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Proof of the Landau–Zener formula

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Abstract

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We consider the time dependent Schrödinger equation in the adiabatic limit when the Hamiltonian is an analytic unbounded operator. It is assumed that the Hamiltonian possesses for any time two instantaneous non-degenerate eigenvalues which display an avoided crossing of finite minimum gap. We prove that the probability of a quantum transition between these two non-degenerate eigenvalues is given in the adiabatic limit by the well-known Landau–Zener formula.

1. Introduction

During the last few years, significant progresses have been made on the rigorous aspects of the adiabatic regime of the Schrödinger equation. This regime is characterized by the singular limit $\varepsilon \to 0$ of the evolution equation

$$i\varepsilon \frac{\partial}{\partial t} U_\varepsilon(t,s) = H(t)U_\varepsilon(t,s), \qquad U_\varepsilon(s,s) = \mathbb{I}, \tag{1.1}$$

where the generator or Hamiltonian $H(t)$, $t \in \mathbb{R}$, is a smooth family of self-adjoint operators defined on the same separable Hilbert space $\mathcal{H}$. It is assumed that for any $t \in \mathbb{R}$, the spectrum of $H(t)$, $\sigma(t)$, is composed of a bounded part $\sigma_1(t)$ separated from another part $\sigma_2(t)$ by a finite gap $g > 0$. The corresponding spectral projectors $Q(t)$ and $(\mathbb{I} - Q(t))$, defined by a Riesz formula, form a smooth orthogonal decomposition of the Hilbert space $\mathcal{H}$ for any $t \in \mathbb{R}$:

$$\mathcal{H} = Q(t) \oplus \big(\mathbb{I} - Q(t)\big)\mathcal{H}. \tag{1.2}$$

In 1987, Avron, Seiler and Yaffe [1] constructed an approximation $V(t,s)$ of the evolution $U_\varepsilon(t,s)$ under very general hypotheses on the unbounded Hamiltonian $H(t)$ generalizing earlier results [4, 18, 24]. This approximation satisfies the estimate

$$\big\|V(t,s) - U_\varepsilon(t,s)\big\| = O\big(\varepsilon|t-s|\big) \tag{1.3}$$

for any t, s in some bounded interval $I \in \mathbb{R}$ and possesses the intertwining property

$$V(t,s)Q(s) = Q(t)V(t,s). \tag{1.4}$$

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As a consequence, the so-called transition probability across the gap, $\mathcal{P}(\varepsilon)$, defined by

$$\mathcal{P}(\varepsilon) = \left\|(\mathbb{I} - Q(t_2))U_\varepsilon(t_2, t_1)Q(t_1)\right\|^2 \tag{1.5}$$

vanishes as ε^2 in the adiabatic limit $\varepsilon \to 0$. They also proved that when all derivatives of the Hamiltonian $H(t)$ vanish at t_1 and t_2, the transition probability tends to zero faster than any power of ε

$$\mathcal{P}(\varepsilon) = \mathrm{O}\big(\varepsilon^n |t_1 - t_2|\big), \quad \forall n. \tag{1.6}$$

Assume now that the Hamiltonian $H(t)$ depends analytically on time t and that it tends sufficiently rapidly to limits $H^\pm$ when $t \to \pm\infty$. In this case we can take the limits $t_1 \to -\infty$, $t_2 \to +\infty$ in (1.5) and the transition probability turns out to be exponentially small:

$$\mathcal{P}(\varepsilon) = \mathrm{O}\big(\exp\{-\tau/\varepsilon\}\big), \quad \tau > 0. \tag{1.7}$$

This result was proven in the matrix case by Joye, Kunz and Pfister [11] and then extended to the unbounded case by Joye and Pfister [13]. Moreover, when the Hamiltonian $H(t)$ is a Hermitian 2×2 analytic matrix, the leading term of the asymptotic expansion of $\mathcal{P}(\varepsilon)$ can be explicitly computed [3, 11], provided some supplementary condition is satisfied (see [11]):

$$\mathcal{P}(\varepsilon) = \exp\{2\mathrm{Im}\,\theta_1\}\exp\{-2\gamma/\varepsilon\}\big(1 + \mathrm{O}(\varepsilon)\big), \quad \gamma > 0. \tag{1.8}$$

This result justifies and generalizes the so-called Dykhne formula [5, 6, 9, 28] for $\mathcal{P}(\varepsilon)$ which is valid if $H(t)$ is a real symmetric 2×2 matrix. The Dykhne formula is obtained from (1.8) by replacing the ε-independent prefactor by 1. The presence of the prefactor $\exp\{2\mathrm{Im}\,\theta_1\}$ in the Hermitian case has been measured experimentally in [32]. Generalizations of this case were also investigated: The full asymptotic expansion of the quantity $\ln\mathcal{P}(\varepsilon)$ is computed in [14] and non-generic situations are considered in [12].

Another important concept in adiabatic dynamics is the notion of superadiabatic evolution, introduced by Berry [2] for two-level systems and generalized by Nenciu [26]. Such an evolution $V_*(t,s)$ is characterized by the fact that it approximates the evolution $U_\varepsilon(t,s)$ for exponentially long times:

$$\left\|V_*(t,s) - U_\varepsilon(t,s)\right\| = \mathrm{O}\big(\exp\{-\tau/\varepsilon\}|t-s|\big), \quad \tau > 0, \tag{1.9}$$

for any $t, s \in I \subset \mathbb{R}$, and by the existence of projectors $Q_*(t)$, $t \in \mathbb{R}$ for which $V_*(t,s)$ possesses the intertwining property

$$V_*(t,s)Q_*(s) = Q_*(t)V_*(t,s). \tag{1.10}$$

In general, V_* and Q_* depend on ε and it can be shown that if

$$\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}H(t)\right|_{t=t_0} = 0, \quad \forall n \leqslant m, \tag{1.11}$$

then

$$Q_*(t_0) = Q(t_0) + \mathrm{O}\big(\varepsilon^m\big). \tag{1.12}$$

The origin of this notion is to be found in [7, 25, 27, 30] where evolutions $V_q(t,s)$ approximating $U_\varepsilon(t,s)$ up to corrections of order ε^q are constructed by means of different iterative schemes. However, the existence of a superadiabatic evolution satisfying the estimate (1.9), which bears some resemblance with the estimates proven in [23] in a classical context, was proven rigorously by Nenciu [26] using a method inspired from [22] when $H(t)$ is analytic. Similar results hold if $H(t)$ belongs to some class of C^∞ operators. Another construction of superadiabatic evolution based on another iterative scheme was proposed later in [15]. This construction allows to improve the estimates on the superadiabatic approximation as a function of the gap g between σ_1 and σ_2. Consequently, Joye and Pfister could solve the following problem. Assume

$$\sigma_1(t) = \{e_1(t), e_2(t)\} \tag{1.13}$$

where $e_j(t)$, $j = 1, 2$, are non-degenerate eigenvalues and let $P_j(t)$ be the associated one-dimensional projectors such that

$$P_1(t) + P_2(t) = Q(t). \tag{1.14}$$

The quantity of interest here is the transition probability between the two levels embedded in the spectrum

$$\mathcal{P}_{21}(\varepsilon) = \lim_{t_1 \to -\infty} \lim_{t_2 \to +\infty} \left\| P_2(t_2) U_\varepsilon(t_2, t_1) P_1(t_1) \right\|^2. \tag{1.15}$$

The question is to compute explicitly the leading term of this transition probability in the limit $\varepsilon \to 0$, as for two-dimensional systems. The idea is to first use a superadiabatic evolution to reduce the initial problem to an effective two-dimensional problem, modulo corrections of order $\mathrm{O}(\exp\{-\tau/\varepsilon\})$. Then, the effective problem is analyzed by the methods developed in [11]. The end result has the same structure as for genuine two-dimensional systems [15]:

$$\mathcal{P}_{21}(\varepsilon) = \exp\{2 \mathrm{Im}\, \theta_1\} \exp\{-2\gamma/\varepsilon\} \left(1 + \mathrm{O}(\varepsilon)\right) + \mathrm{O}\left(\exp\{-\tau/\varepsilon\}\right). \tag{1.16}$$

Here γ coincides with the decay rate of two-dimensional systems and $\mathrm{Im}\, \theta_1$ contains in addition to the expression valid for two-dimensional systems an explicit contribution coming from the global time dependence of the spectral subspace $Q(t)\mathcal{H}$ in $\mathcal{H}$. Of course, in order to have a definite formula, the remainder $\mathrm{O}(\exp\{-\tau/\varepsilon\})$ has to be negligible with respect to the leading term. The estimate of τ as a function of the gap g given in [15],

$$\tau(g) \geqslant cg, \quad c > 0, \tag{1.17}$$

shows that this is the case provided the two levels e_1 and e_2 are sufficiently isolated in the spectrum. Again, in addition to the natural hypotheses quoted above, we need a supplementary technical condition for this result to hold.

In this paper we consider the same general situation under the supplementary assumption that the two levels of interest e_1 and e_2 become nearly degenerate at some time during the evolution. This condition is often referred to as an avoided crossing condition in the physics literature and it occurs in many applications where the adiabatic limit is involved (see [16] and [10] for references). If the avoided crossing takes place at $t = 0$ and if the minimum gap between the levels is of order δ,

$$e_2(t) - e_1(t) \simeq \sqrt{a^2 t^2 + b^2 \delta^2}, \quad t, \delta \ll 0, \tag{1.18}$$

an approximate formula known as the Landau–Zener formula [21, 31], states that the transition probability $\mathcal{P}_{21}(\varepsilon)$ between the two levels e_1 and e_2 depends on the local features of the difference $e_2(t) - e_1(t)$ around $t = 0$ only and that it is given by

$$\mathcal{P}_{21}(\varepsilon) \simeq \exp\left\{ -\frac{\pi\delta^2 b^2}{2a\varepsilon} \right\} \quad \text{as } \varepsilon \to 0. \tag{1.19}$$

Although this formula is believed to hold under quite general conditions, no rigorous proof of it is available in the literature. An important step in that direction was however performed recently by Hagedorn in [8]. By an asymptotics matching procedure, he proved that if the minimum gap δ between $e_2(0)$ and $e_1(0)$ is ε-dependent and closes in the limit $\varepsilon \to 0$ according to the scaling law $\delta = \sqrt{\varepsilon}$, the leading order of the transition probability $\mathcal{P}_{21}(\varepsilon)$ is given by (1.19) with δ^2 replaced by ε. It is the aim of this work to provide a rigorous mathematical status to the Landau–Zener formula for small but finite minimum gap δ, under very general and natural hypotheses. These results have been announced in [16] and [17] where further details and references about the physics behind the Landau–Zener formula can be found.

We also believe that our detailed proof provides a fairly general account of the main methods developed recently in the theory of the exponential suppression of transition probabilities in the adiabatic limit. Indeed, the general idea of the proof is quite simple: We want to apply formula (1.16) and expand the result to the lowest order in δ, where $\delta \simeq e_2(0) - e_1(0)$. However, we have to go through the whole proof of (1.16) in order to control the dependence in δ of the quantities and remainders encountered. The structure of the proof is as follows: After formulating our main result precisely in Section 2 we give in Section 3 a set of basic estimates used throughout this paper which are generalizations of those in [13]. In Section 4 we review the iterative construction of [14, 15] and we give its main properties. Then we show in Section 5 how it yields a superadiabatic evolution V_* and the corresponding projector Q_*, whose dependence in δ is controlled. Following [15] we use this result to reduce the initial problem to a two-dimensional effective problem in Section 6, which we study with the methods of [11] exposed in Section 7. We exploit in particular the presence of the small parameter δ in the problem to show that the above mentioned technical hypothesis needed for (1.16) to hold true is satisfied for δ small enough. We also check that the remainders in the asymptotic formula (1.16) are uniform in δ. We eventually obtain the Landau–Zener formula by inserting a local expression for $e_2(t) - e_1(t)$ given below in condition IV in the result and by expanding the formula to the lowest order in δ.

2. Formalisation of the problem

2.1. Hypotheses

We consider a family of Hamiltonians $H(t, \delta)$, $t \in \mathbb{R}$ and $\delta \geqslant 0$, a small parameter, defined on the same separable Hilbert space $\mathcal{H}$. We suppose that the Hamiltonians $H(t, \delta)$ satisfy the following regularity conditions.

The first one is that the Hamiltonian is analytic in time and sufficiently smooth in t and δ.

I. Self-adjointness, analyticity and smoothness.

There exist a strip $S_\alpha = \{t + is\colon |s| \leqslant \alpha\}$, an interval $I_\Delta = [0, \Delta]$ and a dense domain D in $\mathcal{H}$ such that for each $z \in S_\alpha$ and $\delta \in I_\Delta$

i) *$H(z,\delta)$ is a closed operator defined on D,*
ii) *$H(z,\delta)\varphi$ is holomorphic on S_α, for each $\varphi \in D$ and for each fixed $\delta \in I_\Delta$,*
iii) *$H^*(z,\delta) = H(\bar{z},\delta)$; $H(t,\delta)$ is bounded from below if $t \in \mathbb{R}$,*
iv) *$H(z,\delta)\varphi$ is C^1 as a function of $(z,\delta) \in S_\alpha \times I_\Delta$ for each $\varphi \in D$.*

The next condition states that $H(t,\delta)$ tends sufficiently rapidly to two limiting Hamiltonians as $t \to \pm\infty$. These limiting Hamiltonians also have to be smooth in δ.

II. Behaviour at infinity.
There exist two families of self-adjoint operators $H^{\pm}(\delta)$, defined on D, strongly C^1 in δ and bounded from below and a positive function $b(t)$ tending to zero as $|t| \to \infty$ in an integrable way, independent of δ, such that

$$\sup_{|s|<\alpha} \left\| \left(H(t+\mathrm{i}s,\delta) - H^+(\delta)\right)\varphi \right\| \leqslant b(t)\left(\|\varphi\| + \|H^+(\delta)\varphi\|\right), \quad t>0,$$

and

$$\sup_{|s|<\alpha} \left\| \left(H(t+\mathrm{i}s,\delta) - H^-(\delta)\right)\varphi \right\| \leqslant b(t)\left(\|\varphi\| + \|H^-(\delta)\varphi\|\right), \quad t<0,$$

for all $\varphi \in D$ and $\delta \in I_\Delta$. Moreover, for each $\varphi \in D$,

$$\left\| \frac{\partial}{\partial\delta} H(z,\delta)\varphi \right\| \leqslant N, \quad \forall (z,\delta) \in S_\alpha \times I_\Delta.$$

We shall call such a function $b(t)$ an integrable decay function.

When $\delta = 0$, the derivatives with respect to δ are to be considered as right derivatives. Finally, the last condition expresses the fact that when the parameter $\delta = 0$, the levels e_1 and e_2 display a real crossing at $t = 0$ and when $\delta > 0$, this crossing becomes an avoided crossing.

III. Separation of the spectrum and avoided crossing.
There exists a constant g independent of t and δ such that the spectrum $\sigma(t,\delta)$ of $H(t,\delta)$, $t \in \mathbb{R}$, $\delta \in I_\Delta$, is given by

$$\sigma(t,\delta) = \sigma_1(t,\delta) \cup \sigma_2(t,\delta), \qquad \sigma_1(t,\delta) = \left\{e_1(t,\delta), e_2(t,\delta)\right\},$$

and satisfies

$$\operatorname{dist}\left[\sigma_1(t,\delta), \sigma_2(t,\delta)\right] \geqslant g > 0, \quad \forall t \in \mathbb{R},\ \delta \in I_\Delta.$$

Moreover,

$$e_2(t,\delta) - e_1(t,\delta) > 0, \quad \forall t \in \mathbb{R} \text{ and } \delta > 0,$$

and if $\delta = 0$,

$$e_2(t,0) - e_1(t,0) > 0, \quad \forall t < 0,$$

$$e_2(t,0) - e_1(t,0) < 0, \quad \forall t > 0,$$

$$e_2(0,0) = e_1(0,0),$$

where $t = 0$ is a simple zero of the function $e_2(t,0) - e_1(t,0)$ (see Fig. 1).

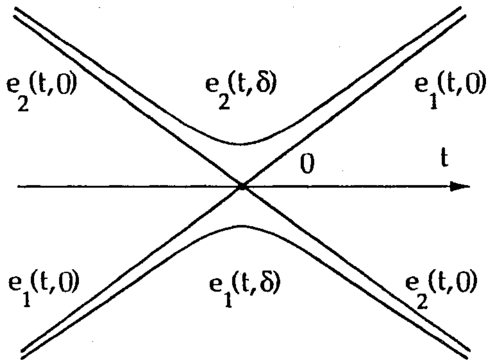

Fig. 1. The levels $e_j(t,\delta)$ and $e_j(t,0)$.

The corresponding one-dimensional projectors are denoted by $P_1(t,\delta)$ and $P_2(t,\delta)$. By condition I, the functions $e_j(z,\delta)$ and operators $P_j(z,\delta)$ are analytic and multivalued in S_α with branch points at the complex eigenvalue crossing points. If the eigenvalue crossing point is real, $e_j(z,\delta=0)$ and $P_j(z,\delta=0)$ are analytic at this point as a consequence of a theorem by Rellich [29], so that the last condition makes sense. It also implies, see Lemma 7.2, that there is a complex eigenvalue crossing point $z_0(\delta)$ together with its complex conjugate in a neighbourhood of $z=0$ if δ is small enough and that $z_0(\delta)$ is a square root type branch point for the eigenvalues. We also define $Q(t,\delta)=P_1(t,\delta)+P_2(t,\delta)$ which is analytic everywhere in S_α.

To investigate the local structure of the Hamiltonian close to the avoided crossing, we need only to consider the restriction of $H(t,\delta)$ to the two-dimensional subspace $Q(t,\delta)\mathcal{H}$. We specify in a fourth condition the generic form of avoided crossings to which the Landau–Zener formula applies. The assumption is that the quadratic form giving the square of the gap between the levels close to $(t,\delta)=(0,0)$ must be positive definite.

IV. Behaviour at the avoided crossing.

i) *There exist constants* $a>0$, $b>0$ *and* c *with* $c^2<a^2b^2$, *such that*

$$e_2(t,\delta)-e_1(t,\delta)=\sqrt{a^2t^2+2ct\delta+b^2\delta^2+R_3(t,\delta)},$$

where $R_3(t,\delta)$ *is a rest of order* 3 *in* (t,δ).

ii) *Let* φ_1 *and* φ_2 *form a basis of* $Q(0,0)\mathcal{H}$. *The matrix elements* $\langle\varphi_j|Q(t,\delta)\varphi_k\rangle$ *and*

$$\langle\varphi_j|H(t,\delta)\,Q(t,\delta)\varphi_k\rangle,\quad k,j=1,2,$$

are C^2 *as functions of the two real variables* (t,δ).

Remark. The point ii) of this condition is automatically satisfied if the Hamiltonian $H(t,\delta)$ is strongly C^2 as an operator-valued function depending on the two real variables (t,δ).

The avoided crossing considered can be rewritten as

$$e_2(t,\delta)-e_1(t,\delta)=\sqrt{a^2t^2+2ct\delta+b^2\delta^2}\,(1+R_1(t,\delta)) \tag{2.1}$$

if $t = O(\delta)$. The minimum gap between the eigenvalues is given at $t_0(\delta) = -c\delta/a^2 + O(\delta^2)$ by

$$e_2\big(t_0(\delta),\delta\big) - e_1\big(t_0(\delta),\delta\big) = \delta\sqrt{b^2 - \frac{c^2}{a^2}}\big(1 + O(\delta)\big). \tag{2.2}$$

2.2. *Main result*

We are interested in the normalized solutions in the limit $t \to +\infty$ of the Schrödinger equation

$$\mathrm{i}\varepsilon\,\frac{\partial}{\partial t}\,\psi_\varepsilon(t) = H(t,\delta)\psi_\varepsilon(t), \quad \psi_\varepsilon(0) = \varphi_0 \in D, \tag{2.3}$$

subject to the boundary condition

$$\lim_{t\to-\infty} \big\|P_1(t,\delta)\psi_\varepsilon(t)\big\| = 1. \tag{2.4}$$

More precisely we want to compute the transition probability to the level e_2 at time $t = \infty$ given by

$$\mathcal{P}_{21}(\varepsilon,\delta) = \lim_{t\to+\infty} \big\|P_2(t,\delta)\psi_\varepsilon(t)\big\|^2 \tag{2.5}$$

in the limit of small ε and δ. Let δ be fixed and let η be a closed loop based at the origin which encloses the complex eigenvalue $z_0(\delta)$ ($\operatorname{Im} z_0(\delta) > 0$) as in Fig. 2. We fix the phases of the normalized eigenvectors $\varphi_1(t,\delta)$ and $\varphi_2(t,\delta)$ of $H(t,\delta)$ associated with $e_1(t,\delta)$ and $e_2(t,\delta)$ by the condition

$$\left\langle \varphi_j(t,\delta)\middle|\frac{\partial}{\partial t}\varphi_j(t,\delta)\right\rangle \equiv 0, \quad \forall t \in \mathbb{R},\ j = 1,2. \tag{2.6}$$

Consider $e_1(0,\delta)$ and $\varphi_1(0,\delta)$ and their analytic continuations along η. If we denote by $\widetilde{e_1}(0,\delta)$ and $\widetilde{\varphi_1}(0,\delta)$ the results of these analytic continuations at the end of the loop η, we have

$$\begin{aligned} &\widetilde{e_1}(0,\delta) = e_2(0,\delta), \\ &\widetilde{\varphi_1}(0,\delta) = \exp\big\{-\mathrm{i}\theta_1(\delta)\big\}\varphi_2(0,\delta), \end{aligned} \tag{2.7}$$

because $z_0(\delta)$ is a square root branch point for the eigenvalues. Note that the term θ_1 is δ-dependent.

