{k} w_i h_i = \sum_{i=1}^{k} w_i (h_i + \alpha_i s)$$

which implies

$$\sum_{i=1}^{k}\alpha_i w_i = \sum_{i=1}^{3}\alpha_i\beta_i w_i = 0. \tag{4}$$

Thus, one has

$$\sum_{i=1}^{3} r_i\frac{\alpha_i}{w_i} = \sum_{i=1}^{3}\beta_i\alpha_i w_i' = \left(\sum_{i=1}^{3}\alpha_i\beta_i w_i\right)' = 0,$$

and (3) is proved.

Consider now the case the cylinder decomposition is unstable, which means that $n \le 2$. If $n = 1$ then X has either a unique horizontal cylinder, or two horizontal cylinders which are exchanged by τ. In both cases, the horizontal direction is clearly parabolic. If $n = 2$, then Theorem 4.13 implies that the ratio μ_1/μ_2 is rational, which means that the horizontal is also parabolic. Proposition 4.12 is then proved for the case D is not a square. □

We end this section with the discussion when D is a square.

Lemma 4.14. *For every $i \in \{1,\dots,k\}$ either h_i is an absolute period, or there exists $j \neq i$ and some integers $x_i, x_j \in \{1,2\}$ such that $x_i h_i + x_j h_j$ is an absolute period. Moreover, if the cylinder decomposition is stable, and α_i, α_j are the coefficients associated to C_i and C_j (by Lemmas 4.10 and 4.11) then $x_i\alpha_i + x_j\alpha_j = 0$.*

Proof. If there is a zero of ω that is contained in both the top and bottom borders of C_i, then h_i is an absolute period. Let us assume that this does not occur. There are two cases.
First case. ω has two zeros P_1, P_2. Note that in this case P_1 and P_2 are exchanged by the Prym involution τ. We can assume that the bottom border of C_i contains P_1, and its top border contains P_2. By connectedness of X, there must exist a cylinder C_j whose bottom border contains P_2 and top border contains P_1. Remark that we must have $i \neq j$ otherwise P_1 is contained in both top and bottom borders of C_i. Let σ_i and σ_j be respectively some saddle connections in C_i and C_j which join P_1 to P_2. Then $c = \sigma_i \cup \sigma_j$ is a simple closed curve in X, and we have $h_1 + h_2 = \mathrm{Im}\omega(c)$.

Second case. ω has 3 zeros. In this case two zeros are permuted by τ, we denote them by P_1, P_2, the third one is fixed by τ, let us denote this one by Q. We can always assume that P_1 is contained in the bottom border of C_i, but not in the top border of C_i.

Assume that the top border of C_i contains P_2, and let σ_i be a saddle connection in C_i which joins P_1 to P_2. If there exists another cylinder whose bottom border contains P_2 and top border contains P_1 then we are done. Otherwise, there must exists a cylinder C_j whose bottom border contains P_2 and top border contains Q. Let $C_{j'}$ be the cylinder which is permuted with C_j by τ, then the top border of $C_{j'}$ contains P_1 and the bottom border of $C_{j'}$ contains Q. In particular, we have $C_{j'} \neq C_i$.

If $C_{j'} = C_j$, then the top border of C_j contains P_1 contradicting our hypothesis. Thus we have $C_{j'} \neq C_j$. Let σ_j be a saddle connection in C_j which joins P_2 to Q, then $\tau(\sigma_j)$ is a saddle connection in $C_{j'}$ that joins Q to P_1. Consequently, $c := \tau(\sigma_j) \cup \sigma_j \cup \sigma_i$ is a simple closed curve in X, and $\mathrm{Im}\omega(c) = h_i + h_j + h_{j'} = h_i + 2h_j$.

We are left with the case where the top border of C_i contains Q. Let $C_{i'}$ be the cylinder which is permuted with C_i by τ. Then the top border of $C_{i'}$ contains P_2 and the bottom border contains Q. By assumption, we have $C_{i'} \neq C_i$. By connectedness of X, there exists a cylinder $C_j \neq C_i$ which contains

P_1 in the top border, and P_2 or Q in the bottom border. If P_2 is contained in the bottom border of C_j then $h_j + h_i + h_{i'} = h_j + 2h_i$ is an absolute period. If Q is an contained in the bottom border of C_j then $h_i + h_j$ is an absolute period. Since $x_i h_i + x_j h_j$ is an absolute period, it is unchanged by the kernel foliation, Lemma 4.11 then implies that $x_i\alpha_i + x_j\alpha_j = 0$. □

4.6.2. *Proof of Proposition 4.12 when D is a square.*

Proof. We first consider the case $n = 3$. Since D is a square, one can normalize, using $\mathrm{GL}^+(2,\mathbb{R})$, so that all the absolute periods of ω belong to $\mathbb{Q}(\imath)$. By Lemma 4.14, one can find (x_1, x_2, x_3) and (y_1, y_2, y_3) with $x_i, y_i \in \{0, 1, 2\}$ such that $x_1h_1 + x_2h_2 + x_3h_3$ and $y_1h_1 + y_2h_2 + y_3h_3$ are absolute periods. The vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) are chosen so that they are not collinear. Since all the absolute periods are in $\mathbb{Q}$, there exists $r \in \mathbb{Q}, r > 0$, such that

$$x_1h_1 + x_2h_2 + x_3h_3 = r(y_1h_1 + y_2h_2 + y_3h_3) \quad \Leftrightarrow \quad \sum_{i=1}^{3}(x_i - ry_i)h_i = 0.$$

Set $r_i := (x_i - ry_i)w_i$, we get

$$\sum_{i=1}^{3} r_i\mu_i = 0.$$

From Lemma 4.14, we also have $\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 = \alpha_1y_1 + \alpha_2y_2 + \alpha_3y_3 = 0$. Thus we have

$$\sum_{i=1}^{3}(x_i - ry_i)\alpha_i = \sum_{i=1}^{3} r_i\frac{\alpha_i}{w_i} = 0.$$

Now let us assume that the horizontal direction is unstable (hence $n \leq 2$). We will show that the horizontal direction is parabolic. Obviously, we only need to consider the case $n = 2$. Recall that we can normalize so that all the absolute periods of ω are in $\mathbb{Q}(\imath)$. In particular, $w_1, w_2 \in \mathbb{Q}$. We will show that both h_1, h_2 are also absolute periods.

First case: ω has two zeros P_1, P_2. Since the cylinder decomposition is unstable, there exists a horizontal saddle connections γ from P_2 to P_1. We can assume that P_1 is contained in the bottom border of C_1. If the top border of C_1 also contains P_1, then h_1 is an absolute period. Otherwise, let σ be a saddle connection joining P_1 to P_2 which is contained in C_1. Since $c := \gamma \cup \sigma$ is a closed curve and $h_1 = \mathrm{Im}\omega(c)$, we conclude that $h_1 \in \mathbb{Q}$. The same arguments show that $h_2 \in \mathbb{Q}$, hence the horizontal direction is parabolic.

Second case: ω has 3 zeros. Let P_1, P_2 denote the zeros which are permuted, and Q be the zero fixed by τ. We first observe that there exists a path from P_1 and P_2 which is a union of horizontal saddle connection. Indeed, by assumption there exists a horizontal saddle connection γ which joins two different zeros. If γ joins P_1 to P_2 then we are done. Otherwise, γ joins Q to either P_1 or P_2. In both case cases, the union of γ and $\tau(\gamma)$ is the desired path. Let us denote this path by η.

Let us assume that P_1 is contained in the bottom border of C_1 but not in the top border. If the top border of C_1 contains P_2, then the union of η and a saddle connection in C_1 joining P_1 to P_2 is a closed curve c such that $\mathrm{Im}\omega(c) = h_1$, which implies $h_1 \in \mathbb{Q}$. If the top border of C_1 contains Q, then let C_3 be the cylinder which is permuted with C_1 by τ. Note that the bottom border of C_3 contains Q, and the top border contains P_2. Let σ_1 be a saddle connection in C_1 joining P_1 to Q, and σ_3 be the image of σ_1 by τ in C_3. The union $c := \eta \cup \sigma_3 \cup \sigma_1$ is then a closed curve such that $\mathrm{Im}\omega(c) = 2h_1$, hence $h_1 \in \mathbb{Q}$. Similar arguments show that $h_2 \in \mathbb{Q}$. The horizontal direction is then parabolic. □

5. Proof of a weaker version of Theorem 1.1

In this section, we prove a weaker version of Theorem 1.1. We say that (X,ω) is not a Veech surface (or the orbit is not closed) for "*the most obvious reason*" if there exists a periodic direction on (X,ω) that is not parabolic (it is a theorem of Veech [Vee89] that on a Veech surface any periodic direction is parabolic). We will prove Theorem 1.1 under this additional assumption.

Theorem 5.1. *Let $(X,\omega) \in \Omega E_D(\kappa)$ and let us denote by $\mathcal{O}$ its $\mathrm{GL}^+(2,\mathbb{R})$-orbit. If $\mathcal{O}$ is not closed for the most obvious reason then $\overline{\mathcal{O}}$ is a connected component of $\Omega E_D(\kappa)$.*

We begin with the following key lemma. The proof is classical, but is included here for completeness.

Lemma 5.2. *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform. We assume that the horizontal direction is completely periodic but not parabolic. Then for all $s \in \mathbb{R}$, the surface $(X,\omega) + (s,0)$ is well defined, and one has:*

$$(X,\omega) + (s,0) \in \overline{U \cdot (X,\omega)}.$$

Before proving the lemma, let us state the following corollary:

Corollary 5.3. *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform. We assume that there exists $(Y,\eta) \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)}$ and $\varepsilon > 0$ such that $(Y,\eta) + (s,0) \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)}$ for all $s \in \mathbb{R}$ with $|s| < \varepsilon$. Then there exists $\varepsilon' > 0$ such that*

$$(Y,\eta) + v \in \overline{\mathrm{GL}^+(2,\mathbb{R}) \cdot (X,\omega)}$$

for any $v \in \mathbb{R}^2$ such that $|v| < \varepsilon'$.

Proof of Lemma 5.2. Let $C_1,\dots,C_k$ be the horizontal cylinders in X. Let n be the number of equivalence classes of cylinders that are permuted by the Prym involution τ. Recall that for all the cases in Table 1, we have $n \leq 3$. Assume that $\{C_1,\dots,C_n\}$ is a representative family for the τ-equivalence classes of cylinders.

Let us consider the case $n = 3$. Lemma 4.4 implies in particular that the cylinder decomposition is stable. The surface is encoded by the topological gluings of the cylinders C_i, and the width, height, and twist of C_i (which will be denoted by w_i, h_i, t_i respectively).

The set of surfaces admitting a cylinder decomposition in the horizontal direction with the same topological gluings, and the same widths and heights of the cylinders as X, is parameterized by the three dimensional torus

$$\mathcal{X} = N(\mathbb{R}) \times N(\mathbb{R}) \times N(\mathbb{R}) / N(w_1\mathbb{Z}) \times N(w_2\mathbb{Z}) \times N(w_3\mathbb{Z}),$$

where $N(A) = \{u_s;\ s \in A\}$.

The horocycle flow u_s preserves the topological decomposition as well as all the parameters, but the twists t_i. The new twists $\widetilde{t_i}$ are given by $\widetilde{t_i} = t_i + sh_i \mod w_i$. Hence surfaces in the U-orbit of (X,ω) are parameterized by the line $\{(t_1,t_2,t_3) + (h_1,h_2,h_3)s,\ s \in \mathbb{R}\}$.

By Kronecker's theorem, the orbit closure $\overline{U \cdot (X,\omega)}$ is a subtorus of $\mathcal{X}$. Since the moduli are not commensurable (the horizontal direction is not parabolic) the dimension of this subtorus is at least two. More precisely, the orbit closure $\overline{U\cdot(X,\omega)}$ consists of the set of all twists $(\widetilde{t_1},\widetilde{t_2},\widetilde{t_3})$ such that the normalized twists $\dfrac{\widetilde{t_i}-t_i}{w_i}$ verify all non-trivial homogeneous linear relations with rational coefficients that are satisfied by the moduli $\mu_i = h_i/w_i$. Let $\mathbb{P}$ be the subspace of $\mathbb{R}^3$ which is defined by all of such rational relations. By assumption, we have $\dim_{\mathbb{R}}\mathbb{P}\geq 2$. But we know from Proposition 4.12 that there exists $(r_1,r_2,r_3)\in\mathbb{Q}^3\setminus\{(0,0,0)\}$ such that $\sum_{i=1}^n r_i\mu_i = 0$. Therefore, we have $\dim_{\mathbb{R}}\mathbb{P}=2$ and

$$\mathbb{P} = \left\{(\widetilde{t_1},\widetilde{t_3},\widetilde{t_3})\in\mathbb{R}^3,\ \sum_{i=1}^3 r_i\left(\frac{\widetilde{t_i}-t_i}{w_i}\right)=0\right\}. \tag{5}$$

It follows that $\overline{U\cdot(X,\omega)}$ is the projection to $\mathcal{X}$ of the plane $\mathbb{P}\subset\mathbb{R}^3$ defined by Equation (5). Hence, all surfaces constructed from the cylinders with the same widths and heights as those of (X,ω) (by the same gluings), and with the twists $\widetilde{t_i}$ satisfying Equation (5) above belong to $\overline{U\cdot(X,\omega)}$.

Recall that in the horizontal kernel foliation leaf, a surface $(X,\omega)+(s,0)$ is still completely periodic (for the horizontal direction), and all the data: topological gluings of the cylinders, widths, heights are preserved, except for the twists (see Lemma 4.10). To be more precise, if C_i^s is the horizontal cylinder in $(X,\omega)+(s,0)$ corresponding to $C_i = C_i^0$, then $t_i(s) = t_i + \alpha_i s$ (where the range of α_i is $\{-1,0,1\}$ or $\{-1,-1/2,0,1/2,1\}$ depending whether ω has 2 or 3 zeros, respectively). It remains to show that $(t_1+\alpha_1 s, t_2+\alpha_2 s, t_3+\alpha_3 s) = (t_1,t_2,t_3)+(\alpha_1,\alpha_2,\alpha_3)s$ belongs to $\mathbb{P}$. But

$$\sum_{i=1}^3 r_i\left(\frac{(t_i+s\alpha_i)-t_i}{w_i}\right) = s\sum_{i=1}^3 r_i\frac{\alpha_i}{w_i} = 0$$

by Equation (3). Thus the lemma is proved for the case $n=3$.

Let us now consider the case $n=2$. Note that if D is not a square then the horizontal direction is parabolic in this case (see Theorem 4.13). Therefore, D must be a square. By Proposition 4.12 we know that the cylinder decomposition is stable, which implies that $(X,\omega)+(s,0)$ is defined for all s. In this case, the closure of $U\cdot(X,\omega)$ can be identified with the torus

$$\mathcal{X}' = N(\mathbb{R})\times N(\mathbb{R})/N(w_1\mathbb{Z})\times N(w_2\mathbb{Z})$$

Using this identification, the horizontal kernel foliation leaf through (X,ω) corresponds to the projection of the affine line $\{(t_1,t_2)+(\alpha_1,\alpha_2)s,\ s\in\mathbb{R}\}$. Hence

$$(X_s,\omega_s) = (X,\omega)+(s,0)\in\overline{U\cdot(X,\omega)},$$

which concludes the proof of Lemma 5.2. □

Proof of Corollary 5.3. We will apply Lemma 5.2 to a transverse direction to $(1:0)$. By Theorem 2.3, let θ be a completely periodic direction on Y which is transverse to the horizontal direction. Up to action of U, we can assume that $\theta=(0:1)$.

By Proposition 4.7 and Proposition 4.9, there exists $\varepsilon'>0$ such that $(Y,\eta)+v$ is well defined, and the direction $(0:1)$ is completely periodic on $(Y,\eta)+v$ for all $v\in(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$. If $s\neq 0$ then the cylinder decomposition of $(Y,\eta)+(s,0)$ in the direction of $(0:1)$ is stable. Moreover, the combinatorial data of this decomposition is unchanged when s varies in the intervals $(-\varepsilon',0)$ and $(0,\varepsilon')$. If the decomposition of (Y,η) in the vertical direction is stable, then we have the same combinatorial data for any $s\in(-\varepsilon',\varepsilon')$.

Let $\{w_i(s)\}_{i=1,\dots,k}$ and $\{h_i(s)\}_{i=1,\dots,k}$ be the widths and heights of the cylinders in the *vertical* direction of $(Y,\eta)+(s,0)$, $s \neq 0$. Note that the functions $w_i(s)$ are constant on each of intervals $(-\varepsilon,0)$ and $(0,\varepsilon)$. However, the set of heights $h_i(s)$ define non constant continuous functions of s. To be more precise, $h_i(s) = h_i + \alpha_i s$, where $\alpha_i \in \{-1,0,1\}$ or $\alpha_i \in \{-1,-1/2,0,1/2,1\}$ depending on whether η has two or three zeros. Obviously, at least two of α_i are different. Hence the set of moduli

$$\mu_i(s) = \frac{h_i + s\alpha_i}{w_i}$$

of cylinders (in the vertical direction) define also non constant continuous functions of s. In particular for almost every s in $(-\varepsilon',0)$ (resp. $(0,\varepsilon')$), the direction $(0:1)$ is completely periodic and not parabolic on $(Y,\eta)+(s,0)$. Applying Lemma 5.2 to the vertical direction on $(Y,\eta)+(s,0)$, we get that, for any $t \in (-\varepsilon',\varepsilon')$ one has

$$(Y,\eta)+(s,t) \in \overline{\mathrm{GL}^+(2,\mathbb{R})\cdot((Y,\eta)+(s,0))}.$$

It follows immediately that we have $(Y,\eta)+v \in \overline{\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)}$ for every $v=(s,t)\in(-\varepsilon',\varepsilon')\times(-\varepsilon',\varepsilon')$. This completes the proof of Corollary 5.3. □

One can now prove the main result of this section.

