

FINITENESS OF TEICHMÜLLER CURVES IN NON-ARITHMETIC RANK 1 ORBIT CLOSURES

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1. INTRODUCTION

1.1. Motivation and results. Closed $GL(2, \mathbb{R})$ orbits of translation surfaces first appeared in the work of Veech in connection to billiards in triangles [Vee89]. Now their study is a sizable industry combining flat geometry, dynamics, Teichmüller theory, algebraic geometry, and number theory. Examples arising from torus covers are abundant, and additionally a few infinite families and sporadic examples have been discovered, but following deep work of McMullen, Möller, Bainbridge and others it is believed that most types of closed $GL(2, \mathbb{R})$ orbits are rare: there should be at most finitely many in each genus.

Recent work of Matheus-Wright demonstrates a new paradigm for proving such finiteness results [MW]. The key step is to show that closed $GL(2, \mathbb{R})$ orbits of the given type are not dense in some larger orbit closure \mathcal{M} . This \mathcal{M} might be either *higher rank* or *rank 1*, but the *Hodge-Teichmüller planes* used by Matheus-Wright are applicable only in the higher rank case. Here we rule out the rank 1 case.

Theorem 1.1. *Each non-arithmetic rank 1 orbit closure contains at most finitely many closed $GL(2, \mathbb{R})$ orbits.*

Theorem 1.1 is new for all Prym eigenform loci in genus 4 and 5 and the Prym eigenform loci in genus 3 in the principal stratum.

A point on a closed $GL(2, \mathbb{R})$ orbit is called a Veech surface. Many strange and fascinating surfaces are known that share some properties with Veech surfaces but are not actually Veech. Most of these lie in rank 1 orbit closures. An open question of Smillie and Weiss asks whether a particular property of Veech surfaces—complete parabolicity—in fact implies that the surface is Veech [SW07]. We give a positive answer in rank 1.

Theorem 1.2. *In a rank 1 orbit closure, every completely parabolic surface is Veech.*

Theorem 1.2 is new even for the much studied eigenform loci in genus 2, and sheds light on the extent to which orbit closures can be computed using cylinder deformations.

1.2. Terminology. Veech surfaces satisfy a remarkable property: in any direction, the straight line flow in that direction is either uniquely ergodic or there is a collection of parallel cylinders in that direction that cover the surface and whose moduli are all rational multiples of each other [Vee89]. (The modulus of a cylinder is defined to be the height divided by the circumference). This motivates the following definition.

Definition 1.3. A translation surface is called *completely parabolic* if for any cylinder direction, there is a collection of parallel cylinders in that direction that cover the surface and whose moduli are all rational multiples of each other.

An orbit closure of translation surfaces is a closed orbit if and only if it contains only completely parabolic surfaces. Relaxing this restriction, one arrives at one definition of a rank 1 orbit closure.

Definition 1.4. A translation surface is called *completely periodic* if for every cylinder direction there is a collection of parallel cylinders in that direction that cover the surface. An orbit closure of translation surfaces is called *rank 1* if it consists entirely of completely periodic translation surfaces. Closed $GL(2, \mathbb{R})$ orbits, or more generally rank 1 orbit closures, are called *arithmetic* if they parameterize torus covers [Wri14].

See Section 2.2 and Theorem 2.5 for equivalent definitions.

1.3. Outline of proof. Both Theorems 1.1 and 1.2 will be deduced from the following result.

Theorem 1.5. *The set of completely parabolic surfaces is not dense in any non-arithmetic rank 1 affine invariant submanifold that is not a closed orbit.*

Theorem 1.5 will be established in Section 3 via a mostly elementary argument, showing that a surface whose Veech group is large and does not have trace field \mathbb{Q} does not admit any small rel deformation that remains completely parabolic. The key tools will be the Zariski density of the Veech group in $SL(2, \mathbb{R})^d$ and a result of Eskin-Mozes-Oh on escape from sub-varieties in Zariski dense groups.

To derive Theorems 1.1 and 1.2 from Theorem 1.5, we will use the recent work of Eskin-Mirzakhani-Mohammadi [EM, EMM].

Theorem 1.6 (Eskin-Mirzakhani-Mohammadi). *Any closed $GL(2, \mathbb{R})$ invariant subset of a stratum is a finite union of affine invariant submanifolds.*

Proof of Theorems 1.1 and 1.2 from Theorem 1.5. We first prove Theorem 1.1. Let us assume that there are infinitely many closed orbits \mathcal{C}_i in a non-arithmetic rank 1 orbit closure \mathcal{N} . By Theorem 1.6, the closure of the union of the \mathcal{C}_i is equal to a finite union of affine invariant submanifolds $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_k$. One of these orbit closures, say \mathcal{M}_1 , must contain infinitely many of the \mathcal{C}_i . Since \mathcal{N} is rank 1 and $\mathcal{M}_1 \subset \mathcal{N}$, it follows that \mathcal{M}_1 is also rank 1. Since \mathcal{N} is non-arithmetic, \mathcal{M}_1 is also a non-arithmetic (Corollary 2.13). Because closed orbits parameterize completely parabolic translation surfaces, this contradicts Theorem 1.5.

