THE LANGEVIN EQUATION FOR A QUANTUM HEAT BATH

Stéphane ATTAL[1] & Alain JOYE[2]

[1] Institut C. Jordan
Université C. Bernard, Lyon 1
21, av Claude Bernard
69622 Villeurbanne Cedex
France

[2] Institut Fourier
Université de Grenoble 1
100, rue des Maths, BP 74
38402 St Martin d'Heres
France

Abstract

We compute the quantum Langevin equation (or quantum stochastic differential equation) representing the action of a quantum heat bath at thermal equilibrium on a simple quantum system. These equations are obtained by taking the continuous limit of the Hamiltonian description for repeated quantum interactions with a sequence of photons at a given density matrix state. In particular we specialise these equations to the case of thermal equilibrium states. In the process, new quantum noises are appearing: thermal quantum noises. We discuss the mathematical properties of these thermal quantum noises. We compute the Lindblad generator associated with the action of the heat bath on the small system. We exhibit the typical Lindblad generator that provides thermalization of a given quantum system.

I. Introduction

The aim of Quantum Open System theory (in mathematics as well as in physics) is to study the interaction of simple quantum systems interacting with very large ones (with infinite degrees of freedom). In general the properties that one is seeking are to exhibit the dissipation of the small system in favor of the large one, to identify when this interaction gives rise to a return to equilibrium or a thermalization of the small system.

There are in general two ways of studying those system, which usually represent distinct groups of researchers (in mathematics as well as in physics).

The first approach is Hamiltonian. The complete quantum system formed by the small system and the reservoir is studied through a Hamiltonian describing the free evolution of each component and the interaction part. The associated unitary group gives rise to a group of *-endomorphisms of a certain von Neumann algebra of observables. Together with a state for the whole system, this constitutes a quantum dynamical system. The aim is then to study the ergodic properties of that quantum dynamical system. This can be performed via the spectral study of a particular generator of the dynamical system: the standard Liouvillian. This is the only generator of the quantum dynamical system which stabilizes the self-dual cone of the associated Tomita-Takesaki modular theory. It has the property to encode in its spectrum the ergodic behavior of the quantum dynamical system. Very satisfactory recent results in that direction were obtained by Jaksic and Pillet ([JP1], [JP2] and [JP3]) who rigorously proved the return to equilibrium for Pauli-Fierz systems, using these techniques.

The second approach is Markovian. In this approach one gives up the idea of modelizing the reservoir and concentrates on the effective dynamics of the small system. This evolution is supposed to be described by a semigroup of completely positive maps. These semigroups are well-known and, under some conditions, admit a generator which is of *Lindblad form*:

$$\mathcal{L}(X) = i[H, X] + \frac{1}{2} \sum_i (2L_i^* X L_i - L_i^* L_i X - X L_i^* L_i).$$

The first order part of $\mathcal{L}$ represents the usual quantum dynamic part, while the second order part of $\mathcal{L}$ carries the dissipation. This form has to be compared with the general form, in classical Markov process theory, of a Feller diffusion generator: a first order differential part which carries the classical dynamics and a second order differential part which represents the diffusion. For classical diffusion, such a semigroup can be realized as resulting of a stochastic differential equation. That is, a perturbation of an ordinary differential equation by classical noise terms such as a Brownian motion usually. In our quantum context, one can add to the small system an adequate Fock space which carries *quantum noises* and show that the effective dynamics we have started with is resulting of a unitary evolution on the coupled system, driven by a quantum Langevin equation. That is, a perturbation of a Schrödinger-type equation by quantum noise terms.

Whatever the approach is, the study of the action of quantum thermal baths is of major importance and has many applications. In the Hamiltonian approach, the model for such a bath is very well-known since Araki-Woods' work ([A-W]). But in the Markovian context, it was not so clear what the correct quantum Langevin equation should be to account for the action of a thermal bath. Some equations have been proposed, in particular by Lindsay and Maassen ([L-M]). But no true physical justification of them has ever been given. Besides, it is not so clear what a "correct" equation should mean?

