Infinite sequence of fixed point free pseudo-Anosov homeomorphisms on a family of genus two surfaces

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ABSTRACT. In this paper we construct an infinite sequence of pseudo-Anosov homeomorphisms without fixed separatrix, and leaving invariant a family of genus two translation surfaces with a single zero. This extends previous work of Los [**Los09**]. The construction uses the Rauzy-Veech induction.

1. Introduction

Counting problems for the number of closed geodesics of given length bound have a long history in differential geometry, in particular in the world of symmetric spaces. Recently, it has been discovered that some techniques and results carry over to geodesics for the Teichmüller metric on the moduli space of curves. Asymptotics for these counting problems are most easily proven for a discretization of the Teichmüller geodesic flow, i.e. using Rauzy-Veech induction. The latter is dened on a cover of the moduli space of Abelian (or quadratic) differentials only. Hence the question arises, whether there are geodesics (i.e. pseudo-Anosov homeomorphims) that spoil the comparison between the two. The first examples of these pseudo-Anosov homeomorphisms were found by Los [**Los09**]. In this paper we provide another construction of such pseudo-Anosov homeomorphisms. We show that that they even can appear in the lowest possible genus and smallest possible stratum.

Let $\mathcal{M}_g$ denote the moduli space of compact genus g Riemann surfaces. Recently, Eskin and Mirzakhani [**EM08**] proved that the number $N(T)$ of closed geodesics in $\mathcal{M}_g$, of length at most T (for the Teichmüller metric) satisfies the following asymptotic property (where T tends to infinity)

$$N(T) \sim \frac{e^{hT}}{hT} \qquad \text{where} \qquad h = 6g - 6.$$

This number $N(T)$ is also the number of conjugacy classes of pseudo-Anosov homeomorphisms of translation length at most $Log(T)$ (see also [**Vee86, Ham07, Raf08**]).

The unit cotangent bundle of $\mathcal{M}_g$ can be viewed as the moduli space of quadratic differentials $\mathcal{Q}_g \to \mathcal{M}_g$. This space is naturally stratified by strata of quadratic differentials with prescribed zeroes of order $\sigma = (k_1, \dots, k_n)$, denoted

$\mathcal{Q}(k_1, \ldots, k_n)$. The Teichmüller geodesic flow $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ acts naturally on these strata and closed loops of length $T > 1$ for this flow projects to geodesics for the Teichmüller metric on $\mathcal{M}_g$ of length $Log(T) > 0$.

Eskin-Mirzakhani's result can be reformulated in terms of counting problem in the strata. Let $N(\sigma, T)$ be the number of closed loops, for the Teichmüller flow in $\mathcal{Q}(\sigma)$, of length at most T. Equivalently $N(\sigma, T)$ is the number of conjugacy classes of pseudo-Anosov homeomorphisms of translation length at most $Log(T)$ with prescribed singularities σ. Then Eskin-Mirzakhani proved that $N(\sigma, T) \sim \dfrac{e^{hT}}{hT}$ where $h = 6g - 6$ for the *principal stratum* (i.e. $\sigma = (1, \ldots, 1)$).

The constant h can be explained as follow. This is a classical fact that the stratum $\mathcal{Q}(\sigma)$ is a complex analytic orbifold of complex dimension $h(\sigma) = 2g + n - \varepsilon$ where $\sum_{i=1}^{n} k_i = 4g - 4$ and $\varepsilon = 1, 2$ depending whether the quadratic differentials are the global square of Abelian differentials or not. A conjecture of Veech is the following [**Vee86**, Section 5] :

$$(1.1) \qquad \lim_{T \to +\infty} \frac{1}{T} \mathrm{Log}\, N(\sigma, T) = h(\sigma).$$

Veech proved that

$$h(\sigma) \leq \liminf_{T \to +\infty} \frac{1}{T} \mathrm{Log}\, N(\sigma, T) \leq \limsup_{T \to +\infty} \frac{1}{T} \mathrm{Log}\, N(\sigma, T) \leq 6h(\sigma)^4.$$

A corollary of Eskin-Mirzakhani's results is a proof of this conjecture for the principal stratum $\mathcal{Q}(1, \ldots, 1)$. In this case $h(1, \ldots, 1) = 6g - 6$.

There exists another similar result in the context of the *Rauzy-Veech induction*. Let us recall this problem here. The strata are not necessarily connected [**KZ03**, **Lan08**]. If $\mathcal{C}$ is a connected component then there is a ramified covering $\widehat{\mathcal{C}}$ of $\mathcal{C}$ consisting of marking a zero and a separatrix. The Rauzy-Veech induction provides a discrete representation of the Teichmüller flow on $\widehat{\mathcal{C}}$ [**Vee82**, **BL09**, **Boi09**] (see Section 2.2 for precise definitions).

As shown by Veech, to each periodic orbit $\gamma \subset \widehat{\mathcal{C}}$ there corresponds a renormalization matrix $V(\gamma) \in SL(h, \mathbb{Z})$ where $h = h(\sigma)$. The period of the orbit γ is the logarithm of the spectral radius of $V(\gamma)$. Again we define by $\widehat{N}(\sigma, T)$, the set of periodic orbits γ for the Teichmüller flow g_t such that $||V(\gamma)|| \leq \exp(T)$ where $||A|| = \max_j \sum_i |A_{ij}|$.