Proof of Theorem 5.1. We will show that any $(Y,\eta) \in \overline{\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)} = \overline{\mathcal{O}}$ has an open neighborhood contained in $\overline{\mathcal{O}}$. Let $\mathbf{B}(\varepsilon) = \{v \in \mathbb{R}^2,\ |v| < \varepsilon\}$.

First case: $(Y,\eta) \in \mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$**.** By assumption, there exists a periodic direction for (X,ω) which is not parabolic. Lemma 5.2 and Corollary 5.3 then imply that there exists $\varepsilon > 0$ such that $(X,\omega)+v \in \overline{\mathcal{O}}$ for any $v \in \mathbf{B}(\varepsilon)$. It follows that for all $g \in \mathrm{GL}^+(2,\mathbb{R})$, $g\cdot((X,\omega)+v) \in \overline{\mathcal{O}}$. In particular, there exists a neighborhood $\mathcal{U}$ of Id in $\mathrm{GL}^+(2,\mathbb{R})$ such that $g\cdot((X,\omega)+v) \in \overline{\mathcal{O}}$, for any $(g,v) \in \mathcal{U}\times\mathbf{B}(\varepsilon)$. But by Proposition 3.2 the set $\{g\cdot((X,\omega)+v),\ (g,v)\in\mathcal{U}\times\mathbf{B}(\varepsilon)\}$ is a neighborhood of (X,ω) in $\Omega E_D(\kappa)$. Hence (X,ω) (and thus (Y,η)) has an open neighborhood contained in $\overline{\mathcal{O}}$.

Second case: $(Y,\eta) \notin \mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$**.** Let $(X_n,\omega_n) = g_n\cdot(X,\omega)$ be a sequence converging to (Y,η) with $g_n \in \mathrm{GL}^+(2,\mathbb{R})$. By Proposition 3.2, there exist $\varepsilon > 0$, and a neighborhood $\mathcal{U}$ of Id in $\mathrm{GL}^+(2,\mathbb{R})$, such that $\mathcal{U}\times\mathbf{B}(\varepsilon)$ is identified with a neighborhood of (Y,η) via the mapping $(g,v)\mapsto g\cdot((Y,\eta)+v)$. Thus for n large enough, there is a pair (a_n,v_n), where $a_n \in \mathcal{U}$, and $v_n \in \mathbf{B}(\varepsilon) \subset \mathbb{R}^2$ such that $(X_n,\omega_n) = a_n\cdot((Y,\eta)+v_n)$. Since (X_n,ω_n) converges to (Y,η), we have $(a_n)_n$ converges to Id, and $(v_n)_n$ converges to 0. Multiplying by a_n^{-1} we get

$$\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega) \ni (X'_n,\omega'_n) = a_n^{-1}\cdot(X_n,\omega_n) = (Y,\eta)+v_n,$$

Without loss of generality, we also assume that the horizontal direction is completely periodic on Y. By Propositions 4.7 and 4.9, we can choose $r > 0$ such that for all $v=(s,t)\in\mathbf{B}(r)$ the surface $(Y,\eta)+v$ also admits a cylinder decomposition in the horizontal direction. When $t \neq 0$ this decomposition is stable with combinatorial data depending only on the sign of t. We can assume that $v_n \in \mathbf{B}(r)$ (for n large enough).

Now, since $(X'_n,\omega'_n) \in \mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$, the first case implies that $\overline{\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)}$ contains a neighborhood of (X'_n,ω'_n). Hence for each n, there exists $\varepsilon_n > 0$ such that for any $v \in \mathbf{B}(\varepsilon_n)$, $(X'_n,\omega'_n)+v \in \overline{\mathcal{O}}$. Now for each n, we choose $\delta_n \in (0,\varepsilon_n)$ small enough such that

(a) $u_n = v_n + (0,\delta_n) \in \mathbf{B}(r)$.

(b) If $v_n = (s_n, t_n)$ with $t_n \neq 0$, then $\delta_n < |t_n|$.

In particular since $u_n \in \mathbf{B}(r)$, (a) implies that $(Y, \eta) + u_n$ also admits a cylinder decomposition in the horizontal direction. Since the ratio of moduli is a continuous (non constant) function of δ_n, one can choose $\delta_n \in (0, \varepsilon_n)$ satisfying (a), (b) and

(c) The horizontal direction is stable and not parabolic for $(Y, \eta) + u_n$,

(d) $\lim_{n\to\infty} \delta_n = 0$.

By construction, we have $\delta_n \in (0, \varepsilon_n)$, hence $(X''_n, \omega''_n) := (X'_n, \omega'_n) + (0, \delta_n) = (Y, \eta) + u_n \in \overline{O}$. Since the horizontal direction is not parabolic on (X''_n, ω''_n), by Lemma 5.2, we derive that for any $s \in \mathbb{R}$, $(X''_n, \omega''_n) + (s, 0) \in \overline{O}$ (see Figure 3). Thus

$$(X''_n, \omega''_n) + (s, 0) \in \overline{O} \qquad \text{for any } s \in (-r/2, r/2).$$

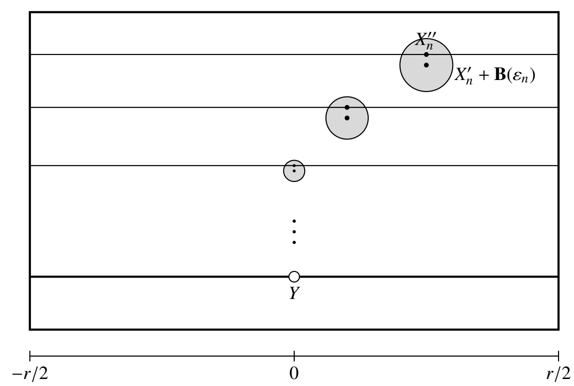

FIGURE 3. The convergence of (X'_n, ω'_n) and (X''_n, ω''_n) to (Y, η) in the kernel foliation leaf of (Y, η).

Since $(\delta_n)_n$ converges to 0, we have $(X''_n, \omega''_n) = (X'_n, \omega'_n) + (0, \delta_n)$ converges to (Y, η). It follows that $(Y, \eta) + (s, 0) \in \overline{O}$ for all $s \in (-r/2, r/2)$. The theorem then follows from Corollary 5.3. □

6. PROOF OF THEOREM 1.1

In this section we complete the proof of Theorem 1.1 in full generality, namely without the assumption that the orbit $O := \mathrm{GL}^+(2, \mathbb{R}) \cdot (X, \omega)$ is not closed "for the most obvious reason". However our proof says nothing about the converse of this assumption, *i.e.* the following question remains open in our setting:

Question. *For an orbit $O := \mathrm{GL}^+(2, \mathbb{R}) \cdot (X, \omega)$, is the property of being not closed equivalent to be not closed "for the most obvious reason"?*

Proof of Theorem 1.1. We begin by fixing some notations and normalization. As usual, let $(X, \omega) \in \Omega E_D(\kappa)$ and let us assume that $O := \mathrm{GL}^+(2, \mathbb{R}) \cdot (X, \omega)$ is not closed. Let $(Y, \eta) \in \overline{O} \setminus O$ be some translation surface in the orbit closure, but not in the orbit itself.

Claim 1. *There exists a sequence $(X_n, \omega_n)_{n\in\mathbb{N}}$ converging to Y such that for every $n \in \mathbb{N}$, $(X_n, \omega_n) = (Y, \eta) + v_n \in O$, $v_n = (x_n, y_n)$, $y_n \neq 0$ and the horizontal direction on Y is completely periodic.*

Proof of the claim. We choose a sequence $(X_n,\omega_n) \in \mathcal{O}$ converging to (Y,η). As in the proof of Theorem 5.1 we can assume that $(X_n,\omega_n) = (Y,\eta) + v_n$ where $v_n = (x_n, y_n)$ converging to $(0,0) \in \mathbb{R}^2$.

Again, up to replacing Y by $R_\theta \cdot Y$ for some suitable θ, without loss of generality, we will also assume that the horizontal direction is completely periodic on Y. If $y_n \neq 0$ infinitely often then the claim follows by taking a subsequence. Otherwise we assume that $y_n = 0$ for every $n > N$. We choose another (transverse) completely periodic direction on Y. We can assume that this direction to be vertical by applying a matrix in U. Note that a matrix in U fixes the vectors $(x_n, 0)$. Then up to replacing (Y,η) and (X_n,ω_n) respectively by $R_{\pi/2}\cdot(Y,\eta)$ and $R_{\pi/2}\cdot(X_n,\omega_n)$ the claim is proved (otherwise $x_n = 0$ for n large enough, thus $(Y,\eta) = (X_n,\omega_n) \in \mathcal{O}$ which is a contradiction to our assumption). □

We choose some $\varepsilon > 0$ so that for any $v = (x,y) \in \mathbb{R}^2$, if $v \in \mathbf{B}(\varepsilon)$ then the horizontal direction on $(Y,\eta) + v$ is periodic, and the cylinder decomposition is stable if $y \neq 0$. We can assume that $v_n \in \mathbf{B}(\varepsilon)$ and $y_n > 0$ for all n, which implies that the combinatorial data of the cylinder decomposition in the horizontal direction of (X_n,ω_n) are the same for all n. Finally we also assume that *all* the horizontal directions on X_n are parabolic (otherwise we are done by Theorem 5.1).

We sketch the idea of the proof. It makes use of the horocycle flow u_s acting on X_n. The key is to show that the actions of the kernel foliation and u_s coincide for a subsequence.

(1) Since *all* surfaces (X_n,ω_n) are horizontally parabolic, we will show that it is always possible to find a "good time" s_n so that $u_{s_n}\cdot X_n = X_n + (x_n,0)$ for some vector $(x_n,0) \in \mathbf{B}(\varepsilon)$.
(2) One can arrange that $(x_n,0)$ converges to some arbitrary vector $(x,0) \in \mathbf{B}(\varepsilon)$.

These two facts correspond, respectively, to Claim 3 and Claim 4 below. Once we achieve this, passing to the limit as $n \to \infty$, we get

$$u_{s_n}\cdot(X_n,\omega_n) = (X_n,\omega_n) + (x_n,0) \longrightarrow (Y,\eta) + (x,0).$$

In other words $(Y,\eta) + (x,0) \in \overline{\mathcal{O}}$ for all $x \in (-\varepsilon,\varepsilon)$. Then Corollary 5.3 applies and this gives some $\varepsilon' > 0$ so that $(Y,\eta) + v \in \overline{\mathcal{O}}$ for any $v \in \mathbf{B}(\varepsilon')$ which proves the theorem.

We now explain how to construct the sequence $(s_n)_{n\in\mathbb{N}}$. As usual, the cylinders on X_n are denoted by $C_i^{(n)}, i = 1,\dots,k$ (the numbering is such that for every $i \in \{1,2,3\}$, $C_j^{(n)} = \tau(C_i^{(n)})$ implies $j = i$ or $j > 3$). The width, height, twist, and modulus of $C_i^{(n)}$ are denoted by $w_i^{(n)}, h_i^{(n)}, t_i^{(n)}, \mu_i^{(n)}$ respectively. Recall that by Proposition 4.7 and Proposition 4.9, we have $w_i^{(n)}$ does not depend on n, therefore we can write $w_i^{(n)} = w_i$. Let us define

$$h_i^\infty = \lim_{n\to\infty} h_i^{(n)}.$$

Since the cylinder decomposition of X_n is stable, we can associate to each family of cylinders $(C_i^{(n)})_n$ a coefficient $\alpha_i \in \{0,\pm 1/2,\pm 1\}$. Recall that the kernel foliation action of a vector $v = (x,y)$ changes the height $h_i^{(n)}$ of $C_i^{(n)}$ to $h_i^{(n)} + \alpha_i y$, hence we can write

$$h_i^{(n)} = h_i^\infty + \alpha_i y_n.$$

Note that the horizontal direction on Y is not necessarily stable, some horizontal cylinders on X_n can be destroyed in the limit (as n tends to infinity). Therefore, some of the limits h_i^∞ may be zero. However, there is at least one cylinder that remains in the limit, say it is $C_3^{(n)}$ (see Figure 4 where the

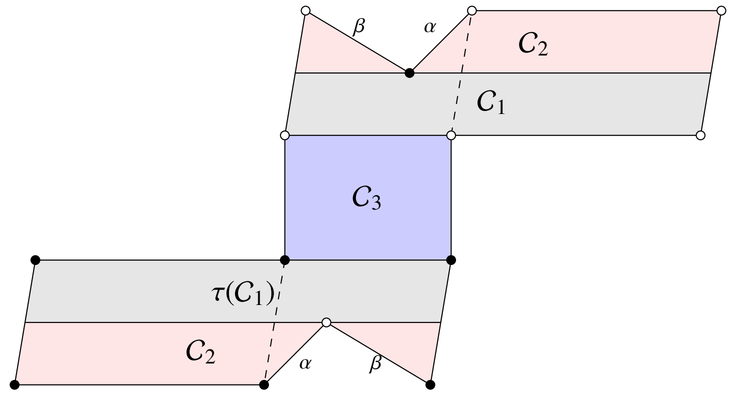

FIGURE 4. Decomposition into four cylinders of $(X_n, \omega_n) = (Y, \eta) + v_n$ near $(Y, \eta) \in \Omega E_D(2,2)$ where $v_n = \int_\alpha \omega$. The cylinders C_2 and C_3 are fixed by the Prym involution τ, while the cylinders C_1 and $\tau(C_1)$ are exchanged. When $v_n \to 0$ the cylinder C_2 is destroyed, while C_3 is remains in the limit (here we have assumed that $h_3 > h_2$).

cylinder $C_2^{(n)}$ is destroyed when performing the kernel foliation). Actually, since (X_n, ω_n) stays in a neighborhood of (Y, η), all the cylinders of (Y, η) persist in (X_n, ω_n). Thus, the number of horizontal cylinders of (X_n, ω_n) is always greater than (Y, η). We denote by C_3 the cylinder on Y corresponding to $C_3^{(n)}$ on X_n, then the height of C_3 is h_3^∞. In particular, we have $h_3^\infty > 0$.

From Equation (4), we have

$$\sum_{i=1}^{3} \beta_i w_i \alpha_i = 0.$$

Since all the α_i can not vanish (otherwise for all $i \in \{1, \dots, k\}$ the upper and lower boundaries of $C_i^{(n)}$ contain the same zero, which means that ω has only one zero), Equation (4) implies that there exist i, j in $\{1,2,3\}$ such that α_i and α_j are non zero and have opposite signs. In particular, there exists $i \in \{1,2,3\}$ such that $\alpha_i \neq 0$ and α_i has the opposite sign to α_3 if $\alpha_3 \neq 0$. In what follows we suppose that α_1 satisfies this condition. By a slight abuse of language, we will say that α_1 and α_3 have opposite signs. Since $\alpha_1 \neq 0$, $(t_1^{(n)}, h_1^{(n)})$ is a relative coordinate. For the surface in Figure 1, ω has three zeros and $(\alpha_1, \alpha_3) = (-1, 1/2)$, and for the one in Figure 4, ω has two zeros and $(\alpha_1, \alpha_3) = (-1, 1)$.

Recall that, by Proposition 4.12, we know that there exists $(r_1, r_2, r_3) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$ such that

$$r_1\mu_1^{(n)} + r_2\mu_2^{(n)} + r_3\mu_3^{(n)} = 0 \quad \text{and} \quad r_1\frac{\alpha_1}{w_1} + r_2\frac{\alpha_2}{w_2} + r_3\frac{\alpha_3}{w_3} = 0.$$

Obviously, we can assume that $(r_1, r_2, r_3) \in \mathbb{Z}^3$. Note that (r_1, r_2, r_3) does not depend on n. Set $\mu_i^\infty = h_i^\infty / w_i$, by continuity we have

$$r_1\mu_1^\infty + r_2\mu_2^\infty + r_3\mu_3^\infty = 0.$$

Claim 2. *We have $r_2 \neq 0$.*

Proof. Suppose that $r_2 = 0$, we have then

$$\begin{cases} r_1\mu_1^{(n)} + r_3\mu_3^{(n)} &= 0 \\ r_1\dfrac{\alpha_1}{w_1} + r_3\dfrac{\alpha_3}{w_3} &= 0 \end{cases}$$

Since $\mu_i^{(n)} > 0, w_i > 0$, and $\alpha_1\alpha_3 \leq 0$, this system with unknowns (r_1, r_3) has a unique solution $r_1 = r_3 = 0$. Thus we have a contradiction. □

From now on, we fix an integral vector $(r_1, r_2, r_3) \in \mathbb{Z}^3$ satisfying Equation (2) and Equation (3), with $r_2 \neq 0$.

Claim 3. *Let $(\tilde{X}, \tilde{\omega}) \in \Omega E_D(\kappa)$ be a surface which admits the same cylinder decomposition as X_n in the horizontal direction. We denote by C_i the cylinder in $\tilde{X}$ which corresponds to the cylinder $C_i^{(n)}$ of X_n. Let $w_i, h_i, t_i, \mu_i, \alpha_i$ be the parameters of C_i. Given two integers k_1, k_3, if the real numbers s and $x(s)$ satisfy*

$$x(s) := \frac{1}{\alpha_3}(sh_3 - r_2k_3w_3) = \frac{1}{\alpha_1}(sh_1 - r_2k_1w_1) \tag{6}$$

then $u_s \cdot (\tilde{X}, \tilde{\omega}) = (\tilde{X}, \tilde{\omega}) + (x(s), 0)$.