A recent work of Attal and Pautrat ([AP1]) is a good candidate to answer that problem. Indeed, consider the setup of a quantum system (such as an atom) having repeated interactions, for a short duration τ, with elements of a sequence of identical quantum systems (such as a sequence of photons). The Hamiltonian evolution of such a dynamics can be easily described. It is shown in [AP1] that in the continuous limit ($\tau \to 0$), this Hamiltonian evolution spontaneously converges to a quantum Langevin equation. The coefficient of the equation being easily computable in terms of the original Hamiltonian. This work has two interesting consequences:

– It justifies the Langevin-type equations for they are obtained without any probabilistic assumption, directly from a Hamiltonian evolution;

– It is an effective theorem in the sense that, starting with a naive model for a quantum field (a sequence of photons interacting one after the other with the small system), one obtains explicit quantum Langevin equations which meet all the usual models of the litterature.

It seems thus natural to apply this approach in order to derive the correct quantum Langevin equations for a quantum heat bath. This is the aim of this article.

We consider a simple quantum system in interaction with a toy model for a heat bath. The toy model consists in a chain of independent photons, each of which in the thermal Gibbs state at inverse temperature β, which are interacting one after the other with the small system. Passing to the continuous interaction limit, one should obtain the correct Langevin equation.

One difficulty here is that in [AP1], the state of each photon needed to be a pure state (this choice is crucial in their construction). This is clearly not the case for a Gibbs state. We solve this problem by taking the G.N.S. (or cyclic) representation associated to that state. If the state space of one (simplified) photon was taken to be n-dimensional, then taking the G.N.S. representation brings us into a n^2-dimensional space. This may seem far too big and give the impression we will need too many quantum noises in our model. But we show that, in all cases, only $2n$ chanels of noise resist to the passage to the limit and that they can be naturally coupled two by two to give rise to n "thermal quantum noises". The Langevin equation then remains driven by n noises (which was to be expected!) and the noises are shown to be actually Araki-Woods representations of the usual quantum noises. Furthermore, the Langevin equation we obtain is very similar to the model given in [L-M].

Altogether this confirms we have identified the correct Langevin equation modelizing the action of a quantum heat bath.

An important point to notice is that our construction does not actually use the fact that the state is a Gibbs-like state, it is valid for any density matrix.

This article is organized as follows. In section II we present the toy model for the bath and the Hamiltonian description of the repeated interaction procedure. In section III we present the Fock space, its quantum noises, its approximation by the toy model and the main result of [AP1]. In section IV we detail the G.N.S. representation of the bath and compute the unitary operator, associated with the total Hamiltonian, in that representation. In section V, applying the continuous limit procedure we derive the limit quantum langevin equation. In the process, we identify particular quantum noises that are naturally appearing and baptize them "thermal quantum noises", in the case of a heat bath. The properties of those thermal quantum noises are studied in section VI; in particular we justify their name. In section VII, tracing out the noise, we compute the Lindblad generator of the induced semigroup on the small system. In section VII, being given any finite dimensional quantum system with its Hamiltonian, we show how to construct a Lindblad generator, representing some interaction with a heat bath, such that the quantum system thermalizes.

II. The toy model

We describe here the physical model of repeated interactions with the bath toy model.

The quantum system (we shall often call "small system") to be put in contact with the bath is represented by a separable Hilbert space $\mathcal{H}_S$, as state space, and a self-adjoint operator H_S, as Hamiltonian.

The toy model for the heat bath is the chain

$$\bigotimes_{k\in\mathbb{N}^*}\mathbb{C}^{N+1}$$

of copies of $\mathbb{C}^{N+1}$, where $N\geq 1$ is a fixed integer. Each copy of $\mathbb{C}^{N+1}$ represents the (simplified) state space of a photon. By this countable tensor product we mean the following. We consider a fixed orthonormal basis $\{e_0,e_1,\ldots,e_N\}$ of $\mathbb{C}^{N+1}$, corresponding to the eigenstates of the photon (e_0 being the ground state); the countable tensor product is taken with respect to the ground state e_0. Together with this structure we consider the associated basic matrices a^i_j, $i,j=1,\ldots,N$, acting on $\mathbb{C}^{N+1}$ by

$$a^i_j\, e_k=\delta_{ik}\, e_j$$

and their natural ampliations to $\otimes_{k\in\mathbb{N}^*}\mathbb{C}^{N+1}$ given by

$$a^i_j(k)=\begin{cases} a^i_j & \text{on the } k\text{-th copy of } \mathbb{C}^{N+1}\\ I & \text{on the other copies.}\end{cases}$$

The Hamiltonian of one photon is the operator

$$H_R=\sum_{i=0}^{N}\gamma_i\, a^0_i a^i_0,$$

where the γ_i's are real numbers. Here notice two points.

