MULTICURVE INTERSECTION DEGREES

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Abstract. We show that every positive integer $d \leq 3g-3$ appears as a multicurve intersection degree on the closed orientable surface of genus $g \geq 2$. As an application, we obtain that the largest degree of a pseudo-Anosov stretch factor obtained via the Thurston–Veech construction on the closed orientable surface of genus $g \geq 2$ equals $6g-6$, validating a claim by Thurston from the 80ies. For $g \geq 3$, we show that the multicurves can be constructed in such a way that the pseudo-Anosov map belongs to the Torelli group.

1. Introduction

1.1. **Multicurve intersection degrees.** Let S be the smooth closed orientable surface of genus $g \geq 2$. A multicurve $\alpha \subset S$ is a disjoint union of finitely many smooth simple closed curves, $\alpha = \alpha_1 \dot\cup \cdots \dot\cup \alpha_n$. Note that our definition allows for parallel multicurve components.

A pair of multicurves $\alpha, \beta \subset S$ fills the surface S if α and β intersect transversally and if the complement $S \setminus (\alpha \cup \beta)$ is a union of topological discs none of which is a bigon. This in particular implies that each pair α_i and β_j of components realises the minimal number of intersection points within their respective isotopy classes.

For a pair $\alpha, \beta \subset S$ of filling multicurves, let $X = (|\alpha_i \cap \beta_j|)_{ij}$ be the matrix encoding the number of intersections of the components of α and β.

The matrix $XX^\top$ is primitive, hence by Perron-Frobenius theory its spectral radius equals its largest eigenvalue and is therefore an algebraic integer. Let d be its algebraic degree. We call the number d the *multicurve intersection degree* of α and β.

The degree d is obviously bounded from above by the rank of the matrix $XX^\top$, which in turn is bounded from above by the maximal number of nonparallel components of a multicurve in S. This number equals the number of curves in a pants decomposition of the surface, which equals $3g-3$.

Our first main result states that this is the only restriction.

Theorem 1. *Every positive integer $d \leq 3g-3$ is realised as a multicurve intersection degree on the closed orientable surface of genus $g \geq 2$.*

Refined versions of this result are conceivable: there are numerous conditions one can impose on the multicurves α and β that correspond to certain geometric situations. Our second main result concerns multicurves whose difference is trivial in homology.

Theorem 2. *Every positive integer $d \leq 3g-3$ is realised as the multicurve intersection degree of a pair of filling multicurves $\alpha, \beta \subset S$ on the closed orientable surface S of genus $g \geq 3$, such that $[\alpha] - [\beta] = 0 \in H_1(S; \mathbb{Z})$.*

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Theorem 2 is again optimal, and it is better than what we could hope to achieve if we asked that both multicurves α and β consist of separating components only. In this case, the maximal number of nonparallel components is $2g-3$, which in turn is an upper bound for the multicurve intersection degree of α and β. Finally, observe that the assumption on the genus is important: for genus two, necessarily α and β are in fact a union of separating curves, and $2g-3 < 3g-3$ (see Section 1.3).

Our proof of Theorem 1 and Theorem 2 is constructive, and even explicit up to the fact that for certain multicurve components of β, we cannot specify how many parallel copies there need to be. The algebraic tools we use are Eisenstein's criterion and Hilbert's irreducibility theorem. See also Section 6 for explicit constructions in small genera.

Obviously, Theorem 2 implies Theorem 1 (at least for $g \geq 3$) but we think it is helpful to present an simpler proof of Theorem 1 first. Indeed, the proof of Theorem 2 is conceptually the same, but with more geometrical ingredients.

The pair of multicurves $\alpha, \beta \subset S$ naturally determines a bipartite graph whose vertices correspond to curve components and the number of edges between each pair of vertices equals the number of intersection points of the respective curve components. The adjacency matrix of this graph is $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$. Clearly, the square root $\sqrt{\mu}$ of the spectral radius μ of $XX^\top$ equals the spectral radius of Ω. We call the algebraic degree of $\sqrt{\mu}$ the *multicurve bipartite degree* of α and β. We prove the following result.

Theorem 3. *Every even integer $2 \leq 2d \leq 6g-6$ is realised as a multicurve bipartite degree on the closed orientable surface of genus $g \geq 2$.*

Our motivation for studying multicurve intersection degrees stems from Teichmüller geometry. More specifically, we are interested in the applications to the theory of pseudo-Anosov maps and their stretch factors.

1.2. **Pseudo-Anosov stretch factors.** A homeomorphism f of S is pseudo-Anosov if there exists a pair of transverse singular measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ and a real number $\lambda > 1$ such that $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$. Thurston's classification states that elements of the mapping class group of S come in three types: reducible, periodic and pseudo-Anosov. The number λ is called the stretch factor. It has several characterisations and is an algebraic integer of degree bounded above by the dimension of the Teichmüller space of S, namely $6g-6$ [Thu88].

An important construction of pseudo-Anosov maps appeared independently in two papers by Thurston and Veech [Thu88, Vee89]. Given a pair of filling multicurves $\alpha, \beta \subset S$, this construction provides pseudo-Anosov mapping classes in the subgroup $\langle T_\alpha, T_\beta \rangle$ of the mapping class group generated by multitwists along the multicurves α and β. This construction is referred to as the *Thurston–Veech construction.* In his seminal 1988 Bulletin paper [Thu88], Thurston provides the upper bound of $6g-6$ on the algebraic degree of a pseudo-Anosov stretch factor $\lambda(f)$ and claims, without proof, that "*the examples of* [Thu88, Theorem 7] *show that this bound is sharp*". The referenced examples are exactly the pseudo-Anosov maps in $\langle T_\alpha, T_\beta \rangle$.

While Strenner precisely determined the set of integers appearing as the algebraic degrees of pseudo-Anosov stretch factors on every closed surface [Str17], Thurston's claim remained open. Based on Theorem 1, we are finally able to substantiate it, even for pseudo-Anosov maps in the Torelli group. Our main application to pseudo-Anosov stretch factors is the following.

Theorem 4. *Every even integer $2 \leq 2d \leq 6g-6$ is realised as the algebraic degree of a pseudo-Anosov stretch factor arising from the Thurston–Veech construction on the closed orientable surface of genus $g \geq 2$. For $g \geq 3$, the pseudo-Anosov maps can be chosen in the Torelli group.*

Proof of Theorem 4. Our proof is based on the following existence result:

Theorem 5. *Let $\alpha, \beta \subset S$ be a pair of filling multicurves having multicurve intersection degree d. For $\varepsilon \in \mathbb{Z} \setminus \{0\}$, there exists $n \in \mathbb{Z}_{>0}$ such that the mapping class $T_\alpha^n \circ T_\beta^{n\varepsilon}$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Assuming Theorem 5, we can immediately conclude the proof of Theorem 4. Setting $\varepsilon = 1$, the first statement follows directly from Theorem 1. For the second statement, we notice that if we impose that the homology class $[\alpha] - [\beta]$ is trivial in $H_1(S;\mathbb{Z})$, then the pseudo-Anosov map $T_\alpha^n \circ T_\beta^{-n}$ acts trivially on homology for every $n \in \mathbb{Z}\setminus\{0\}$. In particular, it is an element of the Torelli group of the surface S. Hence we can apply above criterion with $\varepsilon = -1$ using multicurves α an β provided by Theorem 2 to finish the proof of Theorem 4. □

Remark 6. Specifically, in our construction of multicurves, we can actually prove a stronger result. Namely, if we take our construction of the multicurve intersection degree $1 < d \leq 3g-3$ from Section 4 (proof of Theorem 1), and if we consider the pseudo-Anosov mapping class $T_\alpha \circ T_\beta$, then its stretch factor has degree $2d$. In the setting of Theorem 5, this means that for $\varepsilon = 1$ one can choose $n = 1$. It uses [LL24, Theorem 6] (see Section 6 for details). Unfortunately, for the Torelli case, the criterion from [LL24] does not apply. Hence, we need to use Theorem 5, which provides a slightly less explicit result.

1.3. **The first Johnson subgroup, Johnson filtration.** While Theorem 4 completely answers the question about the sharpness of Thurston's upper bound [Thu88, Theorem 7], refined versions of this question are still open to investigation. There are numerous conditions one can impose on the multicurves α and β. For instance the first Johnson subgroup $K(S)$ is the subgroup of the Torelli group generated by the Dehn twists along separating simple closed curves. Since the rank of any Abelian subgroup of the Torelli group is bounded from above by $2g-3$, this immediately implies that the maximal degree of stretch factors for $K(S)$ is $4g-6$. In the sequel, we will mention one particular other restriction.

