ON THE MINIMUM DILATATION OF BRAIDS ON PUNCTURED DISCS

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Abstract. We find the minimum dilatation of pseudo-Anosov braids on n-punctured discs for $3 \le n \le 8$. This covers the results of Song-Ko-Los ($n = 4$) and Ham-Song ($n = 5$). The proof is elementary, and uses the Lefschetz formula.

1. Introduction

There are many results on the minimum dilatation of pseudo-Anosov homeomorphisms. They includes bounds and specific examples [Pen91, Bri04, Lei04, HK06, Min06, TF06, Ven08, Tsa09, Hir09, AD10, KT10a, KT10b] as well as known values on closed and punctured surfaces [Zhi95, SKL02, Son05, CH08, HS07, Hir09, LT10b]. For the punctured discs, the case with three punctures is classical. Discs with four and five punctures were solved by [SKL02] and [HS07] using train track automata. In this paper we give a simple derivation for four and five punctures, and find the least dilatation for up to eight punctures, using methods introduced in [LT10b].

It is well-known that the braid group B_n is isomorphic to the mapping class group $\mathrm{Mod}(0, n+1)$ of the sphere with $n+1$ punctures (one of which is a marked point), that is the disc with n punctures. We denote by $\sigma_i \in B_n$ the classical generators [Bir75]. We shall prove

Theorem 1.1. *For $3 \le n \le 8$, the minimum dilatation δ_n of pseudo-Anosov n-braids is the Perron root (maximal root) of the following polynomials:*

n	$\delta_n \simeq$	polynomial	braid	stratum
3	2.61803	$X^2 - 3X + 1$	$\sigma_1\sigma_2^{-1}$	$(-1; -1^3)$
4	2.29663	$X^4 - 2X^3 - 2X + 1$	$\sigma_1\sigma_2\sigma_3^{-1}$	$(-1; -1^4, 1)$
5	1.72208	$X^4 - X^3 - X^2 - X + 1$	$\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}$	$(0; -1^5, 1)$
6	1.72208	$X^4 - X^3 - X^2 - X + 1$	$\sigma_2\sigma_1\sigma_2\sigma_1\,(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^2$	$(0; -1^5, 1)$
7	1.46557	$X^7 - 2X^4 - 2X^3 + 1$	$\sigma_4^{-2}(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6)^2$	$(2; -1^7, 1)$
8	1.41345	$X^8 - 2X^5 - 2X^3 + 1$	$\sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7)^5$	$(3, -1^8, 1)$

The notation for strata is explained in Section 2.4. Note that for $n = 6$ the pseudo-Anosov with smallest dilatation is identical to that for $n = 5$, but with a punctured

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degree-1 singularity, as conjectured by Venzke [Ven08]. For $n = 5$ and 7 the braids are part of a sequence described in [HK06, TF06, Ven08]. For $n = 8$ the minimizing braid is the one described by Venzke [Ven08].

In the next section, we introduce the tools we will use to prove Theorem 1.1, and sketch the proof in Section 2.7. The first two cases of the theorem ($n = 3$ and $n = 4$) are detailed in Sections 3 & 4. For the other cases (Sections 5–8) we use a computer to find reciprocal polynomials with a Perron root less than a given constant and for the combinatorics of the Lefschetz numbers; this is straightforward and elementary.[1] In an appendix we give the minimum dilatation for each stratum and provide explicit examples of a braid realizing each minimum (for $3 \leq n \leq 7$).

Presumably the method could be used on discs with more punctures: the limiting steps are (i) the generation of the list of polynomials on the generic stratum (with the most allowable degree-1 singularities); (ii) the eliminination of polynomials 'by hand' using the combinatorics of orbits or the action on singulaties. We also know by examining each stratum (see appendix, Section A.1.3) that sometimes polynomials cannot be ruled out in this manner; in those cases other techniques must be used, such as train track automata [LT10a].

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2. Preliminaries

For basic references on braid groups, mapping class groups, and pseudo-Anosov homeomorphisms see for example [Bir75, FLP79, Thu88].

2.1. **Braid groups.** Let $n \geq 3$ be an integer. The braid group B_n is defined by the presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \text{ and } \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1 \right\rangle.$$

The group B_n is naturally identified with the group of homotopy classes of orientation-preserving homeomorphisms of an n-punctured disc, fixing the boundary pointwise [Bir75]. One can see this as follows. Let $\beta \in B_n$ be a geometric n-braid, sitting in the cylinder $[0, 1] \times D$ with D the unit disc, whose n strands start at the puncture points of $\{0\} \times D$ and end at the puncture points of $\{1\} \times D$. The braid may be considered as the graph of the motion, as time goes from 1 to 0, of n points moving in the disc, starting and ending at the puncture points. It can be proved that this motion extends to a continuous family of homeomorphisms of the disc, starting with the identity and fixed on the boundary at all times. The end map of this isotopy is a homeomorphism $h : D \to D$, which is well-defined up to isotopy fixed at the punctures and the boundary. Conversely, given a homeomorphism $h : D \to D$ representing some element of the

[1] These *Mathematica* [Mat08] functions are included as Electronic Supplementary Material in the file `PseudoAnosovLite.m` and the example notebook `disc5.nb` for the disc with 5 punctures.

mapping class group, we want to get a geometric n-braid. By a well-known trick of Alexander, any homeomorphism of a disc which fixes the boundary is isotopic to the identity, through homeomorphisms fixing the boundary. The corresponding braid is then just the graph of the restriction of such an isotopy to the puncture points. Thus there is a canonical mapping of the braid group B_n to the mapping class group of the sphere with $n+1$ punctures $\mathrm{Mod}(0, n+1)$. Geometrically, the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ are induced by right Dehn half-twists around loops enclosing the punctures p_i and p_{i+1} (up to homotopy).

Obviously, orientation-preserving disc homeomorphisms and orientation-preserving sphere homeomorphisms with a fixed marked point define the same object.

Convention 2.1. *All sphere homeomorphisms in this paper will fix a marked point (regular or singular) on the sphere.*
All punctured surface homeomorphisms in this paper will (globally) fix the punctures.

One can think of punctured surfaces as surfaces with boundary (the boundary consisting then of the punctures).

2.2. **Pseudo-Anosov homeomorphisms.** The classification theorem of Thurston [Thu88] asserts that any orientation-preserving homeomorphism of a compact surface S is relative-isotopic to a finite-order, reducible, or pseudo-Anosov homeomorphism. In this paper we are interested in the last case. A homeomorphism ϕ is pseudo-Anosov if there exists a pair of ϕ-invariant transitive measurable foliations $(\mathcal{F}^s, \mathcal{F}^u)$ on a surface S of genus g that are transverse to each other and have common singularities Σ_i. Furthermore, there must exist a constant $\lambda = \lambda(\phi) > 1$ such that ϕ expands leaves of one foliation and shrinks those of the other foliation with coefficient λ (in the sense of measures). The number λ is a topological invariant called the *dilatation* of ϕ; the number $\log(\lambda)$ is the *topological entropy* of ϕ.