Remark 6.1. *If $\alpha_3 = 0$, we replace Equation* (6) *by the following system*

$$\begin{cases} sh_3 &= r_2k_3w_3 \\ x(s) &= \dfrac{sh_1 - r_2k_1w_1}{\alpha_1}. \end{cases}$$

Proof of the claim. On one hand, the kernel foliation $\tilde{X} + (x, 0)$, for small values of x, maps the twist of the cylinder C_i to $t_i(x) = t_i + \alpha_i x$. On the other hand, the action of u_s on the cylinder C_i maps the twist t_i to the twist $\widetilde{t_i} = t_i + sh_i \mod w_i$. Equation (6) implies

$$sh_1 = \alpha_1 x(s) + r_2k_1w_1 \quad \text{and} \quad sh_3 = \alpha_3 x(s) + r_2k_3w_3$$

which is equivalent to

$$\begin{cases} s\mu_1 &= \dfrac{\alpha_1}{w_1}x(s) + r_2k_1 \\ s\mu_3 &= \dfrac{\alpha_3}{w_3}x(s) + r_2k_3 \end{cases} \tag{7}$$

We see that the twist of the cylinder C_i of $u_s \cdot \tilde{X}$ is $\widetilde{t_i} = t_i + \alpha_i x(s) \mod w_i$, for $i \in \{1, 3\}$. It remains to show that $sh_2 = \alpha_2 x(s) \mod w_2$. Using Equation (2) and Equation (3), (7) implies

$$-r_2s\mu_2 = -r_2\frac{\alpha_2}{w_2}x(s) + r_2(r_1k_1 + r_3k_3).$$

It follows

$$sh_2 = \alpha_2 x(s) - (r_1k_1 + r_3k_3)w_2.$$

Thus we can conclude that $u_s \cdot (\tilde{X}, \tilde{\omega}) = (\tilde{X}, \tilde{\omega}) + (x(s), 0)$. □

Equation (6) above reads

$$s = r_2 \frac{w_1 k_1 \alpha_3 - w_3 k_3 \alpha_1}{h_1 \alpha_3 - h_3 \alpha_1}. \tag{8}$$

Note that since α_1 and α_3 have opposite signs, s is always defined. Reporting this last equation into (6), we derive the relation:

$$x(s) = \frac{r_2}{\alpha_3}\left(\frac{w_1 k_1 \alpha_3 - w_3 k_3 \alpha_1}{h_1 \alpha_3 - h_3 \alpha_1} h_3 - k_3 w_3\right) = \dots = \frac{r_2 h_3 w_1}{h_1 \alpha_3 - h_3 \alpha_1}\left(k_1 - \frac{\mu_1}{\mu_3} k_3\right).$$

We now make the additional assumption that the horizontal direction is parabolic, *i.e* the moduli μ_i are all commensurable. We thus write the last expression as:

$$x(s) = \frac{r_2 h_3 w_1}{h_1 \alpha_3 - h_3 \alpha_1}\left(k_1 - \frac{p}{q} k_3\right), \text{ where } \frac{p}{q} = \frac{\mu_1}{\mu_3} \in \mathbb{Q}.$$

We perform this calculation for each surface X_n, so that given a sequence $(k_1^{(n)}, k_3^{(n)})_n$ we get a sequence

$$x_n = \frac{r_2 h_3^{(n)} w_1^{(n)}}{h_1^{(n)} \alpha_3 - h_3^{(n)} \alpha_1}\left(k_1^{(n)} - \frac{p^{(n)}}{q^{(n)}} k_3^{(n)}\right), \tag{9}$$

where $(p^{(n)}, q^{(n)}) \in \mathbb{Z}^2$ and $\gcd(p^{(n)}, q^{(n)}) = 1$. We want to choose suitable pair of integers $(k_1^{(n)}, k_3^{(n)}) \in \mathbb{Z}^2$ in order to make the sequence $(x_n)_n$ converging to some arbitrary x.

Claim 4. *There exists a constant C independent of n such that, for any $x \in (-\varepsilon, \varepsilon)$, there exists $(k_1^{(n)}, k_3^{(n)}) \in \mathbb{Z}^2$ satisfying the following: if x_n is defined by* (9)*, then*

$$|x_n - x| < \frac{C}{q^{(n)}}.$$

Proof of the claim. For each $n \in \mathbb{N}$, since $p^{(n)}$ and $q^{(n)}$ are co-prime, we can choose $(k_1^{(n)}, k_3^{(n)}) \in \mathbb{Z}^2$ such that

$$\left| k_1^{(n)} - \frac{p^{(n)}}{q^{(n)}} k_3^{(n)} - \frac{h_1^{(n)} \alpha_3 - h_3^{(n)} \alpha_1}{r_2 h_3^{(n)} w_1^{(n)}} x \right| < \frac{1}{q^{(n)}}. \tag{10}$$

As n tends to infinity, the sequence $(h_3^{(n)})_n$ converges to h_3^∞. Since $w_1^{(n)}$ is constant, $h_1^{(n)}\alpha_3 - h_3^{(n)}\alpha_1$ converges to a non-zero constant (since α_1 and α_3 have opposite signs), hence there exists some constant $C > 0$ such that

$$\frac{r_2 h_3^{(n)} w_1^{(n)}}{h_1^{(n)} \alpha_3 - h_3^{(n)} \alpha_1} < C. \tag{11}$$

From (10) and (11) we draw

$$|x_n - x| < \frac{C}{q^{(n)}}$$

that is the desired inequality. The claim is proved. □

In order to conclude the proof of Theorem 1.1, one needs to show that $q^{(n)} \to \infty$. Indeed, we then have that $x_n \longrightarrow x$ and since x was arbitrary, by Claim 3 this shows

$$(Y,\eta) + (x,0) \in \overline{O}, \quad \text{for any } x \in (-\varepsilon,\varepsilon).$$

Then Corollary 5.3 applies and Y has an open neighborhood in $\overline{O}$, which proves the theorem.

We now prove that $q^{(n)} \to \infty$. Recall that

$$\frac{p^{(n)}}{q^{(n)}} = \frac{\mu_1^{(n)}}{\mu_3^{(n)}} = \frac{w_3^{(n)}}{w_1^{(n)}} \cdot \frac{h_1^{(n)}}{h_3^{(n)}} = \frac{w_3}{w_1} \cdot \frac{h_1^\infty + \alpha_1 y_n}{h_3^\infty + \alpha_3 y_n}$$

and $\gcd(p^{(n)}, q^{(n)}) = 1$. Note that since α_1 and α_3 have opposite signs, $\frac{p^{(n)}}{q^{(n)}}$ cannot be a stationary sequence as y_n tends to 0. As n tends to infinity, $p^{(n)}/q^{(n)}$ converges to $p^\infty/q^\infty = \frac{w_3 h_1^\infty}{w_1 h_3^\infty}$. But as we have seen $\frac{p^{(n)}}{q^{(n)}}$ cannot be stationary, therefore there are infinitely many n such that $p^{(n)}/q^{(n)} \neq p^\infty/q^\infty$ which implies that $q^{(n)} \to \infty$. □

In the remaining of this paper, we will apply Theorem 1.1 (more precisely, the techniques used in the proof) to show that, for any D which is not a square, there are at most finitely many closed $\mathrm{GL}^+(2,\mathbb{R})$-orbits in $\Omega E_D(2,2)^{\mathrm{odd}}$. Even though, we only prove the result for this case, it seems very likely that one can also obtain similar results for all strata listed in Table 1. In higher "complexity" (genus and number singularities) the difficulty comes from the increasing number of degenerated surfaces. Along the way, we will give description of surfaces in a partial compactification of $\Omega E_D(2,2)^{\mathrm{odd}}$.

We end this section with a by-product theorem which follows from the same arguments as the proof of Theorem 1.1.

Theorem 6.2. *Let $(Y,\eta) \in \Omega E_D(\kappa)$ be a Prym eigenform (where $\Omega E_D(\kappa)$ has complex dimension 3) satisfying the following properties:*

1. *The horizontal direction is completely periodic on (Y,η).*
2. *There exists a sequence $(X_n,\omega_n) = (Y,\eta) + (x_n,y_n)$ converging to (Y,η) where $y_n \neq 0$, $\forall n \in \mathbb{N}$.*
3. *For every n, X_n is horizontally parabolic.*

Then there exists $\varepsilon > 0$ such that $(Y,\eta) + (x,0) \in \overline{O}$ for all $x \in (-\varepsilon,\varepsilon)$, where $O = \bigcup_n \mathrm{GL}^+(2,\mathbb{R}) \cdot (X_n,\omega_n)$.

7. Preparation of a surgery toolkit

In this section we will describe several useful surgeries for Prym eigenforms. More precisely let us fix a surface (X_0,ω_0) in the following list of strata $\Omega E_D(\kappa)$:

- $\Omega E_D(0,0,0)$ (space a triple tori, Section 7.1),
- $\Omega E_D(4)$ (Section 7.2),
- $\Omega E_D(2)^*$ (set of $(M,\omega) \in \Omega E_D(2)$ with a marked Weierstrass point, Section 7.3).

For each case, we will construct a continuous locally injective map $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$, where $\mathring{D}(\varepsilon) = \{z \in \mathbb{C},\ 0 < |z| < \varepsilon\}$, which induces an embedding of $\mathring{D}(\varepsilon)/(z \sim -z)$ into $\Omega E_D(2,2)^{\mathrm{odd}}$. Up to action $\mathrm{GL}^+(2,\mathbb{R})$, the set $\Psi(\mathring{D}(\varepsilon))$ will be identified to a neighborhood of (X_0, ω_0) in $\Omega E_D(2,2)^{\mathrm{odd}}$.

We now describe these surgeries in details (observe that the second one already appears in [KZ03] as "Breaking up a zero").

7.1. **Space of triples of tori.**

We say that $(X,\omega) \in \mathrm{Prym}(2,2)^{\mathrm{odd}}$ admits a *three tori decomposition* if there exists a triple of homologous saddle connections $\{\sigma_0, \sigma_1, \sigma_2\}$ on X, each of which connects the two zeros of ω, such that (X,ω) can be viewed as a connected sum of three tori which are glued together along the slits corresponding to σ_j. One can reduce the length of saddle connections $\{\sigma_0, \sigma_1, \sigma_2\}$ to zero by moving in the kernel foliation leaf through (X,ω), the limit surface is then the union of three flat tori $(X_j, \omega_j),\ j = 0,1,2$, which are joint at a unique common point P.

Recall that $\mathcal{H}(0)$ is the space of triples (X,ω,P) where X is an elliptic curve, ω a non-zero Abelian differential on X, and P is a marked point of X. We denote by $\mathrm{Prym}(0,0,0)$ the space of triples $\{(X_j,\omega_j,P_j),\ j = 0,1,2\}$, where $(X_j,\omega_j,P_j) \in \mathcal{H}(0)$, such that (X_1,ω_1,P_1) and (X_2,ω_2,P_2) are isometric. The geometric object corresponding to such a triple is the union of the three tori, where we identify P_0, P_1, P_2 to a unique common point. Note that by construction, there exists an involution τ on the "surface" $X := \{(X_j,\omega_j,P_j),\ j = 0,1,2\}$ which preserves X_0 and exchanges X_1 and X_2, we will call τ the Prym involution.

We define $\Omega E_D(0,0,0) \subset \mathrm{Prym}(0,0,0)$ to be the space of all triples $\{(X_j,\omega_j,P_j),\ j = 0,1,2\}$, which can be obtained by collapsing triples of homologous saddle connections associated to three-tori decompositions of surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$. The aim of this section is to show:

Proposition 7.1. *For any triple tori $\{(X_j,\omega_j,P_j),\ j = 0,1,2\}$ in $\Omega E_D(0,0,0)$, there exist $\varepsilon > 0$ and a continuous locally injective map $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ satisfying:*

1. *$\forall z \in \mathring{D}(\varepsilon)$, the surface $(X,\omega) = \Psi(z)$ has a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ decomposing X into three tori such that $\omega(\sigma_j) = z$.*
2. *The map Ψ is two to one and it induces an embedding of $\mathring{D}(\varepsilon)/(z \sim -z)$ into $\Omega E_D(2,2)^{\mathrm{odd}}$.*
3. *Up to action $\mathrm{GL}^+(2,\mathbb{R})$, the set $\Psi(\mathring{D}(\varepsilon))$ can be viewed as a neighborhood of $\{(X_j,\omega_j),\ j = 0,1,2\}$ in $\Omega E_D(2,2)^{\mathrm{odd}}$.*

We postpone the proof of Proposition 7.1 and first provide a description of the space $\Omega E_D(0,0,0)$ (compare with [McM07, Theorem 8.3]).

Proposition 7.2. *Let $\{(X_j,\omega_j,P_j),\ j = 0,1,2\}$ be a triple tori in $\Omega E_D(0,0,0)$ (where X_1, X_2 are exchanged by the Prym involution τ). Then there exist $(e,d) \in \mathbb{Z}^2$, with $d > 0$, and a covering $p : X_1 \to X_0$ of degree d such that*

- *$D = e^2 + 8d$,*
- *$\gcd(e, p_{11}, p_{12}, p_{21}, p_{22}) = 1$, where (p_{ij}) is the matrix of p in some symplectic bases of $H_1(X_0,\mathbb{Z})$ and $H_1(X_1,\mathbb{Z})$.*
- *$p^*\omega_0 = \frac{\lambda}{2}\omega_1$, where λ satisfies $\lambda^2 = e\lambda + 2d$.*

Proof. Let (a_j, b_j) be a symplectic basis of $H_1(X_j,\mathbb{Z})$, where $a_2 = -\tau(a_1), b_2 = -\tau(b_1)$, and set $\hat{a} = a_1+a_2$, $\hat{b} = b_1+b_2$. Then $(a_0, b_0, \hat{a}, \hat{b})$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$ (X is the surface obtained by identifying $P_0 \sim P_1 \sim P_2$). There exists a unique generator T of $\mathcal{O}_D$ such that the matrix of T in the basis $(a_0, b_0, \hat{a}, \hat{b})$ is of the form $T = \begin{pmatrix} e\mathrm{Id}_2 & 2B \\ B^* & 0 \end{pmatrix}$, where $e \in \mathbb{Z}$, $B \in \mathbf{M}_2(\mathbb{Z})$, $B^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot B \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $T^*\omega = \lambda\omega$, with $\lambda > 0$.

Observe that B can be regarded as a map from $H_1(X_1,\mathbb{Z})$ to $H_1(X_0,\mathbb{Z})$. Set $L_0 = \mathbb{Z}\omega_0(a_0) + \mathbb{Z}\omega_0(b_0)$, $L_1 = \mathbb{Z}\omega_1(a_1)+\mathbb{Z}\omega_1(b_1)$. We can identify (X_0, ω_0) and (X_1, ω_1) with $(\mathbb{C}/L_0, dz)$ and $(\mathbb{C}/L_1, dz)$ respectively. The condition $T^*\omega = \lambda\omega$ reads

$$\omega_0(2B(a_1)) = \lambda \cdot \omega_1(a_1) \qquad \text{and} \qquad \omega_0(2B(b_1)) = \lambda \cdot \omega_1(b_1).$$

Hence $\frac{\lambda}{2}L_1$ is a sublattice of L_0. It follows that there exists a covering map $p : \mathbb{C}/L_1 \to \mathbb{C}/L_0$ such that $p^*dz = \lambda/2dz$. The degree of p is given by $d = \det(B) > 0$. Note that T satisfies

$$T^2 = eT + 2\det(B).$$

Since T is a generator of $\mathcal{O}_D$, we have $D = e^2 + 8\det(B)$. As λ is an eigenvalue of T, λ satisfies the same equation. □

Proof of Proposition 7.1. Let $\varepsilon > 0$ be small enough so that the set $D(P_j,\varepsilon) = \{x \in X_j,\ \mathbf{d}(x, P_j) < \varepsilon\}$ is an embedded disk in X_j, $j = 0, 1, 2$. The map Ψ is defined as follows: for any $z \in \mathring{D}(\varepsilon)$, let σ_j be the geodesic segment in X_j whose midpoint is P_j such that $\omega(\sigma_j) = z$ (since $|z| < \varepsilon$, σ_j is an embedded segment). By slitting X_j along σ_j, and gluing X_0, X_1, X_2 along the slits in a cyclic order, we get a surface (X,ω) in $\mathcal{H}(2,2)$. It is easy to check that $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$. We define $(X,\omega) = \Psi(z)$. Since we cannot distinguish the two zeros of ω, one has $\Psi(z) = \Psi(-z)$.

Clearly, any surface in $\Omega E_D(2,2)^{\mathrm{odd}}$ admitting a three-tori decomposition $\{(X'_j, \omega'_j),\ j = 1,2,3\}$ such that $(X'_j, \omega'_j) = (X_j, \omega_j)$, and the length of the slit is smaller than ε belongs to the image of Ψ. The proposition follows immediately from this observation. □

7.2. **Collapsing surfaces to $\Omega E_D(4)$.** This surgery already appears in [KZ03] ("Breaking up a zero"). As in the previous section, our aim is to show:

Proposition 7.3. *For any $(X_0,\omega_0) \in \Omega E_D(4)$, there exist $\varepsilon > 0$ and a continuous locally injective map $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ satisfying:*

(1) $\forall z \in \mathring{D}(\varepsilon)$*, the surface $(X,\omega) = \Psi(z)$ has the same absolute periods as (X_0,ω_0).*
(2) *There exists a saddle connection σ in X joining the zeros of ω and invariant by the Prym involution such that $\omega(\sigma) = z^5$.*
(3) $\Psi(z) = \Psi(-z)$.
(4) *Up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, a neighborhood of $(X_0,\omega_0) \in \Omega E_D(4)$ in $\Omega E_D(2,2)^{\mathrm{odd}}$ is identified with $\Psi(\mathring{D}(\varepsilon))$.*

The constructive proof we will give is on the level of Abelian differentials *i.e.* in $\mathrm{Prym}(2,2)$ and $\mathrm{Prym}(4)$. One can interpret this construction on the level of quadratic differentials *i.e.* $\mathcal{Q}(-1^4, 4)$ and $\mathcal{Q}(-1^3, 3)$, respectively. This last approach is related to the surgery "breaking up a singularity" in [KZ03] (breaking up the zero of degree 3 of the quadratic differential into a pole and a zero of degree 4).