The following question remains open. It was asked to us by Margalit in the more general context of infinite index normal subgroups of the mapping class group.

Question 7. *Which stretch factor degrees appear in the Torelli subgroups? What is the maximal algebraic degree of stretch factors of pseudo-Anosov maps in the Johnson filtration?*

1.4. **Strata of quadratic differentials.** One may impose restrictions on the geometry of the complement $S \setminus (\alpha \cup \beta)$, for example by fixing how often every $2k$-gon is allowed to occur. Via the Thurston–Veech construction, a pair of filling multicurves $\alpha, \beta \subset S$ corresponds to a nonzero quadratic differential on a Riemann surface, and a $2k$-gon in the complement corresponds to a zero of order $k-2$ of the quadratic differential. Furthermore, the multicurve intersection degree for α and β equals the trace field degree of the quadratic differential. Now, the space of nonzero quadratic differentials on a Riemann surface admits a stratification according to the number of zeros of each order, and some of the strata even have multiple connected components.

In previous work, we have shown that in the case where the quadratic differential is the square of an Abelian differential, then the stratum (and even the connected component in case there are multiple) imposes no restrictions on the possible trace field degrees that can be realised via the Thurston–Veech construction [LL24]. We end this introduction by stating this problem for connected components of strata of quadratic differentials.

Question 8. *For a given connected component $\mathcal{C}$ of a stratum of quadratic differentials on Riemann surfaces of genus g, which positive integers $\leq 3g-3$ arise as trace field degrees and which positive integers $\leq 6g-6$ arise as the algebraic degree of a stretch factor of a product of two affine multitwists on a surface belonging to $\mathcal{C}$?*

We note that Question 8 is open also for general pseudo-Anosov maps, that is, pseudo-Anosov maps that are not necessarily a product of two affine multitwists.

1.5. **Odd degree stretch factors.** While Theorem 5 provides the existence of field extensions $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \lambda^{-1})$ of degree two for mapping classes in $\langle T_\alpha, T_\beta \rangle$, realising extensions of degree one seems to be more mysterious. For example, Veech [Vee82] discovered a family of Hecke groups $\langle T_\alpha, T_\beta \rangle = \langle \left(\begin{smallmatrix} 1 & \lambda_q \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -\lambda_q & 1 \end{smallmatrix}\right) \rangle$, where $\lambda_q = 2\cos \pi/q$ for $q \geq 3$. The genus of the surface S is $(q-1)/2$ for odd q. For $q = 7, 9$ one can find stretch factors of degree one over the trace field $\mathbb{Q}(\lambda_q)$: for instance $T_\alpha \circ T_\beta^{-1}$ is an example for $q = 7$, and we refer to [Bou22] for $q = 9$. However, it is conjectured (see [HMTY08, Remark 9]) that stretch factors of degree one over $\mathbb{Q}(\lambda_q)$ do not exist for odd $q \geq 11$.

Organisation. In Section 2 we prove Theorem 5, the new nonsplitting criterion used to deduce Theorem 4 from Theorem 1 and Theorem 2. In Section 3 we introduce an irreducibility criterion for the characteristic polynomial of matrices of the form $XX^\top$ which plays a central role throughout the rest of the article. Using this irreducibility criterion, we give of proof of Theorem 1 in Section 4 and of Theorem 2 in Section 5. Finally, we provide some explicit examples in Section 6.

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2. A nonsplitting criterion

In this section we prove Theorem 5, which is an algebraic criterion that allows us to deduce that the degree of the field extension $\mathbb{Q}(\lambda(f)) : \mathbb{Q}(\lambda(f) + \lambda(f)^{-1})$ equals two for certain f which are a product of multitwists. Compare with [LL24, Theorem 6]. For convenience, we repeat the statement of Theorem 5:

Theorem (Theorem 5)**.** *Let $\alpha, \beta \subset S$ be a pair of filling multicurves having multicurve intersection degree d. For every $\varepsilon \in \mathbb{Z} \setminus \{0\}$, there exists $n \in \mathbb{Z}_{>0}$ such that the mapping class $T_\alpha^n \circ T_\beta^{n\varepsilon}$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Proof of Theorem 5. By the Thurston–Veech construction, there exists a representation $\rho : \langle T_\alpha, T_\beta \rangle \to \mathrm{PSL}_2(\mathbb{R})$ mapping T_α to the matrix $\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$ and T_β to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -r & 1 \end{smallmatrix}\right)$, where $r^2 = \mu$ is the spectral radius of the matrix $XX^\top$ for the multicurves α and β. Furthermore, the stretch factor $\lambda(f)$ of $f \in \langle T_\alpha, T_\beta \rangle$ equals the spectral radius of $\rho(f)$. Now, let us consider the product of multitwists $f = T_\alpha^{2n} \circ T_\beta^{2n\varepsilon}$. A direct computation provides that the trace of $\rho(f)$ equals $\mathrm{tr}(\rho(f)) = 2 - \varepsilon(2nr)^2$. Thus, $\lambda(f) + \lambda(f)^{-1} = |2 - \varepsilon(2nr)^2|$ and hence $\mathbb{Q}(\lambda(f) + \lambda(f)^{-1}) = \mathbb{Q}(\mu) = K$. Note that by assumption, the degree of the field extension $K : \mathbb{Q}$ is d, the multicurve intersection degree of α and β.

Since $\lambda = \lambda(f)$ solves the quadratic equation $t^2 - (\lambda + \lambda^{-1})t + 1 = 0$, λ has degree 1 or 2 over K. All what we need to do is find $n \in \mathbb{Z}_{>0}$ such that $\lambda \notin K$, or equivalently such that the discriminant $D = (2 - \varepsilon(2nr)^2)^2 - 4 = 16 \cdot n^2 \cdot ((n\varepsilon\mu)^2 - \varepsilon\mu)$ of the quadratic equation is not a square in K. We will proceed by contradiction. Let $\mu' = \varepsilon\mu$ and let us assume that $(n\mu')^2 - \mu'$ is a square in $K = \mathbb{Q}(\mu')$ for every $n > 0$. Since the expression is invariant under the transformation $n \mapsto -n$, we can assume the expression is a square for every $n \in \mathbb{Z} \setminus \{0\}$.

Let $P = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$ be the minimal polynomial of μ' over $\mathbb{Q}$. The Thurston–Veech construction implies that μ is an eigenvalue of a square matrix, so is μ' and $a_d = 1$. Thus, μ' and $n^2\mu' - 1$ are algebraic units. The norm of μ' equals $N(\mu') = (-1)^d a_0$. Similarly, the minimal polynomial of $n^2\mu' - 1$ is $n^{2d} P\left(\frac{t+1}{n^2}\right)$. Inspecting the constant term, we have

$$N(n^2\mu' - 1) = (-1)^d \sum_{k=0}^{d} a_k n^{2d-2k}.$$

Altogether this gives $N((n\mu')^2 - \mu') = Q(n^2)$, where

$$Q(t) = a_0 \sum_{k=0}^{d} a_k t^{d-k}.$$

By assumption, $Q(n^2)$ is a square for every $n \in \mathbb{Z} \setminus \{0\}$. We show that $Q(0) = N(-\mu')$ is also a square. Indeed, for any prime integer p, the reduction modulo $n = p$ of $N((n\mu')^2 - \mu')$ gives that $N(-\mu')$ is a quadratic residue. Thus it is also a square in $\mathbb{Z}$. Hence $Q(t) \in \mathbb{Z}[t]$ is a polynomial taking integral square value at every integer specialisation. By a result of Murty [Mur08, Theorem 1], $Q(t^2)$ is the square of a polynomial.

Moreover, we observe that $Q(t) = a_0 \cdot t^d P\left(\frac{1}{t}\right) \in \mathbb{Q}[t]$. In particular $Q\left(\frac{1}{\mu'}\right) = 0$, and since μ' and $\frac{1}{\mu'}$ generate the field extension $K : \mathbb{Q}$, the polynomial Q is irreducible over $\mathbb{Q}$. It is in particular separable. Now each root $0 \neq a \in \mathbb{C}$ of Q gives rise to two distinct roots $\pm\sqrt{a}$ of $Q(t^2)$, and conversely. Thus $Q(t^2)$ is also separable, and cannot be a square. This concludes the proof of the theorem. $\square$

3. An irreducibility criterion

The goal of this section is to present an algebraic criterion that allows us to deduce that certain characteristic polynomials of matrices of the form $XX^\top$ are irreducible.