We have assumed the Hamiltonian H_R to be diagonal in the chosen basis. This is of course not actually a true restriction, for one can always choose such a basis.Note that H_R describe the total energy of a single photon, not the whole field of photon. For this we differ from the model studied in [AJ1].

The second point is that γ_0 is the ground state eigenvalue, it should then be smaller than the other γ_i. One usually assumes that it is equal to 0, but this is not actually necessary in our case, we thus do not specify its value. The only hypothesis we shall make here is that $\gamma_0<\gamma_i$, for all $i=1,\ldots N$. This hypothesis means that the ground eigenspace is simple, it is not actually a necessary assumption, it only simplifies our discussion. At the end of section V we discuss what changes if we leave out this hypothesis.

Finally, notice that the other eigenvalues γ_i need not be simple in our discussion.

When the system and a photon are interacting, we consider the state space $\mathcal{H}_S \otimes \mathbb{C}^{N+1}$ together with the interaction hamiltonian

$$H_I = \sum_{i=1}^{N} \left(V_i \otimes a_i^0 + V_i^* \otimes a_0^i \right),$$

where the V_i's are bounded operators on $\mathcal{H}_S$. This is a usual dipole-type interaction Hamiltonian. The total Hamiltonian for the small system and one photon is thus

$$H = H_S \otimes I + I \otimes H_R + \sum_{i=1}^{N} \left(V_i \otimes a_i^0 + V_i \otimes a_0^i \right).$$

Finally, the state of each photon is fixed to be given by a density matrix ρ which is a function of H_R. We have in mind the usual thermal Gibbs state at inverse temperature β :

$$\rho_\beta = \frac{1}{Z}\, e^{-\beta H_R},$$

where $Z = \mathrm{tr}(e^{-\beta H_R})$, but our construction applies to more general states ρ.

Note that ρ_β is also diagonal in our orthonormal basis. Its diagonal elements are denoted by $\{\beta_0, \beta_1, \ldots, \beta_n\}$.

We shall now describe the repeated interactions of the system $\mathcal{H}_S$ with the chain of photons. The system $\mathcal{H}_S$ is first in contact with the first photon only and they interact together according to the above Hamiltonian H. This lasts for a time length τ. The system $\mathcal{H}_S$ then stops interacting with the first photon and starts interacting with the second photon only. This second interaction is directed by the same Hamiltonian H on the corresponding spaces and it lasts for the same duration τ, and so on... This is mathematically described as follows.

On the space $\mathcal{H}_S \otimes \mathbb{C}^{N+1}$, consider the unitary operator representing the coupled evolution during the time interval $[0, \tau]$:

$$U = e^{-i\tau H}.$$

This single interaction is therefore described in the Schrödinger picture by

$$\rho \mapsto U\, \rho\, U^*$$

and in the Heisenberg picture by

$$X \mapsto U^* X U.$$

After this first interaction, we repeat it but coupling the same $\mathcal{H}_S$ with a new copy of $\mathbb{C}^{N+1}$. This means that this new copy was kept isolated until then; similarly the previously considered copy of $\mathbb{C}^{N+1}$ will remain isolated for the rest of the experience.

The sequence of interactions can be described in the following way: the state space for the whole system is

$$\mathcal{H}_S \otimes \bigotimes_{\mathbb{N}^*} \mathbb{C}^{N+1}.$$

Consider the unitary operator U_k which acts as U on the tensor product of $\mathcal{H}_S$ and the k-th copy of $\mathbb{C}^{N+1}$, and which acts as the identity on all the other copies of $\mathbb{C}^{N+1}$.

The effect of the k-th interaction in the Schrödinger picture is

$$\rho \mapsto U_k \, \rho \, U_k^*,$$

for every density matrix ρ on $\mathcal{H}_S \otimes_{\mathbb{N}}^* \mathbb{C}^{n+1}$. In particular the effect of the k first interactions is

$$\rho \mapsto V_k \, \rho \, V_k^*$$

where $V_k = U_k U_{k-1} \dots U_1$.