Thurston proved that if the foliations are orientable then

(1) The linear map ϕ_* defined on $H_1(S, \mathbb{R})$ has a simple eigenvalue $\rho(\phi_*) \in \mathbb{R}$ such that $|\rho(\phi_*)| > |x|$ for all other eigenvalues x;
(2) $|\rho(\phi_*)| > 1$ is the dilatation λ of ϕ.

In general $\lambda(\phi) \geq \rho(\phi_*)$, with equality if and only if the foliations are orientable.

Remark 2.2. *The number $|\rho(\phi_*)|$ is a* Perron number *(i.e. an algebraic integer $\lambda > 1$ whose conjugates λ' all satisfy $|\lambda'| < \lambda$). If $\rho(\phi_*) > 0$ then $\rho(\phi_*)$ is a Perron root of the polynomial $\chi_{\phi_*}(X)$, where χ_{ϕ_*} is the characteristic polynomial of ϕ_*. If $\rho(\phi_*) < 0$ then $-\rho(\phi_*)$ is a Perron root of the polynomial $\chi_{\phi_*}(-X)$. We also denote by $\rho(P)$ the largest root (in magnitude) of the polynomial P.*

Definition 2.3. *The isotopy class of a braid $\beta \in B_n$ is by definition the isotopy class of the corresponding homeomorphism h of the n-punctured disc (relative to the set $\bigcup_i \Sigma_i \cup \partial D^2$, where Σ_i are the marked points). We say that the braid β is pseudo-Anosov if the isotopy class of h is pseudo-Anosov; in this case we define the dilatation of β as the dilatation of the pseudo-Anosov homeomorphism.*

2.3. **Singularities.** Let Σ be a singularity of the stable (or unstable) foliation determined by ϕ such that there are m leaves passing through Σ ($m \geq 1$, $m \neq 2$). We will say that Σ is a singularity of ϕ of degree $k = m - 2$. We call *separatrices* leaves passing through singularities. For an orientable foliation, if $\Sigma \in S$ is a singularity of ϕ (necessarily of even degree, say $2d$) then there are $2(d+1)$ emanating separatrices: $d+1$ outgoing separatrices and $d+1$ ingoing separatrices. The collection $s = (k_1, \dots, k_m)$ of the degrees of the singularities is called the *singularity data* of the foliations. The Gauss-Bonnet formula gives $\sum_{i=1}^{m} k_i = 4g - 4$.

2.4. **Orientating double cover and strata.** The following construction is classical. Let h be a pseudo-Anosov homeomorphism of the sphere (that fixes a marked point) and let $(\mathcal{F}^s, \mathcal{F}^u)$ be the pair of *nonorientable* invariant foliations on the sphere $\mathbb{P}^1$ determined by h. There exists a canonical (ramified) double covering $\pi : S \to \mathbb{P}^1$ such that the foliations lift to two transverse measured foliations $(\widehat{\mathcal{F}^s}, \widehat{\mathcal{F}^u})$ on S that are orientable.

The homeomorphism h also lifts to a pseudo-Anosov homeomorphism ϕ on S, with the same dilatation. Observe that if we denote by τ the hyperelliptic involution determined by the covering π, then there are two lifts: ϕ and $\tau \circ \phi$ $(= \phi \circ \tau)$. We choose the lift ϕ such that $\rho(\phi_*) > 0$, so that the other lift satisfies $\rho((\tau \circ \phi)_*) < 0$. The covering $\pi : S \to \mathbb{P}^1$ is the minimal (ramified) covering such that the pullback of the foliations $(\mathcal{F}^s, \mathcal{F}^u)$ becomes orientable.

The set of critical values of π (i.e. the image of $\mathrm{Fix}(\tau)$ by π) on the sphere coincides exactly with the set of singularities of odd degree of the foliations.

Convention 2.4. *All surface homeomorphisms ϕ are defined on a surface S equipped with an involution τ. We will always assume that the homeomorphisms ϕ and τ commute (which is fulfilled if the maps are affine with respect to the Euclidian metric determined by the foliations* [LT10b]*).*

We can define strata for a pseudo-Anosov homeomorphism ϕ as follows. If ϕ fixes *globally* a set of r singularities (necessarily of the same degree k), we will use the superscript notation (k^r) for $k, \dots, k$ repeated r times. On the other hand, if ϕ fixes *pointwise* the singularities, we will use the notation $(k, \dots, k)$. For instance, the singularity data $(2^2, 2, 2)$ for ϕ on a genus-3 surface means that ϕ fixes a set consisting of two degree-2 singularities and fixes the other two degree-2 singularities pointwise.

For surfaces, we will allow *fake singularities*, i.e. regular points, and we will use the formulation "singularities of degree 0". One has the following straightforward lemma:

Lemma. *Let h be a pseudo-Anosov homeomorphism on the sphere and let Σ be a degree-k singularity of h. Let ϕ be a lift of h on the orientating double cover. If k is odd then Σ lifts to a single singularity of ϕ of degree $2k+2$; otherwise Σ lifts to two singularities of ϕ of degree k.*

Convention 2.1 prompts us to consider the following two definitions.

Definition 2.5. *A stratum on the sphere is an unordered set of integers $(k_2^{n_2}, \dots, k_m^{n_m})$ such that*

$$\sum_{i=2}^{m} n_i k_i = -k_1 - 4,$$

where $k_i \geq -1$ for any i. We will denote such a stratum by $s = (k_1; k_2^{n_2}, \dots, k_m^{n_m})$, the first element being special — it is the degree of the marked point of Convention 2.1.

Definition 2.6. *Let S be a hyperelliptic surface (of genus $g \geq 1$) and let τ be the hyperelliptic involution. The singularity data of a pseudo-Anosov ϕ determines a stratum on the surface*

$$(k_1^{n_1}, \dots, k_l^{n_l}, \underline{k_{l+1}}^{n_{l+1}}, \dots, \underline{k_m}^{n_m}),$$

where we underline the degree of the singularities that are permuted by the involution τ.

Example 2.7. *Let $(-1; -1^{17}, -1, 4^2, 2, 1^2, 3)$ be the singularity data of a pseudo-Anosov homeomorphism on the sphere. Then the corresponding singularity data for the lifts on the covering surface is $(0, 0^{17}, 0, \underline{4}^4, \underline{2}^2, 4^2, 8)$ and the genus of the covering surface is 10.*

2.5. **Pseudo-Anosov homeomorphisms and the Lefschetz fixed point theorem.** We recall briefly the Lefschetz fixed point formula for homeomorphisms on compact surfaces [Bro71, LT10b]. In the present section, let ϕ be a pseudo-Anosov homeomorphism of a compact surface S with *orientable* invariant measured foliations. If p is a fixed point of ϕ, we define the index of ϕ at p to be the algebraic number $\mathrm{Ind}(\phi, p)$ of turns of the vector $(x, \phi(x))$ when x describes a small loop around p.