Proposition 9. *Let M be a square integer matrix, and let N be the principal submatrix of M obtained by deleting the first row and the first column. If M and N have no common eigenvalue, and if M has a simple eigenvalue ρ, then the characteristic polynomial of $\widetilde{M} = M + ay^pE_{11}$ is an irreducible element of $\mathbb{Z}[t,y]$, for all $p \geq 1$ and for all $0 \neq a \in \mathbb{Z}$.*

Proof. Our goal is to use Eisenstein's criterion on $\chi_{\widetilde{M}} \in \mathbb{Z}[t,y] \cong (\mathbb{Z}[t])\,[y]$, viewing it as a polynomial in the variable y and coefficients in $\mathbb{Z}[t]$. We calculate

$$\chi_{\widetilde{M}}(t,y) = \det(t \cdot \mathrm{Id} - \widetilde{M}) = -y^p a \chi_N(t) + \chi_M(t)$$

and notice that $a\chi_N$ and χ_M are relatively prime in $\mathbb{Z}[t]$. Indeed, χ_M has leading coefficient $+1$ and no root in common with χ_N by our assumption that M and N have no eigenvalue in common. In particular, they have no common factor, which shows that $\chi_{\widetilde{M}} \in (\mathbb{Z}[t])\,[y]$ is primitive. In order to apply Eisenstein's criterion, let $\mu_\rho \in \mathbb{Z}[t]$ be the minimal polynomial of the simple eigenvalue ρ of M. By assumption, μ_ρ divides χ_M exactly once, but it does not divide χ_N since χ_M and χ_N have no common root. In particular, Eisenstein's criterion applies to show that the polynomial $\chi_{\widetilde{M}} \in (\mathbb{Z}[t])\,[y] \cong \mathbb{Z}[t,y]$ is irreducible. □

Remark 10. In the previous statement, one can easily replace $\chi_{\widetilde{M}}(t)$ by $\chi_{\widetilde{M}}(t^n)$ for any integer $n > 0$. Indeed $\chi_M(t^n)$ and $\chi_N(t^n)$ are still coprime and $\mu_\rho(t^n)$ divides $\chi_M(t^n)$ exactly once, so Eisenstein's criterion applies.

Remark 11. Oscillatory matrices satisfy a stronger version of Perron-Frobenius theory, namely *all* the eigenvalues are positive real, simple, and they strictly interlace when taking a principal submatrix [And87]. Hence, Proposition 9 applies very cleanly to this class of matrices.

We use Proposition 9 on the following two cases (Lemma 12 and Lemma 13).

Lemma 12. *For $n \geq 1$, let*

$$N = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right), \quad M = \left(\begin{array}{c|ccc} 0 & \alpha a_1 & \dots & \alpha a_n \\ \hline \alpha a_1 & & & \\ \vdots & & N & \\ \alpha a_n & & & \end{array}\right)$$

be square integer matrices with $a_1 \geq 1$. If M is nonnegative and irreducible, and if $\chi_N \in \mathbb{Z}[t]$ is irreducible, then the characteristic polynomial of $\widetilde{M} = M + ay^2E_{11}$ is irreducible in $\mathbb{Z}[t,y]$ for all $0 \neq a \in \mathbb{Z}$.

Proof. In order to use Proposition 9, we need to show that M has a simple eigenvalue and that M and N share no eigenvalue. The former holds since M is nonnegative and irreducible, and in particular has a Perron-Frobenius eigenvalue which is simple. For the latter, we compute

$$\chi_M(t) = t\chi_N(t) + q(t),$$

where $q(t) \in \mathbb{Z}[t]$ is of degree at most $n-1$. We claim that it is not the zero polynomial either. Indeed, we directly verify

$$q(0) = \det \left(\begin{array}{c|ccc} 0 & -\alpha a_1 & \dots & -\alpha a_n \\ \hline -\alpha a_1 & & & \\ \vdots & & -N & \\ -\alpha a_n & & & \end{array} \right)$$

$$= \det \left(\begin{array}{c|ccc} \alpha^2 a_1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & -N & \\ 0 & & & \end{array} \right) = \pm \alpha^2 a_1 \det N \neq 0.$$

Now if there existed a common root $\lambda \in \mathbb{C}$ of χ_M and χ_N, then λ would also be a root of $q(t)$. But since χ_N is irreducible of degree n and $q(t)$ is a nonzero polynomial of degree at most $n-1$, this is impossible. □

Lemma 13. *For $n, m \geq 1$, let*

$$A = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array} \right), \quad B = \left(\begin{array}{c|ccc} b_1 & b_2 & \dots & b_m \\ \hline b_2 & & & \\ \vdots & & * & \\ b_m & & & \end{array} \right)$$

be square integer matrices of dimension n and m, respectively, with $a_1, b_1 \geq 1$. Furthermore, let $\alpha, \beta \neq 0$ such that

$$M = \left(\begin{array}{c|ccc|ccc} 0 & \alpha a_1 & \dots & \alpha a_n & \beta b_1 & \dots & \beta b_m \\ \hline \alpha a_1 & & & & & & \\ \vdots & & A & & & & \\ \alpha a_n & & & & & & \\ \hline \beta b_1 & & & & & & \\ \vdots & & & & & B & \\ \beta b_m & & & & & & \end{array} \right)$$

is a matrix with integer coefficients. If M is nonnegative and irreducible, and if $\chi_A, \chi_B \in \mathbb{Z}[t]$ are irreducible and distinct, then the characteristic polynomial of $\widetilde{M} = M + ay^2 E_{11}$ is irreducible in $\mathbb{Z}[t, y]$ for all $0 \neq a \in \mathbb{Z}$.

Proof. The proof is similar to the proof of Lemma 12: the only thing to verify is that no eigenvalue of A or of B is also an eigenvalue of M. Again, we compute

$$\chi_M(t) = t\chi_A(t)\chi_B(t) \pm q_1(t)\chi_B(t) \pm q_2(t)\chi_A(t),$$

where $q_1(t) \in \mathbb{Z}[t]$ is of degree at most $n-1$ and $q_2(t) \in \mathbb{Z}[t]$ is of degree at most $m-1$. This is seen by developing the first column of the matrix $tI - M$. The first coefficient is responsible for the summand $t\chi_A(t)\chi_B(t)$, the next n coefficients are responsible for the summand $\pm q_1(t)\chi_B(t)$ and the final m coefficients are responsible for the summand $\pm q_2(t)\chi_A(t)$. We claim that neither among $q_1(t)$ and $q_2(t)$ is the zero polynomial. Indeed, by developing the first column of the matrix $t\mathrm{I} - M$, and evaluating at $t = 0$, we get

$$q_1(0) = \det \left(\begin{array}{c|ccc} 0 & -\alpha a_1 & \dots & -\alpha a_n \\ \hline -\alpha a_1 & & & \\ \vdots & & -A & \\ -\alpha a_n & & & \end{array} \right)$$
$$= \det \left(\begin{array}{c|ccc} \alpha^2 a_1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & -A & \\ 0 & & & \end{array} \right) = \pm \alpha^2 a_1 \det A,$$

which is not zero since χ_A is irreducible. Similarly, $q_2(0) \neq 0$. Now if there existed a common root $\lambda \in \mathbb{C}$ of χ_M and χ_A, then λ would also be a root of either χ_B or q_1. Since χ_A and χ_B are irreducible and distinct, we must have $q_1(\lambda) = 0$. But since χ_A is irreducible of degree n, and $q_1(t)$ is a nonzero polynomial of degree at most $n-1$, this is impossible. Similarly, no root of χ_B can be a root of χ_M, which concludes the proof. □

Remark 14. One could formulate Lemma 13 with $k \geq 2$ blocks $A_1, \dots, A_k$ of respective sizes $n_1, \dots, n_k$, instead of $k = 2$. In this case, all the k characteristic polynomials χ_{A_i} should be irreducible and pairwise distinct. The argument is identical by considering

$$\chi_M(t) = t \prod_{i=1}^{k} \chi_{A_i} + \sum_{i=1}^{k} \pm q_i(t) \prod_{j \neq i} \chi_{A_j},$$

where $q_i(t) \in \mathbb{Z}[t]$ is of degree at most $n_i - 1$ and nonzero.