Then Bufetov [**Buf06**] proved a formula related to Equation (1.1), in the context of the Rauzy-Veech induction (for interval exchanged transformations):

$$\lim_{T \to +\infty} \frac{1}{T} \mathrm{Log}\, \widehat{N}(\sigma, T) = h(\sigma) = 2g + n - 1.$$

It turns out that this result is closely related to Eskin-Mirzakhani's corollary. Indeed periodic orbits for the Teichmüller flow on $\widehat{\mathcal{C}}$ are taken to periodic orbits for the Teichmüller flow on $\mathcal{C}$ (but in general of smaller period). However it is not clear whether Bufetov's result easily imply Eskin-Mirzakhani's corollary.

Pseudo-Anosov homeomorphisms constructed by this induction possess a very remarkable property: they all fix a separatrix adjacent to a zero. But for counting problem, we are only interested with pseudo-Anosov homeomorphisms with large dilatation. Thus Avila and Hubert asked the following natural question:

"Is there an infinite sequence of pseudo-Anosov homeomorphisms, defined on a
family of flat surfaces in the same stratum, and without fixed separatrix?"

A negative answer to that question will thus restrict counting problems on $\mathcal{M}_g$ (and
thus on $\mathcal{Q}_g$) to counting problems for the Rauzy-Veech induction, which is simpler.
In particular Bufetov's result would implies Veech's conjecture. But Los [**Los09**]
answered positively to that question. In this paper we provide a positive answer to
Avila-Hubert's question in genus two, answering also a question of Los in [**Los09**].

THEOREM 1.1. *There exists a sequence of pseudo-Anosov homeomorphisms*
$(\phi_n)_{n \geq 1}$, *fixing a family of quadratic differentials (with a single zero) on a genus
two surface, and without fixed separatrix.*

In contrast to [**Los09**] we will construct our examples by using the Rauzy-Veech
induction. We state Theorem 1.1 more precisely in the next Section. We also give
a generalisation of our construction to arbitrary genera.

2. Background and tools

In this Section we recall very briefly the Rauzy-Veech induction (for more details
see e.g. [**Vee82, MMY05**]). For general properties in the theory of translation
surfaces and pseudo-Anosov homeomorphisms, see say [**Thu88, FLP79, MT02**].
Since our construction deals only with Abelian differentials we will here assume
that all quadratic differentials are globally the square of Abelian differentials. But
we can make all the constructions in general (see [**BL09**]).

2.1. Pseudo-Anosov homeomorphisms.

2.1.1. *Translation surfaces.* A translation surface is a (genus g) Riemann sur-
face X endowed with a holomorphic $1-$form $\omega \in H^1(X, \mathbb{C})$ (we will also say Abelian
differential). We will denote the multiplicity of the zeroes of ω by $\sigma = (k_1, \ldots, k_n)$
where $\sum_{i=1}^n k_i = 2g - 2$.

The moduli space $\mathcal{H}(\sigma)$ of Abelian differentials with prescribed zeroes is a
complex analytic orbifold of complex dimension $2g + n - 1$. The group $\mathrm{SL}_2(\mathbb{R})$ acts
naturally on the moduli space of translation surfaces, and preserves the strata. The
Teichmüller flow is given by the action of the one parameter subgroup $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

2.1.2. *Pseudo-Anosov homeomorphisms.* We will say that a homeomorphism ϕ
is affine with respect to the translation surface (X, ω) if ϕ is a diffeomorphism on
the complementary of the zeroes of ω and if the differential $D\phi$ is a constant matrix
in $\mathrm{SL}_2(\mathbb{R})$.

An affine homeomorphism is pseudo-Anosov if $D\phi$ is hyperbolic i.e. $|\mathrm{Tr}(D\phi)| >
2$; if $\mathrm{Log}(\lambda)$ is the translation length of $D\phi$ then we will say that $\lambda > 1$ is the
dilatation of ϕ.

Up to the $\mathrm{SL}_2(\mathbb{R})-$action we assume that $D\phi = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$. Then, in these
coordinates, the flow g_t has a periodic orbit of length at most $t = \mathrm{Log}(\lambda) > 0$. In
addition the surfaces (X, ω) and $D\phi(X, \omega)$ are isometric i.e. (X, ω) can be obtained
from $D\phi(X, \omega)$ by "cuttings" and "gluings".

The linear map ϕ_* defined on $H_1(X, \mathbb{R})$ has a simple eigenvalue $\rho(\phi_*) \in \mathbb{R}$ which is a Perron number and such that $|\rho(\phi_*)| > 1$ is the dilatation λ of ϕ (see for example [**Thu88**]). In this note, all pseudo-Anosov homeomorphisms ϕ will satisfy $\rho(\phi_*) > 0$.

2.1.3. *Index of closed curves and Lefschetz theorem.* The index of a fixed point of ϕ at p is the algebraic number $\mathrm{Ind}(\phi, p)$ of turns of the vector $(p, \phi(p))$ when p describes a small loop around p. The Lefschetz fixed point theorem asserts that when X is compact, the Lefschetz number satisfies

$$L(\phi) = 2 - \mathrm{Tr}(\phi_*) = \sum_{p=\phi(p)} \mathrm{Ind}(\phi, p).$$

In particular if ϕ fixes a separatrix associated to a degree-k zero of ω then $\mathrm{Ind}(\phi, p) = 1 - 2(k+1)$, otherwise $\mathrm{Ind}(\phi, p) = 1$. Theorem 1.1 can be restated as follow,

THEOREM 2.1. *There exists a sequence $(\phi_n)_{n \geq 1}$, where ϕ_n is a pseudo-Anosov homeomorphism on a sequence of genus two translation surfaces (X_n, ω_n) where ω_n has only one double zero and such that ϕ_n has no fixed points of negative index i.e. $L(\phi_n) = 1$.*

In order to construct the sequence, we will use the so-called Rauzy-Veech induction.