We now prove Theorem 1.2. Suppose \mathcal{N} is a non-arithmetic rank 1 orbit closure, and suppose $(X, \omega) \in \mathcal{N}$ is a completely parabolic surface that is not Veech. Let \mathcal{M} be the orbit closure of (X, ω) . Again we see that \mathcal{M} must be non-arithmetic and rank 1. Because (X, ω) is not Veech, \mathcal{M} is not a closed orbit. Furthermore \mathcal{M} contains a dense set of completely parabolic surfaces given by the orbit of (X, ω) . This contradicts Theorem 1.5. \square

1.4. Context. The projection of a closed $GL(2, \mathbb{R})$ orbit to the moduli space of Riemann surfaces is called a Teichmüller curve, and up to a double cover issue all Teichmüller curves arise in this way. Since the relationship is so close, sometimes we will conflate these notions and refer to closed $GL(2, \mathbb{R})$ orbits as Teichmüller curves.

Marked points on Veech surfaces. The easiest examples of rank 1 orbit closures that are not Teichmüller curves are given by orbit closures of Veech surfaces with finitely many marked points. One can also take covers of these surfaces branched over the marked points and obtain very closely related rank 1 orbit closures of translation surfaces with no marked points.

A Veech surface with a marked point has closed $GL(2, \mathbb{R})$ orbit if and only if the marked point is periodic, i.e., has finite orbit under the action of the affine group. Gutkin-Hubert-Schmidt [GHS03] and Möller [Möl06a] showed that on any non-arithmetic Veech surface there are at most finitely many periodic points. This can be viewed as a special case of Theorem 1.1, and was the inspiration for the present work. More specifically, the paper of Gutkin-Hubert-Schmidt motivated this paper by giving us hope that in rank 1 a general finiteness result could be obtained without using much algebraic geometry.

The finiteness result of Gutkin-Hubert-Schmidt and Möller was used by Hubert-Schmoll-Troubetzkoy to obtain interesting results on the structure of rank 1 orbit closures arising from Veech surfaces with marked points [HST08]. This is the only known instance where a finiteness result for Teichmüller curves has been used to restrict the structure of larger orbit closures.

Rank 1 orbit closures. There are additionally infinitely many rank 1 orbit closures in genus 2 through 5, called the (Prym) eigenform loci. These examples were discovered by McMullen; in the case of genus 2 they were independently discovered in a different guise by Calta [Cal04, McM03a, McM06a]. All currently known rank 1 orbit closures arise from (Prym) eigenform loci or Teichmüller curves via branched covering constructions.

The (Prym) eigenform loci have been the focus of a considerable amount of study [McM, McM14, LNb, LNC].

Arithmetic rank 1 orbit closures never contain non-arithmetic Teichmüller curves, and they always contain a dense set of arithmetic Teichmüller curves. Non-arithmetic orbit closures never contain arithmetic Teichmüller curves [Wri14].

Strange surfaces. The (Prym) eigenform loci and branched covers of Veech surfaces have been a rich source of examples of surfaces with peculiar properties, including surfaces with infinitely generated Veech groups and satisfying the (topological) Veech dichotomy [LNa, McM03b, SW08, HS04].

There are some examples of somewhat less weird but still unusual surfaces not lying in a rank 1 orbit closure, for example there are many completely periodic but not Veech surfaces in $\mathcal{H}(4)^{\text{hyp}}$ [LNa], whose orbits must be dense by [NW14].

Teichmüller curves. A Teichmüller curve is called primitive if it does not arise from a covering construction [Mö106a]. Here is a census of all known primitive Teichmüller curves. In genus 2, 3 and 4 there are infinite families discovered by McMullen [McM03a, McM06b]; in genus two, these were independently discovered in a different guise by Calta [Cal04]. The Veech-Ward-Bouw-Möller Teichmüller curves give finitely many examples in each genus [Vee89, War98, BM10, Wri13, Hoo13]. There is additionally one known primitive Teichmüller curve in genus 3 and one in genus 4 [HS01, KS00].

Primitive Teichmüller curves were classified in genus 2 by McMullen [McM05a, McM06b]. These curves and their friends in genus 3 and 4

have been the focus of an extensive body of research [**Bai07**, **Muk14**, **KM**, **Muk**, **LN14**, **Möl**, **TTZ**].

As part of his classification primitive Teichmüller curves in genus 2, McMullen showed that there is only one primitive Teichmüller curve in $\mathcal{H}(1, 1)$ [**McM06b**]. Before proving this, McMullen established the weaker result that in each non-arithmetic eigenform locus in genus 2 there are at most finitely many Teichmüller curves [**McM05b**]. Theorem 1.1 is a generalization of this result.