Remark 2.8. *The index at a fixed point is easy to calculate for a pseudo-Anosov homeomorphism. Let $\rho(\phi_*)$ be the leading eigenvalue of ϕ_*. If $\rho(\phi_*) < 0$ then $Ind(\phi, p) = 1$ for any fixed point p (regular or singular). If $\rho(\phi_*) > 0$ then let Σ be a fixed degree-$2d$ singularity of ϕ ($d = 0$ for a regular point). It follows that either*

- *ϕ fixes each separatrix of Σ, hence $Ind(\phi, \Sigma) = 1 - 2(d + 1) < 0$, or*
- *ϕ permutes cyclically the outgoing separatrices of Σ, hence $Ind(\phi, \Sigma) = 1$.*

We will use the following corollary (see [LT10b] for a proof).

Corollary 2.9. *Let Σ be a fixed degree-$2d$ singularity of ϕ with $\rho(\phi_*) > 0$. If $Ind(\phi, \Sigma) = 1$ then*

$$\forall\ 1 \leq i \leq d,\ Ind(\phi^i, \Sigma) = 1$$

and

$$Ind(\phi^{d+1}, \Sigma) = 1 - 2(d + 1).$$

Theorem (Lefschetz fixed point theorem)**.** *Let ϕ be a homeomorphism on a compact surface S. Denote by $Tr(\phi_*)$ the trace of the linear map ϕ_* defined on the first homology group $H_1(S, \mathbb{R})$. Then the Lefschetz number $L(\phi) = 2 - Tr(\phi_*)$ satisfies*

$$L(\phi) = \sum_{p=\phi(p)} Ind(\phi, p).$$

For convenience, we isolate the key idea involved in the Lefschetz formula since we will use it often. If ϕ is a pseudo-Anosov homeomorphism on a compact surface S, with

$\rho(\phi_*) > 0$, then

$$2 - Tr(\phi_*) = \left(\sum_{p \text{ singular fixed points of } \phi} \mathrm{Ind}(\phi, p) \right) - \#\{\text{regular fixed points of } \phi\}, \quad \text{and}$$
$$2 + Tr(\phi_*) = \#\{\text{regular fixed points of } \tau \circ \phi\}.$$

In particular

$$2 - Tr(\phi_*) \leq \#\{\text{singular fixed points of } \phi\} - \#\{\text{regular fixed points of } \phi\}.$$

A very useful proposition for calculating Lefschetz numbers which we will use repeatedly without reference is the following.

Proposition 2.10. *Let P be a degree-$2g$ monic reciprocal polynomial*

$$P = X^{2g} + \alpha X^{2g-1} + \beta X^{2g-2} + \gamma X^{2g-3} + \cdots + \gamma X^3 + \beta X^2 + \alpha X + 1.$$

Let ϕ be a homeomorphism such that its characteristic polynomial satisfies $\chi_{\phi_*} = P$. Then

$$\begin{aligned} Tr(\phi_*) &= -\alpha, \\ Tr(\phi_*^2) &= \alpha^2 - 2\beta, \\ Tr(\phi_*^3) &= -\alpha^3 + 3\alpha\beta - 3\gamma. \end{aligned}$$

Proof. Use Newton's formula, as in [LT10b]. □

2.6. **Compatibility with the Lefschetz formula.** Let $P \in \mathbb{Z}[X]$ be a degree-$2g$ monic reciprocal polynomial. Let s be a stratum of the genus-g surface S. Let us assume that there exists a pseudo-Anosov homeomorphism ϕ on S with $\chi_{\phi_*} = P$ and singularity data s. The traces of ϕ_*^m (and so the Lefschetz numbers of iterates of ϕ) are easy to compute in terms of P. This gives algebraic constraints on the number of periodic orbits of ϕ as well as the action of ϕ on the separatrices.

Definition 2.11. *Let P be a degree-$2g$ monic reciprocal polynomial and s be a stratum of the surface S. We will say that P is compatible with s, or that P is admissible, if there are no algebraic obstructions with the Lefschetz formula.*

The following is clear:

Proposition 2.12. *Let h be a pseudo-Anosov homeomorphism on the sphere and let ϕ and $\tau \circ \phi$ be the two lifts on the covering surface S (with singularity data s). If $P = \chi_{\phi_*}$ then the polynomials $P(X)$ and $P(-X)$ are both compatible with s.*

We give two examples to illustrate the above proposition.

2.6.1. *First example.* The polynomial $P = X^2 - 3X + 1$ is compatible with the stratum $s = (0^4, \underline{0}^2)$. Indeed, let us assume there exists an Anosov homeomorphism ϕ on the torus with singularity data s. The Lefschetz numbers of the first five iterates of ϕ are $(-1, -5, -16, -45, -121)$. We can check that the number of periodic orbits of length $m = (1, 2, 3, 4, 5)$ are $(0, 2, 5, 10, 24)$. The same calculation shows that $P(-X)$ is also compatible with s. Of course an Anosov homeomorphism realizing P does actually exists, e.g. $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{Z}) \simeq \mathrm{Mod}(1, 0)$.

2.6.2. *Second example.* The polynomial $P = X^6-3X^5+4X^4-5X^3+4X^2-3X+1$ is compatible with $s = (0^5, 0, 4^2)$, but $P(-X)$ is not. Let us show the latter assertion. Assume there exists ϕ with $\chi_{\phi_*}(X) = P(-X)$. Since $\rho(P) > 0$, one has $\rho(P(-X)) = \rho(\phi_*) < 0$; thus the index at each fixed point (singular and regular) is +1 (see Remark 2.8). We have $L(\phi) = 5$ so that ϕ has 5 fixed points. In particular ϕ has at least 3 regular fixed points. Now let $\psi = \phi^2$. Of course $\rho(\psi_*) > 0$, so the index of ψ at the singularities is at most 1, and the index of ψ at the regular points is -1. Since ϕ has at least three regular fixed points, the same is true for ψ. The Lefschetz formula applied to ψ gives $L(\psi) \leq 2-3 = -1$. But a straightforward calculation produces $L(\psi) = 1$ — a contradiction.

Remark 2.13. *The converse of Proposition 2.12 is not true in general. For instance the polynomials* $P(X) = X^6 - 3X^5 + 2X^4 + 2X^2 - 3X + 1$ *and* $P(-X)$ *are both compatible with the stratum* $s = (0^5, 0, 4^2)$*. Nevertheless there are no pseudo-Anosov homeomorphisms* ϕ *on a genus-3 surface with* $\chi_{\phi_*}(X) = P(X)$ *or* $P(-X)$ *(see* [LT10a]*).*

We end this section by sketching a proof of our result.