4. Proof of Theorem 1

Our goal of this section is to construct, on the closed orientable surface of genus $g \geq 2$ and for every positive integer $d \leq 3g - 3$, a pair of filling multicurves α and β with multicurve intersection degree d. In a first step, we construct the maximal multicurve intersection degree $3g-3$, and in a second step we discuss how to modify our construction in order to realise all smaller multicurve intersection degrees as well.

In order to read off the matrix $XX^\top$ from our figures, we use the following formula for its coefficients, which is a direct consequence of the definition of matrix multiplication:

$$(XX^\top)_{ij} = \sum_k |\alpha_i \cap \beta_k| \cdot |\beta_k \cap \alpha_j|.$$

4.1. Multicurve intersection degree $3g-3$.

We start by realising, on the surface of genus $g \geq 1$ with 2 boundary components, a pair of filling multicurves α and β such that $\chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-1$. We proceed by induction on g.

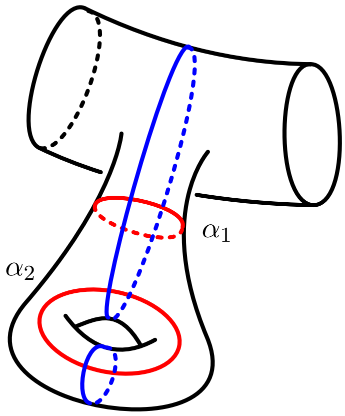

FIGURE 1. Two multicurves α (in red) and β (in blue) on the surface of genus one with two boundary components. The multicurve β contains $y-1$ parallel copies of one of its components.

4.1.1. *For $g = 1$ with two boundary components.* We consider the two multicurves α and β shown in Figure 1, where one of the components of β has $y-1$ parallel copies. Here, we think of y as a variable that we specify later on.
We directly calculate

$$XX^\top = \begin{pmatrix} 4 & 2 \\ 2 & y \end{pmatrix}.$$

Observe that X is a matrix of size $2 \times y$ (the multicurve β has y components). We have $\chi_{XX^\top}(t) = t^2 - (4+y)t + 4(y-1)$ with discriminant $y^2 - 8y + 32$, which is not a square if $y \geq 12$. Indeed, in this case we have

$$(y-3)^2 = y^2 - 6y + 9 > y^2 - 8y + 32 > y^2 - 8y + 16 = (y-4)^2.$$

In particular, for $y \geq 12$ the polynomial $\chi_{XX^\top}$ is irreducible.

4.1.2. *For $g > 1$ and two boundary components.* For the inductive step, assume we have constructed on the surface of genus $g \geq 1$ with 2 boundary components a pair of multicurves α', β' such that the characteristic polynomial $\chi' = \chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-1$. Furthermore, assume that α'_1 is a simple closed curve that encircles all the handles of the surface, as illustrated in Figure 2. Take a surface of genus 1 and two boundary components, as in the case of genus $g = 1$, see Figure 1, and denote its multicurves by α'' and β''. Now glue its right boundary component to the left boundary component of the surface of genus g, and add two new curves α_0 and β_0 to the multicurves. The curve α_0 encircles all the handles of the newly formed surface, and the curve β_0 twice intersects α_0 but no other multicurve component. Again, see Figure 2 for an illustration.
Let A be the matrix $XX^\top$ for the pair of multicurves α', β', and let B be the matrix $XX^\top$ for the pair of multicurves α'', β''. We define the multicurves

$$\alpha = \alpha_0 \cup \alpha' \cup \alpha''$$
$$\beta = \beta_0 \cup \beta' \cup \beta''$$

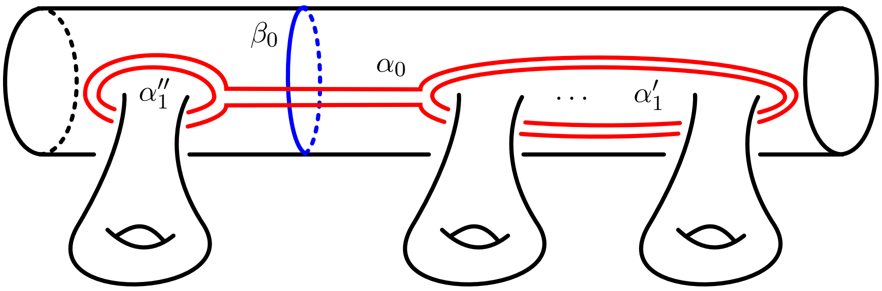

FIGURE 2. Two surfaces of genus g and 1, respectively, and two boundary components, glued together along one of their boundary components. The curves α_1' and α_1'' are shown, each encircling all the handles of their respective surface. The new curve α_0 encircles all the handles of the newly formed surface, and the new curve β_0 runs along the glued boundary component.

A quick computation gives

$$A = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right), \quad B = \begin{pmatrix} 4 & 2 \\ 2 & b \end{pmatrix}.$$

Let us choose b such that χ_B is irreducible and distinct from χ_A. We may assume inductively that $a_1 = 4a$. In the multicurve β, we take $y^2 - a - 1 \geq 1$ parallel copies of β_0, for $y > 0$ large enough. The matrix $XX^\top$ for the multicurves α and β takes the form

$$XX^\top = \left(\begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & A & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array}\right).$$

By Lemma 13, $\chi_{XX^\top} \in \mathbb{Z}[t, y]$ is irreducible (recall that χ_A is irreducible). Hence, by Hilbert's irreducibility theorem, there exist infinitely many specifications of y (and in particular infinitely many specifications of y such that $y^2 - a - 1 > 0$) with $\chi_{XX^\top} \in \mathbb{Z}[t]$ irreducible. This polynomial is of degree

$$3g - 1 + 3 = 3(g + 1) - 1,$$

which is exactly what we wanted to show. Finally, to justify our inductive assumption on the top-left coefficient of the matrix A, note that the top-left coefficient of the matrix $XX^\top$ is again a multiple of 4.

4.1.3. *The closed case for $g \geq 2$.* Take any example of a pair of multicurves α' and β' we constructed on the surface of genus $g - 1 \geq 1$ with two boundary components in Section 4.1.2. Let

$$A = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right)$$

be the matrix $XX^\top$ for the multicurves α' and β', where $a_1 = 4a$. We identify the two boundary components of the surface to increase the genus by one. Let α_0 be a longitude of the created handle, and let β_0 run along the glued boundary. Define the two new multicurves

$$\alpha = \alpha_0 \cup \alpha'$$
$$\beta = \beta_0 \cup \beta',$$

where we take $y^2 - a$ copies of β_0. Then the matrix $XX^\top$ for the multicurves α and β takes the form

$$XX^\top = \left(\begin{array}{c|ccc} y^2 & \frac{a_1}{2} & \dots & \frac{a_n}{2} \\ \hline \frac{a_1}{2} & & & \\ \vdots & & A & \\ \frac{a_n}{2} & & & \end{array}\right),$$

and $\chi_{XX^\top} \in \mathbb{Z}[t, y]$ is irreducible by Lemma 12. By Hilbert's irreducibility theorem, there exist infinitely many specifications of y (and in particular infinitely many specifications of y such that $y^2 - a > 0$) with $\chi_{XX^\top} \in \mathbb{Z}[t]$ irreducible. This polynomial is of degree $3(g-1) - 1 + 1 = 3g - 3$.

4.2. **Multicurve intersection degrees** $< 3g - 3$**.** So far, we have realised the maximal possible multicurve intersection degree $3g-3$. In order to prove Theorem 1 in its full generality, we need to argue that all smaller multicurve intersection degrees are also realised. For this we need a new building block for our surfaces. The surface we need is depicted in Figure 3.