Such a Hamiltonian description of the repeated interaction procedure has no chance to give any non-trivial limit in the continuous limit ($\tau \to 0$) without asking a certain renormalization of the interaction. This renormalization can be thought of as making the Hamiltonian depend on τ, or can be also seen as renormalizing the field operators a_j^0, a_0^i of the photons. As is shown is [AP1] (see the detailed discussion in section III), for our repeated interaction model to give rise to a Langevin equation in the limit, we need the interaction part of the Hamiltonian to be affected by a weight $1/\sqrt{\tau}$. Hence, from now on, the total Hamiltonians we shall consider on $\mathcal{H}_S \otimes \mathbb{C}^{N+1}$ are

$$H = H_S \otimes I + I \otimes H_R + \frac{1}{\sqrt{\tau}} \sum_{i=1}^{N} \left(V_i \otimes a_i^0 + V_i^* \otimes a_0^i \right). \tag{1}$$

In [AJ1], one can find a discussion about this time renormalization and its interpretation in terms of weak coupling limit for repeated quantum interactions.

III. The continuous limit setup

We present here all the elements of the continuous limit result: the structure of the corresponding Fock space, the quantum noises, the approximation of the Fock space by the photon chain and [AP1]'s main theorem.

III.1 The continuous tensor product structure

First, as a guide to intuition, let us make more explicit the structure of the photon chain. We let $T\Phi$ denote the tensor product $\otimes_{\mathbb{N}^*} \mathbb{C}^{N+1}$ with respect to the stabilizing sequence e_0. This simply means that an orthonormal basis of $T\Phi$ is given by the family

$$\{e_\sigma; \sigma \in \mathcal{P}_{\mathbb{N}^*,N}\}$$

where

– the set $\mathcal{P}_{\mathbb{N},N}$ is the set of finite subsets

$$\{(n_1, i_1), \dots, (n_k, i_k)\}$$

of $\mathbb{N}^* \times \{1, \dots, N\}$ such that the n_i's are mutually different;

– e_σ denotes the vector

$$\Omega \otimes \ldots \otimes \Omega \otimes e_{i_1} \otimes \Omega \otimes \ldots \otimes \Omega \otimes e_{i_2} \otimes \ldots$$

where e_{i_1} appears in n_1-th copy of $\mathcal{H}$, where e_{i_2} appears in n_2-th copy of $\mathcal{H}$... Here Ω plays the same role as e_0 in the toy model.

This is for a vector basis on $T\Phi$. From the point of view of operators, we denote by $a^i_j(k)$ the natural ampliation of the operator a^i_j to $T\Phi$ which acts on the copy number k as a^i_j and the identity elsewhere. That is, in terms of the basis e_σ,

$$a^i_j(k)e_\sigma = \mathbb{1}_{(k,i)\in\sigma}\, e_{(\sigma\setminus(k,i))\cup(k,j)}$$

if neither i nor j is zero, and

$$\begin{aligned} a^i_0(k)e_\sigma &= \mathbb{1}_{(k,i)\in\sigma}\, e_{\sigma\setminus(k,i)}, \\ a^0_j(k)e_\sigma &= \mathbb{1}_{(k,0)\in\sigma}\, e_{\sigma\cup(k,j)}, \\ a^0_0(k)e_\sigma &= \mathbb{1}_{(k,0)\in\sigma}\, e_\sigma, \end{aligned}$$

where $(k,0)\in\sigma$ actually means "for any i in $\{1,\ldots,N\}$, $(k,i)\notin\sigma$".

We now describe the structure of the continuous version of the chain of photons. The structure we are going to present here is rather original and not much expanded in the literature. It is very different from the usual presentation of quantum stochastic calculus ([H-P]), but it actually constitutes a very natural language for our purpose: approximation of the atom field by atom chains. This approach is taken from [At1]. We first start with a heuristic discussion.

By a continuous version of the atom chain $T\Phi$ we mean a Hilbert space with a structure which makes it the space

$$\Phi = \bigotimes_{\mathbb{R}^+} \mathbb{C}^{N+1}.$$

We have to give a meaning to the above notation. This could be achieved by invoquing the framework of continous tensor products of Hilbert spaces (see [Gui]), but we prefer to give a self-contained presentation which fits better with our approximation procedure.