2.2. Rauzy-Veech induction. In this Section we recall very briefly the basic construction of pseudo-Anosov homeomorphisms using the Rauzy-Veech induction (for more details see [**Vee82**], §8, and [**Rau79, MMY05**]).

2.2.1. *Interval exchange map.* An interval exchange map is a one-to-one map T from an open interval I to itself that permutes, by translation, a finite partition $\{I_j, \ j = 1, \ldots, d\}$ of I into $d \geq 2$ open subintervals. It is easy to see that T is precisely determined by the following data: a permutation π that encodes how the intervals are exchanged, and a vector λ with positive entries that encodes the lengths of the intervals.

Following [**MMY05**], we will describe an interval exchange by choosing a name to each intervals. In this case, we will speak of *labelled* interval exchanges. Again such objects are encoded by a permutation and a vector. A permutation is a pair of one-to-one maps (π_0, π_1) from a finite alphabet $\mathcal{A}$ to $\{1, \ldots, d\}$ in the following way. In the partition of I into intervals, we denote the interval labelled k, when counted from the left to the right, by $I_{\pi_0^{-1}(k)}$. Once the interval are exchanged, the interval labelled k is $I_{\pi_1^{-1}(k)}$. The permutation π corresponds to the map $\pi = \pi_1 \circ \pi_0^{-1}$. The lengths of the intervals form a vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$.
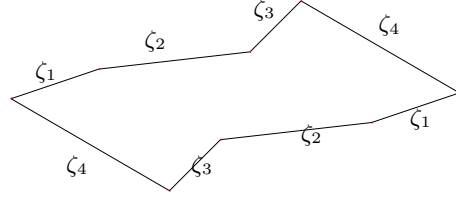
We will usually represent the combinatorial datum $\pi = (\pi_0, \pi_1)$ by a table:

$$\pi = \begin{pmatrix} \pi_0^{-1}(1) & \pi_0^{-1}(2) & \ldots & \pi_0^{-1}(d) \\ \pi_1^{-1}(1) & \pi_1^{-1}(2) & \ldots & \pi_1^{-1}(d) \end{pmatrix}.$$

2.2.2. *Suspension data.* A suspension datum for T is a collection of vectors $\{\zeta_\alpha\}_{\alpha \in \mathcal{A}}$ such that

(1) $\forall \alpha \in \mathcal{A}, \ Re(\zeta_\alpha) = \lambda_\alpha$.
(2) $\forall 1 \leq k \leq d-1, \ Im(\sum_{\pi_0(\alpha) \leq k} \zeta_\alpha) > 0$.
(3) $\forall 1 \leq k \leq d-1, \ Im(\sum_{\pi_1(\alpha) \leq k} \zeta_\alpha) < 0$.

To each suspension datum ζ, we can associate a translation surface $(X, \omega) = X(\pi, \zeta)$ in the following way. Consider the broken line L_0 on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_0^{-1}(j)}$ (in this order) for $j = 1, \ldots, d$ with starting point at the origin. Similarly, we consider the broken line L_1 defined by concatenation of the vectors $\zeta_{\pi_1^{-1}(j)}$ (in this order) for $j = 1, \ldots, d$ with starting point at the origin. If the lines L_0 and L_1 have no intersections other than the endpoints, we can construct a translation surface X by identifying each side ζ_j on L_0 with the side ζ_j on L_1 by a translation. The resulting surface is a translation surface endowed with the form $\omega = dz$. In general one has to use the Veech zippered rectangle's construction to construct such a suspension.

Let $I \subset X$ be the horizontal interval defined by $I = (0, \sum_\alpha \lambda_\alpha) \times \{0\}$. Then the interval exchange map T is precisely the one defined by the first return map to I of the vertical flow on X.

REMARK 2.2. The surface $X(\pi, \zeta)$ is a translation surface with a marked zero (the zero on the left of the interval I) and a marked separatrix (the interval I).

2.2.3. *Rauzy-Veech induction.* The Rauzy-Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to a certain subinterval J of I (see [**Rau79, MMY05**] for details).

We recall very briefly the construction. The *type* of T is 0 if $\lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}$ and 1 if $\lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)}$. We define a subinterval J of I by

$$J = \begin{cases} I \backslash T(I_{\pi_1^{-1}(d)}) & \text{if } T \text{ is of type 0;} \\ I \backslash I_{\pi_0^{-1}(d)} & \text{if } T \text{ is of type 1.} \end{cases}$$

The image of T by the Rauzy-Veech induction $\mathcal{R}$ is defined as the first return map of T to the subinterval J. This is again an interval exchange transformation, defined on d letters (see e.g. [**Rau79**]). Thus this defines two maps $\mathcal{R}_0$ and $\mathcal{R}_1$ by $\mathcal{R}(T) = (\mathcal{R}_\varepsilon(\pi), \lambda')$, where ε is the type of T. It is very easy to compute the new data.