2.7. **Outline of a proof of Theorem 1.1.** We outline the strategy for calculating the minimum dilatation δ_n of pseudo-Anosov n-braids on the disc. Obviously δ_n is the minimum of $\delta(s)$ where s is taken over all the strata on the sphere with n punctures. Since we have a candidate $\rho(P_n)$ for δ_n (given by train track automata, see [LT10a]) it is sufficient to show that $\delta(s) \geq \rho(P_n)$ for each stratum s. The steps are as follows:

(1) Fix a stratum s on the sphere with n punctures. Let h be a pseudo-Anosov homeomorphism on the sphere $\mathbb{P}^1$ that realizes $\delta(s)$. Let us denote by $\pi : S \to \mathbb{P}^1$ the orientating double cover (genus$(S) = g$). The homeomorphism h lifts to two homeomorphisms ϕ and $\tau\circ\phi$ on S, where τ is the hyperelliptic involution of S. (By convention, we choose the lift ϕ so that $\rho(\phi_*) > 0$.) These two homeomorphisms also determine a stratum s' on S. Finally, there exists a reciprocal, monic, degree-$2g$ polynomial $P \in \mathbb{Z}[X]$ with a Perron root $\delta(s)$ and, by Proposition 2.12, both $P(X)$ and $P(-X)$ are compatible with the stratum s'.
(2) We list all degree-$2g$ reciprocal monic polynomials P, with a Perron root $\rho(P)$, $1 < \rho(P) < \rho(P_n)$ (there is a finite number of such polynomials [AY81, Iva88]). If $\delta(s) < \rho(P_n)$ then $\delta(s) = \rho(P)$ for some other P in our list. We will rule out all such polynomials.
(3) Take a polynomial P from our list.
(4) If P is not compatible with stratum s' then go back to step (3) and move on to the next polynomial.
(5) If $P(-X)$ is not compatible with the stratum s' then go back to step (3) and move on to the next polynomial.
(6) We exclude the remaining polynomials using combinatorics of the orbits or the action on the singularities.

Remark 2.14. *One can actually obtain a more precise result: our techniques apply to (almost) all strata of the discs (see the appendix).*

3. Disc with 3 punctures

The only stratum on the sphere with four singularities is $s = (-1; -1^3)$. The corresponding stratum on the associated orientating double cover S is $s' = (0, 0^3)$. Thus the genus of S is 1. The candidate for $\delta(s)$ is the Perron root of $X^2 - 3X + 1$. A direct calculation shows that there are no degree-2 monic reciprocal polynomials with a Perron root strictly less than that of $X^2 - 3X + 1$. The statement is proved.

4. Disc with 4 punctures

There are two strata to consider on the sphere: $s_1 = (0; -1^4)$ and $s_2 = (-1; -1^4, 1)$. We check that the corresponding strata on the orientating double cover S_i are, respectively,

$$s'_1 = (0^4, \underline{0}^2) \qquad \text{and} \qquad s'_2 = (0, 0^4, 4).$$

The candidate for δ_4 is the Perron root of $X^4 - 2X^3 - 2X + 1$.

- For the stratum s_1 the surface upstairs, S_1, has genus 1 so again there is nothing to prove (see Section 3).
- For the stratum s_2 the surface upstairs, S_2, has genus 2. The next lemma shows that there are four degree-4 monic reciprocal polynomials with a Perron root strictly less than our candidate.

Lemma 4.1. *The degree-4 monic reciprocal polynomials with a Perron root strictly less than $\rho(X^4 - 2X^3 - 2X + 1) \simeq 2.29663$ are*

polynomial	*Perron root*
$X^4 - X^3 - X^2 - X + 1$	*1.72208*
$X^4 - 2X^3 + X^2 - 2X + 1$	*1.88320*
$X^4 - X^3 - 2X^2 - X + 1$	*2.08102*
$X^4 - 3X^3 + 3X^2 - 3X + 1$	*2.15372*

Proof. Let $P = X^4 + \alpha X^3 + \beta X^2 + \alpha X + 1$ be a reciprocal Perron polynomial and let Q be such that $X^4 Q(X + X^{-1}) = P$. Observe that λ is a root of P if and only if $t = \lambda + \lambda^{-1}$ is a root of $Q = X^2 - aX + b$ where $\alpha = -a$ and $\beta = b + 2$. Since t is an increasing function of λ, the polynomial Q admits a Perron root t. Let t' be its (real) Galois conjugate so that $|t'| < t$.

If $t' > 0$ then $a = t + t' > 0$. If $t' < 0$ then $|t'| = -t' < t$ thus $t + t' = a > 0$. Hence $a > 0$. Now $2.29663 + (2.29663)^{-1} = 2.73205$. If $a \geq 4$ then the Perron root of Q satisfies

$$2 + \sqrt{4 - b} \leq t = \tfrac{1}{2}(a + \sqrt{a^2 - 4b}) < 2.73205.$$

Thus $4 - 0.73205^2 < b \leq 4$ and we derive $a = b = 4$. Hence $t = 2$ which is a contradiction. Finally $a = 1, 2,$ or 3.

If $a = 1$ then $t = \frac{1}{2}(1 + \sqrt{1 - 4b}) < 2.73205$. Hence $b > -4.73205$ so that $-4 \leq b \leq 0$. The cases $b = 0, -1, -2$ lead to non-Perron roots and the cases $b = -3, -4$ lead to the two polynomials $Q = X^2 - X - 3$ and $Q = X^2 - X - 4$, that is, the polynomials $P = X^4 - X^3 - X^2 - X + 1$ and $P = X^4 - X^3 - 2X^2 - X + 1$ respectively.

The second case $a = 2$ leads to $t = 1 + \sqrt{1-b} < 2.73205$. Hence $b > -2$ and $b = -1$ or $b = 0$. The case $b = 0$ leads to non-Perron roots and the case $b = -1$ leads to the polynomial $Q = X^2 - 2X - 1$, that is, the polynomial $P = X^4 - 2X^3 + X^2 - 2X + 1$.

The third case $a = 3$ leads to $t = \frac{1}{2}(3 + \sqrt{9-4b}) < 2.73205$. Hence $b > 0.732051$, implying $b = 1$. The corresponding polynomial is $Q = X^2 - 3X + 1$, that is, the polynomial $P = X^4 - 3X^3 + 3X^2 - 3X + 1$. □

The next lemma will rule out the first and the third polynomials.

Lemma 4.2. *Let ϕ be a pseudo-Anosov homeomorphism on S_2 with singularity data* $(4, 0)$*. If $\rho(\phi_*) > 0$, then*

$$Tr(\phi_*) \geq 2.$$

Proof. If $\Sigma \in S_2$ is the degree-four singularity then one has

$$2 - \mathrm{Tr}(\phi_*) = \mathrm{Ind}(\phi, \Sigma) - \#\mathrm{Fix}(\phi),$$

where $\mathrm{Fix}(\phi)$ is the set of *regular* fixed points of ϕ. Since $\#\mathrm{Fix}(\phi) \geq 1$ and $\mathrm{Ind}(\phi, \Sigma) \leq 1$, we get the desired inequality. □

This rules out the first and the third polynomials since for those we would have $\mathrm{Tr}(\phi_*) = 1$. Now let us show that the second polynomial is also inadmissible.

Assume the second polynomial is admissible and let ϕ be a pseudo-Anosov homeomorphism such that $\chi_{\phi_*}(X) = X^4 - 2X^3 + X^2 - 2X + 1$. Then $L(\phi) = 0$. Since the singularity data of ϕ is $(4, 0)$, ϕ fixes the degree-four singularity (with positive index) and has only one regular fixed point (which is also fixed by τ by construction). Since $L(\phi^2) = 0$ the same argument applies to ϕ^2 so that ϕ has no period-2 orbits.