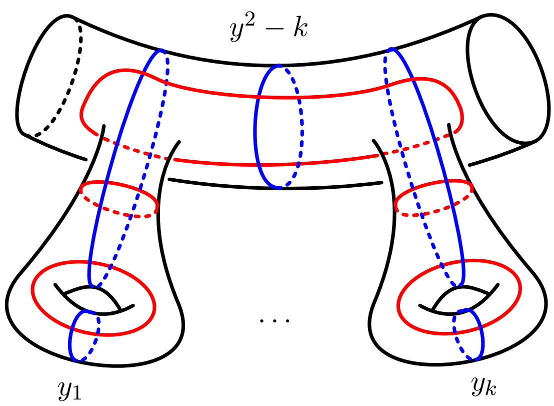

FIGURE 3. A surface of genus k with two boundary components, as well as two multicurves α (in red) and β (in blue). Some components of β have several parallel copies, as indicated by $y_1, \dots, y_k$ and $y^2 - k$.

We denote the red multicurve by α and the blue multicurve by β. The multicurve α has $k+1$ separating components: one for each of the handles that separates the handle, and one that separates all the handles. We denote the component of α that separates all the handles of the surface in Figure 3 by α_1, and we denote the other separating components of α by $\alpha_2, \alpha_4, \ldots, \alpha_{2g-2}$ from left to right. Finally, the remaining nonseparating components of α are $\alpha_3, \alpha_5, \ldots, \alpha_{2g-1}$ from left to right.

In this situation, we have

$$XX^\top = \left(\begin{array}{c|cccc} 4y^2 & v^\top & v^\top & \cdots & v^\top \\ \hline v & B_{y_1} & 0 & & \\ v & 0 & B_{y_2} & & \\ \vdots & & & \ddots & \\ v & & & & B_{y_k} \end{array}\right), \qquad B_{y_i} = \begin{pmatrix} 4 & 2 \\ 2 & y_i \end{pmatrix}, \qquad v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Let $p_{y_i}(t) = t^2 - (4+y_i)t + 4(y_i - 1)$ be the characteristic polynomial of B_{y_i}. We know from Section 4.1.1 that p_{y_i} is irreducible if $y \geq 12$. So, choosing all $y_i \geq 12$ pairwise distinct, Remark 14 guarantees that the polynomial $\chi_{XX^\top}(t,y) \in \mathbb{Z}[t][y]$ is irreducible. By Hilbert's irreducibility theorem, there are infinitely many specifications of y such that $y^2 - k > 0$ and such that $\chi_{XX^\top}(t) \in \mathbb{Z}[t]$ is irreducible and of degree $2k+1$.

Case 1: $2g \leq d < 3g-3$. Assume we want to realise the multicurve intersection degree $3g-3-f$ for $0 < f \leq g-3$. Let $k = f+2 \leq g-1$. Start the inductive procedure as in Section 4.1.2 with the surface from Figure 3 as a starting point, adding $g-1-k$ more handles. The exact same argument yields a surface of genus $g-1$ with two boundary components, and a characteristic polynomial $\chi_{XX^\top} \in \mathbb{Z}[t]$ that is irreducible and of degree $2k+1+3(g-1-k) = 3g-3-k+1$. Closing up the surface exactly as in Section 4.1.3 yields $3g-3-k+2 = 3g-3-f$ as a multicurve intersection degree on the closed orientable surface of genus g.

Case 2: $g < d < 2g$. Assume we want to realise the multicurve intersection degree $2g-f$ for $0 < f \leq g-1$. Take the surface depicted in Figure 3 for $k = g-1$. Now, remove f of the separating curve $\alpha_2, \ldots, \alpha_{2g-2}$. This slightly modifies the matrix $XX^\top$: f of the 2-by-2 blocks on the diagonal are now 1-by-1 blocks, with the single coefficient y_i. Nevertheless, since all the y_i are chosen pairwise distinct, Remark 14 guarantees that the polynomial $\chi_{XX^\top}(t,y) \in \mathbb{Z}[t][y]$ is irreducible. We note that for the coefficients y_i in the 1-by-1 blocks, any positive integer can be chosen. By Hilbert's irreducibility theorem, there are infinitely many specifications of y such that $\chi_{XX^\top}(t) \in \mathbb{Z}[t]$ is irreducible and of degree $2g-1-f$. Closing up the surface as in Section 4.1.3 yields the multicurve intersection degree $2g-f$ on the closed orientable surface of genus g.

Case 3: $1 \leq d \leq g$. This is the case we have already dealt with in [LL24].

Finally, we end this section we a proof of Theorem 3.

Proof of Theorem 3. For every $g \geq 2$ and every integer $g < d \leq 3g-3$, we have constructed a pair of filling multicurves α and β, with a parameter y, such that $\chi_{XX^\top}(t,y) \in \mathbb{Z}[t,y]$ is irreducible. By Remark 10, we may run the same argument to show that also the polynomial $\chi_{XX^\top}(t^2,y) \in \mathbb{Z}[t,y]$ is irreducible. By Hilbert's irreducibility theorem, we find infinitely many specifications of y such that $\chi_{XX^\top}(t^2) \in \mathbb{Z}[t]$ is irreducible of degree $2d$. The leading eigenvalue μ of $XX^\top$

is a root of a characteristic polynomial $\chi_{XX^\top}(t)$, so $\chi_{XX^\top}(t^2) \in \mathbb{Z}[t]$ is the minimal polynomial of $\sqrt{\mu}$. Hence, the multicurve bipartite degree of α and β equals $2d$.

For $1 \leq d \leq g$, instead of the examples constructed in this section, we use the examples in [LL24], as in the third case above. Similarly to Remark 10, one can run the same proof as [LL24, Lemma 10] to show that $\chi_{XX^\top}(t^2, y) \in \mathbb{Z}[t, y]$ is irreducible for these examples. The rest of the proof is then exactly as in the case above. ☐

5. Proof of Theorem 2

The goal of this section is to realise every positive integer $d \leq 3g-3$ as the multicurve intersection degree of a pair of multicurves $\alpha, \beta \subset S$ of the closed orientable surface of genus $g \geq 3$, such that $[\alpha] - [\beta] = 0 \in H_1(S;\mathbb{Z})$.

As before, we start with the maximal degree and then discuss how to adapt the construction in order to realise smaller degrees.

5.1. **Multicurve intersection degree $3g-3$.** We start by realising, on the surface of genus $g \geq 2$ with one boundary component, a pair of filling multicurves α and β such that $\chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-2$, in such a way that their difference is trivial in homology: $[\alpha] - [\beta] = 0 \in H_1(S;\mathbb{Z})$. The construction is done by induction on the genus $g \geq 2$.

5.1.1. *For $g = 2$ with one boundary component.* We consider the two multicurves α and β shown in Figure 4. We first note that $[\alpha] - [\beta] = 0 \in H_1(S;\mathbb{Z})$. Indeed,

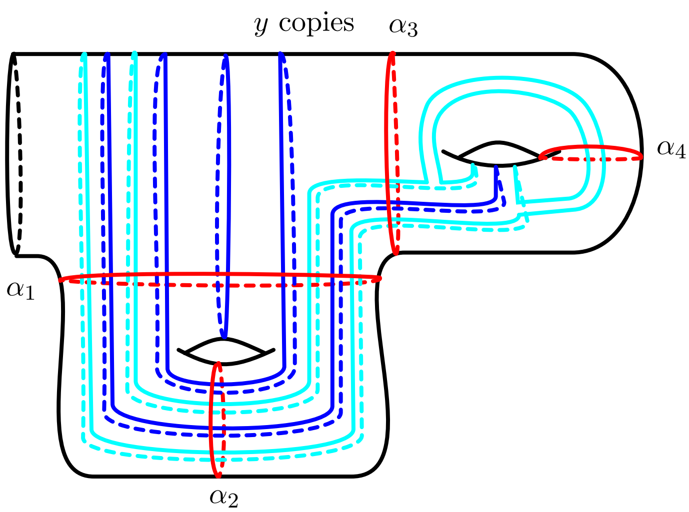

Figure 4. Two multicurves α and β on the surface of genus two with one boundary component. One component of β has y parallel copies.

the components α_1 and α_3 are separating, so they are already trivial in homology. Furthermore, the components α_2 and α_4 have their counterparts in the multicurve β with which they each form a bounding pair. Finally, the component of β of which there are y parallel copies and the component of β drawn in light blue in Figure 4 are separating.

We directly calculate

$$XX^\top = \begin{pmatrix} 84+16y & 40+8y & 40 & 16 \\ 40+8y & 20+4y & 20 & 8 \\ 40 & 20 & 20 & 8 \\ 16 & 8 & 8 & 4 \end{pmatrix},$$

and it is a direct check (by the computer) that the characteristic polynomial of $XX^\top$ is irreducible if $y = 2$ or $y = 3$. This finishes the case $g = 2$ with one boundary component.