Proof of Proposition 7.3. Let $(X_0, \omega_0) \in \Omega E_D(4)$ and let P_0 be the unique zero of ω_0. We consider $0 < \varepsilon < 1$ small enough so that the disk $D(P_0, \varepsilon) = \{x \in X_0, \mathbf{d}(x, P_0) \leq \varepsilon\}$ is embedded into X_0. To define the map Ψ, we will deform the metric structure inside $D(P_0, \varepsilon)$ in a similar manner as in Section 4.3.

Let $D(\varepsilon) := \{v \in \mathbb{R}^2, |v| \leq \varepsilon\}$, and let c denote the center of $D(\varepsilon)$. Let $v \in \mathbb{R}^2 \setminus \{0\}$ be a vector such that $|v| < \varepsilon$. The line in the direction of v through c intersects $\partial D(\varepsilon)$ at two points a and b, the labeling is chosen such that $\overrightarrow{ac} = \overrightarrow{cb} = \frac{\varepsilon}{|v|} v$. Let $D_+(\varepsilon)$ and $D_-(\varepsilon)$ be the two half-discs of $D(\varepsilon)$ that are cut out by $\overline{ab}$. By convention, as one moves from a to b, $D_+(\varepsilon)$ is on the right.

Since the cone angle at P_0 is 10π, the disk $D(P_0, \varepsilon)$ can be constructed from 5 copies of $D_+(\varepsilon)$, denoted by D_{i+}, and 5 copies of $D_-(\varepsilon)$, denoted by D_{i-}, with $i = 1, \dots, 5$. Let $a_{i\pm}, b_{i\pm}, c_{i_\pm}$ denote the points in the boundary of $D_{i\pm}$ that correspond to a, b, c respectively. To obtain $D(P_0, \varepsilon)$, we glue D_{i+} to D_{i-} along the segment $\overline{cb}$, and glue D_{i-} to $D_{(i+1)+}$ along $\overline{ac}$.

Let x and y denote respectively the points in $\overline{ab}$ such that $\overrightarrow{xc} = \overrightarrow{cy} = 1/2v$. As usual, the points in the border of $D_{i\pm}$ corresponding to x and y are denoted by $x_{i\pm}$ and $y_{i\pm}$.

To get a surface (X, ω) in $\Omega E_D(2,2)^{\text{odd}}$ with a saddle connection σ such that $\omega(\sigma) = v$, we first choose a number $k \in \{1, \dots, 5\}$, and then replace $D(P_0, \varepsilon)$ by a domain $\tilde{D}(\varepsilon)$ constructed from $D_{i\pm}$ as follows (see Figure 5 for $k = 2$)

- for $i \notin \{k, k+1, k+2\}$, D_{i+} is glued to D_{i-} along $\overline{xb}$,
- for $i \in \{k, k+1, k+2\}$, D_{i+} is glued to D_{i-} along $\overline{yb}$,
- for $i \notin \{k, k+1\}$, D_{i-} is glued to $D_{(i+1)+}$ along $\overline{ax}$,
- for $i \in \{k, k+1\}$, D_{i-} is glued to $D_{(i+1)+}$ along $\overline{ay}$,
- D_{k+} is glued to $D_{(k+2)-}$ along $\overline{xy}$.

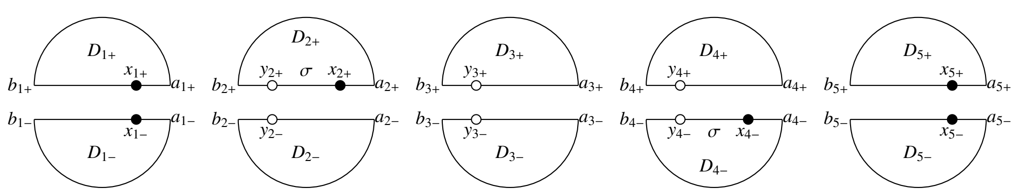

FIGURE 5. Splitting a zero of order 4 into two zeros of order 2.

Observe that all the points $x_{i\pm}$ (resp. $y_{i\pm}$) are identified to give a point with cone angle 6π. There is an involution in $\tilde{D}(\varepsilon)$ that maps D_{i+} to $D_{(i+2)-}$, thus the surface we obtain belongs to $\text{Prym}(2,2)^{\text{odd}}$. By construction, there is a saddle connection σ arising from the identification of $\overline{x_{k+}y_{k+}}$ and $\overline{x_{(k+2)-}y_{(k+2)-}}$. Note that σ is invariant by the involution and and we have $\omega(\sigma) = v$ as desired.

Since we have 5 choices for the pair of half-disks which contain σ in their boundary, we see that there are five surfaces (X, ω) in $\text{Prym}(2,2)$ close to (X_0, ω_0) satisfying the following conditions:

- The absolute periods of ω and ω_0 coincide.
- There exists a saddle connection σ in X, invariant by the Prym involution, joining the two zeros of ω such that $\omega(\sigma) = v$.

Since the absolute periods of ω and ω_0 coincide, the new surfaces actually belong to the same real multiplication locus as (X_0, ω_0), that is $(X, \omega) \in \Omega E_D(2,2)^{\text{odd}}$.

Let z be a complex number such that $z^5 = v$, we define the map Ψ by assigning $\Psi(z)$ to be one of the surfaces constructed above. By analytic continuation, this defines the desired map $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$. Observe that since we cannot distinguish the zeros of ω, the surfaces corresponding to $\pm z$ are the same (with different choices for the orientation of σ). The properties asserted in the statement of the proposition follows immediately from the definition of Ψ. □

Remark 7.4. *The "breaking up a zero" surgery is clearly invertible: we can collapse the two zeros of (X,ω) along σ to get the surface $(X_0,\omega_0) \in \Omega E_D(4)$. More generally, let P, Q denote the zeros of ω, where $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$, and let σ be a saddle connection, that we assume to be horizontal, joining P to Q that is invariant by the involution τ (such a saddle connection always exists, for instance the union of a path of minimal length joining a fixed point of τ to P or Q, and its image by τ). If for any other horizontal saddle connection σ' we have $|\sigma'| > 2|\sigma|$ then one can collapse the zeros of ω along σ by using the kernel foliation (see Section 8). The resulting surface (X_0,ω_0) belongs to $\Omega E_D(4)$. However if σ has twins, that is another saddle connection σ' such that $\omega(\sigma') = \omega(\sigma)$, then the limit surface is no longer in $\Omega E_D(4)$ as we will see in the next section.*

7.3. **Collapsing surfaces to $\Omega E_D(2)^*$.** In this section, we investigate degenerations by shrinking a pair of saddle connections that are exchanged by the Prym involution. Let $\Omega E_{D'}(2)^*$ be the space of triples (X,ω,W), where $(X,\omega) \in \Omega E_{D'}(2)$, and W is a Weierstrass point of X which is not the zero of ω. We will prove

Proposition 7.5. *For any $(X_0,\omega_0,W_0) \in \Omega E_{D'}(2)^*$ there exist $\varepsilon > 0$, $D \in \{D', 4D'\}$, and a continuous locally injective map $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ with the following properties:*

1. *$\forall z \in \mathring{D}(\varepsilon)$ the surface $(X,\omega) = \Psi(z)$ has the same absolute periods as (X_0,ω_0,W_0).*
2. *there exists a pair of saddle connections (σ_1,σ_2) on X that are exchanged by the Prym involution and satisfy $\omega(\sigma_1) = \omega(\sigma_2) = z^3$.*
3. *$\Psi(z) = \Psi(-z)$.*
4. *Up to action of $\mathrm{GL}^+(2,\mathbb{R})$, $\Psi(\mathring{D}(\varepsilon))$ is a neighborhood of (X_0,ω_0,W_0) in $\Omega E_D(2,2)^{\mathrm{odd}}$.*

As for above surgeries, we will describe how one can degenerate some $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ to the boundary of the stratum *i.e.* to $(X_0,\omega_0,W_0) \in \Omega E_{D'}(2)^*$, by using the kernel foliation. The inverse procedure will give the map Ψ of Proposition 7.5. Hence let us show:

Theorem 7.6. *Let (σ_1,σ_2) be a pair of non-homologous saddle connections in X that are exchanged by the Prym involution τ. Suppose that for any other saddle connection σ' joining P to Q in the same direction as σ_1, we have $|\sigma'| > |\sigma_1|$. Then as the length of σ_1 tends to zero (in the leaf of the kernel foliation), (X,ω) tends to a point in the boundary of $\Omega E_D(2,2)^{\mathrm{odd}}$ which is represented by a triple $(X_0,\omega_0,W_0) \in \Omega E_{D'}(2)^*$ for some $D' \in \{D, D/4\}$.*

Observe that we consider θ and $-\theta$ ($\theta \in \mathbb{S}^1$) as two distinct directions. As usual, we choose the orientation for any saddle connection joining P and Q to be *from P to Q*. For the remaining of this section, we fix a pair of saddle connections (σ_1,σ_2) satisfying assumption of Theorem 7.6. We will need of the following:

Lemma 7.7. *Let us construct the translation surface (X',ω') by first cutting (X,ω) along $c = \sigma_1 * (-\sigma_2)$ and then gluing the resulting pair of geodesic segments in each boundary component. Then*

$$(X',\omega') \in \Omega E_{D'}(1,1) \qquad \text{for some} \qquad D' \in \{D, D/4\}.$$

(the involution τ of X descends to the hyperelliptic involution of X').

Proof of Lemma 7.7. We first show that $(X',\omega') \in \mathcal{H}(1,1)$. For that, we remark that the pair of angles specified by these two rays at the zeros P and Q are $(2\pi, 4\pi)$. Since τ sends σ_1 to $-\sigma_2$ and preserves the orientation of X, necessarily the angle 2π at P and the angle 2π at Q belong to the same side of c which prove the first fact.

The surface (X',ω') has two marked segments c_1, c_2, where c_1 is a saddle connection, and c_2 is simply a geodesic segment which has the same length and the same direction as c_1. We denote the endpoints of c_1 (respectively, c_2) by P_1, Q_1 (respectively, P_2, Q_2), where P_1, P_2 correspond to P and Q_1, Q_2 correspond to Q. Note that P_1, Q_1 are the zeros of ω'. We choose the orientation of c_1 (respectively, c_2) to be from P_1 to Q_1 (respectively, from P_2 to Q_2).

With these notations, τ induces an involution τ' on X' such that $\tau'(c_1) = -c_1$ and $\tau'(c_2) = -c_2$. It turns out that τ' has six fixed points on X': these are the four fixed points of τ (none of them are contained in c) and two additional fixed points in c_1 and c_2. By uniqueness τ' is therefore the hyperelliptic involution of X'.

To conclude the proof, one needs to show that (X',ω') is an eigenform. For that we first need to choose a symplectic basis of $H_1(X',\mathbb{Z})$. We proceed as follows (see Figure 6). Let $\alpha_{1,1}, \alpha_{1,2}, \alpha_2, \beta_2$ be the simple closed curves, and $\beta_{1,1}$ and $\beta_{1,2}$ be simple arcs in X' as shown in Figure 6, where $\alpha_{1,2} = -\tau'(\alpha_{1,1})$ and $\beta_{1,2} = -\tau'(\beta_{1,1})$. Let β'_1 denote the simple closed curve which is the concatenation $c_1 \cup \beta_{1,1} \cup c_2 \cup \beta_{1,2}$. Set $\alpha'_1 = \alpha_{1,1}$ (the orientations are chosen so that $(\alpha'_1, \beta'_1, \alpha_2, \beta_2)$ is a symplectic basis of $H_1(X',\mathbb{Z})$).

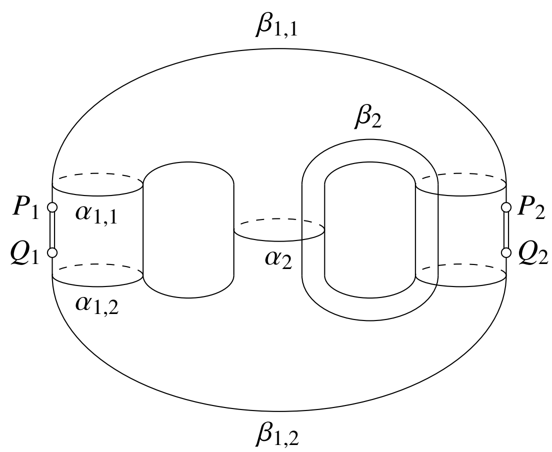

FIGURE 6. Surface in $\mathcal{H}(1,1)$ obtained by cutting and gluing along a pair of saddle connections exchanged by the Prym involution. The hyperelliptic involution τ' exchanges the upper and the lower halves of X'.

Observe that $\beta_{1,1}$, $\beta_{1,2}$ correspond to two simple closed curves in X, and that $\alpha_{1,1}$, $\alpha_{1,2}$ are not homologous in $H_1(X,\mathbb{Z})$. Set $\alpha_1 = \alpha_{1,1} + \alpha_{1,2}, \beta_1 = \beta_{1,1} + \beta_{1,2}$. Then $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$. In this basis, the intersection form is given by the matrix $\left(\begin{smallmatrix} 2J & 0 \\ 0 & J \end{smallmatrix}\right)$.

Since $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$, by definition there exists a unique generator T of $\mathcal{O}_D$ that can be expressed (in the basis $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ of $H_1(X,\mathbb{Z})^-$) by the matrix

$$T = \begin{pmatrix} e & 0 & a & b \\ 0 & e & c & d \\ 2d & -2b & 0 & 0 \\ -2c & 2a & 0 & 0 \end{pmatrix},$$

where $D = e^2 + 8(ad - bc)$, $\gcd(a,b,c,d,e) = 1$ and $T^*\omega = \lambda \cdot \omega$, with $\lambda > 0$. In the symplectic basis $(\alpha'_1,\beta'_1,\alpha_2,\beta_2)$ of $H_1(X',\mathbb{Z})$ we define the endomorphism:

$$T' = \begin{pmatrix} e & 0 & 2a & 2b \\ 0 & e & c & d \\ d & -2b & 0 & 0 \\ -c & 2a & 0 & 0 \end{pmatrix}.$$

It is easy to check that T' is self-adjoint with respect to the symplectic form $\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ and $T'^2 = eT' + 2(ad - bc)\mathrm{Id}$.

We now claim that ω' is an eigenform for T', namely $(T')^*\omega' = \lambda \cdot \omega'$, with $\lambda > 0$. Let (x,y,z,t) be the periods of $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ by ω. The condition $T^*\omega = \lambda\omega$ reads

$$(x,y,z,t) \cdot T = \lambda(x,y,z,t). \tag{12}$$

Now, we have

$$\begin{aligned}
&\omega'(\alpha'_1) = \omega(\alpha_{1,1}) = \tfrac{1}{2}\omega(\alpha_1) = x/2,\\
&\omega'(\beta'_1) = -\omega'(c_1) + \omega'(\beta_{1,1}) + \omega'(c_2) + \omega'(\beta_{1,2}) = \omega(\beta_{1,1}) + \omega(\beta_{1,2}) = \omega(\beta_1) = y,\\
&\omega'(\alpha_2) = \omega(\alpha_2) = z,\\
&\omega'(\beta_2) = \omega(\beta_2) = t.
\end{aligned}$$

By simple computations, we see that (12) implies

$$(x/2,y,z,t) \cdot T' = \lambda(x/2,y,z,t), \tag{13}$$

which means that ω' is an eigenvector for T'. Actually (12) and (13) are equivalent.

Observe that T' generates a self-adjoint subring isomorphic to $\mathcal{O}_D$ in $\mathrm{End}(\mathbf{Jac}(X'))$ for which ω' is an eigenform. In other words $(X',\omega') \in \Omega E_{D'}(1,1)$ for some D' dividing D. The proper subring isomorphic to $\mathcal{O}_{D'}$ is generated by the matrix $T'/k \in \mathrm{End}(\mathbf{Jac}(\mathrm{X}'))$ where $k = \gcd(2a,2b,c,d,e)$. By assumption $\gcd(a,b,c,d,e) = 1$, therefore $k \in \{1,2\}$. Since $D = k^2D'$, the lemma follows. □

We can now proceed to the proof of our results.

Proof of Theorem 7.6. We keep the notations of Lemma 7.7. By construction, there is no obstruction to collapse c_1 along the kernel foliation leaf through (X',ω'), the resulting surface belongs to $\Omega E_{D'}(2)$. Note that when c_1 is shrunken to a point, so is c_2. Since c_2 is invariant by the hyperelliptic involution of X', in the limit c_2 becomes a marked Weierstrass point. □

Proof of Proposition 7.5. The surgery "collapse a pair of saddle connections exchanged by τ", as described above, is invertible: this is the map Ψ of the proposition. Let us give a more precise definition of this map.

We fix a point $(X_0,\omega_0,W_0) \in \Omega E_{D'}(2)^*$, and choose $\varepsilon > 0$ small enough so that the sets $D(P_0,\varepsilon) = \{x \in X_0,\ \mathbf{d}(x,P_0) < \varepsilon\}$, where P_0 is the unique zero of ω_0, and $D(W_0,\varepsilon) = \{x \in X_0,\ \mathbf{d}(x,W_0) < \varepsilon\}$, are two disjoint embedded disks.

Given any vector $v \in \mathbb{R}$, with $|v| < \varepsilon$, we construct a Prym form in Prym(2,2) as follows. We break up the zero P_0 into two zeros to get a surface $(X',\omega') \in \mathcal{H}(1,1)$ having the same absolute periods as

ω, with a marked saddle connection, say σ_1, that is invariant by the hyperelliptic involution and such that $\omega'(\sigma_1) = v$. Note that by assumption σ_1 is disjoint from $D(W_0, \varepsilon)$. Let σ_2 be a geodesic segment in $D(W_0, \varepsilon)$ such that $\omega'(\sigma_2) = v$, and W_0 is the midpoint of σ_2. Cutting X' along σ_1 and σ_2, then regluing the resulting boundary components, we get a new surface $(X, \omega) \in \mathcal{H}(2, 2)$ together with an involution $\tau : X \to X$ induced by the hyperelliptic involution of X'. Since by construction $\tau^*\omega = -\omega$ one has $(X, \omega) \in \mathrm{Prym}(2, 2)$.