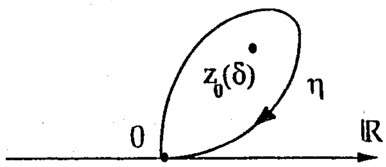

Fig. 2. The loop η and the eigenvalue crossing $z_0(\delta)$.

Theorem 2.1 (Landau–Zener formula). *Let $H(t,\delta)$ be a self-adjoint operator analytic in t satisfying conditions* I *to* III. *Let $\psi_\varepsilon(t)$ be a normalized solution of the Schrödinger equation*

$$\mathrm{i}\varepsilon\,\frac{\partial}{\partial t}\,\psi_\varepsilon(t) = H(t,\delta)\psi_\varepsilon(t), \quad \psi_\varepsilon(0) = \varphi_0 \in D,$$

such that

$$\lim_{t\to-\infty} \|P_1(t,\delta)\psi_\varepsilon(t)\| = 1.$$

If ε *and* δ *are small enough,*

$$\begin{aligned}\mathcal{P}_{21}(\varepsilon,\delta) &= \lim_{t\to+\infty} \|P_2(t,\delta)\psi_\varepsilon(t)\|^2 \\ &= \exp\{2\mathrm{Im}\,\theta_1(\delta)\}\exp\left\{\frac{2}{\varepsilon}\mathrm{Im}\int_\eta e_1(z,\delta)\,\mathrm{d}z\right\}(1+\mathrm{O}(\varepsilon))\end{aligned}$$

where $\mathrm{O}(\varepsilon)$ *is uniformly bounded in* δ *and*

$$\lim_{\delta\to0}\mathrm{Im}\int_\eta e_1(z,\delta)\,\mathrm{d}z = 0,$$

$$\lim_{\delta\to0}\mathrm{Im}\,\theta_1(\delta) = 0.$$

Moreover, if condition IV *is satisfied, we have*

$$\mathcal{P}_{21}(\varepsilon,\delta) = \exp\left\{-\frac{\delta^2\pi}{\varepsilon 2}\left(\frac{b^2}{a}-\frac{c^2}{a^3}\right)(1+\mathrm{O}(\delta))\right\}(1+\mathrm{O}(\delta)+\mathrm{O}(\varepsilon))$$

where $\mathrm{O}(\varepsilon)$, *respectively* $\mathrm{O}(\delta)$, *are uniformly bounded in* δ, *respectively* ε.

Here $\int_\eta e_1(z,\delta)\,\mathrm{d}z$ is the integral along η of the analytic continuation of $e_1(t,\delta)$. We can recover the results obtained by Hagedorn [8] specialized to our setting as a direct corollary.

Proposition 2.1. *If the width* δ *of the avoided crossing is rescaled according to* $\delta = \sqrt{\varepsilon}$, *then*

$$\mathcal{P}_{21}(\varepsilon,\sqrt{\varepsilon}) = \exp\left\{-\frac{\pi}{2}\left(\frac{b^2}{a}-\frac{c^2}{a^3}\right)\right\}(1+\mathrm{O}(\sqrt{\varepsilon})).$$

Remark. As the estimates are uniform in δ, we can set $\delta = 0$ in the above results and we obtain $\mathcal{P}_{21}(\varepsilon,0) = (1+\mathrm{O}(\varepsilon))$, in apparent contradiction with the adiabatic theorem of quantum mechanics. This behaviour is explained by Fig. 1, which shows that the eigenvectors $\varphi_j(t,\delta)$ undergo a change of labels in the limit $\delta\to0$, for $t>0$. Hence $\varphi_1(t,\delta)$ tends to an eigenvector associated with $e_2(t,0)$ as δ tends to 0 so that $\mathcal{P}_{21}(\varepsilon,0)$ is the probability to *stay* on the eigenstate associated with $e_1(t,0)$, which must be close to 1, according to the adiabatic theorem. The transition probability is therefore of order ε, instead of ε^2, as should be the case in presence of a real crossing [4].

The rest of the paper is devoted to the proof of the Landau–Zener formula, as stated in Theorem 2.1.

3. Basic estimates

This paragraph contains the generalization of the preliminaries of [13] when the Hamiltonian H depends on the supplementary parameter δ. The techniques being similar to the ones of [13], we state the main results and give their proofs in the technical Appendix A.

We use the notation

$$R(z,\delta,\lambda) = \big(H(z,\delta) - \lambda\big)^{-1} \tag{3.1}$$

for $\lambda \in T(z,\delta)$, the resolvent set of $H(z,\delta)$.

For $t \in \mathbb{R}$ and $\delta = 0$, we define the two-dimensional projector $Q(t,0)$ by

$$Q(t,0) = -\frac{1}{2\pi i}\oint_\Gamma R(t,0,\lambda)\,d\lambda \tag{3.2}$$

where Γ encircles $\sigma_1(t,0)$.

Lemma 3.1. *Let $t \in \mathbb{R}$ and Γ be as above. We can choose the width α of the strip S_α and the length Δ of the interval I_Δ sufficiently small so that the spectrum of $H(z,\delta)$ is separated in two parts $\sigma_1(z,\delta)$ and $\sigma_2(z,\delta)$ for any $z \in S_\alpha$, $\delta \in I_\Delta$. Moreover, if $|z-t|$ and δ are small enough, the spectral projector $Q(z,\delta)$ corresponding to $\sigma_1(z,\delta)$ is given by*

$$Q(z,\delta) = -\frac{1}{2\pi i}\oint_\Gamma R(z,\delta,\lambda)\,d\lambda \tag{3.3}$$

where Γ encircles $\sigma_1(z,\delta)$.

We assume from now on that α and Δ are so small that the above lemma holds. Let us define limiting projectors by

$$Q(\pm,\delta) = -\frac{1}{2\pi i}\oint_\Gamma R(\pm,\delta,\lambda)\,d\lambda. \tag{3.4}$$

The smoothness and regularity conditions on the Hamiltonian imply the following behaviours for the resolvent and projector.

Lemma 3.2. *For any $z \in S_\alpha$, $\delta \in I_\Delta$ and $\lambda \in T(z,\delta)$, $R(z,\delta,\lambda)$ and $Q(z,\delta)$ are strongly C^1 as functions of $(z,\delta) \in S_\alpha \times I_\Delta$, and $R(\pm,\delta,\lambda)$ and $Q(\pm,\delta)$ are strongly C^1 in $\delta \in I_\Delta$. Moreover, for a fixed δ, $R(z,\delta,\lambda)$ and $Q(z,\delta)$ are holomorphic bounded operators and there exist integrable decay functions $b_{\lambda,\delta}(t)$ and $b(t)$ independent of δ such that if $\lambda \in T(\pm,\delta)$*

$$\big\|R(t+is,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \leqslant b_{\lambda,\delta}(t),$$

$$\left\|\frac{\partial^n}{\partial z^n}R(t+is,\delta,\lambda)\right\| \equiv \big\|R^{(n)}(t+is,\delta,\lambda)\big\| \leqslant b_{\lambda,\delta}(t),$$

$$\big\|Q(t+is,\delta) - Q(\pm,\delta)\big\| \leqslant b(t),$$

$$\big\|Q^{(n)}(t+is,\delta)\big\| \leqslant b(t), \quad t \gtrless 0,\ |t| \gg 1,$$

for any $|s| < r < \alpha$ and for any integer n.

The proofs of these lemmas are given in appendix.

4. Iterative scheme

Consider the iterative scheme defined for any $z \in S_\alpha,\ \delta \in I_\Delta$ as follows:

$$H_0(z,\delta) \equiv H(z,\delta) \tag{4.1}$$

with associated spectral projector $Q_0(z,\delta) \equiv Q(z,\delta)$. Let

$$K_0(z,\delta) = \mathrm{i}\big[Q_0'(z,\delta), Q(z,\delta)\big] \tag{4.2}$$

where $'$ denotes $\partial/\partial z$. We set

$$H_1(z,\delta,\varepsilon) = H(z,\delta) - \varepsilon K_0(z,\delta). \tag{4.3}$$

By perturbation theory, for ε small enough, the spectrum of H_1 is separated into two distinct pieces, one of which is bounded. We denote by $Q_1(z,\delta,\varepsilon)$ the spectral projector associated to the bounded part of spectrum. Defining

$$K_1(z,\delta,\varepsilon) = \mathrm{i}\big[Q_1'(z,\delta,\varepsilon), Q_1(z,\delta,\varepsilon)\big] \tag{4.4}$$

we set

$$H_2(z,\delta,\varepsilon) = H(z,\delta) - \varepsilon K_1(z,\delta,\varepsilon). \tag{4.5}$$

Again, by perturbation theory, we can define a spectral projector associated with the bounded part of the spectrum of H_2 if ε is small enough and we can go on with the construction. At the q-th step we have (dropping the ε dependence in the arguments)

$$\begin{aligned} H_q(z,\delta) &= H(z,\delta) - \varepsilon K_{q-1}(z,\delta), \\ Q_q(z,\delta) &= -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma R_q(z,\delta,\lambda)\,\mathrm{d}\lambda, \\ K_q(z,\delta) &= \mathrm{i}\big[Q_q'(z,\delta), Q_q(z,\delta)\big], \quad \forall q \geqslant 1, \end{aligned} \tag{4.6}$$

with $R_q(z,\delta,\lambda) = (H_q(z,\delta) - \lambda)^{-1}$ for $\lambda \in T_q(z,\delta)$, $T_q(z,\delta)$ being the resolvent set of $H_q(z,\delta)$. Remark that since $K_{q-1}(z,\delta)$ is bounded for any q, $H_q(z,\delta)$ is closed and densely defined on the domain D of $H(z,\delta)$ (see [19, Chapter IV, Theorem 1.1]). We quote from [15] the main proposition regarding this iterative scheme, when the parameter δ is absent. Let $D(z,\eta)$ be the disc

$$D(z,\eta) = \{z' \in \mathbb{C}:\ |z' - z| < \eta\} \subset S_\alpha \tag{4.7}$$

and assume that there exists a simple closed path Γ in the complex plane, counter-clockwise oriented, such that for all $z' \in D(z,\eta)$ the spectrum $\sigma(z')$ of $H(z')$ can be divided into two disjoint parts $\sigma_1(z')$ and $\sigma_2(z')$, with $\sigma_1(z')$ in the interior of Γ. We have the

Proposition 4.1. *Let $D(z,\eta)$ and Γ as above and let a, b and c be constants such that for all integers p, all $z' \in D(z,\eta)$*

$$\text{i)} \qquad \left\| R_0^{(p)}(z',\lambda) \right\| = \left\| \frac{\mathrm{d}^p}{\mathrm{d}z'^p} R_0(z',\lambda) \right\| \leqslant ac^p \frac{p!}{(1+p)^2}, \quad \lambda \in \Gamma,$$

$$\text{ii)} \qquad \left\| K_0^{(p)}(z') \right\| \leqslant bc^p \frac{p!}{(1+p)^2}, \quad \lambda \in \Gamma.$$

Then there exists ε^*, depending on a and b, and there exists a constant d depending on a, b and $|\Gamma|$ such that for $\varepsilon < \varepsilon^*$

$$\left\| K_q^{(p)}(z') - K_{q-1}^{(p)}(z') \right\| \leqslant b\varepsilon^q d^q c^{p+q} \frac{(p+q)!}{(1+p)^2}$$

and

$$\left\| K_q^{(p)}(z') \right\| \leqslant \mathrm{e}bc^p \frac{p!}{(1+p)^2}$$

for all $z' \in D(z,\eta)$, all integers p and q such that

$$p+q \leqslant \left[\frac{1}{\mathrm{e}cd\varepsilon} \right] \equiv N^*.$$

Here $[x]$ is the integer part of x and e *is the basis of the Neperian logarithm.*

We prove in our technical Appendix A a lemma showing that the hypotheses of this proposition are satisfied uniformly in δ under our assumptions I to III:

Lemma 4.1. *There exists a constant N such that*

$$\sup_{t\in\mathbb{R}} \sup_{\substack{z\in D(t,r)\\ \delta\in I_\Delta}} \sup_{\lambda\in\Gamma} \left\| R(z,\delta,\lambda) \right\| \leqslant N. \tag{4.8}$$

We define

$$K(z,\delta) = \mathrm{i}\left[Q'(z,\delta), Q(z,\delta) \right]. \tag{4.9}$$

By Lemma 3.2, there exists an integrable decay function $b(t)$ such that

$$\sup_{\substack{z\in D(t,r)\\ \delta\in I_\Delta}} \left\| K(z,\delta) \right\| \leqslant b(t). \tag{4.10}$$

Hence, using the Cauchy formula in discs $D(t,\eta)$ with $\eta < r$, we have the estimates

$$\begin{aligned} \left\| R^{(p)}(z,\delta,\lambda) \right\| &\leqslant Nc^p \frac{p!}{(1+p)^2}, \\ \left\| K^{(p)}(z,\delta) \right\| &\leqslant b(t)c^p \frac{p!}{(1+p)^2}, \quad \forall z \in D(t,\eta),\ \lambda \in \Gamma_t, \end{aligned} \tag{4.11}$$

for any $t \in \mathbb{R}$, uniformly in $\delta \in I_\Delta$, with $c = 8/r$, provided η is small enough. We can again diminish the width of the strip S_α, so that the above estimates hold uniformly in $z \in S_\alpha$, $\delta \in I_\Delta$. As a consequence Proposition 4.1 holds uniformly in δ for $\varepsilon < \varepsilon^*$ where ε^* is independent of δ.

5. Superadiabatic evolution

In this section we show how the iterative scheme (4.6) yields a superadiabatic approximation V_* of the actual evolution U_ε and consider in particular its dependence in δ. As a consequence of our hypothesis I on $H(t,\delta)$, the operator $U_\varepsilon(t,s)$ (in which we omit the dependence in δ) is a two-parameter family of unitary operators, strongly continuous in t and s and which leave the domain D invariant. For all t_1, t_2, t_3

$$U_\varepsilon(t_1,t_2)U_\varepsilon(t_2,t_3) = U_\varepsilon(t_1,t_3), \quad U_\varepsilon(t_1,t_1) = \mathbb{I}, \tag{5.1}$$

and U_ε is strongly differentiable in t and s on the domain D,

$$\mathrm{i}\varepsilon \frac{\partial}{\partial t} U_\varepsilon(t,s) = H(t,\delta)U_\varepsilon(t,s) \tag{5.2}$$

and

$$\mathrm{i}\varepsilon \frac{\partial}{\partial s} U_\varepsilon(t,s) = -U_\varepsilon(t,s)H(s,\delta) \tag{5.3}$$

(see, e.g., [20, Chapter 2]). From now on we set $s=0$ and we omit this variable in the notation. We define two operators W_{N^*}, Φ_{N^*} by

$$\mathrm{i}W'_{N^*}(t,\delta) = K_{N^*}(t,\delta)W_{N^*}(t,\delta), \qquad W_{N^*}(0,\delta) = \mathbb{I}, \tag{5.4}$$

$$\mathrm{i}\varepsilon\Phi'_{N^*}(t,\delta) = W_{N^*}^{-1}(t,\delta)H_{N^*}(t,\delta)W_{N^*}(t,\delta)\Phi_{N^*}(t,\delta), \qquad \Phi_{N^*}(0,\delta) = \mathbb{I}, \tag{5.5}$$

where $N^*(\varepsilon)$ is δ-independent and is defined in Proposition 4.1. The operators $K_q(t,\delta)$ and $H_q(t,\delta)$ are defined by the iterative scheme (4.6). The operator $W_{N^*}(z,\delta)$ is unitary for real t and it is given by a convergent series since $K_{N^*}(t,\delta)$ is bounded. From Proposition 4.1 we know that $K_{N^*}(z,\delta)$ is analytic for any $z \in S_\alpha$, $\delta \in I_\Delta$ so that the same is true for $W_{N^*}(z,\delta)$. Moreover, there exists a constant w, independent of ε, δ and $z \in S_\alpha$ such that

$$\|W_{N^*}(z,\delta)\| \leqslant w, \qquad \|W_{N^*}^{-1}(z,\delta)\| \leqslant w, \tag{5.6}$$

as is easily seen from the series representing W_{N^*}. For any $z \in S_\alpha$, $W_{N^*}(z,\delta)$ satisfies the same differential equation (5.4) where $'$ means $\partial/\partial z$ and as a consequence, it has the intertwining property [19, 20]

$$W_{N^*}(z,\delta)Q_{N^*}(0,\delta) = Q_{N^*}(z,\delta)W_{N^*}(z,\delta). \tag{5.7}$$

Another important property of W_{N^*} is that it leaves the domain D invariant so that the generator $W_{N^*}^{-1}H_{N^*}W_{N^*}$ of Φ_{N^*} is well defined on D. Moreover, it can be shown that

$$W_{N^*}^{-1}(z,\delta)H_{N^*}(z,\delta)\, W_{N^*}(z,\delta)$$

is analytic in z ([13, Lemma 5.1]). Hence the unitary operator $\Phi_{N^*}(t,\delta)$ shares the properties of an evolution operator for real values of t and it satisfies

$$[\Phi_{N^*}(t,\delta), Q_{N^*}(0,\delta)] = 0, \quad \forall t \in \mathbb{R}. \tag{5.8}$$

We define our superadiabatic evolution by

$$V_*(t,\delta) = W_{N^*}(t,\delta)\Phi_{N^*}(t,\delta). \tag{5.9}$$

To measure the difference between U_ε and V_* we introduce another unitary operator A_* by the identity

$$U_\varepsilon(t) = W_{N^*}(t,\delta)\Phi_{N^*}(t,\delta)A_*(t,\delta). \tag{5.10}$$

In consequence, A_* satisfies the integral equation

$$A_*(t,\delta) = \mathbb{I} + \mathrm{i}\int_0^t V_*^{-1}(s,\delta)\big(K_{N^*}(s,\delta) - K_{N^*-1}(s,\delta)\big)V_*(s,\delta)A_*(s,\delta)\,\mathrm{d}s. \tag{5.11}$$

Now Proposition 4.1 and the definition of N^* imply

$$\begin{aligned}\big\|K_{N^*}(t,\delta) - K_{N^*-1}(t,\delta)\big\| &\leqslant b(t)(\varepsilon dc)^{N^*}N^*! \leqslant b(t)(\varepsilon dcN^*)^{N^*}\\ &\leqslant b(t)\exp\{-N^*\} \leqslant \mathrm{e}b(t)\exp\{-\tau/\varepsilon\}\end{aligned} \tag{5.12}$$

where

$$\tau = \frac{1}{\mathrm{e}cd} > 0 \tag{5.13}$$

is independent of δ. Hence

$$\big\|A_*(t,\delta) - \mathbb{I}\big\| \leqslant \mathrm{e}\left|\int_0^t b(s)\,\mathrm{d}s\right|\exp\{-\tau/\varepsilon\} \tag{5.14}$$

which together with (5.8) and (5.7) yield the

Proposition 5.1. *If conditions* I *to* III *hold, there exist* $\varepsilon^* > 0$ *and* $\tau > 0$ *defined by* (5.13), *both independent of* δ, *such that* $\forall\varepsilon \leqslant \varepsilon^*$

$$\big\|U_\varepsilon(t) - V_*(t,\delta)\big\| = \mathrm{O}\big(\exp\{-\tau/\varepsilon\}\big), \quad \forall t \in \mathbb{R},$$

where the correction term is uniformly bounded in δ *and* $V_*(t,\delta)$ *has the intertwining property*

$$V_*(t,\delta)Q_{N^*}(0,\delta) = Q_{N^*}(t,\delta)V_*(t,\delta), \quad \forall t \in \mathbb{R}.$$

Remarks.