(1) T has type 0, let k be $\pi_1^{-1}(k) = \pi_0^{-1}(d)$ with $k \leq d-1$. Then $\mathcal{R}_0(\pi_0, \pi_1) = (\pi_0', \pi_1')$ where $\pi_0 = \pi_0'$ and

$$\pi_1'^{-1}(j) = \begin{cases} \pi_1^{-1}(j) & \text{if } j \leq k \\ \pi_1^{-1}(d) & \text{if } j = k+1 \\ \pi_1^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(2) T has type 1, let k be $\pi_0^{-1}(k) = \pi_1^{-1}(d)$ with $k \leq d-1$. Then $\mathcal{R}_1(\pi_0, \pi_1) = (\pi_0', \pi_1')$ where $\pi_1 = \pi_1'$ and

$$\pi_0'^{-1}(j) = \begin{cases} \pi_0^{-1}(j) & \text{if } j \leq k \\ \pi_0^{-1}(d) & \text{if } j = k+1 \\ \pi_0^{-1}(j-1) & \text{otherwise.} \end{cases}$$

(3) The new lengths λ' and λ are related by a positive transition matrix $V_{\alpha\beta}$ with $V_{\alpha\beta}\lambda' = \lambda$. If T is of type 0 then let $(\alpha, \beta) = (\pi_0^{-1}(d), \pi_1^{-1}(d))$ otherwise let $(\alpha, \beta) = (\pi_1^{-1}(d), \pi_0^{-1}(d))$. With these notations $V_{\alpha\beta}$ is the

transvection matrix $I + E_{\alpha\beta}$ where $E_{\alpha\beta}$ is the matrix such that all the entries are zeroes except the entry (α, β) which is equal to 1.

Now if we iterate the Rauzy-Veech induction, we will get a sequence of transition matrices $(V_k)_k$. We can write $\mathcal{R}^{(n)}(\pi, \lambda) = (\pi^{(n)}, \lambda^{(n)})$ with $\left(\prod_{k=1}^{n} V_k\right) \lambda^{(n)} = \lambda$.

We can also define the Rauzy-Veech induction on the space of suspensions by

$$\mathcal{R}(\pi, \zeta) = (\mathcal{R}_\varepsilon \pi, V^{-1}\zeta).$$

REMARK 2.3. If $(\pi', \zeta') = \mathcal{R}(\pi, \zeta)$ then the two translation surfaces $X(\pi, \zeta)$ and $X(\pi', \zeta')$ are isometric i.e. they define the same surface in the moduli space.

For a combinatorial datum π, we call the *labelled Rauzy diagram* the graph whose vertices are all combinatorial data that can be obtained from π by the combinatorial Rauzy moves. From each vertices, there are two edges labelled 0 and 1 (the type) corresponding to the two combinatorial Rauzy moves. The dotted lines correspond to moves of type 1 and the other ones to moves of type 0.
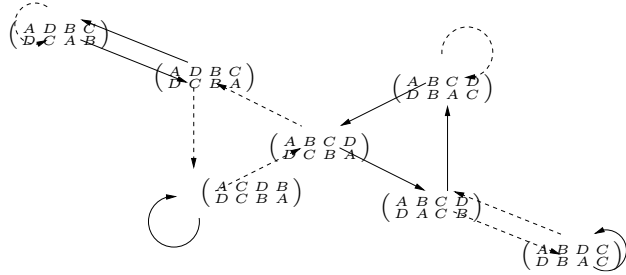
2.2.4. *Unlabelled Rauzy diagrams.* We have previously defined Rauzy induction and Rauzy diagrams, for labelled interval exchanges. One can also define the same for nonlabeled interval exchanges, as it was first, for which the corresponding combinatorial data is just a permutation of $\{1, \ldots, d\}$. These are obtained after identifying (π_0, π_1) with (π'_0, π'_1) if $\pi_1 \circ \pi_0^{-1} = \pi'_1 \circ \pi_0'^{-1}$. Note that the labelled Rauzy diagram is naturally a covering of the unlabelled Rauzy diagram.

Given a closed path γ in the unlabelled Rauzy diagram, one can define a matrix $V(\gamma)$ in the following way: we choose (π_0, π_1) a labelled combinatorial data corresponding to the endpoint of γ. Then we consider $\hat{\gamma}$ a lift of γ in the labelled Rauzy diagram. The path $\hat{\gamma}$ is not necessarily closed, but we can associate to it a matrix $\widehat{V}$ as before. Then V is obtained by multiplying $\widehat{V}$ on the right by a suitable permutation matrix.

2.2.5. *Connected components of the strata.* Let us fix an irreducible permutation π on d letters i.e. $\pi_0^{-1}(\{1, \ldots, k\}) \neq \pi_1^{-1}(\{1, \ldots, k\})$ for $1 \leq k \leq d-1$. Then the set of suspensions over π is a connected space and the map $\zeta \mapsto X(\pi, \zeta)$ is continuous. Thus all surfaces obtained by this construction belong to the same connected component of some strata, say $\mathcal{C}(\pi) \subset \mathcal{H}(\sigma)$. Moreover σ can be computed easily in terms of π. Now if $\mathcal{C}$ is any connected component of a strata then we define

$$\mathcal{T}(\mathcal{C}) = \{(\pi, \zeta),\ \mathcal{C}(\pi) = \mathcal{C},\ \zeta \text{ is a suspension data for } \pi\}.$$

The Rauzy-Veech induction is (almost everywhere) well defined and one-to-one on $\mathcal{T}(\mathcal{C})$. Let $\mathcal{H}(\mathcal{C})$ be the quotient.