The arguments of the proof of Lemma 7.7 actually show that $(X, \omega) \in \Omega E_D(2, 2)$ for some $D \in \{D', 4D'\}$. We then define $\Psi(z) = (X, \omega)$, where z is a complex number such that $v = z^3$ (this condition is due to the fact that we have three choices for the segment σ_1), then extend Ψ to $\mathring{D}(\varepsilon)$ by analytic continuation. It is now straightforward to check that the map Ψ has the desired properties. □

8. Degenerating surfaces of $\Omega E_D(2, 2)^{\mathrm{odd}}$

In this section, we show that the surgeries described in Section 7 are sufficient to describe the all the degenerations (along the kernel foliation) of Prym eigenforms in $\Omega E_D(2, 2)^{\mathrm{odd}}$ having an unstable cylinder decomposition when D is not a square (compare with [LN13c]).

Theorem 8.1. *Assume that D is not a square, and $(X, \omega) \in \Omega E_D(2, 2)^{\mathrm{odd}}$ admits an unstable cylinder decomposition in the horizontal direction. Then there exists a finite interval $[s_{\min}, s_{\max}]$ such that for any $x \in]s_{\min}, s_{\max}[$, the surface $(X, \omega) + (x, 0)$ is well-defined and belongs to $\Omega E_D(2, 2)^{\mathrm{odd}}$. Moreover when x tends to $\partial[s_{\min}, s_{\max}]$, $(X, \omega) + (x, \omega)$ converges to a surface (Y, η) which belongs to*

$$\Omega E_D(0, 0, 0),\ \Omega E_D(4) \text{ or } \Omega E_{D'}(2)^* \ \textit{ with } \ D' \in \{D, D/4\}.$$

We will use the following elementary lemma.

Lemma 8.2. *Let $(X, \omega) \in \Omega E_D(2, 2)^{\mathrm{odd}}$. Assume that one of the following occurs:*

(1) *There exists a non trivial homology class $c \in H_1(X, \mathbb{Z})^-$ such that $\omega(c) = 0$.*
(2) *There exist two twins saddle connections in X joining the two zeros of ω, both of which are invariant by the Prym involution.*
(3) *There exists a triple of twins saddle connections $(\sigma_0, \sigma_1, \sigma_2)$ (that is $\omega(\sigma_0) = \omega(\sigma_1) = \omega(\sigma_2)$), where σ_0 is invariant and (σ_1, σ_2) are exchanged by the Prym involution, such that $c_0 = \sigma_1 * (-\sigma_2)$ is non-separating.*

Then D is a square.

Proof of Lemma 8.2. For the first condition, we set $K = \mathbb{Q}(\sqrt{D})$. If D is not a square then K is a real quadratic field over $\mathbb{Q}$ and, up to a rescaling by $\mathrm{GL}^+(2, \mathbb{R})$, the map $H_1(X, \mathbb{Q})^- \ni c \mapsto \omega(c) \in K(i)$ is an isomorphism of $\mathbb{Q}$-vector spaces. Thus $\omega(c) = 0$ implies $c = 0$ in $H_1(X, \mathbb{Z})^-$.

For the second condition, let σ_1, σ_2 be a pair of twin saddle connections which are both invariant by the Prym involution τ. If $c = \sigma_1 * (-\sigma_2) \in H_1(X, \mathbb{Z})^-$ is separating then by cutting X along σ_1, σ_2 and regluing the segments of the boundary of the two components, we get a pair of translation surfaces each of which has a unique singularity with cone angle 4π. They thus belong to the stratum $\mathcal{H}(1)$. Since this stratum is empty, we get a contradiction. Therefore, c must be non-separating *i.e.* $c \neq 0 \in H_1(X, \mathbb{Z})^-$. One has $\omega(c) = \omega(\sigma_1) - \omega(\sigma_2) = 0$, hence the first condition applies and D is a square.

For the last condition, we set $c_j = \sigma_0 * (-\sigma_j)$, $j = 1, 2$. Remark that we have $\tau(c_1) = -c_2$ and $c_0 = c_2 - c_1$ in $H_1(X,\mathbb{Z})$. Since c_0 is non-separating by assumption, it is a primitive element of $H_1(X,\mathbb{Z})$. Observe that if one of the curves c_1 or c_2 is separating then the other is also separating (as $\tau(c_1) = -c_2$) and in this case $c_0 = c_1 - c_2 = 0 \in H_1(X,\mathbb{Z})$ contradicting the assumption. Hence both c_1, c_2 are non-separating. Let $c = c_1 + c_2$. We have $\tau(c) = -c$ which means that $c \in H_1(X,\mathbb{Z})^-$. If $c = 0 \in H_1(X,\mathbb{Z})$ then $c_2 = -c_1$ *i.e.* $c_0 = c_1 - c_2 = 2c_1$: contradiction with the primitivity of $c_0 \in H_1(X,\mathbb{Z})$. Thus $c \neq 0 \in H_1(X,\mathbb{Z})^-$. Since $\sigma_0, \sigma_1, \sigma_2$ are twin saddle connections, we have

$$\omega(c) = \omega(c_1) + \omega(c_2) = 2\omega(\sigma_0) - \omega(\sigma_1) - \omega(\sigma_2) = 0.$$

Again the first condition applies and D is a square. □

Proof of Theorem 8.1. Let P, Q be the zeros of ω. We denote by $\{\sigma_i,\ i \in I\}$ the set of horizontal saddle connections on X connecting P to Q. Recall that we always define the orientation of such a saddle connection to be from P to Q, it is said to be *positively oriented* if the orientation is from the left to the right, otherwise it is said to be *negatively oriented*. The corresponding holonomy vectors are $\{(s_i, 0) = \omega(\sigma_i) \in \mathbb{R}^2,\ i \in I\}$. For every $i \in I$, σ_i is contained on the lower boundary of a unique cylinder. If σ_i is positively oriented (namely $s_i > 0$) then there exists σ_j in the same lower boundary component as σ_i which is negatively oriented. In particular, all the numbers $\{s_i\}$ cannot have the same sign.

Let us define

$$s_{\min} = -\min\{s_i,\ s_i > 0\} \text{ and } s_{\max} = -\max\{s_i,\ s_i < 0\}.$$

If $(Y, \eta) = (X, \omega) + (x, 0)$ then by construction $\eta(\sigma_i) = (s_i + x, 0)$ and the surface (Y, η) can be constructed from the same cylinders as (X, ω). For all $x \in]s_{\min}, s_{\max}[$, $(X, \omega) + (x, 0)$ is a well-defined surface in $\Omega E_D(2, 2)^{\text{odd}}$ since $s_i + x \neq 0$, proving the first statement.

We now prove the second assertion. Let us analyze the case when x tends to $s_{\min}$ (the case x tends to $s_{\max}$ being similar). Letting $C_{\min} = \{\sigma_i,\ s_i = -s_{\min}\}$ and $C_{\max} = \{\sigma_i,\ s_i = -s_{\max}\}$ (necessarily $|C_{\min}| \leq 3$, and $|C_{\max}| \leq 3$). When $x \to s_{\min}$, only the saddle connections of $C_{\min}$ can collapse to a point. We thus have three cases, parameterized by the number of elements of $C_{\min}$.

(1) $C_{\min} = \{\sigma_{i_0}\}$: the unique saddle connection σ_{i_0} is invariant by τ and $(X, \omega) + (x, 0)$ converges to a surface in $\Omega E_D(4)$.

(2) $C_{\min} = \{\sigma_{i_1}, \sigma_{i_2}\}$: σ_{i_1} and σ_{i_2} are exchanged by τ (otherwise the closed curve $c = \sigma_{i_1} * (-\sigma_{i_2}) \in H_1(X,\mathbb{Z})^-$ represents a non zero element and. Since $\omega(c) = 0$, Lemma 8.2 implies that D is a square). By Theorem 7.6, $(X, \omega) + (x, 0)$ converges to a surface in $\Omega E_{D'}(2)^*$, for some $D' \in \{D, D/4\}$.

(3) $C_{\min} = \{i_0, i_1, i_2\}$: if there are two saddle connections in $\{\sigma_{i_0}, \sigma_{i_1}, \sigma_{i_2}\}$ that are invariant by τ then D must be square (see Lemma 8.2). Hence one can assume that τ preserves σ_{i_0} while it exchanges σ_{i_1} and σ_{i_2}. If the closed curve $c_0 = \sigma_{i_1} * (-\sigma_{i_2})$ is non-separating then D must be a square (again by Lemma 8.2). Thus c_0 is separating and $\{\sigma_{i_0}, \sigma_{i_1}, \sigma_{i_2}\}$ are homologous saddle connections. We only need to show that X decomposes into three tori. Indeed, as x tends to $s_{\min}$ the length of these saddle connections tends to zero, and the limit surface is an element of $\Omega E_D(0, 0, 0)$.

Hence, in view of the above discussion, in order to finish the proof of the theorem, we need to show that, in case (3), the complement of $\sigma_{i_0} \cup \sigma_{i_1} \cup \sigma_{i_2}$ has three connected components, each of which is a one-holed torus.

We begin by observing that $\sigma_{i_1}, \sigma_{i_2}$ determine a pair of angles $(2\pi, 4\pi)$ at P and Q. Since τ exchanges P and Q and preserves the orientation of X, a careful look at the geodesic rays emanating from P and Q shows that the angles 2π at P and the angle 2π at Q belong to the same side of c_0. Cut X along c_0, then glue the two segments in each boundary components together, we then obtain two closed translation surfaces. From the observation above, one of the new surfaces has no singularities, hence it must be a flat torus that will be denoted by (X', ω'). The remaining surface is then a surface (X'', ω'') in $\mathcal{H}(1,1)$.

We have in X' a marked geodesic segment σ' which is the identification of σ_1 and σ_2, we denote the endpoints of this segment by P' and Q' such that P' (resp. Q') corresponds to P (resp. to Q). For (X'', ω''), we denote the zeros of ω'' by P'' and Q'' such that P'' (resp. Q'') corresponds to P (resp. to Q). In X'' we have a pair of twin saddle connections σ_0 and σ'', where σ'' is the identification of σ_1 and σ_2.

The involution τ induces an involution τ' on X' and an involution τ'' on X''. We can consider τ' and τ'' as the restrictions of τ in X' and X'' respectively. Note that τ' exchanges P' and Q' and satisfies $\tau'(\omega') = -\omega'$. Since X' is an elliptic curve, there exists only one such involution. We deduce in particular that τ' has four fixed points in X', one of which is the midpoint of σ', the other three are the fixed points of τ.

Recall that τ has four fixed points in X. Therefore, τ'' has exactly two fixed points, one of which is the midpoint of σ_0 by assumption (recall that σ_0 is invariant by τ), and the other one is the midpoint of σ''. Let ι denote the hyperelliptic involution of X''. Remark that ι has six fixed points. From the observations above, we can conclude that $\tau'' \neq \iota$.

We now claim that $\iota(\sigma_0) = -\sigma''$. Indeed, since ι is in the center of the group $\mathrm{Aut}(X'')$, we have $\iota \circ \tau'' = \tau'' \circ \iota$. Therefore ι preserves the set of fixed points of τ''. If ι fixes the midpoint of σ_0, then it follows that $\iota \circ \tau'' = \mathrm{Id}$, since both ι and τ'' are involutions. Hence $\tau'' = \iota$, and we have a contradiction. Therefore, ι must send the midpoint of σ_0 to the midpoint of σ''. Remark that $\iota^*\omega'' = -\omega''$, which means that ι is an isometry of (X'', ω''). Thus ι maps σ_0 to another saddle connection such that $\omega''(\iota(\sigma_0)) = -\omega''(\sigma_0)$. Since ι exchanges the zeros of ω'', we conclude that $\iota(\sigma_0) = -\sigma''$.

Now, the element in $H_1(X'', \mathbb{Z})$ represented by the closed curve $\sigma_0 \cup \sigma''$ is preserved by ι, which implies that this curve is separating. Cut X'' along $\sigma_0 \cup \sigma''$, then glue the segments in the boundary of each component together, we then get two flat tori (X''_1, ω''_1) and (X''_2, ω''_2) which are exchanged by τ''. This finishes the proof of Theorem 8.1. □

9. Cylinder decomposition of surfaces near $\Omega E_D(4)$ and $\Omega E_D(2)^*$

Let (X_0, ω_0) be a surface in $\Omega E_D(4)$, and $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map in Proposition 7.3.

Proposition 9.1. *Assume that the horizontal direction is completely periodic for (X_0, ω_0). Then there exists $0 < \varepsilon_1 < \varepsilon$ such that for every $(X, \omega) \in \Psi(\mathring{D}(\varepsilon_1))$, the horizontal direction is also completely periodic. Set $R_{(k,5)}(\varepsilon_1) = \{\varrho e^{k\iota\frac{\pi}{5}}, 0 < \varrho < \varepsilon_1\}$, for $k = 0, \dots, 9$, and $\mathring{D}_{(k,5)}(\varepsilon_1) = \{\varrho e^{\iota\theta}, 0 < \varrho < \varepsilon_1, (k-1)\pi/5 < \theta < k\pi/5\}$, for $k = 1, \dots, 10$. Then*

(1) *The cylinder decompositions in the horizontal direction of all surfaces in $\Psi(R_{(k,5)}(\varepsilon_1))$ are* **unstable** *and have the same combinatorial data.*

(2) *The cylinder decompositions in the horizontal direction of all surfaces in* $\Psi(\mathring{D}_{(k,5)}(\varepsilon_1))$ *are* **stable** *and have the same combinatorial data.*

Proof. This proposition follows from similar arguments as Proposition 4.9. Let C_i, $i = 1,\dots,n$, denote the horizontal cylinders of X_0, and γ_i denote the simple closed geodesic in C_i whose distances to the two boundary components of C_i are equal. Choose ε_1 satisfying $0 < \varepsilon_1 < \min\{\varepsilon, 1\}$ small enough so that $D(P_0, \varepsilon_1) = \{x \in X_0,\ \mathbf{d}(x, P_0) < \varepsilon_1\}$, where P_0 is the unique zero of ω_0, is an embedded disk disjoint from the curves γ_i. Note that by the choice of ε_1, we have $\varepsilon_1^5 < \varepsilon_1 < \varepsilon$.

By definition, the surface $\Psi(\varrho e^{\imath\theta})$ has a small saddle connection (of length ϱ^5) in direction 5θ. It follows immediately that the horizontal direction is periodic for the surfaces in $\Psi(R_{(k,5)}(\varepsilon_1))$. Since we have a horizontal saddle connection with distinct endpoints, the corresponding cylinder decomposition is unstable. Clearly, the combinatorial data of the decomposition of $\Psi(z)$ does not change as z varies in $R_{(k,5)}(\varepsilon_1)$ (see Lemma 4.6).

Let us now consider a surface $(X, \omega) = \Psi(z)$, where $z \in \mathring{D}_{(k,5)}(\varepsilon_1)$. We will assume in addition that $z^5 = 2\imath h$ with $0 < h < \varepsilon_1/2$, the general case then follows from Lemma 4.6. Recall that $D(P_0, \varepsilon_1)$ is the union of 10 half-disks $D_{i\pm}$, with $i = 1,\dots,5$, where D_{i+} is a copy of $\{z \in \mathbb{C},\ |z| \leq \varepsilon_1,\ \mathrm{Re}(z) \geq 0\}$ and D_{i-} is a copy of $\{z \in \mathbb{C},\ |z| \leq \varepsilon_1,\ \mathrm{Re}(z) \leq 0\}$. Let $a_{i\pm}, b_{i\pm}, c_{i\pm}$ denote the points in the border of $D_{i\pm}$ that correspond to $-\imath\varepsilon_1, \imath\varepsilon_1, 0$ respectively.

Since the horizontal direction is periodic for (X_0, ω_0), we have a bijection $\pi : \{c_{i+},\ i = 1,\dots,5\} \to \{c_{i-},\ i = 1,\dots,5\}$. The gluing rules then give rise to a permutation σ_a of $\{a_{i+},\ i = 1,\dots,5\}$ and a permutation σ_b of $\{b_{i+},\ i = 1,\dots,5\}$ (see Section 4.3). Now, the surface $(X, \omega) = \Psi(z)$ is obtained from (X_0, ω_0) by replacing the disk $D(P_0, \varepsilon_1)$ by another disk $\tilde{D}(\varepsilon_1)$ constructed from the half-disks $D_{i\pm}$ with a choice of $j \in \{1,\dots,5\}$ and the following gluing rules (see Figure 7 for the case $j = 2$), here we use the convention $i \sim (i-5)$ if $i > 5$,

- D_{i+} is glued to D_{i-} along the segment $\{\mathrm{Re}(z) = 0,\ h \leq \mathrm{Im}(z) < \varepsilon_1\}$ for $i \in \{j, j+1, j+2\}$.
- D_{i+} is glued to D_{i-} along the segment $\{\mathrm{Re}(z) = 0,\ -h \leq \mathrm{Im}(z) < \varepsilon_1\}$ for $i \notin \{j, j+1, j+2\}$.
- D_{i-} is glued to $D_{(i+1)+}$ along the segment $\{\mathrm{Re}(z) = 0,\ -\varepsilon_1 < \mathrm{Im}(z) \leq h\}$ for $i \in \{j, j+1\}$.
- D_{i-} is glued to $D_{(i+1)+}$ along the segment $\{\mathrm{Re}(z) = 0,\ -\varepsilon_1 < \mathrm{Im}(z) \leq -h\}$ for $i \notin \{j, j+1\}$.
- D_{j+} is glued to $D_{(j+2)-}$ along the segment $\{\mathrm{Re}(z) = 0, -h \leq \mathrm{Im}(z) \leq h\}$.