1) In the decomposition $V_*(t,\delta) = W_{N^*}(t,\delta)\Phi_{N^*}(t,\delta)$, the operator $W_{N^*}(t,\delta)$ describes the transitions between the subspaces $Q_{N^*}(0,\delta)\mathcal{H}$ and $(\mathbb{I} - Q_{N^*}(t,\delta))\mathcal{H}$ and $\Phi_{N^*}(t,\delta)$ describes the evolution inside the time-independent subspace $Q_{N^*}(0,\delta)\mathcal{H}$.

2) The superadiabatic evolution V_* satisfies the equation

$$\begin{aligned}\mathrm{i}\varepsilon V_*'(t,\delta) &= \big(H_{N^*}(t,\delta) + \varepsilon K_{N^*}(t,\delta)\big)V_*(t,\delta)\\ &= \big(H(t,\delta) + \varepsilon\big(K_{N^*}(t,\delta) - K_{N^*-1}(t,\delta)\big)\big)V_*(t,\delta), \quad V_*(0,\delta) = \mathbb{I}.\end{aligned} \tag{5.15}$$

3) By perturbation theory and Proposition 4.1 we have

$$H_{N^*}(t,\delta) = H(t,\delta) - \varepsilon K_{N^*-1}(t,\delta) = H(t,\delta) + \mathrm{O}(\varepsilon b(t)) \tag{5.16}$$

so that, for ε small enough, there exist spectral projectors $P_j^{N^*}(t,\delta)$ defined by Riesz formula such that:

$$P_1^{N^*}(t,\delta) + P_2^{N^*}(t,\delta) = Q_{N^*}(t,\delta) \tag{5.17}$$

and

$$\lim_{t\to\pm\infty} \left\| P_j^{N^*}(t,\delta) - P_j(t,\delta) \right\| = 0,$$

$$\lim_{t\to\pm\infty} \left\| Q_{N^*}(t,\delta) - Q(t,\delta) \right\| = 0, \tag{5.18}$$

uniformly in ε and δ.

6. Reduction to an effective problem

Let us write the generator of $\Phi_{N^*}(t,\delta)$ as

$$\widehat{H_*}(t,\delta) = W_{N^*}^{-1}(t,\delta) H_{N^*}(t,\delta) W_{N^*}(t,\delta). \tag{6.1}$$

Its spectral projectors $\widehat{P_j^*}(t,\delta)$, given by

$$\widehat{P_j^*}(t,\delta) = W_{N^*}^{-1}(t,\delta) P_j^{N^*}(t,\delta) W_{N^*}(t,\delta) \tag{6.2}$$

where $P_j^{N^*}(t,\delta)$, the spectral projectors of $H_{N^*}(t,\delta)$, are such that

$$Q_{N^*}(0,\delta) = \widehat{P_1^*}(t,\delta) + \widehat{P_2^*}(t,\delta). \tag{6.3}$$

It is therefore possible to decompose the evolution $\Phi_{N^*}(t)$ as we did for $U_\varepsilon(t)$. We introduce the evolution $\widehat{V_*}(t)$ by

$$\mathrm{i}\varepsilon \frac{\partial}{\partial t} \widehat{V_*}(t) = \left(\widehat{H_*}(t) + \mathrm{i}\varepsilon \sum_{j=1}^{2} \frac{\partial}{\partial t} \widehat{P_j^*}(t) \widehat{P_j^*}(t) \right) \widehat{V_*}(t), \quad \widehat{V_*}(0) = \mathbb{I}, \tag{6.4}$$

and we set

$$\Phi_{N^*}(t) := \widehat{V_*}(t)\widehat{A_*}(t). \tag{6.5}$$

By construction $\widehat{V_*}(t)$ has the intertwining property [19, 20]

$$\widehat{P_j^*}(t)\widehat{V_*}(t) = \widehat{V_*}(t)\widehat{P_j^*}(0), \quad j = 1, 2. \tag{6.6}$$

The operator $\widehat{A_*}(t)$ is the solution of the equation

$$\mathrm{i}\frac{\partial}{\partial t}\widehat{A_*}(t) = -\left(\widehat{V_*}^{-1}(t)\, \mathrm{i} \left(\sum_{j=1}^{2} \frac{\partial}{\partial t} \widehat{P_j^*}(t) \widehat{P_j^*}(t) \right) \widehat{V_*}(t) \right) \widehat{A_*}(t), \quad \widehat{A_*}(0) = \mathbb{I}. \tag{6.7}$$

Since

$$\left\| \sum_{j=1}^{2} \frac{\partial}{\partial t} \widehat{P_j^*}(t) \widehat{P_j^*}(t) \right\|$$

is integrable as $t \to \pm\infty$ the operator $\widehat{A_*}(t)$ has well-defined limits when $t \to \pm\infty$. The reduction to an effective two-dimensional problem is provided by the following proposition.

Proposition 6.1. *Let $\varphi(t)$ be a normalized solution of*

$$\mathrm{i}\varepsilon \frac{\partial}{\partial t} \varphi(t) = H(t,\delta)\varphi(t)$$

satisfying the boundary condition

$$\lim_{t\to-\infty} \|P_1(t,\delta)\varphi(t)\| = 1.$$

If ε is small enough, then

$$\mathcal{P}_{21}(\varepsilon,\delta) = \lim_{t\to+\infty} \|P_2(t,\delta)\varphi(t)\|^2 = \widehat{\mathcal{P}_{21}}(\varepsilon,\delta) + \mathrm{O}\big(\exp\{-\tau/\varepsilon\}\big)$$

where $\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)$ is the transition probability of the following two-dimensional problem in $Q_{N^*}(0,\delta)\mathcal{H}$:
Let $\psi(t)$ be a normalized solution of

$$\mathrm{i}\varepsilon \frac{\partial}{\partial t} \psi(t) = \widehat{H_*}(t,\delta)\psi(t)$$

such that

$$\lim_{t\to-\infty} \|\widehat{P_1^*}(t,\delta)\psi(t)\| = 1.$$

Then

$$\widehat{\mathcal{P}_{21}}(\varepsilon,\delta) := \lim_{t\to+\infty} \|\widehat{P_2^*}(t,\delta)\psi(t)\|^2.$$

The correction term $\mathrm{O}(\exp\{-\tau/\varepsilon\})$ is uniformly bounded in δ.

Proof. Let $\varphi(t)$ with $\varphi(0) = \varphi^*$ be given. We have (using (5.18), (6.2), (6.5), and (6.6))

$$\begin{aligned} 1 &= \lim_{t\to-\infty} \|P_1(t,\delta)\varphi(t)\| \\ &= \lim_{t\to-\infty} \left\| P_1^{N^*}(t,\delta) W_{N^*}(t,\delta) \Phi_{N^*}(t,\delta) A_*(t,\delta)\varphi^* \right\| \\ &= \lim_{t\to-\infty} \left\| W_{N^*}(t,\delta) \widehat{P_1^*}(t,\delta) \Phi_{N^*}(t,\delta) A_*(t,\delta)\varphi^* \right\| \\ &= \lim_{t\to-\infty} \left\| \widehat{P_1^*}(t,\delta) \widehat{V_*}(t,\delta) \widehat{A_*}(t,\delta) A_*(t,\delta)\varphi^* \right\| \\ &= \lim_{t\to-\infty} \left\| \widehat{V_*}(t,\delta) \widehat{P_1^*}(0) \widehat{A_*}(t,\delta) A_*(t,\delta)\varphi^* \right\| \\ &= \left\| \widehat{P_1^*}(0) \widehat{A_*}(-\infty,\delta) A_*(-\infty,\delta)\varphi^* \right\|. \end{aligned} \tag{6.8}$$

Therefore we can write

$$\varphi^* = A_*^{-1}(-\infty,\delta)\widehat{A_*}^{-1}(-\infty,\delta)\psi \tag{6.9}$$

with $\psi \in \widehat{P_1^*}(0,\delta)\mathcal{H}$. By a computation similar to (6.8) we have

$$\begin{aligned}
&\lim_{t\to\infty} \|P_2(t,\delta)\varphi(t)\| \\
&= \lim_{t\to\infty} \|P_2^{N^*}(t,\delta)\varphi(t)\| \\
&= \left\|\widehat{P_2^*}(0,\delta)\widehat{A_*}(\infty,\delta)A_*(\infty,\delta)\varphi^*\right\| \\
&= \left\|\widehat{P_2^*}(0,\delta)\widehat{A_*}(\infty,\delta)A_*(\infty,\delta)A_*^{-1}(-\infty,\delta)\widehat{A_*}^{-1}(-\infty,\delta)\widehat{P_1^*}(0,\delta)\right\|.
\end{aligned} \tag{6.10}$$

Using the equation (5.11) for the unitary operator A_* and the estimate (5.14) we get

$$\begin{aligned}
\mathcal{P}_{21}(\varepsilon,\delta) &= \left\|\widehat{P_2^*}(0,\delta)\widehat{A_*}(\infty,\delta)\widehat{A_*}^{-1}(-\infty,\delta)\widehat{P_1^*}(0,\delta) + O\big(\exp\{-\tau/\varepsilon\}\big)\right\|^2 \\
&= \left\|\widehat{P_2^*}(0,\delta)\widehat{A_*}(\infty,\delta)\widehat{A_*}^{-1}(-\infty,\delta)\widehat{P_1^*}(0,\delta)\right\|^2 + O\big(\exp\{-\tau/\varepsilon\}\big)
\end{aligned} \tag{6.11}$$

where the correction term is uniformly bounded in δ. Then one checks by the same type of computation that

$$\widehat{\mathcal{P}_{21}}(\varepsilon,\delta) = \left\|\widehat{P_2^*}(0,\delta)\widehat{A_*}(\infty,\delta)\widehat{A_*}^{-1}(-\infty,\delta)\widehat{P_1^*}(0,\delta)\right\|^2. \tag{6.12}$$

□

7. Study of the effective problem

From now on we consider $\widehat{H_*}(t,\delta)$ restricted to the two-dimensional subspace $Q_{N^*}(0,\delta)\mathcal{H}$ and we recall that $\widehat{H_*}(z,\delta)$ is analytic in $z \in S_\alpha$ for any $\delta \in I_\Delta$ and $\varepsilon < \varepsilon^*$. As

$$\widehat{H_*}(z,\delta) = W_{N^*}^{-1}(z,\delta)H_{N^*}(z,\delta)W_{N^*}(z,\delta) \tag{7.1}$$

its eigenvalues coincide with the ones of $H_{N^*}(z,\delta)$ which we denote by $e_j^*(z,\delta)$, $j = 1,2$. We define

$$\Delta_{12}^*(t,\delta) = \int_0^t \mathrm{d}s\big(e_1^*(s,\delta) - e_2^*(s,\delta)\big), \quad t \in \mathbb{R}, \tag{7.2}$$

and $\Delta_{12}^*(z,\delta)$ by analytic continuation for $z \in S_\alpha$. Note that $\Delta_{12}^*(z,\delta)$ is multivalued in S_α. Let us consider a set of corresponding eigenvectors of $\widehat{H_*}(t,\delta)$, $\widehat{\varphi_j^*}(t,\delta)$ such that

$$\left\langle \widehat{\varphi_j^*}(t,\delta) \middle| \frac{\partial}{\partial t}\widehat{\varphi_j^*}(t,\delta)\right\rangle \equiv 0. \tag{7.3}$$

Assume that there exists an eigenvalue crossing point $z^*(\delta)$ in the complex plane with $\operatorname{Im} z^*(\delta) > 0$ which is a square root type singularity for $e_j^*(t,\delta)$. As explained in Section 2, if we perform the analytic continuation of $\widehat{\varphi_1^*}(0,\delta)$ along a loop η based at the origin encircling $z^*(\delta)$, we will obtain at the end of the loop a vector $\widetilde{\widehat{\varphi_1^*}}(0,\delta)$ such that

$$\widetilde{\widehat{\varphi_1^*}}(0,\delta) = \exp\{-\mathrm{i}\widehat{\theta_1^*}(\delta)\}\widehat{\varphi_2^*}(0,\delta). \tag{7.4}$$

As $z^*(\delta)$ is an eigenvalue crossing point for $H_{N^*}(z,\delta)$ as well, we can perform the same type of analysis for its eigenvectors.

Lemma 7.1.

i) *Let* $\varphi_j^*(t,\delta)$ *be normalized eigenvectors of* $H_{N^*}(t,\delta)$ *satisfying*

$$\left\langle \varphi_j^*(t,\delta) \middle| \frac{\partial}{\partial t}\varphi_j^*(t,\delta)\right\rangle \equiv 0.$$

Then

$$\varphi_j^*(t,\delta) \equiv W_{N^*}(t,\delta)\widehat{\varphi_j^*}(t,\delta).$$

ii) *Let* $\theta_1^*(\delta)$ *be defined by*

$$\widetilde{\varphi_1^*}(0,\delta) = \exp\{-\mathrm{i}\theta_1^*(\delta)\}\varphi_2^*(0,\delta).$$

Then

$$\exp\{-\mathrm{i}\widehat{\theta_1^*}(\delta)\} = \exp\{-\mathrm{i}\theta_1^*(\delta)\}.$$

Proof. For $t \in \mathbb{R}$, $\widehat{\varphi_j^*}(t,\delta) \in Q_{N^*}(0,\delta)\mathcal{H}$

$$\begin{aligned}
&\left\langle \varphi_j^*(t,\delta) \middle| \frac{\partial}{\partial t}\varphi_j^*(t,\delta)\right\rangle \\
&= \left\langle W_{N^*}(t,\delta)\widehat{\varphi_j^*}(t,\delta) \middle| \frac{\partial}{\partial t}\left(W_{N^*}(t,\delta)\widehat{\varphi_j^*}(t,\delta)\right)\right\rangle = \\
&= \left\langle \widehat{\varphi_j^*}(t,\delta) \middle| W_{N^*}^{-1}(t,\delta)\frac{1}{\mathrm{i}}K_{N^*}(t,\delta)W_{N^*}(t,\delta)\widehat{\varphi_j^*}(t,\delta)\right\rangle \\
&\quad + \left\langle \widehat{\varphi_j^*}(t,\delta) \middle| W_{N^*}^{-1}(t,\delta)W_{N^*}(t,\delta)\frac{\partial}{\partial t}\widehat{\varphi_j^*}(t,\delta)\right\rangle \\
&= \frac{1}{\mathrm{i}}\left\langle \widehat{\varphi_j^*}(t,\delta) \middle| W_{N^*}^{-1}(t,\delta)Q_{N^*}(t,\delta)K_{N^*}(t,\delta)Q_{N^*}(t,\delta)W_{N^*}(t,\delta)\widehat{\varphi_j^*}(t,\delta)\right\rangle \\
&\quad + \left\langle \widehat{\varphi_j^*}(t,\delta) \middle| \frac{\partial}{\partial t}\widehat{\varphi_j^*}(t,\delta)\right\rangle \equiv 0
\end{aligned} \tag{7.5}$$

since $Q_{N^*}(t,\delta)Q_{N^*}'(t,\delta)Q_{N^*}(t,\delta) \equiv 0$.

(ii) is a consequence of (i) and of the analyticity of $W_{N^*}(z,\delta)$, $\forall z \in S_\alpha$. □

The asymptotic computation of the transition probability $\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)$ as $\varepsilon \to 0$ for two-level systems has been studied in details in [11] under the same general hypotheses I to III adapted to two-dimensional systems. The method consists in expanding the solution of the Schrödinger equation $\psi(t)$ along the eigenvectors $\widehat{\varphi_j^*}(t,\delta)$ as

$$\psi(t)=\sum_{j=1}^{2} c_j^*(t,\delta)\exp\left\{-\frac{\mathrm{i}}{\varepsilon}\int_0^t e_j^*(s,\delta)\,\mathrm{d}s\right\}\widehat{\varphi_j^*}(t,\delta), \tag{7.6}$$

where $c_j^*(t,\delta)$ are unknown coefficients to be determined. Then we study the system of equations they satisfy

$$\begin{aligned} c_1^{*\prime}(t,\delta)&=a_{12}^*(t,\delta)\exp\left\{\frac{\mathrm{i}}{\varepsilon}\Delta_{12}^*(t,\delta)\right\}c_2^*(t,\delta),\\ c_2^{*\prime}(t,\delta)&=a_{21}^*(t,\delta)\exp\left\{-\frac{\mathrm{i}}{\varepsilon}\Delta_{12}^*(t,\delta)\right\}c_1^*(t,\delta), \end{aligned} \tag{7.7}$$

where

$$a_{jk}^*(t,\delta)=-\left\langle\widehat{\varphi_j^*}(t,\delta)\middle|\widehat{\varphi_k^*}'(t,\delta)\right\rangle \tag{7.8}$$

which is equivalent to the Schrödinger equation. The boundary conditions are

$$c_1^*(-\infty,\delta)=1,\qquad c_2^*(-\infty,\delta)=0, \tag{7.9}$$

and

$$\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)=\left|c_2^*(-\infty,\delta)\right|^2. \tag{7.10}$$

The idea is to make use of the generic multivaluedness of the eigenvalues and eigenvectors at the eigenvalue crossing point described above by integrating this system of equations along a carefully chosen path in the complex plane going above the crossing point. There are however two supplementary conditions. The first is the genericity condition:

A. There exists an eigenvalue crossing point $z^*(\delta)$ which is a square root type singularity for the eigenvalues $e_j^*(z,\delta)$.

Second, we need a technical condition (although crucial, see [11] for examples) expressed through the function $\Delta_{12}^*(z,\delta)$.

B. There exists a path in the complex plane

$$t\mapsto\gamma_\delta(t)\in S_\alpha\cap\{z\colon\ \operatorname{Im}z>0\}\equiv S_\alpha^+ \tag{7.11}$$

such that

i) $\displaystyle\lim_{t\to\pm\infty}\operatorname{Re}\gamma_\delta(t)=\pm\infty$

ii) $\gamma_\delta(t)$ passes above $z^*(\delta)$

iii) $$\sup_{\varepsilon>0}\ \sup_{s\leqslant t\in\mathbb{R}}\ \exp\left\{\frac{1}{\varepsilon}\operatorname{Im}\left(\Delta_{12}^*(\gamma_\delta(s),\delta)-\Delta_{12}^*(\gamma_\delta(t),\delta)\right)\right\}\leqslant k<\infty. \tag{7.12}$$

(See [11, condition IV and Eq. (2.56)]). We quote from [11] the

Proposition 7.1. *Under conditions* I, II, III, **A** *and* **B**, *there exists* $\varepsilon^*(\delta) > 0$ *such that the transition probability* $\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)$ *for fixed positive* δ *is given by*

$$\widehat{\mathcal{P}_{21}}(\varepsilon,\delta) = \exp\left\{2\operatorname{Im}\theta_1^*(\delta)\right\}\exp\left\{\frac{2}{\varepsilon}\operatorname{Im}\int_\eta e_1^*(z,\delta)\,\mathrm{d}z\right\}(1+\mathrm{O}_\delta(\varepsilon)) \tag{7.13}$$

provided $\varepsilon \leqslant \varepsilon^*(\delta)$.

Remarks. At that stage, $\varepsilon^*(\delta)$ and the bound on the remainder $\mathrm{O}_\delta(\varepsilon)$ are δ-dependent. We prove in the sequel that both the supplementary conditions **A** and **B** are satisfied under our assumptions I to III for δ small enough, $\delta \leqslant \delta^*$, with δ^* independent of ε. Moreover we show that actually the expression $\varepsilon^*(\delta)$ is independent of δ and that the remainder $\mathrm{O}_\delta(\varepsilon)$ is uniformly bounded in δ (Proposition 7.4).