Let $\widehat{\mathcal{C}}$ be the ramified cover over $\mathcal{C}$ obtained by considering the set of triplets (X, ω, l) where $(X, \omega) \in \mathcal{C}$ and l is a separatrix adjacent to a zero of ω. Then the

Veech zippered rectangle's construction provides (almost everywhere) a one-to-one map $Z : \mathcal{H}(\mathcal{C}) \to \hat{\mathcal{C}}$ (see [**Boi09**] for details).

$$\begin{array}{ccc} \mathcal{H}(\mathcal{C}) & \xrightarrow{\ Z\ } & \widehat{\mathcal{C}} \\ & & \downarrow \\ & & \mathcal{C} \quad \subset \quad \mathcal{H}(\sigma) \end{array}$$

A simple calculation shows that $d = \dim_{\mathbb{C}} \mathcal{H}(\sigma) = h(\sigma)$.

One can define the Teichmüller geodesic flow on $\mathcal{H}(\mathcal{C})$ by $g_t(\pi, \zeta) = (\pi, g_t \zeta)$. The Teichmüller flow on $\mathcal{C}$ lifts to a flow g_t on $\widehat{\mathcal{C}}$. It is easy to check that g_t is equivariant with Z i.e. $g_t Z = Z g_t$.

As usual, for counting problems on $\widehat{\mathcal{C}}$, we introduce $\widehat{N}(\sigma, T)$ the number of closed loops, for g_t, in $\widehat{\mathcal{C}}$ of length at most T. It can be shown that $\widehat{N}(\sigma, T)$ is also the set of periodic orbits γ for the Rauzy-Veech induction such that $||V(\gamma)|| \leq \exp(T)$ where $||A|| = \max_j \sum_i |A_{ij}|$.

THEOREM (Bufetov [**Buf06**]). $\lim_{T \to +\infty} \frac{1}{T} \text{Log } \widehat{N}(\sigma, T) = h(\sigma)$.

2.2.6. *Closed loops and pseudo-Anosov homeomorphisms.* We now recall the construction of Veech in order to obtain pseudo-Anosov homeomorphisms. Let π be an irreducible permutation and let γ be a closed loop in the Rauzy diagram associated to π. One can associate to γ a matrix $V(\gamma)$ (see above). Let us assume that V is *primitive* (i.e. there exists k such that for all i, j, the (i, j) entry of V^k is positive) and let $\theta > 1$ be its Perron-Frobenius eigenvalue. We choose a positive eigenvector λ for θ. It can be shown that V is symplectic [**Vee82**] thus let us choose τ an eigenvector for the eigenvalue θ^{-1} with $\tau_{\pi_0^{-1}(d)} > 0$. We form the vector $\zeta = (\lambda, \tau)$. We can show that ζ is a suspension data for π. Thus

$$\mathcal{R}(\pi, \zeta) = (\pi, V^{-1}\zeta) = (\pi, V^{-1}\lambda, V^{-1}\tau) = (\pi, \theta^{-1}\lambda, \theta\tau) = g_t(\pi, \lambda, \tau),$$

where $t = \text{Log}(\theta) > 0$.

Hence the two surfaces $X(\pi, \zeta)$ and $g_t X(\pi, \zeta)$ differ by some element of the mapping class group (see Remark 2.3). In other words there exists a pseudo-Anosov homeomorphism ϕ, with respect to the translation surface $X(\pi, \zeta)$, such that $D\phi = g_t$. In particular the dilatation of ϕ is θ. Note that by construction ϕ fixes the zero on the left of the interval I and also the separatrix adjacent to this zero (namely the interval I). In fact one has:

THEOREM (Veech). Let γ be a closed loop, based at π, in a unlabelled Rauzy diagram and $V = V(\gamma)$ be the associated transition matrix. Let us assume that V is primitive. Let λ be a positive eigenvector for the Perron eigenvalue θ of V and τ be an eigenvector $(\tau_{\pi_0^{-1}(1)} > 0)$ for the eigenvalue θ^{-1} of V. Then

(1) $\zeta = (\lambda, \tau)$ is a suspension datum for $T = (\pi, \lambda)$;
(2) The matrix $A = \left(\begin{smallmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{smallmatrix} \right)$ is the derivative map of a pseudo-Anosov homeomorphism ϕ on $X(\pi, \zeta)$;
(3) The dilatation of ϕ is θ;
(4) All pseudo-Anosov homeomorphisms fixing a separatrix are obtain by this construction.

REMARK 2.4. The renormalization matrix corresponds to the action of the corresponding pseudo-Anosov homeomorphism in relative homology of the underlying surface with respect to the zeroes of the Abelian differential.

REMARK 2.5. Veech's original proof uses at some step the Brouwer fixed point theorem in order to obtain the parameters "a" (related to τ). In turns out that in the construction we have presented the coordinates of the surface can be computed explicitly.

3. The construction

Let $n \geq 1$ be an integer and let $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{smallmatrix}\right)$ be a permutation. The Rauzy diagram associated to π has 7 elements (see the figure in Section 2.2.3) and corresponds to genus two surfaces with a single zero i.e. the stratum $\mathcal{H}(2)$.