Furthermore, McMullen was able to show that every non-Veech surface in these loci must have a saddle connection not parallel to any cylinders [**McM05b**, Theorem 7.5]. This is very much analogous to Theorem 1.2, because both imply the existence of a flat geometry “certificate” that shows that a surface is not Veech. Our techniques also have similarities with those of McMullen. For example, both use that an infinite set of Teichmüller curves must be dense. (McMullen’s work was well before Theorem 1.6 was established, but he had previously established this result in genus 2 [**McM07**].)

A Teichmüller curve is called algebraically primitive if the trace field of its Veech group has degree equal to the genus. Finiteness of algebraically primitive Teichmüller curves was shown in $\mathcal{H}(g - 1, g - 1)^{hyp}$ by Möller [**Möl08**], in $\mathcal{H}(3, 1)$ by Bainbridge-Möller [**BM12**], in general in genus 3 by Bainbridge-Habegger-Möller [**BHM**], and in the minimal stratum in prime genus at least 3 by Matheus-Wright [**MW**]. In the minimal stratum in genus 3, there are only finitely many primitive Teichmüller curves not contained in the Prym locus [**MW**, **NW14**, **ANW**].

There is a partial classification of primitive Teichmüller curves in the Prym eigenform loci in genus three [**LM**]. Before the work of Bainbridge-Habegger-Möller, Lanneau-Nguyen had previously established Theorem 1.1 for many Prym eigenform loci in genus three in genus 3 [**LN_a**], using the strategy that McMullen used in eigenform loci in genus 2 [**McM05b**].

Teichmüller curves have extra-ordinary algebro-geometric properties [**Möl06b**, **Möl06a**, **McM09**], which have been key tools in much previous work on finiteness of Teichmüller curves. Our method will not use any of these algebro-geometric results.

The work of Smillie-Weiss uses flat geometry to characterize and prove finiteness results for Teichmüller curves [**SW10a**, **SW10b**].

Cylinder deformations. For many surfaces, certain recognizable cylinder deformations span the tangent space to the orbit closure of the surface (assuming the set of cylinders in a given direction can be

determined) [Wri15]. However, it is unknown if this holds for all surfaces.

For a completely parabolic surface, there is no indication from the flat geometry that the orbit closure has any non-trivial cylinder deformations. Thus if a completely parabolic but not Veech surface were to exist, the recognizable cylinder deformations wouldn't span the tangent space to its orbit closure.

We write “recognizable” because, for example, if a completely parabolic surface but not Veech surface were to exist, it is possible certain cylinder deformations would stay in its orbit closure. However, without already knowing the orbit closure, there would be no way to guess this from the flat geometry of the surface. In contrast, for a horizontally periodic surface where the moduli are not rational multiples of each other, certain cylinder deformations are easily seen to stay in the horocycle orbit closure [Wri15].

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2. BACKGROUND

We review useful results concerning affine invariant submanifolds, rank and (affine) field of definition. For a general introduction to translation surfaces and their moduli spaces, we refer the reader to the surveys [MT02, Wri, Zor06], or any of the many other surveys listed in [Wri].

Throughout this paper, \mathcal{H}_g will denote the moduli space of Abelian differentials ω on Riemann surfaces X of genus $g \geq 1$. For any non-negative integer partition (k_1, k_2, \dots, k_s) of $2g - 2$ we will denote by $\mathcal{H}(k_1, \dots, k_s)$ the stratum of translation surfaces having s zeros, of orders k_1, \dots, k_s . We will allow translation surface to have finitely many marked points, and by convention we will consider a marked point to be a zero of order 0. The marked points are required to be distinct from each other and the set of true zeros. $\mathrm{GL}(2, \mathbb{R})$ acts on the set of

translation surfaces (by post-composition with the charts of translation atlas). Moreover, this $GL(2, \mathbb{R})$ -action on Abelian differential preserves each stratum. We will denote the stabilizer of (X, ω) by $SL(X, \omega)$, and we will refer to it as the *Veech group*.

For technical reasons we will work with the finite cover \mathcal{H} of the stratum $\mathcal{H}(k_1, k_2, \dots, k_s)$ consisting of surfaces equipped with a level three structure and a labeling of the zeros. The level three structure may be safely ignored. We will write $(X, \omega) \in \mathcal{H}$, and it will be implicit that X carries a level three structure and the requisite number of labelled marked points.

2.1. Affine invariant submanifolds. Let Σ be the set of zeros of $(X, \omega) \in \mathcal{H}$ and let $\gamma_1, \dots, \gamma_n$ be any basis of the relative homology group $H_1(X, \Sigma, \mathbb{Z})$, where $n = 2g + s - 1$. The period coordinate maps defined by

$$(X, \omega) \mapsto \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_n} \omega \right)$$

provide \mathcal{H} with an atlas of charts to \mathbb{C}^n with transition functions in $GL(n, \mathbb{Z})$ (see [Kon97]). If γ is in absolute homology $H_1(X, \mathbb{Z})$, the integral $\int_{\gamma} \omega$ is called an absolute period, and if γ is merely in relative homology $H_1(X, \Sigma, \mathbb{Z})$ it is called a relative period.