We first show that conditions **A** and **B** are satisfied for $Q(z,\delta)H(z,\delta)$ and then, by perturbation, that they are satisfied for $Q_{N^*}(0,\delta)\widehat{H_*}(z,\delta)$ as well. Let us deal with the eigenvalues $e_j(t,\delta)$ of $H(t,\delta)$. Let φ_1 and φ_2 be a basis of the range of $Q(0,0)$. We define for $t\in\mathbb{R}$

$$\begin{aligned}\psi_1(t,\delta) &= \frac{Q(t,\delta)\varphi_1}{\sqrt{\langle\varphi_1|Q(t,\delta)\varphi_1\rangle}},\\ \psi_2(t,\delta) &= \frac{Q(t,\delta)\left(\varphi_2 - \frac{\langle\varphi_1|Q(t,\delta)\varphi_2\rangle}{\langle\varphi_1|Q(t,\delta)\varphi_1\rangle}\varphi_1\right)}{\sqrt{\left\langle\varphi_2\middle|Q(t,\delta)\left(\varphi_2 - \frac{\langle\varphi_1|Q(t,\delta)\varphi_2\rangle}{\langle\varphi_1|Q(t,\delta)\varphi_1\rangle}\varphi_1\right)\right\rangle}}.\end{aligned} \tag{7.14}$$

These vectors form an orthonormal basis of $Q(t,\delta)\mathcal{H}$ for (t,δ) close to $(0,0)$. Moreover, they are continuously differentiable in (t,δ) and they are analytic in t for δ fixed, by assumptions I to III. Without loss of generality we suppose that $e_1(t,\delta)+e_2(t,\delta)=0$, so that we can write

$$H(t,\delta)\big|_{Q(t,\delta)\mathcal{H}} = \mathbf{B}(t,\delta)\cdot\mathrm{s} \tag{7.15}$$

in the basis $\{\psi_1(t,\delta),\psi_2(t,\delta)\}$ with s_j, $j=1,2,3$, the spin-1/2 matrices and with the definitions

$$\begin{aligned}B_1(t,\delta) &= 2\operatorname{Re}\langle\psi_1(t,\delta)|H(t,\delta)\psi_2(t,\delta)\rangle,\\ B_2(t,\delta) &= -2\operatorname{Im}\langle\psi_1(t,\delta)|H(t,\delta)\psi_2(t,\delta)\rangle,\\ B_3(t,\delta) &= 2\langle\psi_1(t,\delta)|H(t,\delta)\psi_1(t,\delta)\rangle.\end{aligned} \tag{7.16}$$

The expressions

$$\frac{\langle\varphi_1|H(t,\delta)Q(t,\delta)\varphi_1\rangle}{\langle\varphi_1|Q(t,\delta)\varphi_1\rangle} \tag{7.17}$$

and

$$\langle\psi_1(t,\delta)|H(t,\delta)\psi_2(t,\delta)\rangle = \frac{\langle\varphi_1|H(t,\delta)Q(t,\delta)\psi_2(t,\delta)\rangle}{\sqrt{\langle\varphi_1|Q(t,\delta)\varphi_1\rangle}} \tag{7.18}$$

have analytic extensions in the complex plane, so that the same is true for their real or imaginary parts considered as real analytic functions on the real axis. Thus the vector field $\mathbf{B}(z,\delta)$ is analytic in $z \in S_\alpha$ for all $\delta \in I_\Delta$ and it is continuously differentiable in z and δ. Moreover, as a consequence of condition II there exist real limits $B_j(\pm\infty,\delta)$, $j = 1,2,3$, which are C^1 in δ and an integrable decay function $b(t)$ independent of δ such that

$$\sup_{|s|<a} \left|B_j(t+\mathrm{i}s,\delta) - B_j(\pm\infty,\delta)\right| \leqslant b(t). \tag{7.19}$$

This is easily seen from the identity

$$H(z,\delta)Q(z,\delta) = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \lambda R(z,\delta,\lambda)\,\mathrm{d}\lambda \tag{7.20}$$

and Lemma 3.2, for example. Hence the eigenvalues of $H(t,\delta)Q(t,\delta)$ are given by the relation

$$e_j(t,\delta) = (-1)^j \frac{1}{2} \sqrt{\rho(t,\delta)} \tag{7.21}$$

where

$$\rho(t,\delta) = \sum_{j=1}^{3} B_j^2(t,\delta) \tag{7.22}$$

is analytic in $z \in S_\alpha$ for any $\delta \in I_\Delta$ and is C^1 in $(z,\delta) \in S_\alpha \times I_\Delta$. Let us define the function $\Delta_{12}(t,\delta)$ by

$$\Delta_{12}(t,\delta) = \int_0^t \mathrm{d}s\big(e_1(s,\delta) - e_2(s,\delta)\big) = -\int_0^t \sqrt{\rho(u,\delta)}\,\mathrm{d}u. \tag{7.23}$$

Lemma 7.2. *For any positive δ small enough there exists a unique eigenvalue crossing point $z_0(\delta)$ such that* $\operatorname{Im} z_0(\delta) > 0$ *and $z_0(\delta)$ is a simple zero of $\rho(z,\delta)$. As a function of δ, $z_0(\delta)$ is continuous and*

$$\lim_{\delta\to 0} z_0(\delta) = 0.$$

Proof. By assumption, $\rho(z,0)$ has a double zero at $z = 0$, since $e_j(z,0)$ is an analytic function. Let $D(0,r)$ be a circle of radius $r > 0$ centered at $z = 0$ and let us consider

$$\rho(z,\delta) = \rho(z,0) + \big(\rho(z,\delta) - \rho(z,0)\big). \tag{7.24}$$

For any r sufficiently small,

$$\left|\rho(z,0)\right| > R > 0, \quad \forall z \in \partial D(0,r), \tag{7.25}$$

and there exists δ small enough such that

$$\left|\rho(z,\delta) - \rho(z,0)\right| < \frac{R}{3}, \quad \forall z \in \partial D(0,r), \tag{7.26}$$

by continuity of $\rho(z,\delta)$ in z and δ and compactness of $\partial D(0,r)$. Applying Rouché's theorem we see that $\rho(z,\delta)$ has as many zeros as $\rho(z,0)$ in $D(0,r)$, counted with their multiplicity. As $\rho(t,\delta) > 0$, $\forall t \in \mathbb{R}$, if $\delta > 0$ and $\rho(\overline{z},\delta) = \overline{\rho}(z,\delta)$ by Schwarz's principle, we conclude that there exists in $D(0,r)$ a unique simple zero $z_0(\delta)$ of $\rho(z,\delta)$ with $\operatorname{Im} z_0(\delta) > 0$. The continuity in δ of $z_0(\delta)$ is proven in a similar way. □

It follows from this lemma that $\Delta_{12}(t,\delta)$ admits an analytic continuation $\Delta_{12}(z,\delta)$ for any $z \in S_\alpha \setminus D(0,r)$. We come to the main proposition of this section.

Proposition 7.2. *There exists a path $\gamma_\delta(t)$, $t \in \mathbb{R}$, passing above $z_0(\delta)$, such that* $\operatorname{Im} \Delta_{12}(z,\delta)|_{\gamma_\delta(t)}$ *is a non-decreasing function of t for a branch of $\Delta_{12}(z,\delta)$ and*

$$\lim_{t\to\pm\infty} \operatorname{Re} \gamma_\delta(t) = \pm\infty,$$

$$\inf_{t\in\mathbb{R}} \operatorname{Im} \gamma_\delta(t) \geqslant h > 0,$$

$$\sup_{t\in\mathbb{R}} |\dot{\gamma}_\delta(t)| \leqslant k$$

where h and k are independent of δ. Such a path will be called a dissipative *path.*

We postpone the proof of this proposition to the end of the section and we use it to compute the transition probability $\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)$ of the effective problem. Consider now the Hamiltonian $\widehat{H_*}(z,\delta)$ given by (7.1) restricted to $Q_{N^*}(0,\delta)\mathcal{H}$. Its eigenvalues coincide with the eigenvalues $e_j^*(z,\delta)$ of $H_{N^*}(z,\delta)$ which can be expressed by means of an analytic function $\rho_*(z,\delta)$, depending on ε as

$$e_j^*(z,\delta) = (-1)^j \frac{1}{2} \sqrt{\rho_*(z,\delta)}, \quad j = 1,2. \tag{7.27}$$

The function $\rho_*(z,\delta)$ is constructed in the same way as $\rho(z,\delta)$, by replacing $Q(t,\delta)$ and $H(t,\delta)$ by $Q_{N^*}(t,\delta)$ and $H_{N^*}(t,\delta)$ in (7.16). By perturbation theory and Proposition 4.1, we can write

$$\rho_*(z,\delta) = \rho(z,\delta) + R_*(z,\delta,\varepsilon) \tag{7.28}$$

where $R_*(z,\delta,\varepsilon)$ is a remainder satisfying

$$|R_*(z,\delta,\varepsilon)| \leqslant \varepsilon b(t), \quad \forall z = t + \mathrm{i}s \in S_\alpha, \tag{7.29}$$

and $b(t)$ is an integrable decay function independent of δ.

Proposition 7.3. *There exists ε^* and δ^*, independent of δ and ε, respectively, such that for all $\varepsilon < \varepsilon^*$, $\delta < \delta^*$*

i) *if $\delta > 0$, there exists a unique complex eigenvalue crossing point $z_0^*(\delta)$ of $e_j^*(z,\delta)$, of square root type, with* $\operatorname{Im} z_0^*(\delta) > 0$ *in S_α,*

ii) *if $\delta = 0$, there exists a unique real eigenvalue crossing point $z_0^*(0)$ of $e_j^*(z,0)$.*

In any case $|z_0^*(\delta)| < r$.

This lemma shows that condition A is satisfied and that $z^*(\delta) \in D(0,r)$ $\forall\delta$ small enough where $D(0,r)$ is the circle of radius $r > 0$ centered at the origin.

Proof. We assume that $\delta > 0$ and we choose ε, independently of δ, in such a way that

$$|\rho_*(z,\delta) - \rho(z,\delta)| \leqslant \frac{R}{3}, \quad \forall z \in S_\alpha \setminus D(0,r),\ \forall\delta \in I_\Delta. \tag{7.30}$$

As

$$\lim_{\varepsilon\to 0} e_j^*(t,\delta) = e_j(t,\delta), \tag{7.31}$$

the real eigenvalue crossing points, if any, must appear by pairs in order to to have $e_1^*(-\infty,\delta) < 0$ and $e_1^*(+\infty,\delta) < 0$. Remember that $H_{N^*}(t,\delta)$ and $H(t,\delta)$ coincide at infinity. To show that actually there is no real eigenvalue crossing point we use

$$\left|\rho_*(z,\delta) - \rho(z,0)\right| \leqslant \frac{2R}{3} < \left|\rho(z,0)\right| \tag{7.32}$$

if $z \in \partial D(0,r)$, see (7.26), (7.30), and we apply Rouché's theorem to

$$\rho_*(z,\delta) = \rho(z,\delta) + \big(\rho_*(z,\delta) - \rho(z,0)\big). \tag{7.33}$$

As there is one double zero of $\rho(z,0)$ in $D(0,r)$, at $z=0$, there are either two simple conjugate zeros $z_0^*(\delta)$ and $\overline{z_0^*(\delta)}$ or only one real double zero of $\rho_*(z,\delta)$ in $D(0,r)$. But the latter case must be excluded because this corresponds to one crossing only. Recall that a real crossing corresponds to a double zero of $\rho^*(z,\delta)$ because of the analyticity of the eigenvalues at that point. If $\delta = 0$, the same type of argument shows that there is one real double zero $z_0^*(0)$ of $\rho_*(z,0)$, in order to ensure $e_1^*(-\infty,0) < 0$ and $e_1^*(+\infty,0) > 0$, which corresponds to one real crossing of eigenvalue. □

With our definitions, we have

$$\Delta_{12}^*(z,\delta) = -\int_0^z \sqrt{\rho_*(u,\delta)}\,\mathrm{d}u \tag{7.34}$$

which yields an analytic function in $S_\alpha^+ \setminus D(0,r)$. The path of integration is the same as the one defining the branch of $\Delta_{12}(z,\delta)$ considered in Proposition 7.2 (see the proof of that proposition). A direct consequence of Propositions 7.2 and 7.3 is that for any $0<\delta<\delta^*$ and $0<\varepsilon<\varepsilon^*$ we can apply Proposition 7.1 to our effective two-level problem. Indeed, we can control the quantity

$$\exp\left\{\frac{1}{\varepsilon}\operatorname{Im}\Big(\Delta_{12}^*\big(\gamma_\delta(s),\delta\big) - \Delta_{12}^*\big(\gamma_\delta(t),\delta\big)\Big)\right\} \tag{7.35}$$

where $s \leqslant t$, uniformly in δ and ε (condition B): It follows from (7.28) and (7.29) that

$$\operatorname{Im}\Delta_{12}^*(z,\delta) = \operatorname{Im}\Delta_{12}(z,\delta) + \mathrm{O}(\varepsilon) \tag{7.36}$$

and by construction $\operatorname{Im}\Delta_{12}(z,\delta)$ is non-decreasing along γ_δ. Hence (7.35) is uniformly bounded in $s \leqslant t$, ε and δ. We define a loop β based at the origin by the path going from 0 to $-r$ along the real axis, from $-r$ to r along $\partial D(0,r)$ and from r back to the origin along the real axis again. By Proposition 7.3, $z_0^*(\delta)$ does not belong to β, for any $\delta > 0$. To obtain the asymptotic formula for $\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)$

$$\widehat{\mathcal{P}_{21}}(\varepsilon,\delta) = \exp\left\{\frac{2}{\varepsilon}\operatorname{Im}\int_\beta e_1^*(z,\delta)\,\mathrm{d}z\right\}\exp\left\{2\operatorname{Im}\theta_1^*(\delta)\right\}\big(1+\mathrm{O}(\varepsilon)\big) \tag{7.37}$$

with a correction term $\mathrm{O}(\varepsilon)$ uniformly bounded in δ, it remains to check that along the path $\gamma_\delta(t)$, we can bound the corresponding coefficients $a_{kj}^*(z,\delta)$ defined in (7.8) uniformly in δ and ε (see [11]).

Lemma 7.3. *There exists an integrable decay function $b(t)$ independent of δ and ε such that*

$$\left|a^*_{kj}(z,\delta)\right| \leqslant b(t), \quad \forall z = t + \mathrm{i}s \in S^+_\alpha \setminus D(0,r).$$

This lemma shows that formula (7.37) is indeed true for our effective two-level problem with a correction term uniformly bounded in δ. Its proof is given in Appendix. Let us express $\widehat{\mathcal{P}_{21}}(\varepsilon,\delta)$ as a function of $H(z,\delta)$ only.

Lemma 7.4.

$$\operatorname{Im}\int_\beta e^*_1(z,\delta)\,\mathrm{d}z = \operatorname{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z + \mathrm{O}(\varepsilon^2),$$

$$\operatorname{Im}\theta^*_1(\delta) = \operatorname{Im}\theta_1(\delta) + \mathrm{O}(\varepsilon),$$

where $\theta_1(\delta)$ is defined by (2.7).

Proof. Let us denote the intersection of $\partial D(0,r)$ with the upper half-plane by C^+_r. We can replace β in the integral of (7.37) by C^+_r without altering the formula, so that we have to evaluate $e^*_1(z,\delta)$, on C^+_r, far from the eigenvalue crossing point $z_0(\delta)$. Moreover, as $\theta^*_1(\delta)$ is given by

$$\widetilde{\varphi^*_1}(0,\delta) = \exp\left\{-\mathrm{i}\theta^*_1(\delta)\right\}\varphi^*_2(0,\delta) \tag{7.38}$$

and as the vectors $\varphi^*_j(0,\delta)$ are normalized on the real axis, $\exp\{\operatorname{Im}\theta^*_1(\delta)\}$ represents the change of norm of the analytic continuation of $\varphi^*_1(z,\delta)$ from $-r$ to r along C^+_r. For any $z \in C^+_r$ we can use perturbation theory to prove the lemma. Indeed, we have by Proposition 4.1

$$\begin{aligned} H_{N^*}(t,\delta) &= H(t,\delta) - \varepsilon K_0(t,\delta) + \varepsilon\big(K_0(t,\delta) - K_{N^*-1}(t,\delta)\big) \\ &= H(t,\delta) - \varepsilon K_0(t,\delta) + \mathrm{O}(\varepsilon^2) \end{aligned} \tag{7.39}$$

where $\mathrm{O}(\varepsilon^2)$ is uniformly bounded in δ. Let $\varphi_1(t,\delta)$ be the eigenvector of $H(t,\delta)$ for $e_1(t,\delta)$. Then

$$e^*_1(t,\delta) = e_1(t,\delta) - \varepsilon\langle\varphi_1(t,\delta)|K_0(t,\delta)\varphi_1(t,\delta)\rangle + \mathrm{O}(\varepsilon^2). \tag{7.40}$$

But the term of first order in ε vanishes identically since

$$\begin{aligned} &P_1(t,\delta)\left[Q'(t,\delta),Q(t,\delta)\right]P_1(t,\delta) \\ &= P_1(t,\delta)Q(t,\delta)\left[Q'(t,\delta),Q(t,\delta)\right]Q(t,\delta)P_1(t,\delta) \equiv 0. \end{aligned} \tag{7.41}$$

By perturbation theory again, we can write the eigenvector $\varphi^*_1(z,\delta)$ associated with $e^*_1(z,\delta)$ as

$$\varphi^*_1(z,\delta) = \varphi_1(z,\delta) + \chi_1(z,\delta) \tag{7.42}$$

with $\|\chi_1(z,\delta)\| = \mathrm{O}(\varepsilon)$ uniformly in δ if $z \in S_\alpha \setminus D(0,r)$. Denoting by $\widetilde{\varphi^*_j}(r,\delta)$ the vector obtained by analytic continuation of $\varphi^*_j(-r,\delta)$ along C^+_r, we have

$$\exp\left\{2\operatorname{Im}\theta^*_1(\delta)\right\} = \left\|\widetilde{\varphi^*_1}(r,\delta)\right\| = \left\|\widetilde{\varphi_1}(r,\delta) + \widetilde{\chi_1}(r,\delta)\right\| \tag{7.43}$$

and similarly

$$\exp\{2\mathrm{Im}\,\theta_1(\delta)\} = \|\widetilde{\varphi_1}(r,\delta)\|. \tag{7.44}$$

Hence

$$\exp\{2\mathrm{Im}\,\theta_1^*(\delta)\} - \exp\{2\mathrm{Im}\,\theta_1(\delta)\} \leqslant \|\widetilde{\chi_1}(r,\delta)\| = \mathrm{O}(\varepsilon) \tag{7.45}$$

and

$$\mathrm{Im}\,\theta_1^*(\delta) = \mathrm{Im}\,\theta_1(\delta) + \mathrm{O}(\varepsilon) \tag{7.46}$$

where $\mathrm{O}(\varepsilon)$ is uniformly bounded in δ. □

Summarizing these considerations we arrive at the conclusion:

Proposition 7.4. *Under conditions* I *to* III, *there exist* $\varepsilon^* > 0$, $\delta^* > 0$ *independent of* δ *and* ε, *respectively, such that for all* $\varepsilon < \varepsilon^*$, $\delta < \delta^*$,

$$\widehat{\mathcal{P}_{21}}(\varepsilon,\delta) = \exp\{2\mathrm{Im}\,\theta_1(\delta)\}\exp\left\{\frac{2}{\varepsilon}\mathrm{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z\right\}(1+\mathrm{O}(\varepsilon))$$

where the term $\mathrm{O}(\varepsilon)$ *is uniformly bounded in* δ.

7.1. Behaviour in δ

Let us now turn to the dependence in δ of these quantities.

Proposition 7.5. *Assume that conditions* I *to* III *hold and consider* $\mathrm{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z$ *and* $\mathrm{Im}\,\theta_1(\delta)$ *defined above. Then*

$$\lim_{\delta\to 0}\mathrm{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z = 0,$$

$$\lim_{\delta\to 0}\mathrm{Im}\,\theta_1(\delta) = 0.$$

Remark. This last proposition implies that for δ small enough

$$\left|2\mathrm{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z\right| < \tau, \tag{7.47}$$

τ being the exponential decay rate of the correction term in Proposition 5.1. Thus we have

$$\mathcal{P}_{21}(\varepsilon,\delta) = \exp\left\{\frac{2}{\varepsilon}\mathrm{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z\right\}\exp\{2\mathrm{Im}\,\theta_1(\delta)\}(1+\mathrm{O}(\varepsilon)) \tag{7.48}$$

for ε and δ small enough. This proves the first assertion of Theorem 2.1.