5.1.2. *For $g > 2$ and one boundary component.* In order to increase the genus by one, we glue a surface of genus one with two boundary components as follows. On this surface, we consider the two multicurves α and β shown in Figure 5. We directly

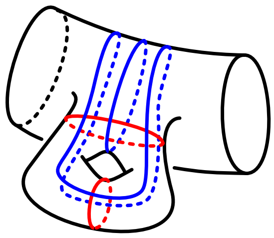

FIGURE 5. Two multicurves α (in red) and β (in blue) on the surface of genus one with two boundary components. The multicurve β has y parallel copies of its separating component.

calculate

$$XX^\top = \begin{pmatrix} 16y+4 & 8y \\ 8y & 4y \end{pmatrix} =: C_y,$$

and $\chi_{XX^\top}(t) = t^2 - (20y+4)t + 16y$ with discriminant $16 \cdot (25y^2 + 6y + 1)$, which is never a square. Indeed, we have

$$(5y)^2 = 25y^2 < 25y^2 + 6y + 1 < 25y^2 + 10y + 1 = (5y+1)^2.$$

In particular, the polynomial $\chi_{XX^\top}$ is irreducible for every positive integer y.

For the inductive step, let $g \geq 2$. Assume we have constructed on the surface of genus g with one boundary component a pair of multicurves α', β' such that the characteristic polynomial $\chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-2$, in such a way that $[\alpha] - [\beta] = 0 \in H_1(S;\mathbb{Z})$. Further, assume that α_1' is a simple closed curve that encircle all the handles of the surface, except for the rightmost. Then, we take such a model surface and glue to its boundary a surface of genus one with two boundary components, as shown in Figure 5, and add two new curves α_0 and β_0 to the multicurves. The curve α_0 encircles all the handles of the newly formed surface, except for the rightmost one, and the curve β_0 runs along the glued boundary components, and twice intersects α_0 but no other component of α, see Figure 6.

The proof of irreducibility is now exactly the same as in the non-Torelli case. The only thing we need to check is that $[\alpha] - [\beta]$ is still trivial in $H_1(S;\mathbb{Z})$. But this is clearly the case, since all the curves we add in the inductive step are separating or come as a bounding pair.

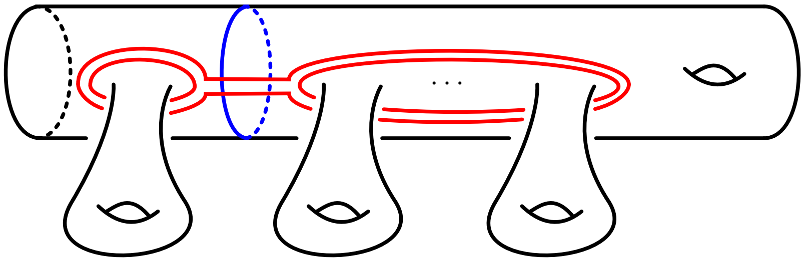

FIGURE 6

5.1.3. *The closed case for $g \geq 4$.* The last step is to make the surfaces closed. We simply glue together two pieces of genera g', g'', where $g' + g'' = g$, and one boundary component together along their boundaries. The same argument as in the inductive step provides irreducible characteristic polynomials of degree

$$3g' - 2 + 3g'' - 2 + 1 = 3g - 3.$$

5.1.4. *The closed case for $g = 3$.* We need a different argument. In this case, we start with the surface of genus two and one boundary component depicted in Figure 4, and close it off to the left by glueing a surface of genus one with one boundary component, see Figure 7. First add the curves α_5 and β_5 with $y^2 - 29$ parallel

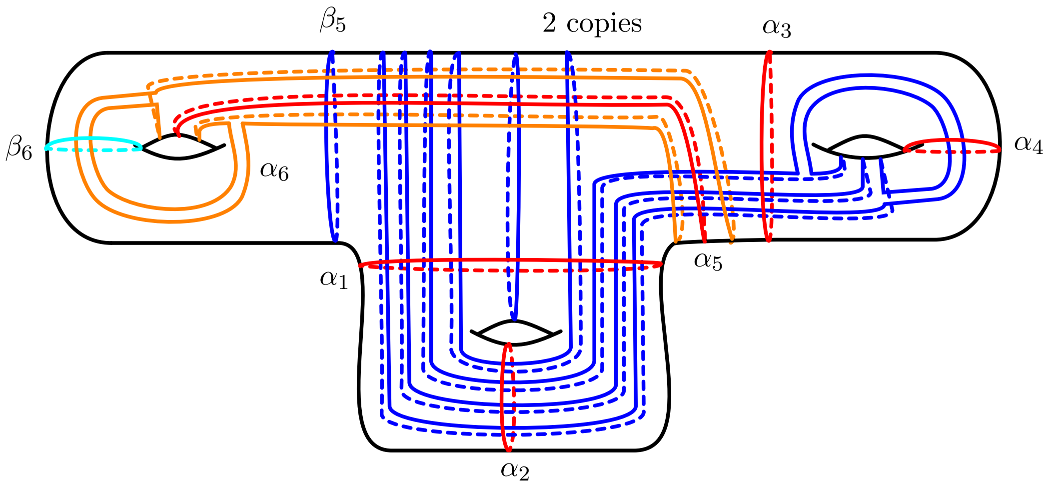

FIGURE 7. Two multicurves α and β on the surface of genus three. There are to new components of α when compared to Figure 4: a nonseparating component (red) that we call α_5 and a separating component (orange) that we call α_6. Similarly, there are two new components of β: a separating component (blue) that we call β_5 and a nonseparating component (light blue) that we call β_6.

copies. The resulting characteristic polynomials is irreducible for infinitely many choices of y by Lemma 12. Repeat the same process with α_6 and β_6 (adjusting the number of parallel copies of β_6 suitably), and we are done.

5.2. **Multicurve intersection degrees $d < 3g-3$.** . We now show how to modify our construction from Section 5.1 in order to realise multicurve intersection degrees smaller than the maximal multicurve intersection degree $3g-3$. As in the non-Torelli case, we need new building blocks to construct our surfaces.

Block 1. Our first block is obtained from the surface depicted in Figure 4, simply by dropping the component α_3. A direct verification yields that for $y = 1, 2$ the characteristic polynomial of $XX^\top$ is irreducible and of degree 3.

Block 2. Our second block is obtained from the surface depicted in Figure 8. The characteristic polynomial of the matrix $XX^\top$ for the multicurves α and β is irreducible and of degree 1. Versions of this block with distinct characteristic polynomial can be obtained by taking y parallel copies of β.

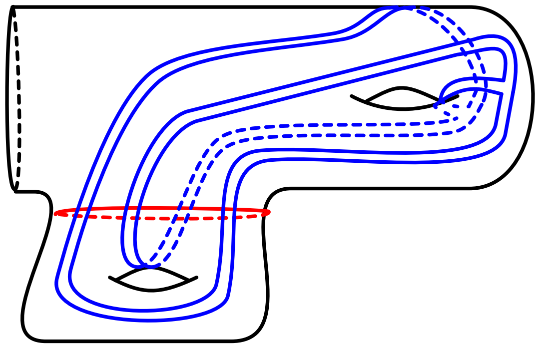

FIGURE 8. Two separating and filling curves α and β on the surface of genus two with one boundary component.