Over each stratum there are flat bundles H_{rel}^1 and H^1 whose fiber over (X, ω) are $H^1(X, \Sigma, \mathbb{C})$ and $H^1(X, \mathbb{C})$ respectively. There is a natural map $p : H_{rel}^1 \rightarrow H^1$.

Definition 2.1. An *affine invariant submanifold* of \mathcal{H} is a properly immersed manifold $\mathcal{M} \hookrightarrow \mathcal{H}$ such that each point of \mathcal{M} has a neighborhood whose image is given by the set of surfaces in an open set satisfying a set of real linear equations in period coordinates. (For notational simplicity, we will treat affine invariant submanifold as subsets of strata, referring to the image of the immersion).

The tangent space to \mathcal{M} gives a flat subbundle $T(\mathcal{M})$ of H_{rel}^1 .

Definition 2.2. A *rel deformation* is a path of translation surfaces $\{(X_t, \omega_t)\}_{t \in [0,1]}$ along which all absolute periods $\int_{\gamma} \omega_t$ stay constant.

Lemma 2.3. *The bundle $\ker(p)$ is a trivial flat bundle over \mathcal{H} .*

Proof. Suppose the zeros of ω are p_1, \dots, p_s . Since the zeros are labelled, the ordering of the p_i can be chosen consistently over \mathcal{H} . At any point $(X, \omega) \in \mathcal{H}$, for each $i = 1, \dots, s-1$, let γ_i be a path on X from p_i to p_{i+1} . The map from the fiber of $\ker(p)$ to \mathbb{C}^{s-1} given by

$$v \mapsto (v(\gamma_i))_{i=1}^{s-1}$$

does not depend on the choice of γ_i and provides an isomorphism between $\ker(p)$ and the trivial bundle \mathbb{C}^{s-1} over \mathcal{H} .

The γ_i cannot be chosen to be global flat sections, but this is not relevant because the map does not depend on the choice of γ_i . The reason it does not depend on the choice of γ_i is because different γ_i differ by absolute homology classes, and by assumption the integral of v over any absolute homology class is zero. \square

Notation. Because $H^1(X, \Sigma, \mathbb{C})$ provides local coordinates for \mathcal{H} at (X, ω) , for any sufficiently small $v \in H^1(X, \Sigma, \mathbb{C})$ there is a nearby point in \mathcal{H} whose period coordinates are given by those of (X, ω) plus v . We will denote this surface by $(X, \omega) + v$.

Lemma 2.4. *Fix $(X, \omega) \in \mathcal{H}$. There is a neighborhood U of 0 in $\ker(p)$ such that on U the map $v \mapsto (X, \omega) + v$ is well defined and injective, and such that if $v \in U$ then $tv \in U$ for all $t \in [0, 1]$. Moreover, if $g \in SL(X, \omega)$ and $v \in U$ then $g((X, \omega) + v) = (X, \omega) + gv$.*

Here g acts on $v \in \ker(p)$ by acting on the real and imaginary parts. That is, $\ker(p) = \mathbb{C}^{s-1}$, and the action is the standard linear action of $GL(2, \mathbb{R})$ on the plane $\mathbb{C} \simeq \mathbb{R}^2$. Generally we will also require $gv \in U$, so that $g((X, \omega) + v)$ is in the image of U under the injective map $v \mapsto (X, \omega) + v$. (We think of U as providing a local coordinate chart for the rel leaf of (X, ω) , and generally we want $g((X, \omega) + v)$ to remain in this coordinate chart.)

Proof. The first claim follow from the fact that period coordinates are indeed local coordinates.

For the final claim, fix $g \in GL(2, \mathbb{R})$ and set $(X_v, \omega_v) = (X, \omega) + v$ and $(X_v^g, \omega_v^g) = g((X, \omega) + v)$.

Now, if γ is any relative cycle,

$$\begin{aligned} \int_{\gamma} \omega_v^g &= g \cdot \int_{\gamma} \omega_v \\ &= g \cdot \left(\int_{\gamma} \omega + v(\gamma) \right) \\ &= g \cdot \int_{\gamma} \omega + g \cdot v(\gamma). \end{aligned}$$

This proves the result, using the triviality of $\ker(p)$. \square

2.2. Rank of an affine invariant submanifold. We define the *rank* of an affine invariant submanifold \mathcal{M} by $\frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M}))$. (Note that

by the work of Avila-Eskin-Möller [AEM], $p(T(\mathcal{M}))$ is symplectic, and thus has even dimension over \mathbb{C}).

The following result was established in general in [Wri15], having previously been known in a number of special cases [LNa, Cal04, McM05b, Vee89]. The simplest known proof, which in fact gives a stronger result, may be found in [Wri].

Theorem 2.5. *Any surface (X, ω) whose orbit closure \mathcal{M} satisfies $\frac{1}{2} \dim_{\mathbb{C}} p(T(\mathcal{M})) = 1$ is completely periodic.*

This theorem is one of two implications showing that the definition of rank 1 in this section is equivalent to the one in the introduction. See [Wri15] for the other implication.