Proof. We have

$$\left|\operatorname{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z\right| = \left|\operatorname{Im}\Delta_{12}\big(z_0(\delta),\delta\big)\right|. \tag{7.49}$$

As $\rho(z,\delta)\to\rho(z,0)$ and $z_0(\delta)\to 0$ when $\delta\to 0$ (Proposition 7.2), we get

$$\lim_{\delta\to 0}\left|\operatorname{Im}\Delta_{12}\big(z_0(\delta),\delta\big)\right| = 0. \tag{7.50}$$

Let us introduce $W(z,t;\delta)$, $z\neq z_0(\delta)$, $t\in\mathbb{R}$, by

$$\begin{aligned}\mathrm{i}W'(z,t;\delta) &= \mathrm{i}\Big(P_1'(z,\delta)P_1(z,\delta)+P_2'(z,\delta)P_2(z,\delta)\\ &\qquad - Q'(z,\delta)\big(\mathbb{I}-Q(z,\delta)\big)\Big)W(z,t;\delta)\\ &\equiv L(z,\delta)W(z,t;\delta),\quad W(t,t;\delta)=\mathbb{I}.\end{aligned} \tag{7.51}$$

The evolution $W(z,t;\delta)$ is a generalization of $W_{N^*}(z,\delta)$ in the sense that it has the intertwining property with $P_j(z,\delta)$, $j=1,2$, and $Q(z,\delta)$ [19, 20]:

$$\begin{aligned}&W(z,t;\delta)P_j(t,\delta) = P_j(z,\delta)W(z,t;\delta),\\ &W(z,t;\delta)Q(t,\delta) = Q(z,\delta)W(z,t;\delta).\end{aligned} \tag{7.52}$$

We have

$$\varphi_j(z,\delta) = W(z,0;\delta)\varphi_j(0,\delta), \tag{7.53}$$

where $\varphi_j(0,\delta)$ satisfies

$$H(0,\delta)\varphi_j(0,\delta) = e_j(0,\delta)\varphi_j(0,\delta),\quad \|\varphi_j(0,\delta)\| = 1. \tag{7.54}$$

As noted previously

$$\|W(r,-r;\delta)\varphi_1(-r,\delta)\| = \exp\{\operatorname{Im}\theta_1(\delta)\} \tag{7.55}$$

where the path of integration of $W(r,-r,\delta)$ from $-r$ to r is along C_r^+. Let us show that

$$W(r,-r;\delta)\to W(r,-r;0) \tag{7.56}$$

strongly as $\delta\to 0$. Consider the identity

$$\begin{aligned}&\big(W^{-1}(z,-r;\delta)W(z,-r;0)-\mathbb{I}\big)\varphi\\ &\quad = \mathrm{i}\int_{-r}^{z} W^{-1}(z',-r;\delta)\big(L(z',0)-L(z',\delta)\big)W(z',-r;0)\varphi\,\mathrm{d}z'\end{aligned} \tag{7.57}$$

where z and the path of integration are along C_r^+. It follows from condition I (see Lemma 3.2), that $L(z,\delta)$ is strongly continuous in z and δ, $\forall z \in S_\alpha^+ \setminus D(0,r)$ so that, by compactness of C_r^+, $L(z,\delta)\varphi$ tends to $L(z,0)\varphi$ uniformly in $z \in C_r^+$ when $\delta \to 0$ and

$$\sup_{\substack{z\in C_r^+\\ \delta\in I_\Delta}} \|W(z,-r;\delta)\| \leqslant w', \qquad \sup_{\substack{z\in C_r^+\\ \delta\in I_\Delta}} \|W^{-1}(z,-r;\delta)\| \leqslant w'. \tag{7.58}$$

Now, the set of vectors

$$\{W(z',-r;0)\varphi;\ z' \in C_r^+\} \tag{7.59}$$

is a compact set in $\mathcal{H}$ because $W(z',-r;0)$ is continuous in z' so that we apply Lemma 3.4 of the introduction of [20] to obtain

$$\lim_{\delta\to 0} \sup_{z'\in C_r^+} \big\|\big(L(z',\delta) - L(z',0)\big)W^{-1}(z',-r;0)\varphi\big\| = 0. \tag{7.60}$$

As a consequence

$$\begin{aligned} &\big\|\big(W(z,-r;0) - W(z,-r;\delta)\big)\varphi\big\| \\ &\leqslant \|W(z,-r;\delta)\|\big\|\big(W^{-1}(z,-r;\delta)W(z,-r;0) - \mathbb{I}\big)\varphi\big\| \\ &\leqslant w'^2\pi r \sup_{z'\in C_r^+} \big\|\big(L(z',\delta) - L(z',0)\big)W^{-1}(z,-r;0)\varphi\big\| \end{aligned} \tag{7.61}$$

showing that $W(z,-r,\delta)$ is strongly continuous in δ on C_r^+. Moreover, we can construct a normalized eigenvector $\varphi_1(-r,\delta)$ of $H(-r,\delta)$ which is continuous in δ by

$$\varphi_1(-r,\delta) = \frac{P_1(-r,\delta)\varphi_1(-r,0)}{\langle\varphi_1(-r,0)|P_1(-r,\delta)\varphi_1(-r,0)\rangle} \tag{7.62}$$

where $H(-r,0)\varphi_1(-r,0) = e_1(-r,0)\varphi_1(-r,0)$. Hence the estimate

$$\begin{aligned} &\|W(z,-r;\delta)\varphi_1(-r,\delta) - W(z,-r;0)\varphi_1(-r,0)\| \\ &\quad\leqslant \big\|\big(W(z,-r;\delta) - W(z,-r;0)\big)\varphi_1(-r,0)\big\| \\ &\quad\quad+ \big\|W(z,-r;\delta)\big(\varphi_1(-r,\delta) - \varphi_1(-r,0)\big)\big\| \end{aligned} \tag{7.63}$$

from which follows that

$$W(z,-r;\delta)\varphi_1(-r,\delta) \to W(z,-r;0)\varphi_1(-r,0) \tag{7.64}$$

as $\delta \to 0$. Since for $\delta = 0$, $W(z,-r;0)$ is analytic for any z in $D(0,r)$, $W(+r,-r;0)$ integrated along C_r^+ coincides with $W(+r,-r;0)$ integrated along the real axis. Thus this operator is unitary and we have $\|W(+r,-r;0)\varphi_1(-r,0)\| = 1$, which together with (7.55) imply

$$\lim_{\delta\to 0} \operatorname{Im}\theta_1(\delta) = 0. \tag{7.65}$$

□

7.2. Expansion in δ

Let us finally turn to the last assertion of Theorem 2.1 which deals with the actual computation of $\operatorname{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z$ and $\operatorname{Im}\theta_1(\delta)$ to the lowest order in δ, when hypothesis IV is fulfilled.

Proposition 7.6. *Under hypothesis* I *to* IV *we have*

$$2\operatorname{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z = -\delta^2\frac{\pi}{2}\left(\frac{b^2}{a}-\frac{c^2}{a^3}\right)(1+\mathrm{O}(\delta)),$$
$$2\operatorname{Im}\theta_1(\delta)=\mathrm{O}(\delta).$$

Proof. By condition IV we have

$$\rho(z,\delta)=a^2z^2+2c\delta z+b^2\delta^2+R_3(z,\delta) \tag{7.66}$$

with

$$c^2<a^2b^2 \tag{7.67}$$

where $R_3(z,\delta)$ is analytic and satisfies

$$|R_3(z,\delta)|\leqslant k\left(|z^2|+\delta^2\right)\left(|z|+\delta\right) \tag{7.68}$$

for k some constant. There will appear several other constants in the sequel, which we shall denote generically by the same letter k. Let $C_{x\delta}$ be the circle centered at the origin of radius $x\delta$, where x is some real parameter independent of δ. We can write

$$\begin{aligned}|a^2z^2+2c\delta z+b^2\delta^2| &= \left|a^2\left(z+\frac{c\delta}{a^2}\right)^2+\left(b^2-\frac{c^2}{a^2}\right)\delta^2\right|\\ &\geqslant \left|a^2\left|z+\frac{c\delta}{a^2}\right|^2-\left|b^2-\frac{c^2}{a^2}\right|\delta^2\right|.\end{aligned} \tag{7.69}$$

If x is large enough, $x\delta>|c|\delta/a^2$, and we have for any $z\in C_{x\delta}$

$$\left|z+\frac{c\delta}{a^2}\right|\geqslant\left||z|-\left|\frac{c\delta}{a^2}\right|\right|\geqslant\left(x-\frac{|c|}{a^2}\right)\delta. \tag{7.70}$$

Thus we can always choose x sufficiently large so that

$$a^2\left|z+\frac{c\delta}{a^2}\right|^2-\left|b^2-\frac{c^2}{a^2}\right|\delta^2\geqslant\left(a^2\left(x-\frac{|c|}{a^2}\right)^2-\left(b^2-\frac{c^2}{a^2}\right)\right)\delta^2\geqslant k\delta^2>0 \tag{7.71}$$

where k is independent of δ and arrive at the conclusion that for any $z\in C_{x\delta}$

$$|a^2z^2+2c\delta z+b^2\delta^2|\geqslant k\delta^2, \tag{7.72}$$

whereas

$$|R_3(z,\delta)| \leqslant k\delta^3 \tag{7.73}$$

on the same circle. By applying Rouché's theorem for δ small enough we have that

$$\zeta_\pm = -\frac{c\delta}{a^2} \pm \mathrm{i}\,\frac{\sqrt{a^2b^2-c^2}}{a^2}\,\delta, \tag{7.74}$$

the zeros of $a^2z^2 + 2c\delta z + b^2\delta^2$, and $z_0(\delta)$, $\overline{z_0(\delta)}$ are in $C_{x\delta}$. Moreover, $\forall z \in C_{x\delta}$,

$$\begin{aligned}\sqrt{\rho(z,\delta)} &= \sqrt{a^2z^2 + 2c\delta z + b^2\delta^2\left(1 + \frac{R_3(z,\delta)}{a^2z^2 + 2c\delta z + b^2\delta^2}\right)}\\ &\equiv \sqrt{a^2z^2 + 2c\delta z + b^2\delta^2}\,(1 + h(z,\delta))\end{aligned} \tag{7.75}$$

where $|h(z,\delta)|_{z\in C_{x\delta}} \leqslant k\delta$, since

$$\left|\frac{R_3(z,\delta)}{a^2z^2 + 2c\delta z + b^2\delta^2}\right| \leqslant k\delta, \quad \forall z \in C_{x\delta}. \tag{7.76}$$

From these last estimates we can write

$$\begin{aligned}2\,\mathrm{Im}\int_\beta e_1(z,\delta)\,\mathrm{d}z &= -\mathrm{Im}\int_{C^+_{x\delta}} \sqrt{\rho(z,\delta)}\,\mathrm{d}z\\ &= -\mathrm{Im}\int_{C^+_{x\delta}} \sqrt{a^2z^2 + 2c\delta z + b^2\delta^2}\,\mathrm{d}z + \mathrm{O}(\delta^3).\end{aligned} \tag{7.77}$$

Finally, we compute by deforming the path of integration to a vertical segment going from $z = \mathrm{Re}\,\zeta_+$ to $z = \zeta_+$ and back to $z = \mathrm{Re}\,\zeta_+$,

$$\begin{aligned}&-\mathrm{Im}\int_{C^+_{x\delta}} \sqrt{a^2z^2 + 2c\delta z + b^2\delta^2}\,\mathrm{d}z\\ &\quad = -2\int_0^{\frac{1}{a^2}\sqrt{a^2b^2-c^2}\,\delta} \sqrt{\left(b^2 - \frac{c^2}{a^2}\right)\delta^2 - y^2a^2}\,\mathrm{d}y = -\delta^2\frac{\pi}{2}\left(\frac{b^2}{a} - \frac{c^2}{a^3}\right).\end{aligned} \tag{7.78}$$

To bound $\mathrm{Im}\,\theta_1(\delta)$ by a term of order δ, we need a little more work. Let us consider the explicit formula for $\mathrm{Im}\,\theta_1(\delta)$ in terms of the matrix elements of $H(t,\delta)$ which is derived in [15]:

Proposition 7.7. *Let $\psi_j(t,\delta)$ and $B_j(t,\delta)$, $j = 1,2,3$, be defined by* (7.14) *and* (7.16), *respectively,* *and assume that conditions* I *to* III *hold. Then*

$$\operatorname{Im}\theta_1(\delta) = \operatorname{Im}\int_\sigma \frac{B_3(z,\delta)\big(B_1(z,\delta)B_2'(z,\delta) - B_2(z,\delta)B_1'(z,\delta)\big)}{2\sqrt{\rho(z,\delta)}\big(B_1^2(z,\delta)+B_2^2(z,\delta)\big)}\,\mathrm{d}z$$
$$+\operatorname{Re}\int_\sigma \Big(\frac{B_3(z,\delta)}{2\sqrt{\rho(z,\delta)}}\big(\langle\psi_1(z,\delta)|\psi_1'(z,\delta)\rangle - \langle\psi_2(z,\delta)|\psi_2'(z,\delta)\rangle\big)$$
$$+\frac{B_1(z,\delta)+\mathrm{i}B_2(z,\delta)}{2\sqrt{\rho(z,\delta)}}\langle\psi_1(z,\delta)|\psi_2'(z,\delta)\rangle$$
$$+\frac{B_1(z,\delta)-\mathrm{i}B_2(z,\delta)}{2\sqrt{\rho(z,\delta)}}\langle\psi_2(z,\delta)|\psi_1'(z,\delta)\rangle\Big)$$

where the path σ encircles $z_0(\delta)$ and contains no zero of $B_1^2(z,\delta)+B_2^2(z,\delta)$.

Our condition IV implies that the analytic functions $B_j(z,\delta)$ defined by (7.16) have the form

$$B_j(z,\delta) = a_j z + b_j\delta + R_2(z,\delta) \tag{7.79}$$

where the real constants a_j and b_j satisfy

$$\sum_{j=1}^{3} a_j^2 = a^2, \qquad \sum_{j=1}^{3} b_j^2 = b^2, \qquad \sum_{j=1}^{3} a_j b_j = c, \tag{7.80}$$

and $R_2(z,\delta)$ is a rest of order two in (z,δ). Again we shall replace the path σ by $C_{x\delta}^+$ since on the real axis, the integrals in Proposition 7.7 do not contribute to $\operatorname{Im}\theta_1(\delta)$. But here some care must be taken for the first integral since the integrand has poles at the zeros of $B_1^2(z,\delta)+B_2^2(z,\delta)$. But this is not the case for the other integrals in which the replacement of σ by $C_{x\delta}^+$ is justified. As on $C_{x\delta}^+$ we have (see (7.72))

$$\big|\sqrt{\rho(z,\delta)}\big| \geqslant k\delta, \quad \big|B_j(z,\delta)\big| \leqslant k\delta, \quad \text{and} \quad \big|\langle\psi_j|\psi_k'\rangle(z,\delta)\big| \leqslant k, \tag{7.81}$$

we immediately obtain

$$\operatorname{Im}\theta_1(\delta) = \operatorname{Im}\int_\sigma \frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}(B_1^2+B_2^2)}\,\mathrm{d}z + \mathrm{O}(\delta). \tag{7.82}$$

To deal with the first term, we introduce

$$\alpha = \sqrt{a_1^2+a_2^2}, \quad \beta = \sqrt{b_1^2+b_2^2}, \quad \text{and} \quad \gamma = a_1b_1 + a_2b_2. \tag{7.83}$$

By the Cauchy–Schwartz inequality these quantities satisfy $|\gamma| \leqslant \alpha\beta$. Actually, we can assume without loss of generality that

$$0 < |\gamma| < \alpha\beta. \tag{7.84}$$

Indeed, the equality $|\gamma| = \alpha\beta$ implies

$$a_1 = yb_1, \qquad a_2 = yb_2 \tag{7.85}$$

for some $y \neq 0$. This cannot be the case for any couple of indices since it would imply $a_3 = yb_3$ as well, in contradiction with the condition $|c| < ab$. Thus we can always perform a change of basis vectors, which amounts to write $H(t,\delta)Q(t,\delta)$ in a new basis $\{S\psi_1(t,\delta), S\psi_2(t,\delta)\}$ instead of $\{\psi_1(t,\delta), \psi_2(t,\delta)\}$, where S is a constant unitary matrix, so that the components of the new field are such that (7.84) is verified. With these definitions and (7.79) we can rewrite

$$B_1^2(z,\delta) + B_2^2(z,\delta) = \alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2 + R_3(z,\delta). \tag{7.86}$$

As previously we have, for $z \in C_{x\delta}^+$

$$\left|\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2\right| \geqslant \left|\alpha^2\left|z + \frac{\gamma\delta}{\alpha^2}\right| - \left|\beta^2 - \frac{\gamma^2}{\alpha^2}\right|\delta^2\right| \tag{7.87}$$

where

$$\alpha^2\left|z + \frac{\gamma\delta}{\alpha^2}\right|^2 \geqslant \alpha^2\left(x - \frac{|\gamma|}{\alpha^2}\right)^2\delta^2 > \left|\beta^2 - \frac{\gamma^2}{\alpha^2}\right|\delta^2, \tag{7.88}$$

provided x is large enough, so that

$$\left|\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2\right| \geqslant k\delta^2. \tag{7.89}$$

Hence,

$$\left|R_3(z,\delta)\right| \leqslant k\delta^3 < k\delta^2 \leqslant \left|\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2\right| \tag{7.90}$$

for δ small enough, $\forall z \in C_{x\delta}^+$. Then it follows from Rouché's theorem that $B_1^2(z,\delta) + B_2^2(z,\delta)$ has as many zeros in $C_{x\delta}^+$ as $\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2$, i.e., two $\zeta_0(\delta)$ and $\overline{\zeta_0(\delta)}$, counting multiplicities. Indeed, the roots of $\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2$ are given by $\xi_\pm$,

$$\xi_\pm = -\frac{\gamma\delta}{\alpha^2} \pm \mathrm{i}\frac{\sqrt{\alpha^2\beta^2 - \gamma^2}}{\alpha^2}\,\delta \tag{7.91}$$

which belong to $C_{x\delta}$ if x is large enough. Note that due to (7.84), $\operatorname{Im}\xi_+ > 0$. Now we can replace the contour of integration σ in (7.82) by $C_{x\delta}^+$, provided we take the residue at $\zeta_0(\delta)$ into account. Consider first the case where $\zeta_0(\delta) \neq \overline{\zeta_0(\delta)}$. Since $\operatorname{Im}\zeta_0(\delta) > 0$, we have

$$\operatorname{Im}\theta_1(\delta) = 2\pi\operatorname{Re}\left(\operatorname{Res}\left(\frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}(B_1^2 + B_2^2)}, \zeta_0(\delta)\right)\right) \tag{7.92}$$
$$+ \operatorname{Im}\int_{C_{x\delta}^+} \frac{(a_3 z + b_3\delta)(a_2b_1 - a_1b_2)\delta + R_3(z,\delta)}{2\sqrt{a^2z^2 + 2c\delta z + b^2\delta^2}\left(\alpha^2z^2 + 2\gamma\delta z + \beta^2\delta^2 + R_3(z,\delta)\right)\left(1 + h(z,\delta)\right)}\,\mathrm{d}z$$

where $|h(z,\delta)| \leqslant k\delta$ (see (7.75)) and $\mathrm{Res}(f, z_0)$ is the residue of $f(z)$ at the point z_0. In view of (7.89) and (7.75), we can estimate the remaining integral by

$$\mathrm{Im}\int_{C_{x\delta}^{+}} \frac{(a_3 z + b_3\delta)(a_2 b_1 - a_1 b_2)\delta}{2\sqrt{a^2z^2 + 2c\delta z + b^2\delta^2}\,(\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2)} + \mathrm{O}(\delta) \tag{7.93}$$

when δ is small. The integrand is now singular at ζ_+ and ξ_+ only, which both belong to $C_{x\delta}^{+}$, when x is large. Thus we can replace the contour of integration $C_{x\delta}^{+}$ by C_R^{+}, the half circle of radius R, which will ultimately tend to infinity, since on the real axis the integral is real. On C_R^{+} we have the estimates

$$|a^2z^2 + 2c\delta z + b^2\delta^2| = |z^2|\left|a^2 + \frac{2c\delta}{z} + \frac{b^2\delta^2}{z^2}\right| \geqslant k(\delta)R^2 \tag{7.94}$$

and

$$|\alpha^2z^2 + 2\gamma\delta z + \beta^2\delta^2| \geqslant k(\delta)R^2 \tag{7.95}$$

which imply

$$\left|\int_{C_R^{+}} \frac{(a_3 z + b_3\delta)(a_2 b_1 - a_1 b_2)\delta}{2\sqrt{a^2z^2 + 2c\delta z + b^2\delta^2}\,(\alpha^2 z^2 + 2\gamma\delta z + \beta^2\delta^2)}\right| \leqslant \frac{k(\delta)}{R}. \tag{7.96}$$