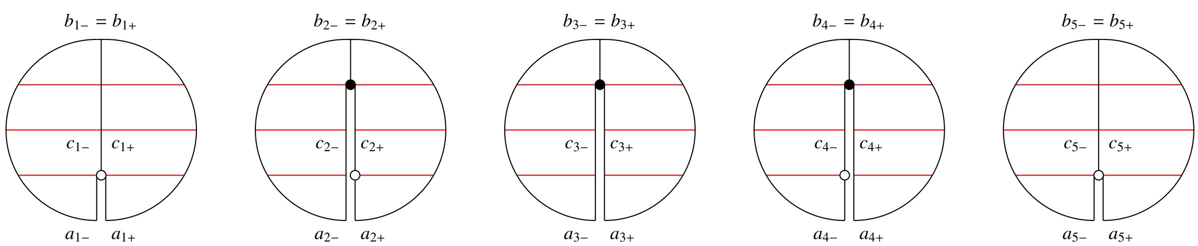

FIGURE 7. Splitting a zero of order 4 to two zeros of order 2 ($j = 2$).

Let P (resp. Q) denote the zero of ω which corresponds to the point $-\imath h \in D_{j+}$ (resp. $\imath h \in D_{j+}$). From the gluing rules, any horizontal geodesic ray emanating from P (reps. Q) ends up at P (resp. Q).

Moreover, those horizontal saddle connections are encoded in the permutations σ_a and σ_b. It follows that (X,ω) admit a stable cylinder decomposition in the horizontal direction.

By the choice of ε_1, (X,ω) has n cylinders associated to the geodesics γ_i, $i=1,\dots,n$, and some additional cylinders which contain some of the points $c_{i\pm}$. The cylinders associated to γ_i are in bijection with the cycles of σ_a and σ_b. For the additional ones, we remark that the gluing rules imply the following identifications:

- c_{i-} is identified with c_{i+} if $i \notin \{j, j+1, j+2\}$,
- c_{i-} is identified with $c_{(i+1)+}$ if $i \in \{j, j+1\}$,
- $c_{(j+2)-}$ is identified with c_{j+}.

Composing these identifications with π, we get a permutation σ_c of the set $\{c_{i+},\ i=1,\dots,5\}$. Clearly, the horizontal cylinders containing some of the points $c_{i\pm}$ are in bijection with the cycles of σ_c. We derive that the permutations $\sigma_a,\sigma_b,\sigma_c$ completely determine the combinatorial data of the cylinder decomposition of (X,ω), hence these combinatorial data depend only on the sector $\mathring{D}_{k,5}(\varepsilon_1)$. The proposition is then proved. □

Remark 9.2. *In general, the topological model of the decomposition of* (X,ω) *changes if we change the sector* $\mathring{D}_{(k,5)}(\varepsilon_1)$.

By a saddle connection on $(X_0,\omega_0,W_0) \in \Omega E_{D'}(2)^*$, we refer to a geodesic segment whose endpoints are in the set $\{P_0, W_0\}$. We consider, by convention, a cylinder in (X_0,ω_0,W_0) as the union of all simple closed geodesics in the same free homotopy class in $X_0 \setminus \{P_0, W_0\}$. Obviously, a direction θ is periodic for (X_0,ω_0,W_0) if and only if it is periodic for (X_0,ω_0), but the associated cylinder decomposition of (X_0,ω_0,W_0) may have one more cylinder than the one of (X_0,ω_0), since a simple closed geodesic passing through W_0 will cut the corresponding cylinder in (X_0,ω_0) into two cylinders in (X_0,ω_0,W_0). The following proposition follows from completely similar arguments as Proposition 9.1.

Proposition 9.3. *Let* (X_0,ω_0,W_0) *be a surface in* $\Omega E_{D'}(2)^*$. *Assume that the horizontal direction is periodic for* (X_0,ω_0,W_0). *Let* $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ *be the map defined in Proposition 7.5. Then there exists* $0<\varepsilon_1<\varepsilon$ *such that for all* $(X,\omega) \in \Psi(\mathring{D}(\varepsilon_1))$, *the horizontal direction is also periodic. Set* $R_{(k,3)}(\varepsilon_1) = \{\varrho e^{k\imath\frac{\pi}{3}}, 0<\varrho<\varepsilon_1\}$, $k=0,\dots,5$, *and* $\mathring{D}_{(k,3)}(\varepsilon_1) = \{\varrho e^{\imath\theta},\ 0<\varrho<\varepsilon_1,\ (k-1)\pi/3 < \theta < k\pi/3\}, k=1,\dots,6$. *We have*

1. *The associated cylinder decomposition of surfaces in* $\Psi(R_{(k,3)}(\varepsilon_1))$ *are unstable and have the same combinatorial data.*
2. *The associated cylinder decomposition of surfaces in* $\Psi(\mathring{D}_{(k,3)}(\varepsilon_1))$ *are stable and have the same combinatorial data.*

Having Propositions 9.1 and 9.3 proved, using the arguments in Section 6 we get

Theorem 9.4. *Let* (X_0,ω_0) *(resp.* (X,ω_0,W_0)*) be a surface in* $\Omega E_D(4)$ *(resp. in* $\Omega E_D(2)^*$*) which is horizontally periodic, and* $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ *be the map defined in Proposition 7.3 (resp. in Proposition 7.5). Let* $\{z_n\}_{n\in\mathbb{N}}$ *be a sequence of complex numbers in a fixed sector* $\mathring{D}_{(k,n)}(\varepsilon_1)$, *where* ε_1 *is the constant in Propositions 9.1 (resp. Proposition 9.3), such that* $z_n \overset{n\to\infty}{\longrightarrow} 0$. *Assume that for all* $n\in\mathbb{N}$, *the horizontal direction is parabolic for the surface* $(X_n,\omega_n) = \Psi(z_n)$. *Then the set*

$$\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathrm{GL}^+(2,\mathbb{R}) \cdot (X_n,\omega_n)$$

is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Sketch of proof. Since the arguments for the two cases are the same, we will only consider the case $(X_0,\omega_0)\in\Omega E_D(4)$. Recall that by definition, all the surfaces in $\Psi(\mathring{D}(\varepsilon))$ belong to the same leaf of the kernel foliation. Set $\mathring{\overline{D}}_{(k,n)}(\varepsilon_1)=\{z=\varrho e^{\iota\theta}\in\mathbb{C},\ 0<\varrho<\varepsilon_1,\ (k-1)\pi/5\le\theta\le k\pi/5\}$. By a slight abuse of notations, if $(X,\omega)=\Psi(z)$, with $z\in\mathring{\overline{D}}_{(k,n)}(\varepsilon_1)$, then we will write $(X,\omega)=(X_0,\omega_0)+z^5$. Using this convention, given z_1,z_2 in $\mathring{\overline{D}}_{(k,n)}(\varepsilon_1)$, we have

$$(X_0,\omega_0)+z_2^5=((X_0,\omega_0)+z_1^5)+(z_2^5-z_1^5),$$

where the expression in the right hand side corresponds to a move in a leaf of the kernel foliation in $\Omega E_D(2,2)^{\mathrm{odd}}$.

By assumption, we can write $(X_n,\omega_n)=(X_0,\omega_0)+(s_n,t_n)$, with $(s_n,t_n)\overset{n\to\infty}{\longrightarrow}(0,0)$, $t_n\neq 0$, and (X_n,ω_n) admits a parabolic cylinder decomposition in the horizontal direction. By Proposition 9.1, we know that the topological data and the widths of the cylinders in this decomposition are the same for all n. Thus, the arguments in Section 6 allows us to conclude that $(X_0,\omega_0)+(x,0)\in\overline{\mathcal{O}}$, for all $x\in(-\varepsilon_1^5,\varepsilon_1^5)$.

Pick a point $x\in(-\varepsilon_1^5,\varepsilon_1^5)\setminus\{0\}$, and set $(X,\omega)=(X_0,\omega_0)+(x,0)$, we see that there exists $\varepsilon_0>0$ such that $(X,\omega)+(s,0)\in\overline{\mathcal{O}}$ for all $s\in(-\varepsilon_0,\varepsilon_0)$. Corollary 5.3 then allows us to conclude that $(X,\omega)+v\in\overline{\mathcal{O}}$ for any $v\in\mathbb{R}^2$, with v small enough. We can then choose v such that $(X,\omega)+v\in\Psi(\mathring{D}_{(k,n)}(\varepsilon_1))$ and the horizontal direction is not parabolic for $(X,\omega)+v$. The theorem then follows from Theorem 5.1. □

10. The set of Veech surfaces is not dense

In this section we will prove the following theorem:

Theorem 10.1. *If D is not a square then for any connected component $\mathscr{C}$ of $\Omega E_D(2,2)^{\mathrm{odd}}$, there exists an open subset $\mathcal{U}\subset\mathscr{C}$ which contains no Veech surfaces.*

10.1. **Cylinder decomposition and prototypes.** We first prove the following lemma, which says that if we have a three tori decomposition such that the direction of the slits is periodic, then up to $\mathrm{GL}^+(2,\mathbb{R})$, the surface belongs to the real kernel foliation leaf of some "prototypical surface" in a finite family.

Lemma 10.2. *Let $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$ be an eigenform with a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ so that (X,ω) admits a three tori decomposition into tori (X_j,ω_j), $j=0,1,2$. Assume that (X,ω) is periodic in the direction of σ_0. Let $(\widetilde{a}_j,\widetilde{b}_j)$ be a basis of $H_1(X_j,\mathbb{Z})$ with $\widetilde{a}_j$ parallel to σ_j, and $\tau(\widetilde{a}_1)=-\widetilde{a}_2$, $\tau(\widetilde{b}_1)=-\widetilde{b}_2$, where τ is the Prym involution. Then there exists a tuple $(w,h,t,e)\in\mathbb{Z}^4$ satisfying*

$$(\mathcal{P}_D(0,0,0))\ \begin{cases} w>0,h>0,0\le t<\gcd(w,h),\gcd(w,h,t,e)=1,\\ D=e^2+8wh\end{cases}$$

such that up to the action of $\mathrm{GL}^+(2,\mathbb{R})$ and Dehn twists, we have

$$\begin{aligned}\omega(\mathbb{Z}\widetilde{a_0}\oplus\mathbb{Z}\widetilde{b_0}) &= \lambda\cdot\mathbb{Z}^2,\\ \omega(\mathbb{Z}\widetilde{a_j}\oplus\mathbb{Z}\widetilde{b_j}) &= \mathbb{Z}(w,0)\oplus\mathbb{Z}(t,h) \qquad \text{for } j=1,2,\end{aligned}$$

where $\lambda\in\mathbb{Q}(\sqrt{D})$ is the unique positive root of the equation $\lambda^2-e\lambda-2wh=0$.

Proof. We include a sketch of this result (compare with [LN13, Proposition 4.2]). Set $\widetilde{a} = \widetilde{a_1} + \widetilde{a_2}$ and $\widetilde{b} = \widetilde{b_1} + \widetilde{b_2}$. We have $(\widetilde{a_0}, \widetilde{b_0}, \widetilde{a}, \widetilde{b})$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$. The restriction of the intersection form is given by the matrix $\left(\begin{smallmatrix} J & 0 \\ 0 & 2J \end{smallmatrix}\right)$.

Since $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$, let us denote by T a generator of the order $\mathcal{O}_D$. In the above coordinates, since T is self-adjoint, T has the following form (up to replacing T by $T - f\cdot \mathrm{Id}$)

$$T = \begin{pmatrix} e & 0 & 2w & 2t \\ 0 & e & 2c & 2h \\ h & -t & 0 & 0 \\ -c & w & 0 & 0 \end{pmatrix},$$

for some $(w,h,t,e,c) \in \mathbb{Z}^5$. Since ω is an eigenform, we have $T^*\omega = \lambda\cdot\omega$ for some λ (that can be chosen to be positive by changing T to $-T$). Now up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, one can always assume that $\omega(\mathbb{Z}\widetilde{a_0} \oplus \mathbb{Z}\widetilde{b_0}) = \lambda\cdot\mathbb{Z}^2$. Now in our coordinates, $\mathrm{Re}(\omega) = (\lambda, 0, x, y)$ and $\mathrm{Im}(\omega) = (0,\lambda,0,z)$, for some $x,y,z > 0$. Reporting into the equation $T^*\omega = \lambda\cdot\omega$, we draw $x = 2w$, $y = 2t$, $z = 2h$ and $c = 0$. Since T satisfies the quadratic equation $T^2 - eT - 2wh\mathrm{Id} = 0$, we get $D = e^2 + 8wh$. We can renormalize further using Dehn twists so that $0 \leq t < \gcd(w,h)$. Finally properness of $\mathcal{O}_D$ implies $\gcd(w,h,t,e) = 1$. All the conditions of $\mathcal{P}_D(0,0,0)$ are now fulfilled and the lemma is proved. □

Definition 10.3. *For each D, let* $\mathcal{P}_D(0,0,0)$ *denote the set*

$$\{(w,h,t,e) \in \mathbb{Z}^4,\ (w,h,t,e) \textit{ satisfies } (\mathcal{P}_D(0,0,0))\}.$$

We call an element of $\mathcal{P}_D(0,0,0)$ *a* prototype. *The set of prototypes is clearly finite.*

10.2. **Switching decompositions.** Let (X,ω) be a surface in $\Omega E_D(2,2)^{\mathrm{odd}}$ which admits a three-tori decomposition by a triple of saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$. We also assume that the direction of σ_j is periodic. Let (X_j,ω_j) and $(\widetilde{a}_j, \widetilde{b}_j)$ be as in Lemma 10.2. We wish now to investigate the situation where X admits other three-tori decompositions.

By Proposition 7.2, for any primitive element $b_0 \in H_0(X_0,\mathbb{Z})$, there exists a unique primitive element $b_j \in H_1(X_j,\mathbb{Z})$, $j = 1,2$ such that

$$\omega(b_j) = \frac{2\beta_j}{\lambda}\omega(b_0)$$

with $\beta_j \in \mathbb{N}$. This is because $L(X_j,\omega_j)$ is a sublattice of $\frac{2}{\lambda}L(X_0,\omega_0)$ ($L(X_j,\omega_j)$ is the lattice associated to (X_j,ω_j), see Proposition 7.2), hence it contains a vector parallel to $2/\lambda\omega_0(b_0)$. We call b_j the *shadow* of b_0 in X_j.

The following lemma provides us with a sufficient condition of the existence of many other three-tori decompositions. Its proof is inspired from [McM05b, Theorem 5.3].

Lemma 10.4. *Let* b_0 *be a primitive element of* $H_1(X_0,\mathbb{Z}) \setminus \{\pm\widetilde{a_0}\}$ *and let* b_j *be the shadows of* b_0 *in* X_j, $j = 1,2$. *Set* $c = b_0 + b_1 + b_2$. *Then there exists* $s_0 > 0$ *such that if the ratio* $s = |\sigma_0|/|\widetilde{a_0}|$ *is smaller than* s_0, *then the surface* (X,ω) *admits a three-tori decomposition by a triple of saddle connections* $\{\delta_0,\delta_1,\delta_2\}$ *such that* $\delta_j * (-\sigma_j) = c$.

Proof. For $v_1 = (x_1,y_1), v_2 = (x_2,y_2)$ in $\mathbb{R}^2$, let us define $v_1 \wedge v_2 = \det\left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right)$. By assumption, we have $b_0 \notin \mathbb{Z}\widetilde{a_0}$, hence $|\omega(b_0)\wedge\omega(\widetilde{a_0})| > 0$. Since $\omega(b_j)$ is parallel to $\omega(b_0)$, and $\omega(\widetilde{a_j})$ is parallel to $\omega(\widetilde{a_0})$, we also have $|\omega(b_j)\wedge\omega(\widetilde{a_j})| > 0$.

Choose s_0 small enough so that if $0 < s < s_0$, then $0 < s|\omega(b_j)\wedge\omega(\widetilde{a_j})| < \mathbf{Area}(X_j)$. Assume that $|\sigma_j| < s_0|\widetilde{a_j}|$ for $j = 0,1,2$. Note that $|\sigma_0| = |\sigma_1| = |\sigma_2|$, and $|\widetilde{a_1}| = |\widetilde{a_2}| = w/\lambda|\widetilde{a_0}|$.

Let $\hat{\sigma}_j$ be the marked geodesic segment corresponding to $\{\sigma_0, \sigma_1, \sigma_2\}$ in the torus X_j, and let γ_j be a simple closed geodesic representing the homology class $b_j \in H_1(X_j, \mathbb{Z})$. By assumption, we have $0 < |\omega(\gamma_j) \wedge \omega(\hat{\sigma}_j)| < \mathbf{Area}(X_j)$, hence γ_j intersects $\hat{\sigma}_j$ at at most one point. Thus the union of all the geodesics representing b_j which intersect $\hat{\sigma}_j$ is an embedded cylinder $\hat{C}_j$ in X_j.

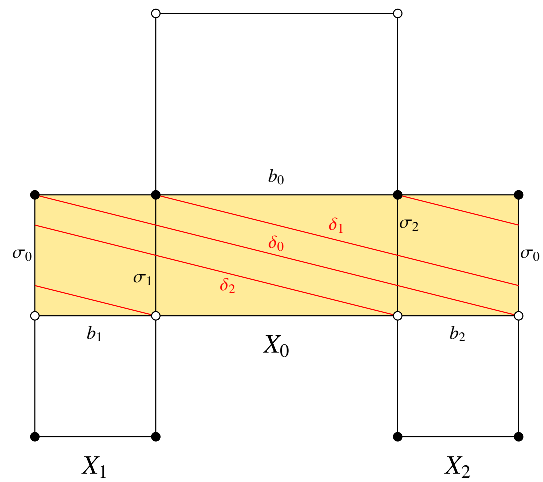

FIGURE 8. Switching three-tori decomposition.