When \mathcal{M} is a rank 1 submanifold, $p(T(\mathcal{M}))$ is spanned by the real and imaginary parts of the absolute cohomology class of ω . Hence $\text{span}(\text{Re}(\omega), \text{Im}(\omega))$ is locally constant, because $p(T(\mathcal{M}))$ is a flat subbundle. One thinks of $\text{span}(\text{Re}(\omega), \text{Im}(\omega))$ as the directions coming from $GL(2, \mathbb{R})$, because this subspace is always contained in $T(\mathcal{M})$ as a consequence of $GL(2, \mathbb{R})$ invariance.

Proposition 2.6 ([Wri]). *An orbit closure \mathcal{M} is rank 1 if and only if for each (X, ω) , there is a neighborhood U of the identity in $GL(2, \mathbb{R})$ and a neighborhood V of 0 in $\ker(p) \cap T(\mathcal{M})$ such that the map from $U \times V \rightarrow \mathcal{M}$ given by $(g, v) \mapsto g(X, \omega) + v$ is a diffeomorphism onto a neighborhood of (X, ω) in \mathcal{M} .*

2.3. Stable directions. Let \mathcal{M} be an affine invariant submanifold. We say that a surface $(X, \omega) \in \mathcal{M}$ is \mathcal{M} -stably periodic in some direction if the surface is periodic in this direction and if all saddle connections in this direction remain parallel on all sufficiently nearby surfaces in \mathcal{M} . A periodic direction that is not \mathcal{M} -stable will be called \mathcal{M} -unstable.

Remark 2.7. If (X, ω) is horizontally \mathcal{M} -stably periodic then there is an open subset U of $\ker(p) \cap T(\mathcal{M})$ containing 0 such that for any $v \in U$, $(X, \omega) + v$ is also horizontally \mathcal{M} -stably periodic with the same cylinder diagram (compare to [LNc]).

Remark 2.8. \mathcal{M} -stability can be described in the language of [Wri15] by saying that the twist space is equal to the cylinder preserving space.

Remark 2.9. If \mathcal{M} is rank 1 and $(X, \omega) \in \mathcal{M}$ is horizontally periodic and U is a neighborhood of 0 in $\ker(p) \cap T(\mathcal{M})$ on which the map $v \mapsto (X, \omega) + v$ is well defined and injective, then there is an open dense subset $V \subset U$ such that for any $v \in V$, $(X, \omega) + v$ is horizontally \mathcal{M} -stably periodic. This is because the condition of being \mathcal{M} -unstable

periodic gives rise to linear equations that do not vanish identically on $T(\mathcal{M}) \cap \ker(p)$.

2.4. Holonomy field, trace field and (affine) field of definition.

For more details on the results in this section, see [KS00, Wri14].

The *holonomy field* of a translation surface is the smallest subfield of \mathbb{R} such that the absolute periods are a vector space of dimension two over this field [KS00]. The *trace field* of a translation surface is the subfield of \mathbb{R} generated by the traces of elements of its Veech group. This is also $\mathbb{Q}[\lambda + \lambda^{-1}]$ where λ is the largest eigenvalue of *any* hyperbolic element of $SL(X, \omega)$. When $SL(X, \omega)$ contains a hyperbolic, the trace field and the holonomy field coincide.

The (affine) field of definition of an affine invariant submanifold \mathcal{M} (introduced in [Wri14]), denoted $\mathbf{k}(\mathcal{M})$, is defined to be the smallest subfield of \mathbb{R} such that locally \mathcal{M} can be described by homogeneous linear equation in period coordinates with coefficients in $\mathbf{k}(\mathcal{M})$. It is a number field [Wri14] and is always totally real [Fil].

Lemma 2.10. *If \mathcal{M} is an affine rank 1 invariant submanifold then any translation surface $(X, \omega) \in \mathcal{M}$ has holonomy field equal to $\mathbf{k}(\mathcal{M})$.*

Proof. By a result in [Wri14], $\mathbf{k}(\mathcal{M})$ is the intersection of the holonomy fields of all translation surfaces in \mathcal{M} . By definition, the holonomy field is invariant under both the $GL(2, \mathbb{R})$ -action and rel deformations. Since \mathcal{M} is rank 1, any two points in \mathcal{M} may be connected by using the $GL(2, \mathbb{R})$ -action and rel deformations. This proves the lemma. \square

Definition 2.11. We will say that \mathcal{M} is arithmetic if $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$.

Remark 2.12. In the rank 1 case, it is not hard to show that this definition of arithmetic coincides with the one given in the introduction.