Taking the limit $R \to \infty$ we are left with

$$\mathrm{Im}\,\theta_1(\delta) = 2\pi\mathrm{Re}\left(\mathrm{Res}\left(\frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}(B_1^2 + B_2^2)}, \zeta_0(\delta)\right)\right) + \mathrm{O}(\delta). \tag{7.97}$$

The residue is given here by the formula

$$\begin{aligned}\left.\frac{B_3(B_1B_2' - B_2B_1')}{4\sqrt{\rho}(B_1B_1' + B_2B_2')}\right|_{\zeta_0(\delta)} &= \varepsilon_1 \left.\frac{(B_1B_2' - B_2B_1')}{4(B_1B_1' + B_2B_2')}\right|_{\zeta_0(\delta)} \\ &= \varepsilon_1\varepsilon_2 \left.\frac{\mathrm{i}(B_2B_2' + B_1B_1')}{4(B_1B_1' + B_2B_2')}\right|_{\zeta_0(\delta)} = \pm\frac{\mathrm{i}}{4}\end{aligned} \tag{7.98}$$

where we have used the fact that

$$B_1^2(\zeta_0(\delta),\delta) + B_2^2(\zeta_0(\delta),\delta) = 0, \tag{7.99}$$

so that

$$\sqrt{\rho(\zeta_0(\delta),\delta)} = \sqrt{B_3^2(\zeta_0(\delta),\delta)} = \varepsilon_1 B_3(\zeta_0(\delta),\delta) \tag{7.100}$$

where $\varepsilon_1 = \pm 1$ and

$$B_1(\zeta_0(\delta),\delta) = \varepsilon_2 \mathrm{i} B_2(\zeta_0(\delta),\delta) \tag{7.101}$$

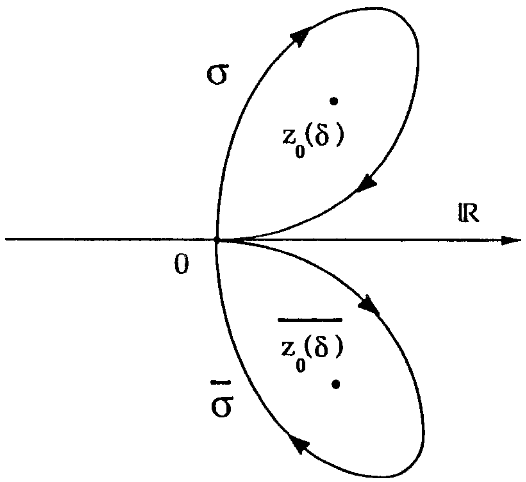

Fig. 3. The integration path $\sigma \cup \overline{\sigma}$.

with $\varepsilon_2 = \pm 1$ as well. Hence

$$\operatorname{Im}\theta_1(\delta) = \mathrm{O}(\delta). \tag{7.102}$$

Consider now the case $\zeta_0(\delta) = \overline{\zeta_0(\delta)}$. We come back to (7.82) and we use the fact that $B_j(\overline{z}, \delta) = \overline{B_j}(z, \delta)$ by Schwartz's principle, and that $z_0(\delta)$ is a simple zero $\rho(z, \delta)$, to write

$$\operatorname{Im}\theta_1(\delta) = \frac{1}{2}\operatorname{Im}\int_{\sigma\cup\overline{\sigma}} \frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}(B_1^2 + B_2^2)} + \mathrm{O}(\delta) \tag{7.103}$$

where $\sigma \cup \overline{\sigma}$ form a closed path surrounding $z_0(\delta)$ and $\overline{z_0(\delta)}$ (see Fig. 3). By the same argument as before, we have

$$\operatorname{Im}\theta_1(\delta) = \pi \operatorname{Re}\left(\operatorname{Res}\left(\frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}(B_1^2 + B_2^2)}, \zeta_0(\delta)\right)\right) + \mathrm{O}(\delta). \tag{7.104}$$

The residue is now given by

$$2\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}}\right)\frac{1}{\frac{\mathrm{d}^2}{\mathrm{d}z^2}(B_1^2 + B_2^2)}\bigg|_{\zeta_0(\delta)}, \tag{7.105}$$

since $\zeta_0(\delta)$ is a double zero of $B_1^2 + B_2^2$. Moreover, as it is located on the real axis, this implies

$$B_1(\zeta_0(\delta)) = B_2(\zeta_0(\delta)) = 0. \tag{7.106}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{B_3(B_1B_2' - B_2B_1')}{2\sqrt{\rho}}\right)\bigg|_{\zeta_0(\delta)} = 0 \tag{7.107}$$

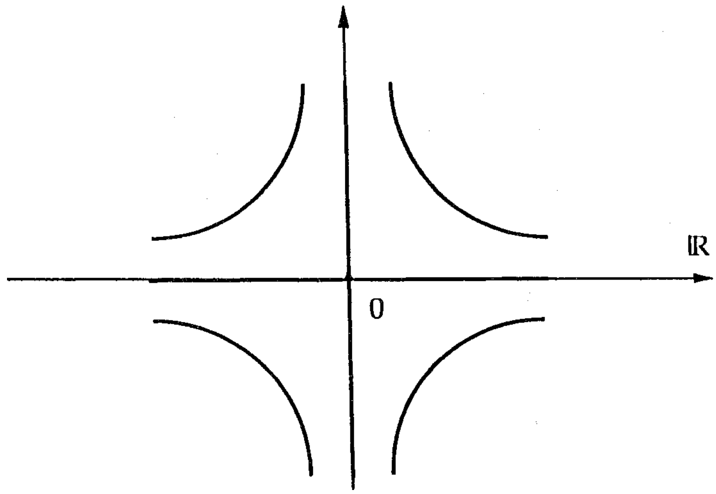

Fig. 4. The level lines Imz^2 =cst.

and

$$\operatorname{Im}\theta_1(\delta) = \mathrm{O}(\delta). \tag{7.108}$$

This last assertion ends the proof of Proposition 7.6. □

To bring the proof of Theorem 2.1 to an end, it remains to show the existence of the dissipative path γ_δ of Proposition (7.2).

7.3. Existence of a dissipative path γ_δ

Proof of Proposition 7.2. To prove the existence of a dissipative path γ_δ for $\Delta_{12}(z,\delta)$, we first show that there exists a dissipative path γ_0 for $\Delta_{12}(z,0)$. When $\delta = 0$, the function

$$\Delta_{12}(z,0) = \int_0^z \big(e_1(u,0) - e_2(u,0)\big)\,\mathrm{d}u = -\int_0^z \sqrt{\rho(u,0)}\,\mathrm{d}u \tag{7.109}$$

is analytic in a neighbourhood of the real axis and behaves as z^2 close to the origin. We select the branch of the square root by requiring $\Delta_{12}(t,0) > 0$ if $t < 0$. The Stokes lines given by the level lines

$$\operatorname{Im}\Delta_{12}(z,0) = 0 \tag{7.110}$$

are homeomorphic to the lines depicted in Fig. 4 in a neighbourhood of $z = 0$. As a consequence, there exist in this neighbourhood two points z_1 and z_2 above the real axis such that

$$\begin{aligned}\operatorname{Im}\Delta_{12}(z_1,0) &= -\chi,\\ \operatorname{Im}\Delta_{12}(z_2,0) &= +\chi\end{aligned} \tag{7.111}$$

with $\chi > 0$ small, which are connected by the level line

$$\mathrm{Re}\,\Delta_{12}(z,0) = \mathrm{Re}\,\Delta_{12}(z_1,0). \tag{7.112}$$

Then, the idea is to take χ small enough, and to complete this segment on the left by the level line $\mathrm{Im}\,\Delta_{12}(z,0) = -\chi$ and on the right by $\mathrm{Im}\,\Delta_{12}(z,0) = +\chi$ which connect z_1 to $-\infty$ in S_α and z_2 to $+\infty$ in S_α. If we can find such a χ, we have at hand a path $\gamma_0(t)$, whose parametrization can be chosen such that $\gamma_0(t_1) = z_1$, $\gamma_0(t_2) = z_2$ which is dissipative for $\Delta_{12}(z,0)$ (see Fig. 5). Indeed, we have for any path

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\,\Delta_{12}(\gamma_0(t),0) = -\mathrm{Re}\,\dot{\gamma}_0(t)\,\mathrm{Im}\sqrt{\rho(\gamma_0(t),0)} - \mathrm{Im}\,\dot{\gamma}_0(t)\,\mathrm{Re}\sqrt{\rho(\gamma_0(t),0)} \tag{7.113}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\,\Delta_{12}(\gamma_0(t),0) = -\mathrm{Re}\,\dot{\gamma}_0(t)\,\mathrm{Re}\sqrt{\rho(\gamma_0(t),0)} + \mathrm{Im}\,\dot{\gamma}_0(t)\,\mathrm{Im}\sqrt{\rho(\gamma_0(t),0)}. \tag{7.114}$$

Thus, if we choose for $t \in [t_1, t_2]$

$$\begin{aligned} \mathrm{Re}\,\dot{\gamma}_0(t) &= -\mathrm{Im}\sqrt{\rho(\gamma_0(t),0)}, \\ \mathrm{Im}\,\dot{\gamma}_0(t) &= -\mathrm{Re}\sqrt{\rho(\gamma_0(t),0)} \end{aligned} \tag{7.115}$$

then equation (7.114) is identically equal to 0 and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}\,\Delta_{12}(\gamma_0(t),0) = \left|\sqrt{\rho(\gamma_0(t),0)}\right|^2 > d > 0. \tag{7.116}$$

We can continue this path on the left and on the right as described by using the following

Lemma. *For any $\mu > 0$, there exists $\nu > 0$ such that on*

$$F_\pm = \{z\colon\ \mathrm{Re}\,z \gtrless \pm\mu,\ |\mathrm{Im}\,z| \leqslant \nu\}$$

the function $\Delta_{12}(z,0)$ is bijective.

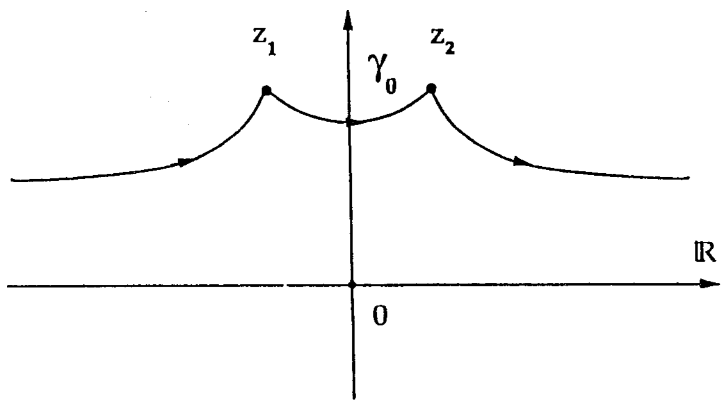

Fig. 5. The dissipative path γ_0.

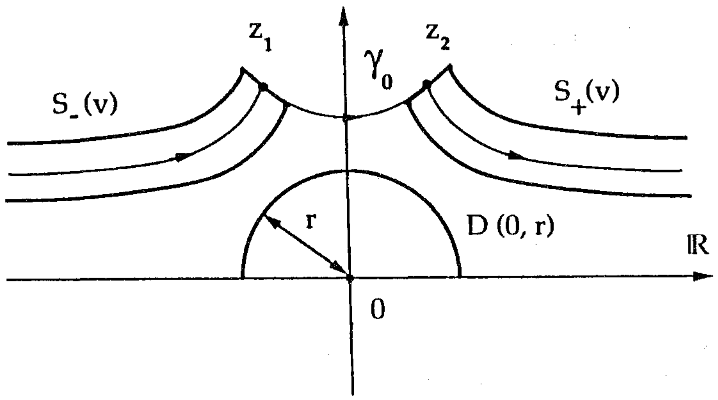

Fig. 6. The disc $D(0,r)$ and the tubular neighbourhoods $S_-(v)$ and $S_+(v)$ of γ_0.

Proof. Let $\mu > 0$. By continuity of $\rho(z,0)$ and condition III, we can chose ν sufficiently small to insure $\operatorname{Re}\sqrt{\rho(z,0)} > R > 0$ for any $z \in F_-$. Let us consider the rectangle $R_-(L)$ whose border is defined by

$$\partial R_-(L) = \partial\big(F_- \setminus \{z\colon\ \operatorname{Re} z \leqslant -L\}\big). \tag{7.117}$$

Along its horizontal segments we have that

$$\operatorname{Re}\Delta_{12}(t \pm \mathrm{i}\nu) = \operatorname{Re}\Delta_{12}(-\mu \pm \mathrm{i}\nu) + \int_{-\mu}^{t} \mathrm{d}x \operatorname{Re}\sqrt{\rho(x \pm \mathrm{i}\nu)} \tag{7.118}$$

is strictly monotonic. Similarly, along its vertical segments

$$\operatorname{Im}\Delta_{12}(-\mu \pm \mathrm{i}s) = \operatorname{Im}\Delta_{12}(-\mu) \pm \int_0^s \mathrm{d}y \operatorname{Re}\sqrt{\rho(-\mu \pm \mathrm{i}y)} \tag{7.119}$$

and $\operatorname{Im}\Delta_{12}(-L \pm \mathrm{i}s)$ are strictly monotonic as well. Thus the image by $\Delta_{12}(z,0)$ of $\partial R_-(L)$ is a simple closed curve so that we can apply the argument principle which shows that $\Delta_{12}(z,0)$ is bijective on $R_-(L)$. Since the length L of the rectangle is arbitrary, this proves the first assertion of the lemma. We proceed similarly for the positive part of the real axis and F_+. □

We shall assume from now on that the width α of the strip S_α is smaller than ν. Now that we have constructed a dissipative path for $\Delta_{12}(z,0)$, we show that there exists a dissipative path for $\Delta_{12}(z,\delta)$ close to it. Let $D(0,r)$ be the disc centered at the origin whose radius r is such that $D(0,r) \cap \gamma_0 = \varnothing$ and let $S_+(v)$ and $S_-(v)$ be tubular neighbourhoods of $\gamma_0(t)$ for $t > t_2$ and $t < t_1$, respectively, defined by their boundaries. These boundaries are given by the level lines

$$\begin{aligned}\partial S_-(v) = &\big\{z\colon\ \operatorname{Re}\Delta_{12}(z,0) \geqslant \operatorname{Re}\Delta_{12}(z_1,0),\ \operatorname{Im}\Delta_{12}(z,0) = -\chi \pm v\big\}\\ &\cup \big\{z\colon\ \operatorname{Re}\Delta_{12}(z,0) = \operatorname{Re}\Delta_{12}(z_1,0),\ \big|\operatorname{Im}\Delta_{12}(z,0) + \chi\big| \leqslant v\big\}\end{aligned} \tag{7.120}$$

and $\partial S_+(v)$ is defined similarly (see Fig. 6). We choose v sufficiently small so that

$$S_\pm(v) \cap D(0,r) = \varnothing. \tag{7.121}$$

Consider the multivalued function

$$\Delta_{12}(z,\delta) = -\int_0^z \sqrt{\rho(u,\delta)}\,\mathrm{d}u. \tag{7.122}$$

When restricted to

$$S_\alpha^+ \setminus D(0,r) \equiv \left(S_\alpha \setminus D(0,r)\right) \cap \{z:\ \operatorname{Im} z \geqslant 0\}, \tag{7.123}$$

$\Delta_{12}(z,\delta)$ is an analytic univalued function provided δ is so small that

$$|z_0(\delta)| < r. \tag{7.124}$$

We fix a branch of $\Delta_{12}(z,\delta)$ by requiring that the path of integration in (7.122) follows the real axis from 0 to $-r$ and that $\Delta_{12}(t,\delta) > 0$ for $t < -r$.

Lemma 7.6. *Let $\Delta_{12}(z,0)$ and $\Delta_{12}(z,\delta)$ be defined as above, and let $z \in S_\alpha^+ \setminus D(0,r)$.*

$$\lim_{\delta\to 0} \operatorname{Im} \Delta_{12}(z,\delta) = \operatorname{Im} \Delta_{12}(z,0) \quad \textit{uniformly in } z \in S_\alpha^+ \setminus D(0,r).$$

Proof. We first show that $\rho(z,\delta)$ tends to $\rho(z,0)$ uniformly in z. Let $\varepsilon > 0$ and consider

$$\begin{aligned} |\rho(z,\delta) - \rho(z,0)| \leqslant{}& |\rho(z,\delta) - \rho(\pm\infty,\delta)| + |\rho(\pm\infty,\delta) - \rho(\pm\infty,0)| \\ &+ |\rho(\pm\infty,0) - \rho(z,0)|. \end{aligned} \tag{7.125}$$

It follows from (7.19) that there exists $T(\varepsilon) > 0$ such that for any $t \gtreqless T(\varepsilon)$

$$\begin{aligned} &|\rho(t+\mathrm{i}s,\delta) - \rho(\pm\infty,\delta)| < \frac{\varepsilon}{3}, \\ &|\rho(\pm\infty,0) - \rho(t+\mathrm{i}s,0)| < \frac{\varepsilon}{3}. \end{aligned} \tag{7.126}$$

Since $\rho(\pm\infty,\delta)$ is continuous in δ, there exists $\delta_1(\varepsilon)$ such that $\delta < \delta_1(\varepsilon)$ implies

$$|\rho(z,\delta) - \rho(z,0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \tag{7.127}$$

for any $|t| \geqslant T(\varepsilon)$. Now the set

$$S_\alpha \setminus \left(D(r,0) \cup D_\pm(T(\varepsilon))\right) \tag{7.128}$$

where

$$D_\pm(T(\varepsilon)) = \{z:\ \operatorname{Re} z \gtrless T(\varepsilon)\} \tag{7.129}$$

is a compact set, so that $\rho(z,\delta)$ is uniformly continuous in (z,δ) for z in this set and $\delta \in I_\Delta$. Thus there exists $\delta_2(\varepsilon, T(\varepsilon))$ such that if $|\operatorname{Re} z| \leqslant T(\varepsilon)$,

$$|\rho(z,\delta) - \rho(z,0)| < \varepsilon \tag{7.130}$$

if $\delta < \delta_2(\varepsilon, T(\varepsilon))$. Since $S_\alpha^+ \setminus D(0,r)$ is simply connected and contains no zero of $\rho(z,\delta)$ for any small δ (see (7.124)), the analytic function $\sqrt{\rho(z,\delta)}$ tends to $\sqrt{\rho(z,0)}$ uniformly in $z \in S_\alpha^+ \setminus D(0,r)$, provided we select the suitable branches for the square roots. Our choice is $\sqrt{\rho(t,\delta)}$ and $\sqrt{\rho(t,0)}$ positive if $t < -r$. Consider now

$$\left|\operatorname{Im}\Delta_{12}(z,\delta) - \operatorname{Im}\Delta_{12}(z,0)\right| = \left|\operatorname{Im}\int_0^z \left(\sqrt{\rho(u,\delta)} - \sqrt{\rho(z,0)}\right) \mathrm{d}u\right|. \tag{7.131}$$

Let $z = t + \mathrm{i}s \in S_\alpha^+ \setminus D(0,r)$. If $t \leqslant -r$ we can choose a path of integration going from 0 to $t < -r$ along the real axis and then vertically to $t + \mathrm{i}s$. If $t \geqslant -r$ we take a path from $-r$ to t following the boundary of $D(0,r)$ and the real axis, if necessary, and then a vertical path to $t + \mathrm{i}s$, see Fig. 7. Along the second path for $t > r$, for example, we have

$$\begin{aligned}\left|\operatorname{Im}\Delta_{12}(z,\delta) - \operatorname{Im}\Delta_{12}(z,0)\right| &\leqslant \pi r \sup_{\theta\in[0,\pi]} \left|\sqrt{\rho(r\exp\{\mathrm{i}\theta\},\delta)} - \sqrt{\rho(r\exp\{\mathrm{i}\theta\},0)}\right| \\ &\quad + \alpha \sup_{|s|\leqslant\alpha} \left|\sqrt{\rho(t+\mathrm{i}s,\delta)} - \sqrt{\rho(t+\mathrm{i}s,0)}\right|\end{aligned} \tag{7.132}$$

where the second member tends to zero uniformly in $z = t + \mathrm{i}s$ as δ tends to zero. The result is the same when $t \leqslant r$. □