Block 3. Take a surface as depicted in Figure 9. We denote the red multicurve by α and the blue multicurve by β. The multicurve α has $k+1$ separating components: one for each of the handles that separates the handle, and one that separates all the handles. We denote the component of α that separates all the handles of the surface in Figure 9 by α_1, and we denote the other separating components of α by $\alpha_2, \alpha_4, \ldots, \alpha_{2g-2}$ from left to right. Finally, the remaining nonseparating components of α are $\alpha_3, \alpha_5, \ldots, \alpha_{2g-1}$ from left to right. In this situation, we have

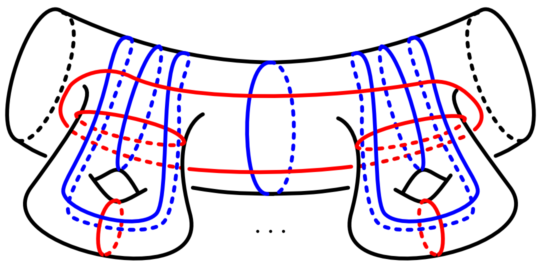

FIGURE 9. A surface of genus k with two boundary components, as well as two multicurves α (in red) and β (in blue). The separating components of β can have several parallel copies: the ones separating the handles have $y_1, \ldots, y_k$ copies, and the separating component in the middle has $y^2 - 4k - y_1 - \cdots - y_k$ copies.

$$XX^\top = \left(\begin{array}{c|cccc} 4y^2 & v_{y_1}^\top & v_{y_2}^\top & \cdots & v_{y_k}^\top \\ \hline v_{y_1} & C_{y_1} & 0 & & \\ v_{y_2} & 0 & C_{y_2} & & \\ \vdots & & & \ddots & \\ v_{y_k} & & & & C_{y_k} \end{array}\right), \; C_{y_i} = \begin{pmatrix} 16y_i+4 & 8y_i \\ 8y_i & 4y_i \end{pmatrix}, \; v_{y_i} = \begin{pmatrix} 16y_i+4 \\ 8y_i \end{pmatrix}.$$

By Remark 14, $\chi_{XX^\top}(t,y) \in \mathbb{Z}[t][y]$ is irreducible. By Hilbert's irreducibility theorem, there are infinitely many specifications of y such that $y^2 - 4k - y_1 - \cdots - y_k > 0$ and such that $\chi_{XX^\top}(t) \in \mathbb{Z}[t]$ is irreducible and of degree $2k+1$. Note that as in the non-Torelli case, we can drop the separating components of α winding around one handle one by one in order to decrease the degree, again reducing a 2-by-2 block to a 1-by-1 block for each component dropped in this way. The irreducibility argument remains the same. We can in this way construct all degrees $k+1 \leq d \leq 2k+1$ for the surface of genus k and 2 boundary components.

5.2.1. *Realising multicurve intersection degrees* $3g-6 \leq d < 3g-3$. Using a block of type 1 or 2 instead of our standard starting surface depicted in Figure 4, we can reduce the multicurve intersection degree by 1 or 3, respectively. Since we use such block on both sides of the surface in our construction, this gives the possibility to reduce the degree by any among the numbers 1,2,3,4 or 6. In particular, we can clearly realise the multicurve intersection degrees $3g-3, 3g-4$ and $3g-5$. This argument works for $g \geq 4$.

In case of $g = 3$, we need a separate argument. The idea is to copy our example of maximal degree from Figure 7, but leave out first α_3 and then also α_1. We start from the multicurves depicted in Figure 4 and drop α_3. Letting $y = 2$, we then get

$$XX^\top = \begin{pmatrix} 116 & 56 & 16 \\ 56 & 28 & 8 \\ 16 & 8 & 4 \end{pmatrix},$$

which has irreducible characteristic polynomial. We can now close off the surface by glueing a torus with one boundary component and add more components to α and β, in the same way as in Figure 7. The exact same argument we used to realise degree 6 now yields degree 5 instead.

In order to realise degree 4 for $g = 3$, we note that if we start from the multicurves depicted in Figure 7 and drop the components $\alpha_1, \alpha_3, \alpha_6$ as well as β_5, β_6, then the matrix $XX^\top$ for $\alpha_5, \alpha_2, \alpha_4$ is exactly the matrix as above:

$$XX^\top = \begin{pmatrix} 116 & 56 & 16 \\ 56 & 28 & 8 \\ 16 & 8 & 4 \end{pmatrix},$$

with irreducible characteristic polynomial. If we add back α_6 and β_6 with $y^2 - 116$ parallel copies, the resulting characteristic polynomials is irreducible for infinitely many choices of y by Lemma 12, realising degree 4. Note that all in all, we have dropped the components α_1, α_3 and β_6, which are all separating. Therefore, we have not changed the homology classes.

5.2.2. *Realising multicurve intersection degrees* $g \leq d \leq 3g-6$. We start by constructing a surface of genus $g-2$ with two boundary components, which we then close off in a second step.

Using surfaces of the type depicted in Figure 5 and applying the inductive step procedure, we can construct a surface of genus $g-2 \geq 1$ and two boundary components, as well as filling multicurves α and β with intersection degree $3(g-2)-1 = 3g-7$. Using at some point in the inductive procedure a block of type 3 of genus $k \leq g-2$, as depicted in Figure 9, we can reduce the degree by up to $2k-2 \leq 2g-6$, realising multicurve intersection degrees from $g-1$ to $3g-7$ on the surface of genus $g-2$ with two boundary components. Now we close the surface, as depicted in Figure 10, adding the new components α_0 and β_0 to the multicurves α and β, respectively.

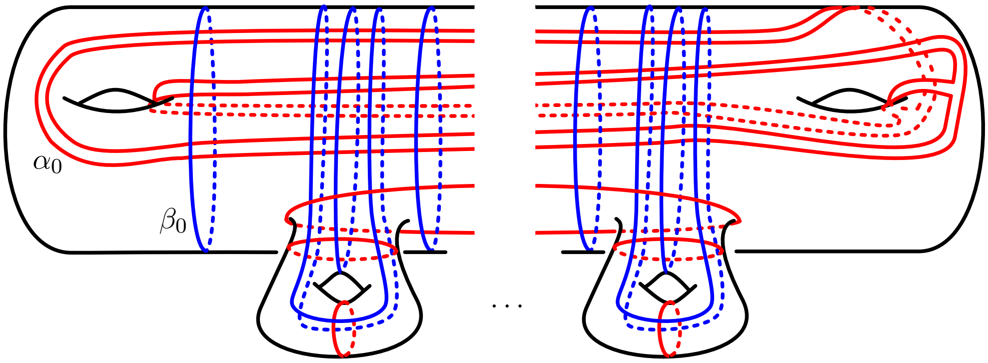

FIGURE 10. Two separating curves α_0 and β_0. There are ρ parallel copies of β_0.

We obtain the matrix

$$XX^\top = \left(\begin{array}{c|ccc} 64\rho + 16a_1 & 4a_1 & \dots & 4a_n \\ \hline 4a_1 & & & \\ \vdots & & A & \\ 4a_n & & & \end{array}\right),$$

where A is the matrix $XX^\top$ before adding the curves α_0 and β_0. Since $a_1 = 4a$, we can set $\rho = y^2 - a$ to have the top-left coefficient $64y^2$, which is exactly the form of the matrix in Lemma 12. Finishing the argument as usual, we can realise the multicurve intersection degrees $g \leq d \leq 3g-6$ for $g \geq 3$.

5.2.3. *Realising multicurve intersection degrees* $1 \leq d < g$. Realising multicurve intersection degree one is clearly achieved by taking a pair of separating filling curves on the surface S.

For $2 \leq d < g$, let us define $f = g-1-d$. We start with a surface block of type 3 of genus $g-2$, where we deleted all the components of α that are separating. We also remove the component of β in the middle of Figure 9. Furthermore, we let the $f+1 \leq g-2$ first of the parameters y_i be equal to 1. Then we close off the surface as in the previous case, adding one component α_0 to α and one component β_0 to β, compare with Figure 10. Assume there are ρ parallel copies of β_0. We get

$$XX^\top = \begin{pmatrix} 64(\rho - g + 2) + 256\delta & 32y_1 & 32y_2 & \cdots & 32y_{g-2} \\ 32y_1 & 4y_1 & & & \\ 32y_2 & & 4y_2 & & \\ \vdots & & & \ddots & \\ 32y_{g-2} & & & & 4y_{g-2} \end{pmatrix},$$

where $\delta = y_1 + \cdots + y_{g-2}$. We choose ρ such that $64(\rho - g + 2) + 256\delta = 64y^2$. To simplify the calculations, we let $z_i = 4y_i$ for $i = 1, \ldots, g-2$. The matrix becomes

$$XX^\top = \begin{pmatrix} 64y^2 & 8z_1 & 8z_2 & \cdots & 8z_{g-2} \\ 8z_1 & z_1 & & & \\ 8z_2 & & z_2 & & \\ \vdots & & & \ddots & \\ 8z_{g-2} & & & & z_{g-2} \end{pmatrix}.$$

By Lemma 9 in [LL24], the characteristic polynomial of $XX^\top$ equals

$$p(t, y, \mathbf{z}) = -64y^2 \prod_{i=1}^{g-2}(t - z_i) + t\prod_{i=1}^{g-2}(t - z_i) - \sum_{i=1}^{g-2} 64z_i^2 \prod_{j \neq i}(t - z_j).$$

If all the z_i are pairwise distinct, this polynomial is irreducible as a polynomial in t, y by Lemma 10 in [LL24]. However, we chose that the first $f+1$ coefficients $y_1, \ldots, y_{f+1}$ are equal to 1 and the other $y_i \neq 1$ and pairwise distinct. In particular, the polynomial $p(t, y)$ factors as $(t-4)^f \tilde{p}(t, y)$, where $\tilde{p}(t, y)$ is of degree $g - 1 - f = d$ in the variable t and with pairwise distinct z_i. In particular, Lemma 10 in [LL24] implies that $\tilde{p}(t, y) \in \mathbb{Z}[t, y]$ is irreducible. Hilbert's irreducibility theorem guarantees the existence of infinitely many positive specifications of y such that the resulting polynomial is irreducible in $\mathbb{Z}[t]$.