Let us make out an idea of what it should look like by mimicking, in a continuous time version, what we have described in $T\Phi$.

The countable orthonormal basis $e_\sigma, \sigma\in\mathcal{P}_{\mathbb{N}^*,N}$ is replaced by a continuous orthonormal basis $d\chi_\sigma$, $\sigma\in\mathcal{P}_{\mathbb{R}^+,N}$, where $\mathcal{P}_{\mathbb{R}^+,N}$ is the set of finite subsets of $\mathbb{R}^+\times\{1,\ldots,N\}$. With the same idea as for $T\Phi$, this means that each copy of $\mathbb{C}^{N+1}$ is equipped with an orthonormal basis $\{\Omega, d\chi^1_t,\ldots,d\chi^N_t\}$ (where t is the parameter attached to the copy we are looking at).

Recall the representation of an element f of $T\Phi$:

$$f = \sum_{\sigma\in\mathcal{P}_{\mathbb{N}^*,N}} f(\sigma)\, e_\sigma,$$

$$||f||^2 = \sum_{\sigma\in\mathcal{P}_{\mathbb{N}^*,N}} |f(\sigma)|^2,$$

it is replaced by an integral version of it in Φ:

$$f = \int_{\mathcal{P}_{\mathbb{R}^+,N}} f(\sigma)\, d\chi_\sigma,$$

$$||f||^2 = \int_{\mathcal{P}_{\mathbb{R}^+,N}} |f|^2 \, d\sigma.$$

This last integral needs to be explained: the measure $d\sigma$ is a "Lebesgue measure" on $\mathcal{P}_{\mathbb{R}^+,N}$, as will be explained later.

From now on, the notation $\mathcal{P}$ will denote, depending on the context, the set $\mathcal{P}_{\mathbb{N}^*,N}$ or $\mathcal{P}_{\mathbb{R}^+,N}$.

A good basis of operators acting on Φ can be obtained by mimicking the operators $a^i_j(k)$ of $T\Phi$. We have here a set of infinitesimal operators $da^i_j(t)$, $i,j \in \{0,1,\ldots,N\}$, acting on the "t-th" copy of $\mathbb{C}^{N+1}$ by:

$$\begin{aligned}
da^0_0(t)\, d\chi_\sigma &= d\chi_\sigma\, dt\, \mathbb{1}_{t\notin\sigma}\\
da^0_i(t)\, d\chi_\sigma &= d\chi_{\sigma\cup\{(t,i)\}}\, \mathbb{1}_{t\notin\sigma}\\
da^i_0(t)\, d\chi_\sigma &= d\chi_{\sigma\setminus\{(t,i)\}}\, dt\, \mathbb{1}_{(t,i)\in\sigma}\\
da^i_j(t)\, d\chi_\sigma &= d\chi_{(\sigma\setminus\{(t,i)\})\cup\{(t,j)\}}\, \mathbb{1}_{(t,i)\in\sigma}
\end{aligned}$$

for all $i,j \in \{1,\ldots,N\}$. We shall now describe a rigourous setup for the above heuristic discussion.

We recall the structure of the bosonic Fock space Φ and its basic structure (cf [At1] for more details and [At2] for a complete study of the theory and its connections with classical stochastic processes).

Let $\Phi = \Gamma_s(L^2(\mathbb{R}^+,\mathbb{C}^N))$ be the symmetric (or bosonic) Fock space over the space $L^2(\mathbb{R}^+,\mathbb{C}^N)$. We shall give here a very efficient presentation of that space, the so-called *Guichardet interpretation* of the Fock space.

Let $\mathcal{P}$ $(=\mathcal{P}_{\mathbb{R}^+,N})$ be the set of finite subsets $\{(s_1,i_1),\ldots,(s_n,i_n)\}$ of $\mathbb{R}^+ \times \{1,\ldots,N\}$ such that the s_i are two by two different. Then $\mathcal{P} = \cup_k \mathcal{P}_k$ where $\mathcal{P}_k$ is the subset of $\mathcal{P}$ made of k-elements subsets of $\mathbb{R}^+ \times \{1,\ldots,N\}$. By ordering the $\mathbb{R}^+$-part of the elements of $\sigma \in \mathcal{P}_k$, the set $\mathcal{P}_k$ can be identified with the increasing simplex $\Sigma_k = \{0 < t_1 < \cdots < t_k\} \times \{1,\ldots,N\}$ of $\mathbb{R}^k \times \{1,\ldots,N\}$. Thus $\mathcal{P}_k$ inherits a measured space structure from the Lebesgue measure on $\mathbb{R}^k$ times the counting measure on $\{1,\ldots,N\}$. This also gives a measure structure on $\mathcal{P}$ if we specify that on $\mathcal{P}_0 = \{\emptyset\}$ we put the measure $\delta_\emptyset$. Elements of $\mathcal{P}$ are often denoted by σ, the measure on $\mathcal{P}$ is denoted by $d\sigma$. The σ-field obtained this way on $\mathcal{P}$ is denoted by $\mathcal{F}$.