Now let us count the number of fixed points of $\tau \circ \phi$. These are fixed points of ϕ^2. Indeed if $\tau \circ \phi(p) = p$ then $\phi(p) = \tau(p)$ and $\phi^2(p) = \tau \circ \phi(p) = p$. By the above discussion, $\tau \circ \phi$ has only one regular fixed point (hence also fixed by τ). Thus $\tau \circ \phi$ has two fixed points in total (one singular and one regular). But

$$L(\tau \circ \phi) = 2 - \mathrm{Tr}((\tau \circ \phi)_*) = 2 + \mathrm{Tr}(\phi_*) = 4.$$

Since $L(\tau \circ \phi)$ is the number of fixed points (singular and regular, see Remark 2.8) of $\tau \circ \phi$, we get a contradiction.

Finally, to finish the proof of the $n = 4$ case let us show that the fourth polynomial is inadmissible. Assume it is admissible, which implies $L(\phi) = -1$. Then ϕ must fix the singularity with positive index (since otherwise $L(\phi) \leq -5$) and has two regular fixed points. Since $L(\phi^2) = -1$ as well, the same argument shows that ϕ^2 has two regular fixed points (the same as ϕ) and so ϕ has no period-2 orbits.
As we have seen, fixed points of $\tau \circ \phi$ are also fixed points of ϕ^2; thus $\tau \circ \phi$ has three fixed points in total (one singular and two regular). But

$$3 = L(\tau \circ \phi) = 2 - \mathrm{Tr}((\tau \circ \phi)_*) = 2 + \mathrm{Tr}(\phi_*) = 5,$$

which is a contradiction. Hence $\delta(s_2) = \rho(X^4 - 2X^3 - 2X^2 + 1)$ and Theorem 1.1 for $n = 4$ is proved.

5. Disc with 5 punctures

There are four strata to consider on the sphere:

$$s_1 = (1; -1^5), \qquad s_2 = (0; -1^5, 1), \qquad s_3 = (-1; -1^5, 2) \qquad \text{and} \qquad s_4 = (-1; -1^5, 1^2).$$

The corresponding strata on the orientating double cover S_i are

$$s_1' = (0^5, 4), \qquad s_2' = (0^5, 4, \underline{0}^2), \qquad s_3' = (0^5, 0, \underline{2}^2) \quad \text{and} \quad s_4' = (0^5, 0, 4^2).$$

The candidate for δ_5 is the Perron root of $X^4 - X^3 - X^2 - X + 1$. For the first three cases, there is nothing to prove since by Lemma 4.1 there are no degree-4 monic reciprocal polynomials having a Perron root less than δ_5. We use a computer for the last case (analogous to Lemma 4.1). This is straightforward.

The surface S_4 has genus 3. There are 9 degree-6 monic reciprocal polynomials with a Perron root strictly less than our candidate, namely:

polynomial	Perron root
$X^6 + X^5 - X^4 - 3X^3 - X^2 + X + 1$	1.32472
$X^6 - X^4 - X^3 - X^2 + 1$	1.40127
$X^6 - X^5 + X^4 - 3X^3 + X^2 - X + 1$	1.46557
$X^6 - X^5 - X^3 - X + 1$	1.50614
$X^6 - X^5 - X^4 + X^3 - X^2 - X + 1$	1.55603
$X^6 - 2X^5 + 3X^4 - 5X^3 + 3X^2 - 2X + 1$	1.56769
$X^6 - X^4 - 2X^3 - X^2 + 1$	1.58235
$X^6 - 2X^5 + 2X^4 - 3X^3 + 2X^2 - 2X + 1$	1.63557
$X^6 - X^5 + X^4 - 4X^3 + X^2 - X + 1$	1.67114

We will use a simple algebraic criterion in order to eliminate these polynomials.

Lemma 5.1. *Let ϕ be a pseudo-Anosov homeomorphism on S_4 with singularity data $(4, 4, 0)$, with $\rho(\phi_*) > 0$. Then $L(\phi^m) \leq 1$ for any $m \geq 1$. In addition, if $L(\phi) \geq 0$ then*

$$L(\phi^3) \leq -11.$$

Proof of the lemma. The first remark $L(\phi^m) \leq 1$ is trivial: positive indices arise only from singularities, so that $L(\phi^m) \leq 2$. But since ϕ fixes a regular point for any m the first statement holds.

Now let us assume in addition that $L(\phi) \geq 0$. Since ϕ fixes a regular point, if ϕ permutes the two singularities then $L(\phi) < 0$, a contradiction. Hence, ϕ fixes the two singularities, with at least one of the singularities having positive index. Assume that the other singularity is fixed with *negative* index. Then $L(\phi) \leq 1 - 5 - \#\{\text{Regular fixed points}\} < 0$ which is again a contradiction. Hence the index at the two singularities is positive. Thus the third power of ϕ fixes the singularities and their separatrices, and

$$L(\phi^3) = -5 - 5 - \#\{\text{Regular fixed points of } \phi^3\} \leq -11$$

which proves the lemma. □

The above lemma can be restated as follows:

Lemma 5.2. *Let* $P = X^6 + \alpha X^5 + \beta X^4 + \gamma X^3 + \beta X^2 + \alpha X + 1$ *be a degree-6 monic reciprocal polynomial. If there exists a pseudo-Anosov homeomorphism* ϕ *with* $\rho(\phi_*) > 0$, $\chi_{\phi_*} = P$, *and with singularity data* $(4^2, 0)$, *then* $2 + \alpha \leq 1$ *and* $2 - \alpha^2 + 2\beta \leq 1$. *Moreover if* $2 + \alpha \geq 0$ *then* $2 + \alpha^3 - 3\alpha\beta + 3\gamma \leq -11$.

None of the 9 polynomials above satisfies this algebraic criterion; thus Theorem 1.1 for $n = 5$ is proved.

6. Disc with 6 punctures

The techniques of the previous sections can also be applied to the case $n = 6$. However the complexity becomes huge so we rely on a set of *Mathematica* [Mat08] functions to test whether a polynomial P is compatible with a given stratum. This is straightforward: we simply try all possible permutations of the singularities and separatrices (there is a finite number of these), and calculate the contribution to the Lefschetz numbers for each iterate of ϕ. We then check whether the deficit in the Lefschetz numbers can be exactly compensated by regular periodic orbits. If not, the polynomial P cannot correspond to a characteristic polynomial of some pseudo-Anosov homeomorphism on that stratum. If it can, we also test the polynomial $P(-X)$ corresponding to $\rho(\phi_*) < 0$.

6.1. **Puncturing a singularity of higher degree.** So far we have considered strata on the sphere where only singularities of degree -1 or the marked singularity (corresponding to the disc's boundary) are punctured. In general, it is possible to have higher-degree punctured singularities. This does not yield any new pseudo-Anosov homeomorphisms: for instance, taking the stratum $(0; -1^6, 2)$ of the sphere (corresponding to a stratum on the disc with 6 punctures) and puncturing the degree-2 singularity gives a stratum of the sphere corresponding to a stratum on the disc with 7 punctures. However, the pseudo-Anosov homeomorphisms on this new stratum are identical to the original since the degree-2 singularity must be fixed anyways.