Corollary 2.13. *If $\mathcal{M}' \subset \mathcal{M}$ are rank 1, then $\mathbf{k}(\mathcal{M}) = \mathbf{k}(\mathcal{M}')$. In particular \mathcal{M} is arithmetic if and only if \mathcal{M}' is arithmetic.*

Lemma 2.14. *If $(X, \omega) \in \mathcal{M}$ is a completely parabolic surface, then there exists $g \in GL(2, \mathbb{R})$, arbitrarily close to the identity, such that $g(X, \omega)$ has absolute and relative periods in $\mathbf{k}[i]$.*

Proof. This follows from the Thurston-Veech construction [Thu88, Vee89]. Alternatively, by the definition of holonomy field there is $g \in GL(2, \mathbb{R})$ arbitrarily close to the identity such that $g(X, \omega)$ has absolute periods in $\mathbf{k}[i]$. By [KS00, Theorem 30] or [McM03b, Theorem 9.4], if $SL(X, \omega)$ has a hyperbolic element, then the relative and absolute periods span the same \mathbb{Q} vector subspace of \mathbb{C} . \square

Suppose that \mathbf{k} is a totally real field of degree d over \mathbb{Q} . We denote the field embeddings of \mathbf{k} into \mathbb{R} by ι_1, \dots, ι_d . If a subgroup $\Gamma \subset SL(2, \mathbb{R})$ has totally real trace field \mathbf{k} of degree d over \mathbb{Q} (Γ is not necessarily a subgroup of $SL(2, \mathbf{k})$) then there is a natural injective group homomorphism $\rho : \Gamma \rightarrow SL(2, \mathbb{R})^d$. This homomorphism is defined up to conjugation by the condition that for all $\gamma \in \Gamma$, we have $\rho(\gamma) = (g_1, \dots, g_d)$, where $tr(g_i) = \iota_i(tr(\gamma))$ for all i .

The next proposition is a consequence of [Gen12, Proposition 2.1 and Corollary 2.2], which is slightly stronger.

Proposition 2.15. *Let $\Gamma \subset SL(2, \mathbb{R})$ be a non-elementary subgroup having a totally real trace field \mathbf{k} of degree d over \mathbb{Q} . If there exists $g \in \Gamma$ with eigenvalue λ such that $\mathbf{k} = \mathbb{Q}[\lambda + \lambda^{-1}]$, then the image $\rho(\Gamma)$ of Γ in $SL(2, \mathbb{R})^d$ is Zariski dense.*

Proof. We provide a sketch for convenience.

Let \mathfrak{g} be the Lie algebra of the Zariski closure of $\rho(\Gamma) \subset SL(2, \mathbb{R})^d$. It suffices to show $\mathfrak{g} = \bigoplus_{i=1}^d \mathfrak{sl}(2, \mathbb{R})$.

Step 1: Let $\pi_i : \bigoplus_{i=1}^d \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ be the projection onto the i -th factor. For each i , we have that $\pi_i(\mathfrak{g}) = \mathfrak{sl}(2, \mathbb{R})$. This follows since the Adjoint action of $\rho(\Gamma)$ on each coordinate of $\bigoplus_{i=1}^d \mathfrak{sl}(2, \mathbb{R})$ has no invariant subspaces, and \mathfrak{g} is an invariant subspace for the Adjoint action of $\rho(\Gamma)$ on $\bigoplus_{i=1}^d \mathfrak{sl}(2, \mathbb{R})$.

Step 2: For each i , we have that \mathfrak{g} contains an element which is non-zero only in coordinate i . This follows from the previous result, and the following observation: if $g \in \Gamma$ has an eigenvalue λ with $\mathbf{k} = \mathbb{Q}[\lambda + \lambda^{-1}]$, then $\text{Ad}(g)$ has a simple eigenvector that is non-zero only in coordinate i . Basic linear algebra gives that the projection onto the line spanned by this eigenvector can be written as a polynomial in $\text{Ad}(g)$.

Step 3: Again using that $\text{Ad}(\rho(\Gamma))$ acting on each $\mathfrak{sl}(2, \mathbb{R})$ coordinate has no invariant subspaces, we get the result. \square

Corollary 2.16. *If (X, ω) is contained in a non-arithmetic rank 1 orbit closure and has two non-commuting parabolic, then $\rho(SL(X, \omega))$ is Zariski dense in $SL(2, \mathbb{R})^d$.*

3. PROOF OF THEOREM 1.5

For the duration of this section, in order to find a contradiction, we assume that \mathcal{M} is a non-arithmetic rank 1 affine invariant submanifold, and completely parabolic surfaces are dense in \mathcal{M} . Let $\mathbf{k} = \mathbf{k}(\mathcal{M})$ be the (affine) field of definition of \mathcal{M} . The definition of non-arithmetic means that $\mathbf{k} \neq \mathbb{Q}$. We denote the field embeddings of \mathbf{k} into \mathbb{R} by ι_1, \dots, ι_d .