Recall that (X, ω) is obtained from X_0, X_1, X_2 by slitting and regluing along $\hat{\sigma}_j$. As a consequence, we see that the union of the cylinders $\hat{C}_j$, $j = 0, 1, 2$, is an embedded cylinder C whose core curves represent the homology class $c = b_0 + b_1 + b_2$. Let δ_j be the image of σ_j under a Dehn twist in C, then $\{\delta_j,\ j = 0, 1, 2\}$ is also a triple of homologous saddle connections which decompose X into three tori (see Figure 8). By definition, we have $\delta_j * (-\sigma_j) = c$, and the lemma follows. □

Using the same notations as in Lemma 10.4. Let (X'_j, ω'_j), $j = 0, 1, 2$, denote the tori in the decomposition specified by $\{\delta_0, \delta_1, \delta_2\}$ (X'_0 is the torus which is fixed by τ). We regard X_j and X'_j as subsurfaces of X. The following elementary lemma provides us with an explicit basis of $H_1(X'_0, \mathbb{Z})$, its proof is left to the reader.

Lemma 10.5. *Let a_0 be a primitive element of $H_1(X_0, \mathbb{Z})$ such that (a_0, b_0) is a basis of $H_1(X_0, \mathbb{Z})$. Then we have $H_1(X'_0, \mathbb{Z}) = \mathbb{Z} \cdot (a_0 + c) + \mathbb{Z} \cdot b_0$.*

Next, we have

Lemma 10.6. *Let (X, ω) be a surface in $\Omega E_D(2, 2)^{\mathrm{odd}}$ satisfying the hypothesis of Lemma 10.4. Let a_0 be a primitive element of $H_1(X_0, \mathbb{Z})$ such that (a_0, b_0) is a basis of $H_1(X_0, \mathbb{Z})$. There exists $(p, q) \in \mathbb{Z}^2$ such that $\widetilde{a}_0 = pa_0 + qb_0$. Set $\beta = 2\beta_1 + 2\beta_2 = 4\beta_1 \in \mathbb{Z}$, where $\omega(b_j) = (2\beta_j/\lambda)\omega(b_0)$. If the direction of δ_0 is completely periodic, then we have*

$$s = \frac{\lambda + \beta}{(rp + p - q)\lambda + p\beta} \tag{14}$$

with $r \in \mathbb{Q}$.

Proof. We know that the saddle connections $\{\delta_0, \delta_1, \delta_2\}$ decompose X into three tori X'_0, X'_1, X'_2, where X'_0 is preserved by τ. By Lemma 10.5 we have $H_1(X'_0, \mathbb{Z}) = \mathbb{Z}\cdot(a_0+b_0+b_1+b_2)+\mathbb{Z}\cdot b_0$. Set $A = \omega(a_0+b_0+b_1+b_2), B = \omega(b_0)$, then we have $L(X'_0) = \mathbb{Z}A + \mathbb{Z}B$, where $L(X'_0)$ is the lattice associated to X'_0. Set $v = \omega(\sigma_0)$, $w = \omega(\delta_0)$. We have

$$A = \omega(a_0) + \omega(b_0) + \frac{\beta}{\lambda}\omega(b_0) = \omega(a_0) + (1+\frac{\beta}{\lambda})B.$$

Thus

$$\omega(a_0) = A - (1+\frac{\beta}{\lambda})B.$$

Since $\widetilde{a}_0 = pa_0 + qb_0$, we have

$$v = s\omega(\widetilde{a}_0) = s(p\omega(a_0)+q\omega(b_0)) = s(p(A-(1+\frac{\beta}{\lambda})B)+qB) = s(pA+(q-p(1+\frac{\beta}{\lambda}))B).$$

Now

$$\begin{aligned} w &= v + \omega(b_0+b_1+b_2) \\ &= spA + s(q-p(1+\frac{\beta}{\lambda}))B + (1+\frac{\beta}{\lambda})B \\ &= spA + (sq + (1-sp)(1+\frac{\beta}{\lambda}))B. \end{aligned}$$

The direction of δ_0 is periodic if and only if w is parallel to a vector in the lattice $\mathbb{Z}A+\mathbb{Z}B$, which is equivalent to

$$r = \frac{sq+(1-sp)(1+\frac{\beta}{\lambda})}{sp} = \frac{sq\lambda + (1-sp)(\lambda+\beta)}{sp\lambda} \in \mathbb{Q}.$$

It follows

$$srp\lambda = sq\lambda + (\lambda+\beta) - sp(\lambda+\beta),$$

or equivalently

$$s = \frac{\lambda+\beta}{rp\lambda - q\lambda + p(\lambda+\beta)} = \frac{\lambda+\beta}{(rp+p-q)\lambda + p\beta}.$$

□

We can now prove

Proposition 10.7. *Let (X,ω) be a surface in $\Omega E_D(2,2)^{\mathrm{odd}}$, where D is not a square. Assume that there exists a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ which decompose (X,ω) into three tori, and the direction of σ_j is periodic. Set $s = \dfrac{|\sigma_0|}{|\widetilde{a}_0|}$, where $\widetilde{a}_0$ is a simple closed geodesic parallel to σ_0 in the torus which is preserved by the involution. Then there exists a constant $s_0 > 0$ depending only on D such that if $s < s_0$ then (X,ω) is not a Veech surface.*

Proof. Let $(\widetilde{a}_j, \widetilde{b}_j)$, $j = 0,1,2$, be as in Lemma 10.2. Let (e,w,h,t) be the prototype in $\mathcal{P}_D(0,0,0)$ which is associated to the cylinder decomposition in the direction of σ_0. Set $(a_0,b_0) = (\widetilde{a}_0, \widetilde{b}_0)$, and $(a'_0, b'_0) = (\widetilde{a}_0+\widetilde{b}_0, \widetilde{a}_0+2\widetilde{b}_0)$. Let b_j (resp. b'_j) be the shadow of b_0 (resp. b'_0) in X_j, $j=1,2$. We have

$$\omega(b_1+b_2) = \frac{\beta}{\lambda}\omega(b_0),\ \omega(b'_1+b'_2) = \frac{\beta'}{\lambda}\omega(b'_0),$$

where $\beta,\beta' \in \mathbb{N}$ are determined by the prototype (e,w,h,t). From Lemma 10.4, there exists $s_1 > 0$ such that if $s < s_1$, then (X,ω) admits three-tori decompositions by the triples of saddle connections $\{\delta_j,\ j=0,1,2\}$ and $\{\delta'_j,\ j=0,1,2\}$, where δ_0 and δ'_0 satisfy

$$\delta_0 * (-\sigma_0) = b_0 + b_1 + b_2 \in H_1(X,\mathbb{Z}), \text{ and } \delta'_0 * (-\sigma_0) = b'_0 + b'_1 + b'_2 \in H_1(X,\mathbb{Z}).$$

By definition, we have $\widetilde{a}_0 = a_0 = 2a'_0 - b'_0$. Assume that (X,ω) is a Veech surface, then the directions of δ and δ' must be periodic, hence, from Lemma 10.6, we have

$$s = \frac{\lambda+\beta}{(r+1)\lambda+\beta} = \frac{\lambda+\beta'}{(2r'+3)\lambda+2\beta'} \tag{15}$$

with $r,r' \in \mathbb{Q}$. Set $R = r+1, R' = 2r'+3$, we see that the equation (15) is equivalent to

$$R'\lambda^2 + (R'\beta+2\beta')\lambda + 2\beta\beta' = R\lambda^2 + (R\beta'+\beta)\lambda + \beta\beta'$$

Using $\lambda^2 = e\lambda + 2wh$, we get

$$\begin{array}{rcl} R'(e\lambda+2wh) + (R'\beta+2\beta')\lambda + 2\beta\beta' & = & R(e\lambda+2wh) + (\beta+R\beta')\lambda + \beta\beta' \\ \Leftrightarrow (R'e+R'\beta+2\beta')\lambda + (2whR'+2\beta\beta') & = & (Re+\beta+R\beta')\lambda + (2whR+\beta\beta') \end{array}$$

It follows

$$\left\{ \begin{array}{l} R'(e+\beta) + 2\beta' = R(e+\beta') + \beta \\ 2whR' + 2\beta\beta' = 2whR + \beta\beta' \end{array} \right.$$

or

$$\left\{ \begin{array}{l} R(e+\beta') - R'(e+\beta) = 2\beta' - \beta \\ R - R' = \dfrac{\beta\beta'}{2wh}. \end{array} \right. \tag{16}$$

We first remark that $\beta \neq \beta'$, otherwise Equation(15) would imply that $(R-R')\lambda = \beta$, and hence $R-R' \notin \mathbb{Q}$ since $\beta \neq 0$. It follows that the linear system (16) has a unique solution. Let s_2 be the value of s corresponding to this solution which given by Equation (15). It follows that if $s < \min\{s_1,s_2\}$ then the directions of δ_0 and δ'_0 cannot be both periodic, hence (X,ω) cannot be a Veech surface. Since the set $\mathcal{P}_D(0,0,0)$ is finite, the proposition follows. □

The next proposition is a direct consequence of Proposition 10.7.

Proposition 10.8. *Let* $\{(X_j,\omega_j,P_j),\ j=0,1,2\}$ *be an element of* $\Omega E_D(0,0,0)$*, and* $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ *be the map in Proposition 7.1. Then there exists* $0<\delta<\varepsilon$ *such that if* $(X,\omega) \in \Psi(\mathring{D}(\delta))$*, then* (X,ω) *is not a Veech surface.*

Proof. Let ℓ_0 be the length of the shortest simple closed geodesic in the torus (X_0,ω_0). Let s_0 be the constant in Proposition 10.7. Pick $\delta < \min\{\varepsilon, s_0\ell_0\}$. By definition, if $(X,\omega) = \Psi(z)$, then we have a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ which decompose X into three tori such that $\omega(\sigma_j) = z$. Assume that $z \in \mathring{D}(\delta)$, we claim that (X,ω) is not a Veech surface. We have two cases:

- z is not parallel to any vector in the lattice L_0 associated to (X_0,ω_0). In this case, the direction of σ_j is not periodic, hence (X,ω) is not a Veech surface.

- z is parallel to some vector in L_0. In this case, (X,ω) admits a decomposition into three cylinders, which correspond to the tori X_0, X_1, X_2, in the direction of z. Let v be the primitive vector in L_0 in the same direction as z, then the width of the cylinder corresponding to X_0 is $|v|$. By assumption, we have
$$\frac{|\sigma_0|}{|v|} \leq \frac{|\sigma_0|}{\ell_0} < s_0.$$
Therefore, (X,ω) cannot be a Veech surface by Proposition 10.7

□

Using Proposition 10.8, we can now prove the theorem announced at the beginning of the section.

Proof of Theorem 10.1. Fix a connected component $\mathscr{C}$ of $\Omega E_D(2,2)^{\mathrm{odd}}$. By the main result of [LN13c], we know that there exists a surface $(X,\omega) \in \mathscr{C}$ which admits a three-tori decomposition by a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$.

By moving in the kernel foliation leaves, we can assume that the direction of σ_j is periodic on (X,ω). By Lemma 10.2, we get a corresponding prototype (w,h,t,e) in $\mathcal{P}_D(0,0,0)$. Set $L_0 = \mathbb{Z}(\lambda,0) + \mathbb{Z}(0,\lambda)$, $L_1 = L_2 = \mathbb{Z}(w,0) + \mathbb{Z}(t,h)$, and $(X_j,\omega_j) = (\mathbb{C}/L_j, dz),\ j = 0,1,2$. Let P_j be the projection of $0 \in \mathbb{C}$ in X_j. Then the triple $\{(X_j,\omega_j,P_j),\ j=0,1,2\}$ belongs to $\Omega E_D(0,0,0)$. Note that we obtain this triple of tori as the limit surface when $\sigma_0,\sigma_1,\sigma_2$ are collapsed.

Let $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map in Proposition 7.1. It is easy to see that $\Psi(\mathring{D}(\varepsilon)) \subset \mathscr{C}$. From Proposition 10.8, we know that there exists $0 < \delta < \varepsilon$ such that the set $\mathcal{V} = \Psi(\mathring{D}(\delta))$ does not contain any Veech surface. As a consequence the set $\mathcal{U} = \mathrm{GL}^+(2,\mathbb{R}) \cdot \mathcal{V}$ does not contain any Veech surface either. It is easy to see that $\mathcal{U}$ is an open subset of $\mathscr{C}$. The theorem is then proved. □

11. Finiteness of closed orbits

In this section we will prove our main second main result, namely:

Theorem 11.1. *If D is not a square then the number of closed* $\mathrm{GL}^+(2,\mathbb{R})$*-orbits in* $\Omega E_D(2,2)^{\mathrm{odd}}$ *is finite.*

We first show a useful finiteness result up to the kernel foliation for surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$. Recall that (X,ω) admits an unstable cylinder decomposition in the horizontal direction if and only if this direction is periodic, and there exists (at least) one horizontal saddle connection whose endpoints are distinct zeros of ω.

Theorem 11.2. *If D is not a square then there exists a finite family $\mathbb{P}_D$ of surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ such that if $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ admits an unstable cylinder decomposition, then up to rescaling by* $\mathrm{GL}^+(2,\mathbb{R})$*, one has*
$$(X,\omega) = (X_i,\omega_i) + (x,0) \qquad \textit{for some } (X_i,\omega_i) \in \mathcal{P}_D.$$

If we label the zeros of ω by P and Q, we always choose the orientation for any saddle connection joining P and Q to be *from P to Q*: this defines in a unique way the surface $(X,\omega) + (x,0)$.

Proof of Theorem 11.2. By [McM05a], for any $D' \equiv 0,1 \mod 4, D' > 0$, the set $\Omega E_{D'}(2)^*$ is a finite union of $\mathrm{GL}^+(2,\mathbb{R})$-closed orbits. More precisely there exists a finite family $\mathbb{P}_{D'}(2)$ of surfaces (*prototypical splittings*) such that any $(X,\omega) \in \Omega E_{D'}(2)^*$ which is horizontally periodic belongs to the P-orbit (here $\mathrm{P} = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{GL}^+(2,\mathbb{R})\}$) of some surface in $\mathbb{P}_{D'}(2)$.

In [LN13], we have proved the same result for the stratum $\Omega E_D(4)$: there exists a finite family $\mathbb{P}_D(4)$ of surfaces such that any horizontally periodic surface $(X,\omega)\in\Omega E_D(4)$ belongs to the P-orbit of a surface in $\mathbb{P}_D(4)$. The corresponding statement for the stratum $\Omega E_D(0,0,0)$ is Lemma 10.2. Let $\mathbb{P}_D(0,0,0)$ be the set of corresponding surfaces in $\Omega E_D(0,0,0)$. We will call the surfaces in the families $\mathbb{P}_{D'}(2)$, $\mathbb{P}_D(4)$, $\mathbb{P}_D(0,0,0)$ *prototypical surfaces*.

Given a discriminant $D>0$, for each prototypical surface X_∞ in these finite families $\mathbb{P}_D(0,0,0)$, $\mathbb{P}_D(4)$ and $\mathbb{P}_{D'}(2)$, where $D'\in\{D,D/4\}$, we apply, respectively, Propositions 7.1, 7.3 and 7.5. This furnishes a map $\Psi:\mathring{D}(\varepsilon)\to\Omega E_D(2,2)^{\mathrm{odd}}$ where $\varepsilon>0$.

By construction, the surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ whose horizontal kernel foliation leaf contains X_∞ (*i.e* X_∞ is a limit of the real kernel foliation leaf through such surfaces) and close enough to X_∞ are contained in the set $\Psi(R_{(k,n)}(\varepsilon))$, where $n\in\{1,3,5\}$, $k\in\{0,\dots,2n-1\}$, depending on the space to which X_∞ belongs. For each prototypical surface, and each admissible pair (k,n), we pick a surface in $\Psi(R_{(k,n)}(\varepsilon))$. Let $\mathbb{P}_D$ denote this (finite) family. Note that for all the surfaces in this family, the cylinder decomposition in the horizontal direction is unstable.

Now, thanks to Theorem 8.1, if $(X,\omega)\in\Omega E_D(2,2)^{\mathrm{odd}}$ admits an unstable cylinder decomposition, then up to action of $GL^+(2,\mathbb{R})$, the horizontal kernel foliation leaf through (X,ω) contains some prototypical surface. Therefore (X,ω) belongs to the same horizontal kernel foliation leaf of a surface in the family $\mathbb{P}_D$, and the theorem follows. □

We have now all necessary tools to prove our main result.

Proof of Theorem 11.1. Let $\{(X_i,\omega_i),\ i\in I\}$ be a family of Veech surfaces that generates an infinite family of closed $GL^+(2,\mathbb{R})$-orbits in $\Omega E_D(2,2)^{\mathrm{odd}}$. We will show that the set

$$\mathcal{O}=\bigcup_{i\in I}GL^+(2,\mathbb{R})\cdot(X_i,\omega_i)$$

is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$ contradicting Theorem 10.1.

Since the direction of any saddle connection on a Veech surface is periodic, each surface in the family $\{(X_i,\omega_i),\ i\in I\}$ admits infinitely many unstable cylinder decompositions. Therefore, we can assume that each of the surfaces (X_i,ω_i) belongs to the horizontal kernel foliation leaf of one surface in the family $\mathbb{P}_D$ of Theorem 11.2. Since the set $\mathbb{P}_D$ is finite, there exists a surface $(X,\omega)\in\mathbb{P}_D$ and an infinite subfamily $I_0\subset I$ such that $(X_i,\omega_i)=(X,\omega)+(x_i,0)$ for any $i\in I_0$. By Theorem 8.1, $x_i\in]a,b[$, where a,b does not depend on i.

The compactness of the interval $[a,b]$ implies the existence of a subsequence $\{i_k\}_{k\in\mathbb{N}}\subset I_0$ such that $\{x_{i_k}\}$ converges to some $x\in[a,b]$. The sequence $(X_{i_k},\omega_{i_k})=(X,\omega)+(x_{i_k},0)$ thus converges to $(Y,\eta):=(X,\omega)+(x,0)$. If $x\in]a,b[$ then (Y,η) belongs to $\Omega E_D(2,2)^{\mathrm{odd}}$. However if $x\in\{a,b\}$ then by Theorem 8.1, (Y,η) belongs to a boundary component of $\Omega E_D(2,2)^{\mathrm{odd}}$, namely $\Omega E_D(4),\Omega E_{D'}(2)^*$ with $D'\in\{D,D/4\}$, or $\Omega E_D(0,0,0)$. We distinguish separately the four cases below.