Another example is to puncture the degree-1 singularity of the stratum $(0; -1^5, 1)$ (arising from a stratum of the disc with 5 punctures; see [Ven08]). This produces a braid on the disc with 6 punctures with dilatation 1.72208; namely $\sigma_2\sigma_1\sigma_2\sigma_1\,(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^2$. We will show in the next section that $\delta_6 \simeq 1.72208$. For this reason, the pseudo-Anosov braid with the least dilatation on the disc with 6 punctures actually arises from a stratum of the disc with 5 punctures.

Puncturing a degree-1 singularity in the stratum $(-1; -1^5, 1, 1)$ does not give new pseudo-Anosov homeomorphisms, since this merely eliminates the ones that permute the two degree-1 singularities. However, the least dilatation on the stratum $(-1; -1^5, 1, 1)$ with a punctured degree-1 singularity can be larger than the unpunctured case. We do not consider such cases here. See [HS07, Ven08] for more details.

Thus we reduce Theorem 1.1, case $n = 6$ to the following:

Theorem 6.1. *Let s_i be a stratum of the sphere where the punctures are only singularities of degree one or regular points. Then $\delta(s_i) > \rho(X^4 - X^3 - X^2 - X + 1) \simeq 1.72208$.*

6.2. **Strata of the six punctured disc.** In Table 1 we present the list of all possible strata of the six-punctured disc with the corresponding strata on the surface S, and the list of polynomials with Perron root strictly less than $\rho(X^4 - X^3 - X^2 - X - 1)$, that of the candidate polynomial. Under "# polynomials" we give the number of polynomials P with Perron root strictly less than our candidate, and under "# compatible" we give how many of these are both compatible ($P(X)$ and $P(-X)$) with the Lefschetz formula for that stratum.

case	stratum on $\mathbb{P}^1$	stratum on S	genus of S	# polynomials	# compatible
s'_1	$(2; -1^6)$	$(0^6, \underline{2}^2)$	2	0	0
s'_2	$(1; -1^6, 1)$	$(0^6, 4, 4)$	3	9	0
s'_3	$(0; -1^6, 2)$	$(0^6, \underline{0}^2, \underline{2}^2)$	2	0	0
s'_4	$(-1; -1^6, 3)$	$(0^6, 0, 8)$	3	9	0
s'_5	$(0; -1^6, 1^2)$	$(0^6, 4^2, \underline{0}^2)$	3	9	0
s'_6	$(-1; -1^6, 1, 2)$	$(0^6, 0, 4, \underline{2}^2)$	3	9	0
s'_7	$(-1; -1^6, 1^3)$	$(0^6, 0, 4^3)$	4	148	2

TABLE 1. Strata of the six-punctured disc.

6.2.1. *The cases s_2, s_4, s_5 and s_6.* We use a *Mathematica* [Mat08] program to eliminate the polynomials using the Lefschetz formula, and find that there are no such polynomials.

6.2.2. *The case s_7.* Since the number of polynomials is large, we use a *Mathematica* [Mat08] program to eliminate the polynomials using the Lefschetz formula. Among the 148 polynomials, there are only two polynomials that satisfy this criterion, namely

polynomial	Perron root
$X^8 - 2X^7 + 2X^6 - 4X^5 + 5X^4 - 4X^3 + 2X^2 - 2X + 1$	1.59937
$X^8 - 3X^7 + 4X^6 - 7X^5 + 10X^4 - 7X^3 + 4X^2 - 3X + 1$	1.67114

We will rule out these two polynomials by considering the ϕ-action on the singularities.

For the first polynomial, assume there exists ϕ such that $\chi_{\phi_*} = P$. We first show that ϕ fixes only one degree-4 singularity. Recall that ϕ fixes (at least) one regular point. If ϕ permutes the singularities then $L(\phi) \leq -1$, contradicting $L(\phi) = 0$. If ϕ fixes pointwise the singularities (necessarily with positive index) then it has three regular fixed points. Hence ϕ^2 has at least three regular fixed points and still fixes pointwise the singularities (with positive index). Thus

$$L(\phi^2) \leq 3 - 3 = 0$$

contradicting $L(\phi^2) = 2$. We have proved that ϕ fixes only one singularity and so the same is true for $\tau \circ \phi$. Since $L(\tau \circ \phi) = 4$ one can conclude that ϕ has 3 regular fixed points. This implies that $L(\phi) \leq 1 - 3 < 0$ contradicting again $L(\phi) = 0$.

For the second polynomial, since $L(\phi) = 1$ and $L(\phi^2) = 0$ the same argument shows that ϕ fixes only one degree-4 singularity, so the same is true for $\tau \circ \phi$. Since $L(\tau \circ \phi) = 5$ one can conclude that $\tau \circ \phi$ has 4 regular fixed points. In particular ϕ has at least two regular fixed points and so $L(\phi) \leq -1$, which is not possible.

Theorem 1.1 for $n = 6$ is proved.

7. Disc with 7 punctures

Again the techniques of the previous sections can be applied to the seven-punctured disc. Table 2 gives the list of all possible strata of the seven-punctured disc with the corresponding strata on the surface S, and the number of polynomials with a dilatation smaller than the candidate polynomial ($\rho(X^7 - 2X^4 - 2X^3 + 1) = \delta_7 \simeq 1.46557$). The only stratum we have to check is s_{12}. The two compatible polynomials are

polynomial	Perron root
$X^{10} - 4X^9 + 5X^8 - X^7 - 2X^6 + 2X^5 - 2x^4 - x^3 + 5x^2 - 4x + 1$	1.40127
$X^{10} - 2X^9 + x^7 + x^6 - 2x^5 + x^4 + x^3 - 2x + 1$	1.45799

There are no pseudo-Anosov homeomorphisms that realize these polynomials. Indeed, if we assume there is one, then the same arguments as in the previous section give a contradiction on the number of periodic points (periodic orbits of length 14 and of length 7, respectively). We leave the details to the reader.

case	stratum on $\mathbb{P}^1$	stratum on S	genus of S	# polynomials	# compatible
s'_1	$(3; -1^7)$	$(0^7, 8)$	3	2	0
s'_2	$(2; -1^7, 1)$	$(0^7, 4, \underline{2}^2)$	3	2	0
s'_3	$(1; -1^7, 2)$	$(0^7, 4, \underline{2}^2)$	3	2	0
s'_4	$(0; -1^7, 3)$	$(0^7, 8, \underline{0}^2)$	3	2	0
s'_5	$(-1; -1^7, 4)$	$(0^7, 0, \underline{4}^2)$	3	2	0
s'_6	$(1; -1^7, 1^2)$	$(0^7, 4^2, 4)$	4	21	0
s'_7	$(0; -1^7, 1, 2)$	$(0^7, 4, \underline{0}^2, \underline{2}^2)$	3	2	0
s'_8	$(-1; -1^7, 1, 3)$	$(0^7, 0, 4, 8)$	4	21	0
s'_9	$(-1; -1^7, 2^2)$	$(0^7, 0, \underline{2}^4)$	3	2	0
s'_{10}	$(0; -1^7, 1^3)$	$(0^7, 0, 4^3)$	4	21	0
s'_{11}	$(-1; -1^7, 1^2, 2)$	$(0^7, 0, 4^2, \underline{2}^2)$	4	21	0
s'_{12}	$(-1; -1^7, 1^4)$	$(0^7, 0, 4^4)$	5	227	2

Table 2. Strata of the seven-punctured disc.