We identify any element $\sigma \in \mathcal{P}$ with a family $\{\sigma_i,\, i \in \{1,\ldots,N\}\}$ of (two by two disjoint) subsets of $\mathbb{R}^+$ where

$$\sigma_i = \{s \in \mathbb{R}^+; (s,i) \in \sigma\}.$$

The *Fock space* Φ is the space $L^2(\mathcal{P},\mathcal{F},d\sigma)$. An element f of Φ is thus a measurable function $f : \mathcal{P} \to \mathbb{C}$ such that

$$||f||^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma < \infty.$$

Finally, we put Ω to be the *vacuum vector* of Φ, that is, $\Omega(\sigma)=\delta_\emptyset(\sigma)$.

One can define, in the same way, $\mathcal{P}_{[a,b]}$ and $\Phi_{[a,b]}$ by replacing $\mathbb{R}^+$ with $[a,b]\subset\mathbb{R}^+$. There is a natural isomorphism between $\Phi_{[0,t]}\otimes\Phi_{[t,+\infty[}$ and Φ given by $h\otimes g\mapsto f$ where $f(\sigma)=h(\sigma\cap[0,t])g(\sigma\cap[t,+\infty[)$. This is, with our notations, the usual exponential property of Fock spaces. Note that in the sequel we identify $\Phi_{[a,b]}$ with a subspace of Φ, the subspace

$$\{f\in\Phi; f(\sigma)=0 \text{ unless } \sigma\subset[a,b]\}.$$

We now define a particular family of curves in Φ, which is going to be of great importance here. Define $\chi_t^i\in\Phi$ by

$$\chi_t^i(\sigma)=\begin{cases}\mathbb{1}_{[0,t]}(s) & \text{if } \sigma=\{(s,i)\}\\ 0 & \text{otherwise.}\end{cases}$$

Then notice that for all $t\in\mathbb{R}^+$ we have that χ_t^i belongs to $\Phi_{[0,t]}$. We actually have much more than that:

$$\chi_t^i-\chi_s^i\ \in\ \Phi_{[s,t]} \text{ for all } s\le t.$$

This last property can be checked immediately from the definitions, and it is going to be of great importance in our construction. Also notice that χ_t^i and χ_s^j are orthogonal elements of Φ as soon as $i\neq j$. One can show that, apart from trivialities, the curves $(\chi_t^i)_{t\ge0}$ are the only ones to share these properties.

These properties allow to define the so-called *Ito integral* on Φ. Indeed, let $g=\{(g_t^i)_{t\ge0},\ i\in\{1,\dots,N\}\}$ be families of elements of Φ indexed by both $\mathbb{R}_+$ and $\{1,\dots,N\}$, such that

i) $t\mapsto\|g_t^i\|$ is measurable, for all i,

ii) $g_t^i\in\Phi_{[0,t]}$ for all t,

iii) $\sum_{i=1}^N\int_0^\infty\|g_t^i\|^2\,dt<\infty$,

then one says that g is *Ito integrable* and we define its *Ito integral*

$$\sum_{i=1}^N\int_0^\infty g_t^i\,d\chi_t^i$$

to be the limit in Φ of

$$\sum_{i=1}^N\sum_{j\in\mathbb{N}}\widetilde{g}^i_{t_j}\otimes\left(\chi^i_{t_{j+1}}-\chi^i_{t_j}\right)\tag{2}$$

where $\mathcal{S}=\{t_j,\ j\in\mathbb{N}\}$ is a partition of $\mathbb{R}^+$ which is understood to be refining and to have its diameter tending to 0, and $(\widetilde{g}^i_\cdot)_i$ is an Ito integrable family in Φ, such that for each i, $t\mapsto\widetilde{g}^i_t$ is a step process, and which converges to $(g^i_\cdot)_i$ in $L^2(\mathbb{R}^+\times\mathcal{P})$.