We consider the closed loop γ in the Rauzy diagram based at π defined by the following Rauzy moves

$$\gamma = 0 \to (1 \to 1)^n \to 0 \to 1^n \to 0 \to (1 \to 1 \to 1)^n,$$

where $(a \to b)^n$ means the concatenation on the path $a \to b$ n times. Then the associated transition matrix $V(\gamma)$ is

$$V(\gamma) = \begin{pmatrix} 1 & n & n & n \\ 0 & 1+n & n(2+n) & n(1+n) \\ 0 & 0 & 1+n & n \\ 1 & 1+n & 1+n & 1+n \end{pmatrix}.$$

One has the following easy proposition.

PROPOSITION 3.1. The matrix V is primitive and its characteristic polynomial χ_V satisfies $\chi_V(X^3) = P(X)Q(X)$ where $P(X) = X^4 - X^3 - nX^2 - X + 1$.

Let $\theta = \theta_n > 1$ be the maximal real root of the polynomial $P(X)$ defined in Proposition 3.1. Then θ^3 is the Perron Frobenius eigenvalue of $V(\gamma)$.

In order to construct the flat surface associated to γ, one has to calculate an eigenvector λ of $V(\gamma)$ for the eigenvalue θ^3 and an eigenvector τ of $V(\gamma)$ for the eigenvalue θ^{-3}. It is not difficult to obtain

$$\begin{aligned} &\lambda_1 = n\theta, &&\lambda_2 = n\theta^2, &&\tau_1 = n\theta^{-1}, &&\tau_2 = n\theta^{-2}, \\ &\lambda_3 = n, &&\lambda_4 = \theta^3 - (1+n), &&\tau_3 = n, &&\tau_4 = \theta^{-3} - (1+n). \end{aligned}$$

We define the flat surface (X_n, ω_n) by the following way. For $i = 1, \ldots, 4$, we form the vectors $\zeta_i = \left(\begin{smallmatrix} \lambda_i \\ \tau_i \end{smallmatrix}\right)$. The surface is then obtained by gluing the edges with respect to the labelling (see the figure in Section 2.2.2).

REMARK 3.2. If A is the hyperbolic matrix $\left(\begin{smallmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{smallmatrix}\right)$ then by Veech's theorem A^3 stabilises the translation surface (X_n, ω_n) i.e. $A^3(X_n, \omega_n)$ and (X_n, ω_n) are isometric.

We shall actually prove more.

THEOREM 3.3. *The matrix A stabilises the translation surface (X_n, ω_n).*

Theorem 3.3 immediately implies our main result. We defer the proof of Theorem 3.3 in the next Section.

PROOF OF THEOREM 2.1. Let $\phi_n : X_n \to X_n$ be the pseudo-Anosov homeomorphism given by $D\phi_n = A$. Let us prove that ϕ_n has no fixed point of negative index. By Remark 2.4 the action on homology of ϕ_n is $V(\gamma)$. Hence the Lefschetz number is $L(\phi_n) = 2 - \mathrm{Tr}((\phi_n)_*) = 2 - 1 = 1$. The theorem is proven.   □

4. A proof of Theorem 3.3

To simplify let $(X, \omega) := (X_n, \omega_n)$ and $(X', \omega') = A(X, \omega)$. We only need to show that (X', ω') and (X, ω) define the same translation surface i.e. differ by some elements of the mapping class group. We label the vertices of the polygon that define (X, ω) in the following way. Let $P_1 = \binom{0}{0}$; We denote by P_i the complex number $\sum_{j=1}^{i-1} \zeta_j$ for $2 \leq i \leq 5$, $-\sum_{j=1}^{9-i} \zeta_{5-j}$ for $6 \leq i \leq 8$. The coordinates P_i' of the surface (X, ω) are easily calculated in terms of those of (X, ω), namely $P_i' = A \cdot P_i$.

We now cut the polygons that define (X, ω) and (X', ω') into $2n + 2$ pieces, as defined by Figure 1. Since this partition will be invariant by the hyperelliptic involution (rotation by 180°), we only give the coordinates of the decomposition for a half of the pieces:

$$Q_i := P_2 + \frac{i}{n+1}\overrightarrow{P_2 P_3} \qquad \text{for } i = 1, \ldots, n,$$

$$R_i := P_2 + \frac{i}{n}\overrightarrow{P_2 P_3} \qquad \text{for } i = 1, \ldots, n-1,$$

and

$$Q_i' = P_5' + \frac{i}{n+1}\overrightarrow{P_5' P_4'} \qquad \text{for } i = 1, \ldots, n,$$

$$R_i' = P_3' + \frac{i}{n}\overrightarrow{P_3' P_4'} \qquad \text{for } i = 1, \ldots, n-1.$$

In order to prove that the surfaces (X, ω) and (X', ω') are isometric, it suffices to show that we can "cut" the surface (X, ω) according to the decomposition and recover the surface (X', ω'). In other words we only have to prove that (we used the symmetry of the decomposition):

(a) $\overrightarrow{R_i R_i'} = \overrightarrow{Q_{i+1} Q_{i+1}'}$ for all $i = 1, \ldots, n-1$ and,
(b) $\overrightarrow{P_1 P_2'} = \overrightarrow{P_2 P_3'} = \overrightarrow{Q_1 Q_1'} = \overrightarrow{P_7 P_5'} = \overrightarrow{P_8 P_6'}$.