Lemma 3.1. *There exists a completely parabolic surface $(X, \omega) \in \mathcal{M}$ that is \mathcal{M} -stably periodic in the horizontal direction, with all relative periods in $\mathbf{k}[i]$ and satisfying the following property. There exist a pair of horizontal cylinders C_1, C_2 on (X, ω) such that the ratio of the moduli of C_1 and C_2 is not constant under small rel deformations in \mathcal{M} .*

Proof. Given any periodic direction on any surface in \mathcal{M} , a generic small deformation of this surface will be \mathcal{M} -stably periodic in the same direction. Thus, there is a surface $(X', \omega') \in \mathcal{M}$ with an \mathcal{M} -stably periodic direction. There is small neighborhood U of (X', ω') where the cylinders in this direction persist, and the direction stays \mathcal{M} -stably periodic. Since we have assumed that completely parabolic surfaces are dense in \mathcal{M} , there must be a completely parabolic surface (X, ω) in U . After applying an element in $GL(2, \mathbb{R})$, we may assume that the \mathcal{M} -stably periodic direction on (X, ω) is horizontal and that the absolute and hence also relative periods lie in $\mathbf{k}[i]$; compare to the proof of Lemma 2.14.

For any non-trivial imaginary rel deformation of an \mathcal{M} -stably horizontally periodic surface, there is a cylinder that increases in height, and another that decreases in height. Two such cylinders can be chosen as C_1 and C_2 . Since rel deformations do not change circumference, the ratio of moduli of C_1 and C_2 changes under the imaginary rel deformation. \square

Let $(X, \omega) \in \mathcal{M}$, and $C_1, C_2 \subset X$ be as in Lemma 3.1. For v in a neighborhood of 0 in $\ker(p) \cap T(\mathcal{M})$, define $f(v)$ to be the ratio of the moduli of C_1 and C_2 . Note that f is the quotient of two degree 1 polynomials in the *imaginary coordinates* of v with coefficients in $\mathbf{k}(\mathcal{M})$. More precisely, if the heights and circumferences of C_i at (X, ω) are denoted by h_i and c_i respectively, then there are linear functionals α_1, α_2 with coefficients in \mathbf{k} such that

$$f(v) = \frac{c_2}{c_1} \cdot \frac{h_1 + \alpha_1(\mathrm{Im}(v))}{h_2 + \alpha_2(\mathrm{Im}(v))}.$$

Using this formula as a definition, we extend f to all of $\ker(p) \cap T(\mathcal{M})$.

Remark 3.2. For large v , $f(v)$ may not have geometric meaning. There are several reasons for this: the cylinders C_1 and C_2 may not persist on $(X, \omega) + v$, or worse yet, $(X, \omega) + v$ may not even be well defined (zeroes may collide). Furthermore, if a path $(X, \omega) + tv$ ceases to be \mathcal{M} -stably periodic at some $t = t_0$, then $f(tv)$ may not represent the ratio of the moduli of C_1 and C_2 for $t > t_0$, even if C_1 and C_2 persist. This is because some zero can “hit” the boundary of the C_i . (The heights of cylinders are only piecewise linear functions.)

In the sequel we set

$$P = \{v \in \ker(p) \cap T(\mathcal{M}) : f(tv) = f(0) \text{ for all } t \in \mathbb{R}\}.$$

Observe that since f is the ratio of two degree one polynomials, P is a real hyperplane of $\ker(p) \cap T(\mathcal{M})$.

Lemma 3.3. *Let $v \in \ker(p) \cap T(\mathcal{M}) \cap H^1(X, \Sigma, \mathbf{k}[i])$ be a vector satisfying $v \notin P$. There exists a collection of $d - 1$ polynomials (depending on v) of degree at most 2 in the $4d$ entries of $\rho(g)$ such that for any $g \in \mathrm{SL}(2, \mathbf{k})$, if any of these polynomials are non-zero at $\rho(g)$, then $f(gv) \notin \mathbb{Q}$. Moreover, these polynomials do not vanish identically on $\rho(\mathrm{SL}(2, \mathbf{k}))$.*

Proof. The condition that $f(gv) \in \mathbb{Q}$ is equivalent to the $d - 1$ equations

$$\iota_i(f(gv)) = \iota_{i+1}(f(gv)), \quad i = 1, \dots, d - 1.$$

Since f is the ratio of two degree one polynomials, by clearing denominators in each of these equations we get a system of $d - 1$ quadratic equations in the coefficients of $\rho(g)$ that vanish whenever $f(gv) \in \mathbb{Q}$.

It remains to show that these polynomials do not vanish identically on $\rho(\mathrm{SL}(2, \mathbf{k}))$. For this, consider the matrices

$$a_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

for $t \in \mathbf{k}^*$. Since $f(v)$ depends only on the imaginary parts of v , we get that for any $t \in \mathbf{k}^*$, $f(a_tv) = f(tv)$. Since $v \notin P$ the function $t \mapsto f(tv)$ is not constant. Recall that $\mathbf{k} \neq \mathbb{Q}$, hence any non-constant function $\mathbf{k} \rightarrow \mathbb{R}$ defined as the ratio of two degree one polynomials cannot always take rational values. This proves the lemma. \square

For any $v \in \ker(p) \cap T(\mathcal{M}) \cap H^1(X, \Sigma, \mathbf{k}[i])$ with $v \notin P$, we will denote the algebraic subvariety of $\mathrm{SL}(2, \mathbb{R})^d$ defined by the polynomials in Lemma 3.3 by $\mathcal{X}(v)$. In particular $\rho(\mathrm{SL}(2, \mathbf{k})) \not\subset \mathcal{X}(v)$.