As a consequence of this lemma we can assume that δ is small enough so that we have

$$\left|\rho(z,\delta) - \rho(z,0)\right| \leqslant \frac{R}{3}, \quad \forall z \in S_\alpha \setminus D(0,r), \tag{7.133}$$

where $0 < R = \inf_{z\in S_\alpha \setminus D(0,r)} \rho(z,0)$ and

$$\left|\operatorname{Im}\Delta_{12}(z,\delta) - \operatorname{Im}\Delta_{12}(z,0)\right| < \frac{v}{3}, \quad \forall z \in S_\alpha^+ \setminus D(0,r). \tag{7.134}$$

Hence the level line

$$\operatorname{Im}\Delta_{12}(z,\delta) = \operatorname{Im}\Delta_{12}(z_1,\delta) \tag{7.135}$$

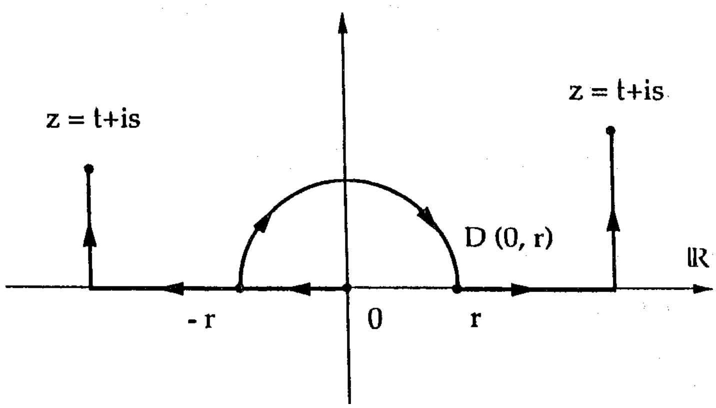

Fig. 7. Particular integration paths.

cannot cross the level lines

$$\operatorname{Im} \Delta_{12}(z,0) = -\chi \pm v \tag{7.136}$$

since this would imply

$$\left| \operatorname{Im} \Delta_{12}(z_1,\delta) - \operatorname{Im} \Delta_{12}(z_1,0) \right| = v > \frac{v}{3}. \tag{7.137}$$

Moreover, the line

$$\operatorname{Im} \Delta_{12}(z,\delta) = \operatorname{Im} \Delta_{12}(z_1,\delta) \tag{7.138}$$

cannot cross the segment

$$\{z\colon \operatorname{Re} \Delta_{12}(z,0) = \operatorname{Re} \Delta_{12}(z_1,0),\ \left| \operatorname{Im} \Delta_{12}(z,0) + \chi \right| \leqslant v\} \tag{7.139}$$

if δ is small, except at $z = z_1$. Indeed, for δ small enough $\Delta'_{12}(z_1,\delta) \neq 0$, so that $\Delta_{12}(z,\delta)$ is bijective in a δ-independent neighbourhood V of z_1. Moreover $\Delta_{12}(z,\delta)$ tends to $\Delta_{12}(z,0)$ which has the same property in V so that we can conclude. Note that a level line

$$\operatorname{Im} \Delta_{12}(z,\delta) = \text{cst} \tag{7.140}$$

is given by the solution $\gamma(t)$ of the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \Delta_{12}(\gamma(t),\delta) = 0, \tag{7.141}$$

i.e.,

$$\begin{aligned} \operatorname{Re} \dot\gamma(t) &= \operatorname{Re} \sqrt{\rho(\gamma(t),\delta)}, \\ \operatorname{Im} \dot\gamma(t) &= -\operatorname{Im} \sqrt{\rho(\gamma(t),\delta)}. \end{aligned} \tag{7.142}$$

Thus

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re} \Delta_{12}(\gamma(t),\delta) \right| = \left| \rho(\gamma(t),\delta) \right| > R > 0 \tag{7.143}$$

which implies that $|\operatorname{Re} \Delta_{12}(\gamma(t),\delta)|$ is strictly increasing along $\gamma(t)$. Hence the level line

$$\operatorname{Im} \Delta_{12}(z,\delta) = \operatorname{Im} \Delta_{12}(z_1,\delta)$$

leads from z_1 to $-\infty$ in $S_-(v)$. Moreover, $|\dot\gamma(t)| = |\sqrt{\rho(\gamma(t),\delta)}|$ is uniformly bounded in δ. Finally, we have along $\gamma_0(t)$ for $t \in [t_1, t_2]$

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \Delta_{12}(\gamma_0(t),\delta) = -\left(\operatorname{Re} \dot\gamma_0(t) \operatorname{Im} \sqrt{\rho(\gamma_0(t),\delta)} + \operatorname{Im} \dot\gamma_0(t) \operatorname{Re} \sqrt{\rho(\gamma_0(t),\delta)} \right) \tag{7.144}$$

which is strictly greater than zero if δ is sufficiently small, since $\sqrt{\rho(z,\delta)} \to \sqrt{\rho(z,0)}$ and by construction

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \Delta_{12}(\gamma_0(t),0) > d > 0 \quad \text{(see (7.116).}$$

Hence, the path γ_δ defined by

$$\gamma_\delta = \begin{cases} \operatorname{Im} \Delta_{12}(z,\delta) = \operatorname{Im} \Delta_{12}(z_1,\delta) & \text{from } -\infty \text{ to } z_1, \\ \gamma_0 & \text{from } z_1 \text{ to } z_2, \\ \operatorname{Im} \Delta_{12}(z,\delta) = \operatorname{Im} \Delta_{12}(z_2,\delta) & \text{from } z_2 \text{ to } +\infty \end{cases} \tag{7.145}$$

is dissipative for $\Delta_{12}(z,\delta)$ and has all the properties announced in the proposition. □

This completes the proof of Theorem 2.1 as well. □

A. Technicalities

Let us introduce different norms. Let $\varphi \in D$. We define for $z \in S_\alpha$ and $\delta \in I_\Delta$

$$\|\varphi\|_{z,\delta} = \|\varphi\| + \|H(z,\delta)\varphi\|, \qquad \|\varphi\|_{\pm,\delta} = \|\varphi\| + \|H^{\pm}(\delta)\varphi\|. \tag{A.1}$$

The domain D equipped with any of these norms is a Banach space we shall denote by $X_{z,\delta}$, respectively, $X_{\pm,\delta}$. By the closed graph theorem we have for any $z, z' \in S_\alpha$ and $\delta, \delta' \in I_\Delta$

$$H(z,\delta) \in \mathcal{B}(X_{z',\delta'}, \mathcal{H}), \tag{A.2}$$

the set of bounded linear operators from $X_{z',\delta'}$ to $\mathcal{H}$. Similarly

$$H(z,\delta) \in \mathcal{B}(X_{\pm,\delta'}, \mathcal{H}), \quad H^{\pm}(\delta) \in \mathcal{B}(X_{z',\delta'}, \mathcal{H}), \quad H^{\pm}(\delta) \in \mathcal{B}(X_{\pm,\delta'}, \mathcal{H}). \tag{A.3}$$

We denote the norms in these spaces of bounded operators by

$$|||\cdot|||_{z',\delta'} \quad \text{and} \quad |||\cdot|||_{\pm,\delta'}. \tag{A.4}$$

The norms in $X_{z,\delta}$ are related by

$$\|\varphi\|_{z,\delta} \leqslant \big(1 + |||H(z,\delta)|||_{z',\delta'}\big)\|\varphi\|_{z',\delta'} \tag{A.5}$$

where z or z' can also be replaced by $+$ or $-$.

Lemma A.1. *Under the assumptions* I *and* II, *there exists a constant M, independent of $z, z' \in S_\alpha$ and $\delta, \delta' \in I_\Delta$ such that*

$$\max\big(|||H(z,\delta)|||_{z',\delta'}, |||H(z,\delta)|||_{\pm,\delta'}, |||H^{\pm}(\delta)|||_{z',\delta'}, |||H^{\pm}(\delta)|||_{\pm,\delta'}\big) < M$$

and there exists an integrable decay function $b(t)$ and a positive constant B, both uniform in δ, such that for all $\varphi \in D$

$$\left\|\frac{\partial}{\partial z}H(z,\delta)\varphi\right\| \equiv \|H'(z,\delta)\varphi\| \leqslant b(t)\|\varphi\|_{z',\delta'},$$

$$\left\|\frac{\partial}{\partial \delta}H(z,\delta)\varphi\right\| \leqslant B\|\varphi\|_{z',\delta'},$$

$$\left\|\frac{\partial}{\partial \delta}H^{\pm}(\delta)\varphi\right\| \leqslant B\|\varphi\|_{z',\delta'}$$

for any $z = t + \mathrm{i}s$, $z' \in S_\alpha$ and $\delta, \delta' \in I_\Delta$.

The proof of this lemma is given at the end of the appendix.

It follows from Lemma A.1 that

$$\begin{aligned}\|(H(z,\delta)-H(t,0))\varphi\| &\leqslant \|(H(z,\delta)-H(t,\delta))\varphi\|+\|(H(t,\delta)-H(t,0))\varphi\| \\ &\leqslant (|z-t|b(t)+\delta B)\|\varphi\|_{t,0}\end{aligned} \tag{A.6}$$

so that, for $\lambda \in T(t,0)$,

$$\begin{aligned}&\|(H(z,\delta)-H(t,0))R(t,0,\lambda)\| \\ &\quad\leqslant (|z-t|b(t)+\delta B)(\|R(t,0,\lambda)\|+\|H(t,0)R(t,0,\lambda)\|) \\ &\quad\equiv (|z-t|b(t)+\delta B)d(t,\lambda).\end{aligned} \tag{A.7}$$

Now, if $(|z-t|b(t)+\delta B)d(t,\lambda)<1$ we have the identity

$$R(z,\delta,\lambda)-R(t,0,\lambda)=-R(z,\delta,\lambda)(H(z,\delta)-H(t,0))R(t,0,\lambda) \tag{A.8}$$

hence $\lambda \in T(z,\delta)$ as well and

$$\|R(z,\delta,\lambda)\| \leqslant \frac{\|R(t,0,\lambda)\|}{1-(|z-t|b(t)+\delta B)d(t,\lambda)}, \tag{A.9}$$

$$\|R(z,\delta,\lambda)-R(t,0,\lambda)\| \leqslant \|R(t,0,\lambda)\|\frac{(|z-t|b(t)+\delta B)d(t,\lambda)}{1-(|z-t|b(t)+\delta B)d(t,\lambda)}. \tag{A.10}$$

Similarly, if $|\operatorname{Re} z| \gg 1$, we use condition II, Lemma A.1 and (A.5) to write

$$\begin{aligned}\|(H(z,\delta)-H^{\pm}(0))\varphi\| &\leqslant \|(H(z,\delta)-H^{\pm}(\delta))\varphi\|+\|(H^{\pm}(\delta)-H^{\pm}(0))\varphi\| \\ &\leqslant b(t)\|\varphi\|_{\pm,\delta}+\delta B\|\varphi\|_{0,0} \\ &\leqslant (1+M)(b(t)+\delta B)\|\varphi\|_{\pm,0}\end{aligned} \tag{A.11}$$

where $z=t+\mathrm{i}s$. Thus if $\lambda \in T(\pm,0)$ and $(1+M)(b(t)+\delta B)d(\pm,\lambda)<1$ where

$$d(\pm,\lambda)=\|R(\pm,0,\lambda)\|+\|H^{\pm}(0)R(\pm,0,\lambda)\|, \tag{A.12}$$

then $\lambda \in T(z,\delta)$ as well and

$$\|R(z,\delta,\lambda)\| \leqslant \frac{\|R(\pm,0,\lambda)\|}{1-(1+M)(b(t)+\delta B)d(\pm,\lambda)}, \tag{A.13}$$

$$\|R(z,\delta,\lambda)-R(\pm,0,\lambda)\| \leqslant \|R(\pm,0,\lambda)\|\frac{(1+M)(b(t)+\delta B)d(\pm,\lambda)}{1-(1+M)(b(t)+\delta B)d(\pm,\lambda)}. \tag{A.14}$$

Proof of Lemma 3.1. From (A.14), there exist $T > 0$ and Γ_+, $\Gamma_- \in S_\alpha$ such that

$$Q(t,0) = -\frac{1}{2\pi i}\oint_{\Gamma_\pm} R(t,0,\lambda)\,d\lambda \tag{A.15}$$

if $t \gtrless \pm T$. Then from (A.10), if $t \in [-T,T]$, the path used in (3.2) can be chosen locally independently of t. Thus by compactness of $[-T,T]$, we can define $Q(t,0)$ for any $t \in \mathbb{R}$ by choosing Γ among a finite set $\{\Gamma_j\colon\ j = 1,\ldots,n\}$ with $\Gamma_1 = \Gamma_-$ and $\Gamma_n = \Gamma_+$. The length of these paths is bounded by $|\Gamma|$ and they satisfy $\mathrm{dist}[\Gamma_j, \sigma(t)] \geqslant \eta > 0$. As a consequence, if $\lambda \in \Gamma_j \in T(t,0)$ for some time t,

$$d(t,\lambda) \leqslant 1 + \frac{1+|\lambda|}{\eta} \leqslant K < \infty, \quad \forall t \in \mathbb{R}. \tag{A.16}$$

Then by choosing α and Δ so small that

$$\Big(\alpha \sup_{t\in\mathbb{R}} b(t) + \Delta B\Big) K < 1 \tag{A.17}$$

and

$$(1+M)\big(b(T) + \Delta B\big) K < 1 \tag{A.18}$$

for T large, we have that $\Gamma_j \subset T(t+is,\delta)$, respectively, $\Gamma_\pm \subset T(t+is,\delta)$, as well so that the spectrum is still separated in two pieces. □

It is now possible to give the

Proof of Lemma 4.1. Let $t \in \mathbb{R}$ and let $\Gamma_t \in \{\Gamma_j\colon\ j = 1,\ldots,n\}$ such that $\Gamma_t \in T(t,0)$. By the choice (A.17) of α and Δ, we have that $\Gamma_t \in T(z,\delta)$ $\forall \delta \in I_\Delta$ and $\forall z \in D(t,r)$, provided $r < \alpha$. Thus it follows from (A.9) and (A.13) that there exist an N such that

$$\sup_{t\in\mathbb{R}}\ \sup_{\substack{z\in D(t,r)\\ \delta\in I_\Delta}}\ \sup_{\lambda\in\Gamma_t} \big\|R(z,\delta,\lambda)\big\| \leqslant N. \tag{A.19}$$

□

For later purposes we define

$$d(t,\delta,\lambda) = \big\|R(t,\delta,\lambda)\big\| + \big\|H(t,\delta)R(t,\delta,\lambda)\big\|. \tag{A.20}$$

It follows from the foregoing that for any $\lambda \in \Gamma_j \subset T(t,\delta)$

$$d(t,\delta,\lambda) \leqslant \overline{K} < \infty, \tag{A.21}$$

where $\overline{K}$ is independent of δ, λ and t. We assume that α and Δ are so small that the preceding lemmas hold.

Proof of Lemma 3.2. For a fixed $z \in S_\alpha$ and $\lambda \in T(z,\delta)$ we have the strong derivative (see, e.g., [20, Chapter II, Paragraph 1]).

$$\frac{\partial}{\partial\delta} R(z,\delta,\lambda) = -R(z,\delta,\lambda)\frac{\partial}{\partial\delta} H(z,\delta)R(0,0,\mathrm{i})\big(H(0,0)-\mathrm{i}\big)R(z,\delta,\lambda) \tag{A.22}$$

where the bounded operators $R(z,\delta,\lambda)$, $(\partial/\partial\delta)H(z,\delta)R(0,0,\mathrm{i})$ and $(H(0,0)-\mathrm{i})R(z,\delta,\lambda)$ are strongly continuous in z and δ. Indeed, this is easily seen for $R(z,\delta,\lambda)$ by considering identities analogous to (A.10) and this is true by hypothesis for $(\partial/\partial\delta)H(z,\delta)R(0,0,\mathrm{i})$. Finally, $(H(0,0)-\mathrm{i})R(z,\delta,\lambda)$ is the inverse of the bounded operator $(H(z,\delta)-\lambda)R(0,0,\mathrm{i})$ which is continuous in norm, as can be seen from estimates of the type (A.6). Thus, by Lemma 3.7 of the introduction of [20], $(H(0,0)-\mathrm{i})R(z,\delta,\lambda)$ is bounded and even continuous in norm. Hence the strong continuity of $(\partial/\partial\delta)R(z,\delta,\lambda)$ and of $(\partial/\partial z)R(z,\delta,\lambda)$, by similar considerations. These properties are true for the projector $Q(z,\delta)$ as well by passing the derivatives under the integral of the formula in Lemma 3.1. We now turn to the second part of the lemma. Consider the identity

$$R(z,\delta,\lambda) - R(\pm,\delta,\lambda) = -R(z,\delta,\lambda)\big(H(z,\delta) - H^{\pm}(\delta)\big)R(\pm,\delta,\lambda). \tag{A.23}$$

With condition II and Lemma A.1 we obtain

$$\big\|R(z,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \leqslant \big\|R(\pm,\delta,\lambda)\big\| \frac{b(t)d(\pm,\delta,\lambda)}{1-b(t)d(\pm,\delta,\lambda)} \tag{A.24}$$

for $z = t + \mathrm{i}s$, $|t|$ large with the definition

$$d(\pm,\delta,\lambda) = \big\|R(\pm,\delta,\lambda)\big\| + \big\|H^{\pm}(\delta)R(\pm,\delta,\lambda)\big\|. \tag{A.25}$$

This defines the integrable decay function $b_{\lambda,\delta}(t)$. The estimate on the derivatives are consequences of the Cauchy formula. We define the projector $Q(z,\delta)$ by using the finite set of paths Γ_j, introduced in the proof of Lemma 3.1. If $\lambda \in \Gamma_j$ we obtain from (A.24) and (A.21)

$$\big\|R(z,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \leqslant \frac{\overline{K}^2 b(t)}{1-b(t)\overline{K}} \leqslant K' b(t) \tag{A.26}$$

if $|t|$ is large enough. This estimate and the Cauchy formula finish the proof of the lemma. □

Proof of Lemma A.1. By definition

$$|||H(z,\delta)|||_{z',\delta'} = \sup_{\varphi\in D} \frac{\|H(z,\delta)\varphi\|}{\|\varphi\|_{z',\delta'}}. \tag{A.27}$$

We first show that

$$|||H(z,\delta)|||_{z',\delta'} \leqslant M(z',\delta'), \tag{A.28}$$

where $M(z',\delta')$ is independent of z and δ. As $H(z,\delta)$ is strongly C^1 in $\mathcal{B}(X_{z',\delta'},\mathcal{H})$, $\|H(z,\delta)\varphi\|$ is continuous in $(z,\delta) \in S_\alpha \times I_\Delta$, so that

$$\big\|H(z,\delta)\varphi\big\| \leqslant M_1(\varphi), \quad \forall (z,\delta) \in \omega \times I_\Delta, \tag{A.29}$$

where $\omega = \{z \in S_\alpha\colon |\operatorname{Re} z| \leqslant T\}$ is compact. By applying the uniform boundedness principle [20] we obtain the estimate

$$\|H(z,\delta)\varphi\| \leqslant M_1(z',\delta')\|\varphi\|_{z',\delta'}, \quad \forall (z,\delta) \in \omega \times I_\Delta. \tag{A.30}$$