Equation (a) can be reformulated

$$(4.1) \quad \overrightarrow{P_2 P_3'} + \frac{i}{n}(\overrightarrow{P_3' P_4'} - \overrightarrow{P_2 P_3}) = \overrightarrow{P_2 P_5'} + \frac{i+1}{n+1}(\overrightarrow{P_5' P_4'} - \overrightarrow{P_2 P_3}) \qquad \forall\, i = 1, \ldots, n-1.$$

A useful remark for the next is $\theta^{-1} = 1 + n\theta + \theta^2 - \theta^3$.
Since $\overrightarrow{P_2 P_3'} = \overrightarrow{P_3' P_4'} = \zeta_3$, $\overrightarrow{P_2 P_3} = A^{-2}\zeta_3$, $\overrightarrow{P_2 P_5'} = \overrightarrow{P_2 P_3'} + \overrightarrow{P_3' P_5'} = \zeta_3 + A(\zeta_3 + \zeta_4)$, and $\overrightarrow{P_5' P_4'} = -A\zeta_4$, Equation (4.1) can be rewritten as

$$\zeta_3 + \frac{i}{n}(A\zeta_3 - A^{-2}\zeta_3) = \zeta_3 + A(\zeta_3 + \zeta_4) + \frac{i+1}{n+1}(-A\zeta_4 - A^{-2}\zeta_3),$$

or

$$\frac{i}{n}(\zeta_3 - A^{-3}\zeta_3) - \zeta_3 - \zeta_4 - \frac{i+1}{n+1}(-\zeta_4 - A^{-3}\zeta_3) = 0.$$

Then this is straightforward to check, indeed:

$$\frac{i}{n}\left(\binom{n}{n} - \binom{\theta^3 n}{\theta^{-3} n}\right) - \binom{n}{n} - \binom{\theta^3 - 1 - n}{\theta^{-3} - 1 - n} - \frac{i+1}{n+1}\left(-\binom{\theta^3 - 1 - n}{\theta^{-3} - 1 - n} - \binom{\theta^3 n}{\theta^{-3} n}\right) = \cdots = 0.$$
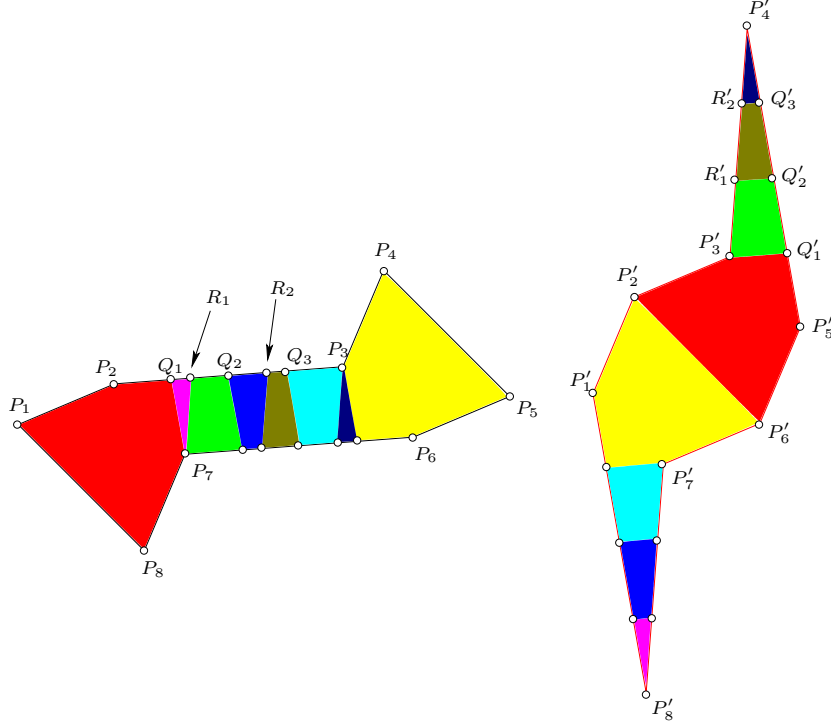
FIGURE 1. A decomposition of the surface (X, ω) and its image by the linear matrix A_n (for $n = 3$).

Equation (b) can be reformulated by

$$\overrightarrow{P_1 P_2'} = P_2' - P_1 = A\zeta_1 = \zeta_3$$
$$\overrightarrow{P_2 P_3'} = A(\zeta_1 + \zeta_2) - \zeta_1 = \zeta_3$$
$$\overrightarrow{P_7 P_5'} = A(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4) + \zeta_4 + \zeta_3 = \zeta_3$$
$$\overrightarrow{P_8 P_6'} = -A(\zeta_4 + \zeta_3 + \zeta_2) + \zeta_4 = \zeta_3,$$

and

$$\overrightarrow{Q_1 Q_1'} = \overrightarrow{P_2 P_5'} + \frac{1}{n+1}\left(\overrightarrow{P_5' P_4'} - \overrightarrow{P_2 P_3'}\right) = \zeta_3 + A\left(\zeta_3 + \zeta_4 - \frac{1}{n+1}(\zeta_4 + A^{-3}\zeta_3)\right) =$$

$$= \zeta_3 + A\left(\left(\begin{smallmatrix} \theta^3-1 \\ \theta^{-3}-1 \end{smallmatrix}\right) - \frac{1}{n+1}(\left(\begin{smallmatrix} \theta^3-1-n \\ \theta^{-3}-1-n \end{smallmatrix}\right) + \left(\begin{smallmatrix} n\theta^3 \\ n\theta^{-3} \end{smallmatrix}\right))\right) = \zeta_3.$$

This ends the proof.