Note that by assumption we always have that $\widetilde{g}^i_{t_j}$ belongs to $\Phi_{[0,t_j]}$ and $\chi^i_{t_{j+1}}-\chi^i_{t_j}$ belongs to $\Phi_{[t_j,t_{j+1}]}$, hence the tensor product symbol in (2).

Also note that, as an example, one can take

$$\widetilde{g}^i_t=\sum_{t_j\in\mathcal{S}}\frac{1}{t_{j+1}-t_j}\int_{t_j}^{t_{j+1}}P_{t_j}g_s^i\,ds\,\mathbb{1}_{[t_j,t_{j+1}[}(t)$$

if $t \in [t_j, t_{j+1}]$, where P_t denotes the orthogonal projection onto $\Phi_{[0,t]}$.

One then obtains the following properties ([At1], Proposition 1.4), where $\vee\sigma$ means $\max\{s \in I\!R^+ ; (s,k) \in \sigma \text{ for some } k\}$ and where $\sigma-$ denotes the set $\sigma \backslash (\vee\sigma, i)$ if $(\vee\sigma, i) \in \sigma$.

Theorem 1. – *The Ito integral $I(g) = \sum_i \int_0^\infty g_t^i \; d\chi_t^i$, of an Ito integrable family $g = (g^i_\cdot)_{i=1}^N$, is the element of Φ given by*

$$I(g)(\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g^i_{\vee\sigma}(\sigma-) & \text{otherwise.} \end{cases}$$

It satisfies the Ito isometry formula*:*

$$||I(g)||^2 = \Big\| \sum_{i=1}^N \int_0^\infty g_t^i \; d\chi_t^i \Big\|^2 = \sum_{i=1}^N \int_0^\infty \left|\left|g_t^i\right|\right|^2 \, dt \; . \tag{3}$$

■

In particular, consider a family $f = (f_i)_{i=1}^N$ which belongs to $L^2(\mathcal{P}_1) = L^2(I\!R^+ \times \{1, \ldots, N\})$, then the family $(f_i(t)\Omega)$, $t \in I\!R^+$, $i = 1, \ldots, N$, is clearly Ito integrable. Computing its Ito integral we find that

$$I(f) = \sum_{i=1}^N \int_0^\infty f_i(t)\Omega \, d\chi_t^i$$

is the element of the first particle space of the Fock space Φ associated with the function f, that is,

$$I(f)(\sigma) = \begin{cases} f_i(s) & \text{if } \sigma = \{(s,i)\} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f \in L^2(\mathcal{P}_n)$, one can easily define the *iterated Ito integral* on Φ:

$$I_n(f) = \int_{\mathcal{P}_n} f(\sigma) \, d\chi_\sigma$$

by iterating the definition of the Ito integral:

$$I_n(f) = \sum_{i_1, \ldots, i_n \in \{1, \ldots, N\}} \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_{i_1, \ldots, i_n}(t_1, \ldots, t_n)\Omega \; d\chi_{t_1}^{i_1} \ldots d\chi_{t_n}^{i_n}.$$

We obtain this way an element of Φ which is actually the representant of f in the n-particle subspace of Φ, that is

$$[I_n(f)](\sigma) = \begin{cases} f_{i_1, \ldots, i_n}(t_1, \ldots, t_n) & \text{if } \sigma = \{(t_1, i_1) \cup \ldots \cup (t_n, i_n)\} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for any $f \in \mathcal{P}$ we put

$$\int_{\mathcal{P}} f(\sigma) \, d\chi_\sigma$$

to denote the series of iterated Ito integrals

$$f(\emptyset)\Omega + \sum_{n=1}^\infty \sum_{i_1, \ldots, i_n = 1}^N \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_{i_1, \ldots, i_n}(t_1, \ldots, t_n)\Omega \, d\chi_{t_1}^{i_1} \ldots d\chi_{t_n}^{i_n}.$$

We then have the following representation ([At1], Theorem 1.7).