Suppose z does not belong to ω. Then by condition II and by the uniform boundedness principle again we have

$$\begin{aligned}\|H(z,\delta)\varphi\| &\leqslant b(t)\big(\|\varphi\| + \|H^\pm(\delta)\varphi\|\big) + \|H^\pm(\delta)\varphi\| \\ &\leqslant \sup_{t\in\mathbb{R}} b(t)\big(\|\varphi\| + M^\pm(\varphi)\big) + M^\pm(\varphi) \leqslant M_2(\varphi)\end{aligned} \tag{A.31}$$

for some $M_2(\varphi)$. Here we have used the compactness of I_Δ. As a consequence, there exists $M_2(z',\delta')$ such that

$$\|H(z,\delta)\varphi\| \leqslant M_2(z',\delta')\|\varphi\|_{z',\delta'} \tag{A.32}$$

and it remains to take $M(z',\delta') = \max(M_1(z',\delta'), M_2(z',\delta'))$ to obtain (A.28). Note that z' can be replaced by $+$ or $-$ in (A.28). Using the Cauchy formula, we immediately get

$$\|H'(z,\delta)\varphi\| \leqslant N(z',\delta')\|\varphi\|_{z',\delta'} \tag{A.33}$$

so that

$$\big\|\big(H(z_1,\delta) - H(z_2,\delta)\big)\varphi\big\| \leqslant |z_1 - z_2| N(z',\delta')\|\varphi\|_{z',\delta'} \tag{A.34}$$

for any z_1, z_2 in a convex subset of S_α. We need a similar estimate for the variations of δ. By assumption, $(\partial/\partial\delta)H(z,\delta)\varphi$ is continuous in $(z,\delta) \in S_\alpha \times I_\Delta$ and we show that

$$(\partial/\partial\delta)H(z,\delta)R(z',\delta',\lambda)$$

is bounded as an operator from $\mathcal{H}$ to $\mathcal{H}$, if $\lambda \in T(z',\delta')$. Indeed, by the closed graph theorem $H(z,\delta)R(z',\delta',\lambda)$ is bounded and strongly continuously differentiable in δ, so that Banach–Steinhaus theorem [20] implies that $(\partial/\partial\delta)H(z,\delta)R(z',\delta',\lambda)$ is bounded as well. Thus we have

$$\begin{aligned}\left\|\frac{\partial}{\partial\delta} H(z,\delta)\varphi\right\| &= \left\|\frac{\partial}{\partial\delta}H(z,\delta)R(z',\delta',\lambda)\big(H(z',\delta') - \lambda\big)\varphi\right\| \\ &\leqslant \left\|\frac{\partial}{\partial\delta} H(z,\delta)R(z',\delta',\lambda)\right\|\big(\|H(z',\delta')\varphi\| + |\lambda|\|\varphi\|\big) \\ &\leqslant \left\|\frac{\partial}{\partial\delta} H(z,\delta)R(z',\delta',\lambda)\right\|\big(1 + |\lambda|\big)\|\varphi\|_{z',\delta'}\end{aligned} \tag{A.35}$$

so that

$$\frac{\partial}{\partial\delta} H(z,\delta) \in \mathcal{B}(X_{z',\delta'}, \mathcal{H}). \tag{A.36}$$

Then, by condition II and the uniform boundedness principle again, there exists $\widetilde{N}(z',\delta')$ such that

$$\left\| \frac{\partial}{\partial\delta} H(z,\delta)\varphi \right\| \leqslant \widetilde{N}(z',\delta')\|\varphi\|_{z',\delta'} \tag{A.37}$$

which implies

$$\|(H(z,\delta_1) - H(z,\delta_2))\varphi\| \leqslant |\delta_1 - \delta_2|\widetilde{N}(z',\delta')\|\varphi\|_{z',\delta'}. \tag{A.38}$$

From (A.34) and (A.38) the estimate

$$\|(H(z_1,\delta_1) - H(z_2,\delta_2))\varphi\| \leqslant (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z',\delta')\|\varphi\|_{z',\delta'} \tag{A.39}$$

follows, where $C(z',\delta') = \max(N(z',\delta'), \widetilde{N}(z',\delta'))$. Putting $(z',\delta') = (z_1,\delta_1)$ we get

$$\begin{aligned} \|\varphi\|_{z_2,\delta_2} &\leqslant \|\varphi\|_{z_1,\delta_1} + (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)\|\varphi\|_{z_1,\delta_1}, \\ \|\varphi\|_{z_1,\delta_1} &\leqslant \|\varphi\|_{z_2,\delta_2} + (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)\|\varphi\|_{z_1,\delta_1} \end{aligned} \tag{A.40}$$

hence

$$\begin{aligned} &\frac{\|\varphi\|_{z_2,\delta_2}}{1 + (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)} \\ &\quad \leqslant \|\varphi\|_{z_1,\delta_1} \leqslant \frac{\|\varphi\|_{z_2,\delta_2}}{1 - (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)}. \end{aligned} \tag{A.41}$$

These relations show that the application $|||\cdot|||_{z,\delta}$ is continuous in (z,δ). Let A belong to $\mathcal{B}(X_{z,\delta},\mathcal{H})$.

$$\begin{aligned} \|A\varphi\| &\leqslant |||A|||_{z_2,\delta_2}\|\varphi\|_{z_2,\delta_2} \\ &\leqslant |||A|||_{z_2,\delta_2}(1 + (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1))\|\varphi\|_{z_1,\delta_1} \end{aligned} \tag{A.42}$$

so that

$$|||A|||_{z_1,\delta_1} \leqslant |||A|||_{z_2,\delta_2}(1 + (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)). \tag{A.43}$$

Similarly

$$|||A|||_{z_2,\delta_2} \leqslant |||A|||_{z_1,\delta_1}\frac{1}{1 - (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)} \tag{A.44}$$

so that

$$\big||||A|||_{z_2,\delta_2} - |||A|||_{z_1,\delta_1}\big| \leqslant |||A|||_{z_1,\delta_1}\frac{(|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)}{1 - (|z_1 - z_2| + |\delta_1 - \delta_2|)C(z_1,\delta_1)} \tag{A.45}$$

which tends to zero as $(z_2, \delta_2) \to (z_1, \delta_1)$. Finally, from (A.45), (A.28) and (A.39) follows the estimate

$$\begin{aligned}
\Big| |||H(z_1,\delta_1)|||_{z_1',\delta_1'} - |||H(z_2,\delta_2)|||_{z_2',\delta_2'} \Big| &\leqslant \Big| |||H(z_2,\delta_2)|||_{z_1',\delta_1'} - |||H(z_2,\delta_2)|||_{z_2',\delta_2'} \Big| \\
&+ \Big| |||H(z_1,\delta_1)|||_{z_1',\delta_1'} - |||H(z_2,\delta_2)|||_{z_1',\delta_1'} \Big| \\
&\leqslant |||H(z_2,\delta_2)|||_{z_1',\delta_1'} \frac{(|z_2'-z_1'| + |\delta_2'-\delta_1'|)C(z_1',\delta_1')}{1-(|z_2'-z_1'| + |\delta_2'-\delta_1'|)C(z_1',\delta_1')} \\
&+ |||H(z_1,\delta_1) - H(z_2,\delta_2)|||_{z_1',\delta_1'} \\
&\leqslant M(z_1',\delta_1') \frac{(|z_2'-z_1'| + |\delta_2'-\delta_1'|)C(z_1',\delta_1')}{1-(|z_2'-z_1'| + |\delta_2'-\delta_1'|)C(z_1',\delta_1')} \\
&+ \big(|z_2-z_1| + |\delta_2-\delta_1|\big)C(z_1',\delta_1').
\end{aligned} \tag{A.46}$$

The last term in this inequality tends to zero as $(z_2, \delta_2) \to (z_1, \delta_1)$ and $(z_2', \delta_2') \to (z_1', \delta_1')$ which shows that the application $|||H(z,\delta)|||_{z',\delta'}$ in continuous as a function of (z, z', δ, δ'). Thus, on the compact set $\omega^2 \times I_\Delta^2$ we have

$$\sup_{(z,z',\delta,\delta') \in \omega^2 \times I_\Delta^2} |||H(z,\delta)|||_{z',\delta'} \leqslant M' \tag{A.47}$$

It remains to control this application when $|z|, |z'| \to \infty$ in S_α. If $|\,\mathrm{Re}\,| \geqslant T$, T large, we have by condition II

$$\|\varphi\|_{z,\delta} \leqslant \big(1 + b(\pm T)\big) \|\varphi\|_{\pm,\delta} \leqslant K_1 \|\varphi\|_{\pm,\delta} \tag{A.48}$$

for K_1 some constant, and similarly

$$\|\varphi\|_{\pm,\delta} \leqslant \frac{\|\varphi\|_{z,\delta}}{1 - b(\pm T)} \leqslant K_2 \|\varphi\|_{z,\delta}. \tag{A.49}$$

Moreover, from (A.38)

$$\begin{aligned}
\|\varphi\|_{\pm T,\delta} &\leqslant \big(1 + \Delta \widetilde{N}(\pm T, 0)\big) \|\varphi\|_{\pm T,0} \leqslant K_3 \|\varphi\|_{\pm T,0} \\
\|\varphi\|_{\pm T,\delta} &\leqslant \frac{\|\varphi\|_{\pm T,0}}{(1 - \Delta \widetilde{N}(\pm T, 0))} \leqslant K_4 \|\varphi\|_{\pm T,0}
\end{aligned} \tag{A.50}$$

provided Δ is small enough. Thus if $|\,\mathrm{Re}\, z| \geqslant T$, $|\,\mathrm{Re}\, z'| \leqslant T$ we can write with (A.47)

$$\begin{aligned}
\|H(z,\delta)\varphi\| &\leqslant \big(1 + b(\pm T)\big) \|\varphi\|_{\pm,\delta} \leqslant \big(1 + b(\pm T)\big) K_2 \|\varphi\|_{\pm T,\delta} \\
&\leqslant \big(1 + b(\pm T)\big) K_2 \big(1 + M'\big) \|\varphi\|_{\pm T',\delta'}
\end{aligned} \tag{A.51}$$

showing that

$$|||H(z,\delta)|||_{z',\delta'} \leqslant \big(1+b(\pm T)\big)K_2\big(1+M'\big). \tag{A.52}$$

Now if $|\operatorname{Re} z'| \geqslant T$ and $|\operatorname{Re} z| \leqslant T$

$$\|H(z,\delta)\varphi\| \leqslant M'\|\varphi\|_{\pm T,\delta'} \leqslant M'K_1\|\varphi\|_{\pm,\delta'} \leqslant M'K_1K_2\|\varphi\|_{z',\delta'} \tag{A.53}$$

from which follows

$$|||H(z,\delta)|||_{z',\delta'} \leqslant M'K_1K_2. \tag{A.54}$$

Finally if both $|\operatorname{Re} z|$ and $|\operatorname{Re} z'|$ are $\geqslant T$, we use (A.50) as well to get

$$\begin{aligned}\|H(z,\delta)\varphi\| &\leqslant \big(1+b(\pm T)\big)K_2\|\varphi\|_{\pm T,\delta} \leqslant \big(1+b(\pm T)\big)K_2K_3K_4\|\varphi\|_{\pm T,\delta'}\\ &\leqslant \big(1+b(\pm T)\big)K_2K_3K_4K_1K_2\|\varphi\|_{z',\delta'}.\end{aligned} \tag{A.55}$$

Gathering these estimates, we eventually obtain

$$\sup_{(z,z',\delta,\delta')\in S_a^2\times I_\Delta^2} |||H(z,\delta)|||_{z',\delta'} \leqslant M < \infty. \tag{A.56}$$

The result is true if z, or z' or both are replaced either by $+$ or $-\infty$.

The last assertions of the lemma are proven as follows. By condition II and (A.56) we have for $z = t + \mathrm{i}s$ and any (z',δ')

$$\big\|\big(H(z,\delta) - H^{\pm}(\delta)\big)\varphi\big\| \leqslant b(t)\|\varphi\|_{\pm,\delta} \leqslant b(t)(1+M)\|\varphi\|_{z',\delta'} \tag{A.57}$$

and we conclude by the Cauchy formula that

$$\big\|H'(z,\delta)\varphi\big\| \leqslant \bar{b}(t)\|\varphi\|_{z',\delta'} \tag{A.58}$$

for any $z = t + \mathrm{i}s$ with $|s| < r < a$, where $\bar{b}(t)$ is another integrable decay function. Similarly, (A.37) implies

$$\left\|\frac{\partial}{\partial\delta} H(z,\delta)\varphi\right\| \leqslant \widetilde{N}(0,0)\|\varphi\|_{0,0} \tag{A.59}$$

so that by (A.56) again

$$\left\|\frac{\partial}{\partial\delta} H(z,\delta)\varphi\right\| \leqslant \widetilde{N}(0,0)M\|\varphi\|_{z',\delta'}. \tag{A.60}$$

This finishes the proof of the lemma. □

B. Proof of Lemma 7.3

We consider here the coefficients $a^*_{kj}(z,\delta)$ defined by (7.8). In order to deal with this term we introduce inside $Q_{N^*}(0,\delta)\mathcal{H}$ the evolution $\widehat{W}_*'(z,\delta)$ by

$$\mathrm{i}\widehat{W}_*'(z,\delta) = \widehat{K}_*(z,\delta)\widehat{W}_*(z,\delta), \qquad \widehat{W}_*(0,\delta) = Q_{N^*}(0,\delta) \equiv \mathbb{I}_{Q_{N^*}(0,\delta)\mathcal{H}}, \tag{B.1}$$

with

$$\widehat{K_*}(z,\delta) = Q_{N^*}(0,\delta)\mathrm{i}\left[\widehat{P_1^*}'(z,\delta), \widehat{P_1^*}(z,\delta)\right]Q_{N^*}(0,\delta). \tag{B.2}$$

As above we can write

$$\widehat{\varphi_j^*}(z,\delta) = \widehat{W_*}(z,\delta)\widehat{\varphi_j^*}(0,\delta) \tag{B.3}$$

where $\widehat{\varphi_j^*}(0,\delta)$, $j = 1,2$, is a pair of normalized eigenvectors of $\widehat{H_*}(0,\delta)Q_{N^*}(0,\delta)$ so that the coefficients $a_{kj}^*(z,\delta)$ take the form

$$a_{kj}^*(z,\delta) = \mathrm{i}\left\langle \varphi_k^*(0,\delta)\middle|\widehat{W_*}^{-1}(z,\delta)\widehat{K_*}(z,\delta)\widehat{W_*}(z,\delta)\varphi_j^*(0,\delta)\right\rangle. \tag{B.4}$$

As

$$\widehat{P_j^*}(z,\delta) = W_{N^*}^{-1}(z,\delta)P_j^{N^*}(z,\delta)W_{N^*}(z,\delta) \tag{B.5}$$

we have

$$\begin{aligned}\widehat{P_j^*}'(z,\delta) &= W_{N^*}^{-1}(z,\delta)P_j^{N^*\prime}(z,\delta)W_{N^*}(z,\delta)\\ &\quad + \frac{1}{\mathrm{i}}W_{N^*}^{-1}(z,\delta)\left[P_j^{N^*}(z,\delta), K_{N^*}(z,\delta)\right]W_{N^*}(z,\delta).\end{aligned} \tag{B.6}$$

The operators are restricted to $Q_{N^*}(0,\delta)\mathcal{H}$, so that the last term vanishes,

$$\begin{aligned}&Q_{N^*}(0,\delta)W_{N^*}^{-1}(z,\delta)\left[P_j^{N^*}(z,\delta), K_{N^*}(z,\delta)\right]W_{N^*}(z,\delta)Q_{N^*}(0,\delta)\\ &\quad \equiv W_{N^*}^{-1}(z,\delta)\left[P_j^{N^*}(z,\delta), Q_{N^*}(z,\delta)K_{N^*}(z,\delta)Q_{N^*}(z,\delta)\right]W_{N^*}(z,\delta) \equiv 0,\end{aligned} \tag{B.7}$$

and we obtain using (5.6)

$$\begin{aligned}&\left\|\widehat{P_j^*}(z,\delta)\right\| \leqslant w^2\left\|P_j^{N^*}(z,\delta)\right\|\\ &\left\|\widehat{K_*}(z,\delta)\right\| \leqslant 4w^2\left\|P_1^{N^*}(z,\delta)\right\|\,\left\|P_1^{N^*\prime}(z,\delta)\right\|, \quad z \neq z_0(\delta),\end{aligned} \tag{B.8}$$

where w is independent of z, δ and ε. It remains to show that $P_1^{N^*}(z,\delta)$ is uniformly bounded in ε and δ along the dissipative path $\gamma_\delta(t)$. By construction (Proposition 7.2), $\gamma_\delta \in S_\alpha^+ \setminus D(0,r)$ if r is small enough, where there is no eigenvalue crossing point of $e_j^*(z,\delta)$ and $e_j(z,\delta)$. Hence for any $z \in S_\alpha^+ \setminus D(0,r)$ the projections $P_j^{N^*}(z,\delta)$ are given by means of a Riesz formula

$$P_j^{N^*}(z,\delta) = -\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_j} R_{N^*}(z,\delta,\lambda)\,\mathrm{d}\lambda \tag{B.9}$$

where γ_j, encircles both $e_j^*(z,\delta)$ and $e_j(z,\delta)$, a finite distance away from the spectra of $H_{N^*}(z,\delta)$ and $H(z,\delta)$. By an argument similar to the one given in Lemma 3.1, we see that we can pick γ_j, for any $z \in S_\alpha^+ \setminus D(0,r)$, among a finite set of paths which are all of finite length and bounded away from the spectra of $H_{N^*}(z,\delta)$ and $H(z,\delta)$. Then in view of

$$R_{N^*}(z,\delta,\lambda) - R(z,\delta,\lambda) = -R_{N^*}(z,\delta,\lambda)\big(H_{N^*}(z,\delta) - H(z,\delta)\big)R(z,\delta,\lambda) \tag{B.10}$$

and Proposition 4.1 we obtain the estimate

$$\big\|R_{N^*}(z,\delta,\lambda)\big\| \leqslant \big\|R(z,\delta,\lambda)\big\| \frac{1}{1-\varepsilon\,\frac{\mathrm{e}}{\mathrm{e}-1}\,b(t)} \tag{B.11}$$

provided ε is small enough, and $\lambda \in T(z,\delta)$. By the continuity in norm of $R(z,\delta,\lambda)$ in δ, there exists δ^* such that $\|R(z,\delta,\lambda)\|$ is uniformly bounded in $\delta < \delta^*$ if $\lambda \in \gamma_j$, and $z \in S_\alpha^+ \setminus D(0,r)$. When $z = t + is$, with $|t|$ large, we use (A.24),

$$\big\|R(z,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \leqslant \big\|R(\pm,\delta,\lambda)\big\| \frac{b(t)d(\pm,\delta,\lambda)}{1-b(t)d(\pm,\delta,\lambda)} \tag{B.12}$$

and the fact that if $\lambda \in \gamma_j \subset T(\pm,0)$, there exists a constant $\overline{k}$ independent of δ such that

$$d(\pm,\delta,\lambda) \leqslant \overline{k}. \tag{B.13}$$

Hence the estimate

$$\big\|R(z,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \leqslant \overline{k}\,\frac{b(t)\overline{k}}{1-b(t)\overline{k}} \leqslant k'b(t). \tag{B.14}$$

As a consequence of (B.10), (B.11), (B.14) and $R(\pm,\delta,\lambda) = R_{N^*}(\pm,\delta,\lambda)$ we have

$$\begin{aligned} &\big\|R_{N^*}(z,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \\ &\quad\leqslant \big\|R_{N^*}(z,\delta,\lambda) - R(z,\delta,\lambda)\big\| + \big\|R(z,\delta,\lambda) - R(\pm,\delta,\lambda)\big\| \\ &\quad\leqslant \big\|R(z,\delta,\lambda)\big\|^2 \frac{\varepsilon\,\frac{\mathrm{e}}{\mathrm{e}-1}\,b(t)}{1-\varepsilon\,\frac{\mathrm{e}}{\mathrm{e}-1}\,b(t)} + k'b(t) \leqslant k''b(t) \end{aligned} \tag{B.15}$$

where k'' is independent of ε and δ. Thus we eventually obtain

$$\big\|\widehat{K}_*(z,\delta)\big\| \leqslant kb(t), \quad \forall z = t + is \in S_\alpha^+ \setminus D(0,r), \tag{B.16}$$

and

$$\big\|\widehat{W}_*(z,\delta)\big\| \leqslant \widehat{w}, \quad \forall \delta \leqslant \delta^*, \tag{B.17}$$

so that

$$\big|a_{kj}^*(z,\delta)\big| \leqslant \widehat{w}^2 kb(t) \tag{B.18}$$

where $b(t)$ is an integrable decay function. □

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