8. Disc with 8 punctures

Again the techniques of the previous sections can be applied to the eight-punctured disc. Table 3 gives the list of all strata of the eight-punctured disc with the corresponding strata on the surface S, and the number of polynomials with a dilatation smaller than the candidate polynomial ($\rho(X^8 - 2X^5 - 2X^3 + 1) = \delta_8 \simeq 1.41345$), as given by [Ven08]. There are five polynomials to check, and these are readily eliminated by examining the detailed action on singularities. The method is identical to the previous sections so we omit the detailed argument here.

case	stratum on $\mathbb{P}^1$	stratum on S	genus of S	# polynomials	# compatible
s'_1	$(4;-1^8)$	$(0^8,\underline{4}^2)$	3	2	0
s'_2	$(3;-1^8,1)$	$(0^8,4,8)$	4	15	0
s'_3	$(2;-1^8,2)$	$(0^8,\underline{2}^2,\underline{2}^2)$	3	2	0
s'_4	$(1;-1^8,3)$	$(0^8,4,8)$	4	15	0
s'_5	$(0;-1^8,4)$	$(0^8,\underline{0}^2,\underline{4}^2)$	3	2	0
s'_6	$(-1;-1^8,5)$	$(0^8,0,12)$	4	15	1
s'_7	$(2;-1^8,1^2)$	$(0^8,\underline{2}^2,4^2)$	4	15	0
s'_8	$(1;-1^8,1,2)$	$(0^8,4,4,\underline{2}^2)$	4	15	0
s'_9	$(0;-1^8,1,3)$	$(0^8,\underline{0}^2,4,8)$	4	15	0
s'_{10}	$(-1;-1^8,1,4)$	$(0^8,0,4,\underline{4}^2)$	4	15	0
s'_{11}	$(0;-1^8,2^2)$	$(0^8,\underline{0}^2,\underline{2}^4)$	3	2	0
s'_{12}	$(-1;-1^8,2,3)$	$(0^8,0,\underline{2}^2,8)$	4	15	0
s'_{13}	$(1;-1^8,1^3)$	$(0^8,4,4^3)$	5	129	2
s'_{14}	$(0;-1^8,1^2,2)$	$(0^8,\underline{0}^2,4^2,\underline{2}^2)$	4	15	0
s'_{15}	$(-1;-1^8,1^2,3)$	$(0^8,0,4^2,8)$	5	129	0
s'_{16}	$(-1;-1^8,1,2^2)$	$(0^8,0,4,\underline{2}^4)$	4	15	0
s'_{17}	$(0;-1^8,1^4)$	$(0^8,\underline{0}^2,4^4)$	5	129	2
s'_{18}	$(-1;-1^8,1^3,2)$	$(0^8,0,4^3,\underline{2}^2)$	5	129	0
s'_{19}	$(-1;-1^8,1^5)$	$(0^8,0,4^5)$	6	1096	0

Table 3. Strata of the eight-punctured disc.

Appendix A. Minimum dilatation for each stratum

In this appendix, we give the minimum dilatation for each stratum and give explicit examples of a braid realizing each minimum, for $3 \le n \le 7$. We first detail the $n = 5$ case (Section A.1) and then give the other cases without proof since the techniques are the same (Section A.2).

A.1. **Strata for the disc with 5 punctures.** Here we give an alternative proof to that of Ham & Song [HS07]. There are four strata to consider on the sphere:

$$s_1 = (1;-1^5), \qquad s_2 = (0;-1^5,1), \qquad s_3 = (-1;-1^5,2) \qquad \text{and} \qquad s_4 = (-1;-1^5,1^2).$$

The corresponding strata on the orientating double cover S_i are

$$s'_1 = (0^5, 4), \qquad s'_2 = (0^5, 4, \underline{0}^2), \qquad s'_3 = (0^5, 0, \underline{2}^2) \quad \text{and} \quad s'_4 = (0^5, 0, 4^2).$$

The candidate dilatations for each stratum are given in Table 5. We will examine each stratum in turn.

A.1.1. *The strata s_1 and s_2.* The candidate dilatation for these strata is the Perron root of $X^4 - X^3 - X^2 - X + 1$. Lemma 4.1 implies there are no degree-4 monic reciprocal polynomials with a Perron root strictly less than that of the candidate.

A.1.2. *The stratum s_3.* The genus of S_3 is 2. Again Lemma 4.1 implies there are three degree-4 monic reciprocal polynomials with a Perron root strictly less than our candidate for $\delta(s_3)$. The next lemma will rule out these three polynomials.

Lemma A.1. *Let ϕ be a pseudo-Anosov homeomorphism on S with singularity data $(2,2,0)$. Assume in addition that $\rho(\phi_*) > 0$. If $Tr(\phi_*) \leq 2$ then*

$$Tr(\phi_*^2) \geq 9.$$

We first show how to rule out the polynomials and then we will prove the lemma. For each polynomial P, let us assume there exists ϕ with $\chi_{\phi_*} = P$; then we can see that $\mathrm{Tr}(\phi_*) \geq 2$, so that we should have $\mathrm{Tr}(\phi_*^2) \geq 9$. But Proposition 2.10 shows that $\mathrm{Tr}(\phi_*^2) \leq 5$. We now prove the lemma.

Proof of Lemma A.1. First of all observe that $\mathrm{Tr}(\phi_*) \leq 2$ if and only if $L(\phi) \geq 0$, and $\mathrm{Tr}(\phi_*^2) \geq 9$ if and only if $L(\phi^2) \leq -7$. Thus let us assume $L(\phi) \geq 0$. Since ϕ fixes a regular point, ϕ fixes the two degree-2 singularities with positive index (otherwise we would have $L(\phi) < 0$). Hence ϕ^2 fixes the two singularities and their separatrices, so that the index of ϕ^2 at the singularities is -3. Now

$$L(\phi^2) = -3 - 3 - \#\mathrm{Fix}(\phi^2) \leq -7,$$

where $\mathrm{Fix}(\phi^2)$ is the set of regular fixed points of ϕ^2. The lemma is proved. □

A.1.3. *The stratum s_4.* The surface S_4 has genus 3. There are 41 degree-6 monic reciprocal polynomials with a Perron root strictly less than our candidate $\delta(s_4)$. Among these polynomials, there are only 3 that satisfy the conclusion of Lemma 5.1, namely

$$\begin{aligned} P_1 &= X^6 - 3X^5 + 2X^4 + 2X^2 - 3X + 1, \\ P_2 &= X^6 - 3X^5 + 4X^4 - 5X^3 + 4X^2 - 3X + 1, \quad \text{and} \\ P_3 &= X^6 - 4X^5 + 6X^4 - 6X^3 + 6X^2 - 4X + 1. \end{aligned}$$

We discuss now the compatibility of the three polynomials P_1, P_2, and P_3.

First of all $P_2(-X)$ is not compatible with the stratum s'_4 (see Example 2.6.2).