REMARK 4.1. Our construction can be generalised to arbitrary genera in the following way. For any even integer $s = 2g \geq 4$, let us consider the permutation $\pi_s = \left(\begin{smallmatrix} 1 & 2 & 3 & \dots & s-1 & s \\ s & s-1 & \dots & 3 & 2 & 1 \end{smallmatrix}\right)$. The corresponding Rauzy diagram has $2^{s-1} - 1$ elements and correspond to the hyperelliptic component of the stratum $\mathcal{H}(2g - 2)$. Then let us consider the closed loop γ in the Rauzy diagram based at π_s defined by the

following Rauzy moves:

$$\gamma = 0 \rightarrow (1 \rightarrow \cdots \rightarrow 1)^n \rightarrow 0 \rightarrow (1 \rightarrow \cdots \rightarrow 1)^n \rightarrow \cdots \rightarrow$$
$$\rightarrow (1 \rightarrow 1)^n \rightarrow 1^n \rightarrow 0 \rightarrow (1 \rightarrow \cdots \rightarrow 1)^n,$$

where the first parentheses correspond to $s-2$ moves of type 1, the next one to $s-3$ moves of type 1, etc. The last parentheses correspond to $s-1$ moves of type 1. The characteristic polynomial χ_V of the associated transition matrix $V(\gamma)$ satisfies

$\chi_V(X^{g+1}) = P(X)Q(X)$ where $P(X)$ is a degree $2g$ polynomial with coefficient in X equal to -1.

With these notations we can show

THEOREM 4.2. *For each $g \geq 2$, there exists a sequence of pseudo-Anosov homeomorphisms $(\phi_n)_{n \geq 1}$ on a family of genus g translation surfaces (X_n, ω_n) with a single zero, and without fixed separatrix i.e. $L(\phi_n) = 1$.*

References

[AY81] P. ARNOUX and J. C. YOCCOZ – "Construction de difféomorphismes pseudo-Anosov (French)", *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), pp. 75–78.

[BL09] C. BOISSY and E. LANNEAU – "Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials", *Ergodic Theory Dynam. Systems* **29** no. 3 (2009), pp. 767–816.

[Boi09] C. BOISSY – "Classification of Rauzy classes in the moduli space of quadratic differentials", *preprint arXiv:0904.3826* (2009).

[Bro71] R. BROWN – "The Lefschetz fixed point theorem", *Scott, Foresman and Co., Glenview, Ill.-London* (1971).

[Buf06] A. I. BUFETOV – "Logarithmic asymptotic for the number of periodic orbits of the Teichmueller flow on Veech's space of zippered rectangles", preprint arXiv:math/0511035 (2006).

[EM08] A. ESKIN and M. MIRZAKHANI – "Counting closed geodesics in Moduli space", preprint, arXiv:0811.2362, (2008).

[FLP79] A. FATHI, F. LAUDENBACH and V. POÉNARU – "Travaux de Thurston sur les surfaces", *Astérisque* **66–67** (1979).

[Ham07] U. HAMENSTÄDT – "Bernoulli Measures for the Teichmüller Flow." Preprint ArXiV math.DS/0607386 (2007).

[KZ03] M. KONTSEVICH, and A. ZORICH – "Connected components of the moduli spaces of Abelian differentials with prescribed singularities", *Invent. Math.* **153** (2003), no. 3, pp.631–678.

[Lan08] E. LANNEAU – "Connected components of the strata of the moduli spaces of quadratic differentials with prescribed singularities", *Ann. Sci. École Norm. Sup.* (4) **41** (2008), pp. 1–56.

[Los09] J. LOS – "Infinite sequences of fixed point free pseudo-Anosov homeomorphisms.", *Erg. Th. Dyn. Sys*, to appear (2009).

[Ler04] F. LE ROUX – "Homéomorphismes de surfaces: théorèmes de la fleur de Leau-Fatou et de la variété stable. (French)", *Astérisque* **292** (2004).

[MMY05] S. MARMI, P. MOUSSA and J.-C. YOCCOZ – "The cohomological equation for Roth type interval exchange transformations", *Journal of the Amer. Math. Soc.* **18** (2005), pp. 823–872.

[MT02] H. MASUR and S. TABACHNIKOV–"Rational billiards and flat structures", *Handbook of dynamical systems*, **1A**, *North-Holland, Amsterdam* (2002), pp. 1015–1089.

[Rau79] G. RAUZY – "Échanges d'intervalles et transformations induites", *Acta Arith.* **34** (1979), pp. 315–328.

[Raf08] K. RAFI – "Closed geodesics in the thin part of moduli space", preprint (2008).

[Thu88] W. THURSTON – "On the geometry and dynamics of homeomorphisms of surfaces", *Bull. A.M.S.* **19** (1988), pp. 417–431.

[Vee82] W. Veech – "Gauss measures for transformations on the space of interval exchange maps", *Ann. of Math. (2)* **115** (1982), no. 1, pp. 201–242.

[Vee86] ――― "The Teichmüller geodesic flow", *Ann. of Math.(2)* **124** (1986), no. 3, pp. 441–530.

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