Lemma 3.4. *There exists $R > 0$ such for any $v \in \ker(p) \cap T(\mathcal{M}) \cap H^1(X, \Sigma, \mathbf{k}[i])$ with $v \notin P$, there exists $g \in \mathrm{SL}(X, \omega)$ such that $\|g\| \leq R$ and $\rho(g) \notin \mathcal{X}(v)$ i.e. $f(gv) \notin \mathbb{Q}$.*

The non-trivial part of Lemma 3.4 is the control of the size of g . This is done with the following result of Eskin-Mozes-Oh. The statement presented here is slightly stronger than [EMO05, Lemma 3.2] (see [BG08, Lemma 2.5] where the statement below appears).

Theorem 3.5 (Eskin-Mozes-Oh). *For any $k > 0$, there is an $N > 0$ such that for any integer $m \geq 1$, any field \mathbf{k} , any algebraic subvariety $\mathcal{X} \subset \mathrm{GL}(m, \mathbf{k})$ defined by at most k polynomials of degree at most k ,*

and any family $S \subset GL(m, \mathbf{k})$ that generates a subgroup which is not contained in \mathcal{X} , we have $S^N \not\subset \mathcal{X}$.

Here S^N denotes the word ball of radius N .

Proof of Lemma 3.4. By Corollary 2.16, we have that $\rho(SL(X, \omega))$ is Zariski dense in $SL(2, \mathbb{R})^d$. Let $S \subset SL(X, \omega)$ be a finite set such that $\rho(S)$ generates a Zariski dense subgroup Γ of $SL(2, \mathbb{R})^d$.

Let v be as in the statement of the lemma. By Lemma 3.3 we know that $\rho(SL(2, \mathbf{k}))$ is not contained in the subvariety $\mathcal{X}(v)$ of $SL(2, \mathbb{R})^d \subset GL(2d, \mathbb{R})$. Thus the subgroup Γ is not contained in this variety. Theorem 3.5 then implies that there is an N independent of v such that there exists an element $g \in SL(X, \omega)$ of word length in S at most N , such that $\rho(g)$ does not belong to $\mathcal{X}(v)$, which means that $f(gv) \notin \mathbb{Q}$ by Lemma 3.3.

If $M = \max_{h \in S} \|h\|$, we may set $R = M^N$, and hence $\|g\| \leq R$. \square

Corollary 3.6. *For all sufficiently small $v \in \ker(p) \cap T(\mathcal{M}) \cap H^1(X, \Sigma, \mathbf{k}[i])$, if $v \notin P$ then $(X, \omega) + v$ is not completely parabolic.*

Proof. Let R be the constant given by Lemma 3.4. Let U be the neighborhood of 0 in $\ker(p) \cap T(\mathcal{M})$ given by Lemma 2.4 so that the map $v \mapsto (X, \omega) + v$ is well defined and injective, and if $g \in SL(X, \omega)$ and $v \in U$ then $g((X, \omega) + v) = (X, \omega) + gv$. By making U smaller if necessary, we can assume that the cylinders C_1 and C_2 persist at $(X, \omega) + v$ for all $v \in U$ and that $f(v)$ gives the ratio of the moduli of C_1 and C_2 at $(X, \omega) + v$.

Now consider any v as above that is small enough so that $Rv \in U$. If $v \notin P$, Lemma 3.4 furnishes an element $g \in SL(X, \omega)$ such that $f(gv) \notin \mathbb{Q}$ and $\|g\| \leq R$. But $g((X, \omega) + v) = (X, \omega) + gv$. Since the function f is the ratio of the moduli of C_1 and C_2 , we see that $(X, \omega) + gv$ has a pair of parallel cylinders whose ratio of moduli is not rational. Since $(X, \omega) + gv$ belongs to the $SL(2, \mathbb{R})$ orbit of $(X, \omega) + v$, we see that $(X, \omega) + v$ is not completely parabolic. \square

Conclusion of the proof of Theorem 1.5. Recall that the set of completely parabolic surfaces is $GL(2, \mathbb{R})$ invariant. Since \mathcal{M} is rank 1, if completely parabolic surfaces are dense, then there is a neighborhood U of 0 in $\ker(p) \cap T(\mathcal{M})$ such that $(X, \omega) + v$ is completely parabolic for a dense set of $v \in U$.

By construction, (X, ω) has relative periods in $\mathbf{k}[i]$. Hence if $(X, \omega) + v$ is completely parabolic then it must have relative periods in $\mathbf{k}[i]$, and hence v has coordinates in $\mathbf{k}[i]$. By Corollary 3.6, if $v \notin P$ is small enough and has coordinates in $\mathbf{k}[i]$, then $(X, \omega) + v$ is not completely

parabolic. Since P is a hyperplane, this contradicts the fact that the set of completely parabolic surfaces is dense in \mathcal{M} . Theorem 1.5 is proved. \square

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