For the third polynomial, both $P_3(X)$ and $P_3(-X)$ are admissible, which means the Lefschetz formula does not rule out P_3. Let us assume that there exists ϕ, with singularity data s'_4, such that $\chi_{\phi_*} = P_3$. We will get a contradiction using the ϕ-action on the singularities. More precisely we will show ϕ permutes the singularities whereas $\tau \circ \phi$ fixes them, which is impossible (since τ fixes the singularities).

First assume that the two singularities are fixed by ϕ. Since $L(\phi) = -2$ the index at these singularities is positive, otherwise $L(\phi) \leq -5 + 1 = -4$. Now ϕ^3 fixes the singularities with negative index, so that $L(\phi^3) \leq -5 - 5 = -10$. This is a contradiction with $L(\phi^3) = -8$. Thus ϕ has to permutes the two singularities. Now let us show that $\tau \circ \phi$ fixes the two singularities. If not, then since $L(\tau \circ \phi) = 6$ the homeomorphism $\tau \circ \phi$ has 6 regular fixed points, so that $(\tau \circ \phi)^2 = \phi^2$ also has 6 regular fixed points. Hence $L(\phi^2) \leq 2 - 6 = -4$ but $L(\phi^2) = 1$: we have a contradiction.

Finally, the first polynomial P_1 cannot be ruled out using the Lefschetz formula. Indeed it is compatible with the stratum s'_4 and there are no contradictions with the ϕ-action on the singularities. Thus the Lefschetz formula does not help to calculate $\delta(s_4)$ and we need a more elaborate argument to conclude. Using train track automata, one can actually prove $\delta(s_4) \simeq 2.01536$ [HS07, LT10a].

A.2. **Minimum dilatation for each stratum.** We give the minimum dilatation for each stratum and provide explicit examples of a braid realizing each minimum. We have indicated by a star the cases where we cannot conclude using the Lefschetz formula only. For these cases one can conclude using train track automata (see [HS07] for $n = 5$ and [LT10a] for $n = 6, 7$).

Note that in the tables we denote Δ_k the braid $\sigma_1 \cdots \sigma_k$. We used Toby Hall's implementation of the Bestvina–Handel algorithm [Hal, BH95] to verify that the braids correspond to pseudo-Anosov homeomorphisms.

case	$\delta(s) \simeq$	polynomial	braid
s_1	2.61803	$X^2 - 3X + 1$	$\sigma_1\sigma_2^{-1}$

case	$\delta(s) \simeq$	polynomial	braid
s_1	2.61803	$X^3 - 2X^2 - 2X + 1$	$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3^{-1}\Delta_3$
s_2	2.29663	$X^4 - 2X^3 - 2X + 1$	$\sigma_1\sigma_2\sigma_3^{-1}$

TABLE 4. Discs with 3 and 4 punctures.

case	$\delta(s) \simeq$	polynomial	braid
s_1	1.72208	$X^4 - X^3 - X^2 - X + 1$	$\Delta_3\Delta_4\sigma_3^{-1}$
s_2	1.72208	$X^5 - 2X^3 - 2X^2 + 1$	$\sigma_1^2\Delta_4^2$
s_3	2.15372	$X^5 - 2X^4 - 2X + 1$	$\Delta_3\sigma_4^{-1}$
s_4 *	2.01536	$X^6 - X^5 - 4X^3 - X + 1$	$\sigma_1\sigma_2\sigma_4^{-1}\sigma_3^{-1}$

TABLE 5. Disc with 5 punctures.

case	$\delta(s)$	polynomial	braid
s_1	1.88320	$X^5 - X^4 - X^3 - X^2 - X + 1$	$\Delta_5\sigma_4\sigma_5$
s_2	1.83929	$X^6 - X^4 - 4X^3 - X^2 + 1$	$\sigma_5\sigma_4^{-1}\Delta_5^2$
s_3	1.88320	$X^6 - 2X^4 - 2X^3 - 2X^2 + 1$	$\sigma_1^2\sigma_4\Delta_5^2$
s_4 *	2.08102	$X^6 - 2X^5 - 2X + 1$	$\Delta_4\sigma_5^{-1}$
s_5 *	2.08102	$X^7 - X^6 - 2X^5 - 2X^2 - X + 1$	$\sigma_4\sigma_5^2\sigma_4\Delta_5^2$
s_6	1.88320	$X^7 - X^6 - 2X^4 - 2X^3 - X + 1$	$\Delta_3\sigma_5^{-1}\sigma_4^{-1}$
s_7 *	2.17113	$X^8 - 2X^7 + X^6 - 4X^5 + 4X^4 - 4X^3 + X^2 - 2X + 1$	$\Delta_3(\sigma_3\sigma_4\sigma_5)^{-2}$

TABLE 6. Disc with 6 punctures.

case	$\delta(s)$	polynomial	braid
s_1	1.55603	$X^6 - X^5 - X^4 + X^3 - X^2 - X + 1$	$\sigma_3\sigma_4\sigma_5\sigma_6\sigma_2\sigma_3\sigma_4\Delta_3\Delta_6$
s_2	1.46557	$X^7 - 2X^4 - 2X^3 + 1$	$\sigma_4^{-2}\Delta_6^2$
s_3	1.46557	$X^7 - 2X^4 - 2X^3 + 1$	$\sigma_6^2\Delta_6^2$
s_4	1.55603	$X^7 - 2X^5 - 2X^2 + 1$	$\sigma_5^2\Delta_6^3$
s_5 *	2.04249	$X^7 - 2X^6 - 2X + 1$	$\sigma_4^{-2}\Delta_6$
s_6	1.61094	$X^8 - X^7 - 2X^5 + 2X^4 - 2X^3 - X + 1$	$\sigma_2^{-1}\sigma_3\sigma_4\sigma_5\Delta_6^2$
s_7 *	2.47541	$X^8 - 3X^7 + 2X^6 - 2X^5 + 2X^3 - 2X^2 + 3X - 1$	$\Delta_3\sigma_3(\sigma_3\sigma_4\sigma_5\sigma_6)^{-1}$
s_8 *	1.80979	$X^8 - X^7 - 2X^5 - 2X^3 - X + 1$	$\Delta_4\sigma_6^{-1}\sigma_5^{-1}$
s_9	1.75488	$X^8 - X^7 - 4X^4 - X + 1$	$\Delta_3\sigma_6^{-1}\sigma_5^{-1}\sigma_4^{-1}$
s_{10}	1.61094	$X^9 - X^7 - 2X^6 - 2X^3 - X^2 + 1$	$\sigma_5^{-1}\sigma_4^{-1}\sigma_3\sigma_4\sigma_5\sigma_6\Delta_6^3$
s_{11} *	2.04249	$X^9 - 2X^8 + X^7 - 2X^6 - 2X^3 + X^2 - 2X + 1$	$\sigma_4\sigma_5\sigma_6\sigma_3\sigma_4\sigma_5\sigma_2^{-1}\sigma_1^{-1}\Delta_6^{-1}$
s_{12} *	2.21497	$X^{10} - 2X^9 - X^7 - X^3 - 2X + 1$	$\sigma_2\sigma_1^2\sigma_2\Delta_6^{-2}$

TABLE 7. Disc with 7 punctures.

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