6. Explicit pseudo-Anosov maps

In this section, we show that in our construction of multicurves in Theorem 1, we can actually prove that the degree of the stretch factor of $T_\alpha \circ T_\beta$ equals two over the trace field. It uses the nonsplitting criterion of [LL24, Theorem 6] that we recall below.

6.1. Even degrees stretch factors.

Theorem 15 ([LL24], Theorem 6)**.** *Let $\alpha, \beta \subset S$ be a pair of filling multicurves. Let X be their geometric intersection matrix, let d be their multicurve intersection degree and let $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$. If $\dim(\Omega) > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I) > \dim(\Omega) - 2d$, then the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Here, $\sigma(A)$ and $\mathrm{null}(A)$ denote the signature and the nullity, respectively, of the matrix A.

Theorem 16. *Let α and β be an example of a pair of multicurves described in Section 4, realising a multicurve intersection degree $1 \leq d \leq 3g-3$. Then the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

For the case $1 \leq d \leq g$, this is shown in [LL24].

Proof of Theorem 16. According to Theorem 15, all there is to show is

$$\dim(\Omega) > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I) > \dim(\Omega) - 2d. \tag{1}$$

We now make a case distinction depending on d.

Case 1: $2g \leq d \leq 3g-3$. We consider the submatrix Ω' of Ω that is obtained by deleting all the rows and columns corresponding to components of the multicurve α

that have been added during the inductive step or closing up of the surface. Furthermore, if $d < 3g - 3$, we also remove the component of α encircling multiple handles of the starting surface, that is, the surface depicted in Figure 3.

A base change by a permutation matrix brings $\Omega' + 2I$ into block diagonal form with $g - 1$ blocks corresponding to genus one surface pieces as depicted in Figure 1, and a block of the form $2I$. For a block of the former type, and for $y > 4$, we directly calculate that the nullity is zero and the signature equals the dimension of the block minus two. Already, this implies that certainly the signature of $\Omega + 2I$ is not equal to its dimension, and it only remains to verify the lower bound in Equation (1).

By construction, if the genus equals $g \geq 2$, we have $g - 1$ surface pieces as in Figure 1. This in particular implies that $\sigma(\Omega') = \dim(\Omega') - 2g + 2$. Furthermore, we have $\dim(\Omega) - \dim(\Omega') = d - 2g + 2$. The latter equality follows from that fact that the number of components of α in our construction is exactly d, and there are two components per surface pieces as in Figure 1. We now calculate

$$\begin{aligned}\sigma(\Omega + 2I) &\geq \sigma(\Omega') - (\dim(\Omega) - \dim(\Omega')) \\ &= (\dim(\Omega') - 2g + 2) - (d - 2g + 2) \\ &= \dim(\Omega') - d \\ &> \dim(\Omega) - 2d,\end{aligned}$$

which implies Equation (1), so we are done for this case.

Case 2: $g < d < 2g$. We consider the submatrix Ω' of Ω that is obtained by deleting two rows and two columns corresponding to components of the multicurve α: the one corresponding to the component encircling multiple handles in Figure 3 and the one obtained from closing the surface. Recall that we have removed $f = 2g - d$ separating curves $\alpha_2, \ldots, \alpha_{2g-2}$.

A base change by a permutation matrix brings $\Omega' + 2I$ into block diagonal form with $g - 1 - f$ blocks corresponding to surface pieces as in Figure 1, f blocks corresponding to surface pieces as in Figure 1 but with the separating component of α removed, and a block of the form $2I$.

For a block of the first type, and for $y > 4$, recall from the previous case that the nullity is zero and the signature equals the dimension of the block minus two. For a block of the second type, the sum of the nullity and the signature equals the dimension of the block if $y \leq 3$, and it equals the dimension of the block minus two if $y > 3$. We may assume that for at least one block of the second type, we have $y = 3$. This is enough to ensure that $\dim(\Omega) > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I)$, so again we only need to verify the lower bound in Equation (1).

By construction, if the genus equals $g \geq 2$, we have $g-1$ surface pieces as in Figure 1. Having at least one piece with $y \leq 3$, this implies that $\sigma(\Omega') > \dim(\Omega') - 2g + 2$. Furthermore, we have $\dim(\Omega) - \dim(\Omega') = 2$. We now calculate

$$\begin{aligned}\sigma(\Omega + 2I) &\geq \sigma(\Omega') - (\dim(\Omega) - \dim(\Omega')) \\ &> (\dim(\Omega') - 2g + 2) - 2 \\ &= \dim(\Omega') - 2g \\ &= \dim(\Omega) - 2g + 2 \geq \dim(\Omega) - 2d,\end{aligned}$$

which implies Equation (1) also in the case $g < d < 2g$, so we are done. □

6.2. **Explicit multicurves.** We conclude this section by giving explicit computations supporting a conjecture on the irreducibility of the characteristic polynomials constructed in Theorem 1 for specific values of y. In the inductive step of Section 4.1.2, one uses a map

$$\phi_k : \begin{array}{ccc} M_k(\mathbb{Z}) \times \mathbb{Z} & \longrightarrow & M_{k+3}(\mathbb{Z}) \\ (C, y) & \mapsto & \left(\begin{array}{c|c|c} 4y^2 & * & * \\ \hline * & C & \\ \hline * & & A \end{array}\right) \end{array}, \qquad \text{with } A = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 12 \end{smallmatrix}\right).$$

For $g > 1$ we inductively construct the $(3g-1) \times (3g-1)$ matrix M_g with the maps ϕ_{3i-1} for $i = 1, \ldots, g-1$:

$$M_g = \phi_{3(g-1)-1}(\phi_{3(g-2)-1}(\ldots \phi_{3\cdot 2-1}(\phi_{3\cdot 1-1}(B, y^{(1)}), y^{(2)}), \ldots, y^{(g-2)}), y^{(g-1)}),$$

with $B = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 13 \end{smallmatrix}\right)$ and suitable parameters $y^{(i)}$ given by Hilbert's irreducibility theorem. The condition $y^2 > \frac{1}{4}c_{11} + 1$ appearing in the construction is obviously equivalent to $(y^{(i)})^2 > (y^{(i-1)})^2 + 1$. Finally, following Section 4.1.3 the matrix $XX^\top$ for the multicurves α and β on the closed surface of genus $g+1$ takes the form

$$N_g = \left(\begin{array}{c|c} y^2 & * \\ \hline * & M_g \end{array}\right)$$

with the condition $y^2 > \frac{1}{4}(M_g)_{11} = (y^{(g)})^2$.

By computer assistance, one immediately checks the following proposition.

Proposition 17. *For any $1 < g \leq 200$, if $y^{(i)} = i + 1$ for $i = 1, \ldots, g-1$, then the characteristic polynomial χ_{M_g} is irreducible over $\mathbb{Q}$. Moreover for $y = g + 1$, χ_{N_g} is irreducible over $\mathbb{Q}$.*

Together with Theorem 16, this gives explicit examples of pseudo-Anosov maps realizing the upper bound $6g - 6$ in Theorem 4 for every $1 < g \leq 201$. We don't know whether χ_{M_g} and χ_{N_g} are actually irreducible for every $g > 200$ with the parameters $y^{(i)} = i + 1$ chosen as in Proposition 17.

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