Case $(Y,\eta)\in\Omega E_D(2,2)^{\mathrm{odd}}$.
Let θ be a periodic direction on (Y,η) that is different from $(\pm1,0)$. Set

$$(Y^\theta,\eta^\theta):=R_{-\theta}\cdot(Y,\eta),\ \text{and}\ (X^\theta_{i_k},\omega^\theta_{i_k})=R_{-\theta}\cdot(X_{i_k},\omega_{i_k}),$$

where $R_{-\theta}=\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$. Note that (Y^θ,η^θ) is horizontally periodic, and we have

$$(X^\theta_{i_k},\omega^\theta_{i_k})=(Y^\theta,\eta^\theta)+v_k,$$

where $v_k = R_{-\theta} \cdot (x - x_{i_k}, 0)$. Thus we have $v_k \overset{k\to\infty}{\longrightarrow} (0,0)$. Note that, since $\theta \neq (\pm 1, 0)$, v_k does not belong to $\mathbb{R} \times \{0\}$.

By Propositions 4.7 and 4.9, for k large enough, $(X^\theta_{i_k}, \omega^\theta_{i_k})$ admits a stable cylinder decomposition in the horizontal direction. Moreover, we can assume that the cylinder decompositions of $(X^\theta_{i_k}, \omega^\theta_{i_k})$ in the horizontal direction share the same combinatorial data, and the same widths of cylinders. Finally, since $(X^\theta_{i_k}, \omega^\theta_{i_k})$ are Veech surfaces, the horizontal direction is parabolic. The assumptions of Theorem 6.2 are therefore fulfilled, and we derive that there exists $\varepsilon_1 > 0$ such that $(Y^\theta, \eta^\theta) + (s, 0) \in \overline{O}$ for all $s \in (-\varepsilon_1, \varepsilon_1)$. It follows from Corollary 5.3, that there exists $\varepsilon'_1 > 0$, such that $(Y^\theta, \eta^\theta) + v \in \overline{O}$ for any $v \in \mathbb{R}^2$ such that $|v| < \varepsilon'_1$. One can find a vector v with $|v| < \varepsilon'$ such that the surface $(Y^\theta, \eta^\theta) + v$ is horizontally periodic but not parabolic. By Theorem 5.1, the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of $(Y^\theta, \eta^\theta) + v$ is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$. Since this $\mathrm{GL}^+(2,\mathbb{R})$-orbit is contained in $\overline{O}$, we conclude that $\overline{O}$ contains a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Case $(Y, \eta) \in \Omega E_D(4)$**.**
In this case (Y, η) is a Veech surface. Choose a periodic direction θ for (Y, η) that is different from $(\pm 1, 0)$. We define (Y^θ, η^θ) and $(X^\theta_{i_k}, \omega^\theta_{i_k})$ as in the previous case.

Let $\Psi : \mathring{D}(\varepsilon) \to \Omega E_D(2,2)^{\mathrm{odd}}$ be the map in Proposition 7.3 associated to (Y^θ, η^θ). Recall that by construction, the set $\Psi(R_{(k,5)}(\varepsilon))$ consists of surfaces in $\Omega E_D(2,2)^{\mathrm{odd}}$ close to (Y^θ, η^θ) which have a small *horizontal* saddle connection invariant by the Prym involution.

By the choice of θ, $(X^\theta_{i_k}, \omega^\theta_{i_k})$ is not contained in $\Psi(R_{(k,5)}(\varepsilon))$ for any $k \in \{0, \dots, 9\}$. Thus, there must exist $k \in \{1, \dots, 10\}$ such that the sector $\Psi(\mathring{D}_{(k,5)}(\varepsilon))$ contains infinitely many elements of the family $\{(X^\theta_{i_k}, \omega^\theta_{i_k})\}$. Note that every surface in $\Psi(\mathring{D}_{(k,5)}(\varepsilon))$ admits a stable cylinder decomposition in the horizontal direction with the same combinatorial data and the same widths of cylinders (see Proposition 9.1). By assumption, the horizontal direction is parabolic for all $(X^\theta_{i_k}, \omega^\theta_{i_k})$. Thus Theorem 9.4 allows us to conclude that O is dense in a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Case $(Y, \eta) \in \Omega E_{D'}(2)^*$**.**
In particular (Y, η) is a Veech surface (viewed as a surface of $\Omega E_{D'}(2)$). The same arguments as the case $(Y, \eta) \in \Omega E_D(4)$ show that $\overline{O}$ contains a component of $\Omega E_D(2,2)^{\mathrm{odd}}$.

Case $(Y, \eta) \in \Omega E_D(0,0,0)$**.**
In this case (X, ω) has a triple of horizontal saddle connections $\{\sigma_0, \sigma_1, \sigma_2\}$ that decompose the surface into a connected sum of three tori, and (Y, η) can be viewed as the limit when the length of σ_j goes to zero. By Proposition 10.8, there is no Veech surface in the neighborhood of (Y, η). Thus this case does not occur.

From above discussion, we draw that O is always dense a component of $\Omega E_D(2,2)^{\mathrm{odd}}$, but this is a contradiction with Theorem 10.1. The proof of Theorem 11.1 is now complete. □

Appendix A. Existence of Veech surfaces in infinitely many Prym eigenform loci

It follows from the work of McMullen [McM06a] that there exists only finitely many $\mathrm{GL}^+(2,\mathbb{R})$ closed orbits in the union $\bigcup_{D \text{ not a square}} \Omega E_D(1,1)$ (see [LMöl13] for a similar result in $\Omega E_D(1,1,2)$).

However the situation is different in $\Omega E_D(2,2)^{\mathrm{odd}}$. We will show that for infinitely many discriminants D that are not squares, the locus $\Omega E_D(2,2)^{\mathrm{odd}}$ contains at least one $\mathrm{GL}^+(2,\mathbb{R})$ closed orbit (the fact that $\Omega E_{D_1}(2,2)^{\mathrm{odd}}$ and $\Omega E_{D_2}(2,2)^{\mathrm{odd}}$ are disjoint if $D_1 \neq D_2$ will be proved in [LN13c]). Remark that the corresponding Veech surfaces we found are not primitive, they are double coverings of surfaces in $\Omega E_D(2)$. It is unknown to the authors if there exists any primitive Veech surface in $\bigcup_{D \text{ not a square}} \Omega E_D(2,2)^{\mathrm{odd}}$.

Following [McM05a] we say that a quadruple of integers (w,h,t,e) is a *splitting prototype* of discriminant D if the conditions below are fulfilled:

$$\begin{cases} w > 0, h > 0,\ 0 \leq t < \gcd(w,h), \\ \gcd(w,h,t,e) = 1, \\ D = e^2 + 4wh, \\ 0 < \lambda := \frac{e+\sqrt{D}}{2} < w. \end{cases}$$

To each splitting prototype one can associate a Veech surface $(X,\omega) \in \Omega E_D(2)$ as follows (see Figure 9).

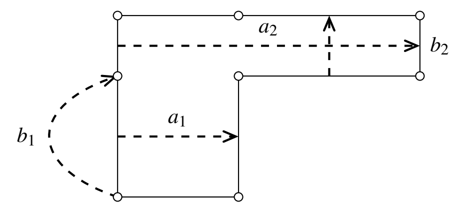

FIGURE 9. Prototypical splitting of type $(w,h,0,e)$ where $\omega(a_1) = (\lambda,0)$, $\omega(b_1) = (0,\lambda)$, $\omega(a_2) = (w,0)$ and $\omega(b_2) = (0,h)$. Parallel edges are identified to obtain a surface $(X,\omega) \in \Omega E_D(2)$

Define a pair of lattices in $\mathbb{C}$ by $\Lambda_1 = \mathbb{Z}(\lambda,0) \oplus \mathbb{Z}(0,\lambda)$ and $\Lambda_2 = \mathbb{Z}(w,0) \oplus \mathbb{Z}(t,h)$ (recall that $\lambda := \frac{e+\sqrt{D}}{2} > 0$). We construct the corresponding tori $(E_i,\omega_i) = (\mathbb{C}/\Lambda_i, dz)$ and the genus two surface (X,ω) where $X = E_1 \# E_2$ and $\omega = \omega_1 + \omega_2$.

Geometrically, the surface (X,ω) is made of two horizontal cylinders whose core curves are denoted by a_1 and a_2 (see [McM05a] and Figure 9 for details).

Let $\{a_1,b_1,a_2,b_2\}$ be the symplectic basis of $H_1(X,\mathbb{Z})$ such that $\omega(a_1) = (\lambda,0)$, $\omega(b_1) = (0,\lambda)$, $\omega(a_2) = (w,0)$ and $\omega(b_2) = (t,h)$. A generator of the order $\mathcal{O}_D$ is given (in the above basis) by the following matrix

$$T = \begin{pmatrix} e & 0 & w & t \\ 0 & e & 0 & h \\ h & -t & 0 & 0 \\ 0 & w & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that T is a self-adjoint with respect to the intersection form of $H_1(X,\mathbb{Z})$, $T^2 = eT + wh\mathrm{Id}$, and T satisfies $T^*\omega = \lambda\omega$. It follows that T generates a proper subring in $\mathrm{End}(\mathbf{Jac}(X))$ for which ω is an eigen vector. Thus $(X,\omega) \in \Omega E_D(2)$, and therefore (X,ω) is a Veech surface (see [McM06] for more details).

Theorem A.1. *Let (w,h,t,e) be a splitting prototype for a discriminant D, and (X,ω) be the associated Veech surface in $\Omega E_D(2)$. Let (Y_1,η_1) and (Y_2,η_2) be two surfaces in $\mathcal{H}(2,2)$ constructed from*

(w, h, t, e) as shown in Figure 10. Then both (Y_1, η_1) and (Y_2, η_2) are Veech surfaces in some Prym eigenform loci in $\mathcal{H}(2,2)^{\mathrm{odd}}$. More specifically, we have

(i) $(Y_1, \omega_1) \in \Omega E_{4D}(2,2)^{\mathrm{odd}}$ *if h is odd, otherwise* $(Y_1, \eta_1) \in \Omega E_D(2,2)^{\mathrm{odd}}$,
(ii) $(Y_2, \omega_2) \in \Omega E_{4D}(2,2)^{\mathrm{odd}}$ *if w is odd, otherwise* $(Y_2, \eta_2) \in \Omega E_D(2,2)^{\mathrm{odd}}$.

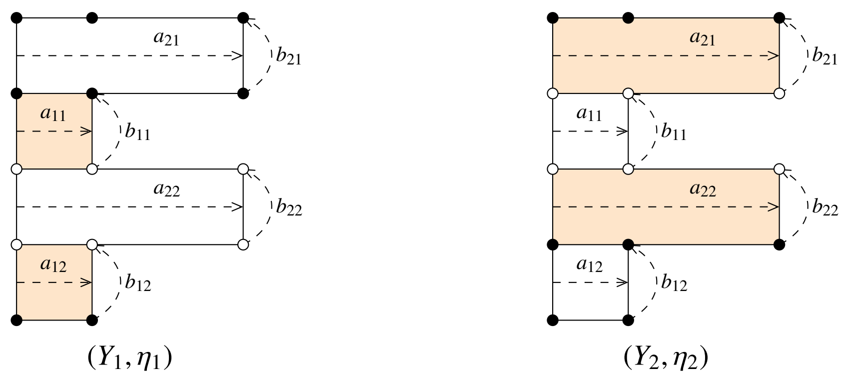

FIGURE 10. Double coverings of a surface in $\Omega E_D(2)$: $\eta_i(a_{11}) = \eta_i(a_{12}) = \lambda, \eta_i(b_{11}) = \eta_i(b_{12}) = \imath\lambda$, $\eta_i(a_{21}) = \eta_i(a_{22}) = w, \eta_i(b_{21}) = \eta_i(b_{22}) = t + \imath h$, $i = 1, 2$. The cylinders fixed by the Prym involution are colored.

Remark A.2.

- *In general, the Teichmüller disks generated by (Y_1, ω_1) and by (Y_2, ω_2) are different, for instance when h is odd, and w is even.*
- *If $D \equiv 5 \mod 8$, then it is easy to see that e, w, h are all odd. Therefore, in both construction (Y_i, η_i) belongs to $\Omega E_{4D}(2,2)^{\mathrm{odd}}$.*

Proof. It is easy to see that both (Y_1, η_1) and (Y_2, η_2) are double coverings of (X, ω), the deck transformation sends a_{ij} to a_{ij+1} and b_{ij} to b_{ij+1} (here we use the convention $(i3) \sim (i1)$). Since (X, ω) is a Veech surface both (Y_1, ω_1) and (Y_2, ω_2) are Veech surfaces (see [GJ00] and [MT02]).

Remark that Y_i has an involution τ_i that exchanges the zeros of η_i such that $\tau_i^*\eta_i = -\eta_i$, in Figure 10 the cylinders fixed by τ_i are colored. It follows that (Y_i, η_i) belongs to the Prym locus $\mathrm{Prym}(2,2) \subset \mathcal{H}(2,2)^{\mathrm{odd}}$ ($\mathrm{Prym}(2,2)$ consists of double coverings of quadratic differentials in $\mathcal{Q}(-1^4, 4)$). By some standard arguments (see [LN13] and [McM06]), we can conclude that (Y_i, η_i) is a Prym eigenform, thus (Y_i, η_i) is contained in some locus $\Omega E_{\widetilde{D}}(2,2)^{\mathrm{odd}}$. It remains to determine the discriminant $\widetilde{D}$.

Set $H_1(Y_i, \mathbb{Z})^- = \{\alpha \in H_1(Y_i, \mathbb{Z}) \,|\, \tau_i(\alpha) = -\alpha\}$. Since $(Y_i, \eta_i) \in \mathrm{Prym}(2,2)$, we have $H_1(Y_i, \mathbb{Z})^- \simeq \mathbb{Z}^4$. We choose a basis of $H_1(Y_i, \mathbb{Z})^-$ as follows:

- for (Y_1, η_1), set $\alpha_1 = a_{11} = a_{12}$ and $\alpha_2 = a_{21} + a_{22}$, we choose $\beta_1 = b_{11} + b_{12}$ and $\beta_2 = b_{21} + b_{22}$. In particular the restriction of the symplectic form has the following matrix $\left(\begin{smallmatrix} J & 0 \\ 0 & 2J \end{smallmatrix}\right)$.
- for (Y_2, η_2), set $\alpha_1 = a_{11} + a_{12}, \alpha_2 = a_{21} = a_{22}, \beta_1 = b_{11} + b_{12}, \beta_2 = b_{21} + b_{22}$. In this basis, the restriction of the intersection form to $H_1(Y_2, \mathbb{Z})^-$ is given by $\left(\begin{smallmatrix} 2J & 0 \\ 0 & J \end{smallmatrix}\right)$.

In the above bases, the coordinates of η_i are the following:

$$\mathrm{Re}(\eta_1) = (\lambda, 0, 2w, 2t) \qquad \text{and} \qquad \mathrm{Im}(\eta_1) = (0, 2\lambda, 0, 2h).$$

$$\mathrm{Re}(\eta_2) = (2\lambda, 0, w, 2t) \qquad \text{and} \qquad \mathrm{Im}(\eta_2) = (0, 2\lambda, 0, 2h).$$

Let $\widetilde{T}_1$ be the following self-adjoint endomorphism of $H_1(Y_1,\mathbb{Z})^-$ (given in the basis $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$):

$$\widetilde{T}_1 = \begin{pmatrix} 2e & 0 & 4w & 4t \\ 0 & 2e & 0 & 2h \\ h & -2t & 0 & 0 \\ 0 & 2w & 0 & 0 \end{pmatrix}.$$

Similarly, let $\widetilde{T}_2$ be the self-adjoint endomorphism of $H_1(Y_2,\mathbb{Z})^-$ (given in the basis $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$) by the following matrix

$$\widetilde{T}_2 := \begin{pmatrix} 2e & 0 & w & 2t \\ 0 & 2e & 0 & 2h \\ 4h & -4t & 0 & 0 \\ 0 & 2w & 0 & 0 \end{pmatrix}$$

It is straightforward to check that $\widetilde{T}_i^*\eta_i = (2\lambda)\cdot\eta_i$ thus η_i is an eigenform of $\widetilde{T}_i$. Remark that both $\widetilde{T}_i$ satisfy $\widetilde{T}_i^2 - 2e\widetilde{T}_i - 4wh\mathrm{Id} = 0$, which implies that $\widetilde{T}_i$ generates a self-adjoint subring of End(Prym(Y_i)) isomorphic to $\mathcal{O}_{D'}$, where $D' = (2e)^2 + 16wh = 4(e^2 + 4wh) = 4D$.

There exists a unique proper subring of End(Prym(Y_i)) for which η_i is an eigenform, this proper subring is isomorphic to a quadratic order $\mathcal{O}_{\widetilde{D}_i}$. Clearly, this subring must contain $\widetilde{T}_i$, hence it is generated by $\widetilde{T}_i/k_i$, where $k_1 = \gcd(2e, 4w, 2h, 2w, h, 4t, 2t) = \gcd(2e, 2w, h, 2t)$, and $k_2 = \gcd(2e, w, 2h, 2t)$. Since $\gcd(w, h, t, e) = 1$ we have $k_i \in \{1, 2\}$. Note that $4D = k_i^2\widetilde{D}_i$, therefore $\widetilde{D}_i = 4D$ if $k_i = 1$, and $\widetilde{D}_i = D$ if $k_i = 2$. We can now conclude by noticing that $k_1 = 1$ if and only if h is odd, and $k_2 = 1$ if and only if w is odd. □

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