Theorem 2. [Fock space chaotic representation property] – *Any element* f *of* Φ *admits a* Fock space chaotic representation

$$f = \int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma \tag{4}$$

satisfying the isometry formula

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2\ d\sigma. \tag{5}$$

This representation is unique.

The above theorem is the exact expression of the heuristics we wanted in order to describe the space

$$\Phi = \bigotimes_{\mathbb{R}^+} \mathcal{H}.$$

Indeed, we have, for each $t \in \mathbb{R}^+$, a family of elementary orthonormal elements $\{\Omega, d\chi_t^1, \ldots, d\chi_t^N\}$ (a basis of $\mathcal{H}$) whose (tensor) products $d\chi_\sigma$ form a continuous basis of Φ (formula (4)) and, even more, form an orthonormal continuous basis (formula (5)).

III.2 The quantum noises

The space Φ we have constructed is the natural space for defining *quantum noises.* These quantum noises are the natural, continuous-time, extensions of the basis operators $a^i_j(n)$ we met in the atom chain $T\Phi$.

As indicated in the heuristic discussion above, we shall deal with a family of infinitesimal operators $da^i_j(t)$ on Φ which act on the continuous basis $d\chi_\sigma$ in the same way as their discrete-time counterparts $a^i_j(n)$ act on the e_σ. The integrated version of the above heuristic infinitesimal formulas easily gives an exact formula for the action of the operators $a^i_j(t)$ on Φ:

$$\begin{aligned}
[a^0_i(t)f](\sigma) &= \sum_{\substack{s\in\sigma_i \\ s\le t}} f(\sigma\setminus(s,i)), \\
[a^i_0(t)f](\sigma) &= \int_0^t f(\sigma\cup(s,i))\ ds, \\
[a^i_j(t)f](\sigma) &= \sum_{\substack{s\in\sigma_i \\ s\le t}} f\left((\sigma\setminus(s,i))\cup(s,j)\right) \\
[a^0_0(t)f](\sigma) &= t\, f(\sigma)
\end{aligned}$$

for $i, j \neq 0$.

All these operators, except $a^0_0(t)$, are unbounded, but note that a good common domain to all of them is

$$\mathcal{D} = \left\{ f\in\Phi\ ;\ \int_{\mathcal{P}} |\sigma|\ |f(\sigma)|^2\ d\sigma < \infty \right\}.$$

This family of operators is characteristic and universal in a sense which is close to the one of the curves χ^i_t. Indeed, one can easily check that in the decomposition of $\Phi \simeq \Phi_{[0,s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t,+\infty[}$, the operators $a^i_j(t) - a^i_j(s)$ are all of the form

$$I \otimes (a^i_j(t) - a^i_j(s))_{|\Phi_{[s,t]}} \otimes I.$$

This property is fundamental for the definition of the quantum stochastic integrals and, in the same way as for $(\chi^i_\cdot)$, these operator families are the only ones to share that property (cf [Coq]).

This property allows to consider Riemann sums:

$$\sum_k H_{t_k} \left(a^i_j(t_{k+1}) - a^i_j(t_k)\right) \tag{6}$$

where $\mathcal{S} = \{0 = t_0 < t_1 < \ldots < t_k < \ldots\}$ is a partition of $I\!R^+$, where $(H_t)_{t\geq 0}$ is a family of operators on Φ such that

– each H_t is an operator of the form $H_t \otimes I$ in the tensor product space $\Phi = \Phi_{[0,t]} \otimes \Phi_{[t,+\infty[}$ (we say that H_t is a *t-adapted operator* and that $(H_t)_{t\geq 0}$ is an *adapted process of operators*),

– $(H_t)_{t\geq 0}$ is a *step process*, that is, it is constant on intervals:

$$H_t = \sum_k H_{t_k} 1\!\!1_{[t_k,t_{k+1}]}(t).$$

In particular, the operator product $H_{t_k}\left(a^i_j(t_{k+1}) - a^i_j(t_k)\right)$ is actually a tensor product of operators

$$H_{t_k} \otimes \left(a^i_j(t_{k+1}) - a^i_j(t_k)\right).$$

Thus this product is commutative and does not impose any new domain constraint on the operators apart from the ones attached to the operators H_t and $a^i_j(t_{k+1}) - a^i_j(t_k)$ themselves. The resulting operator associated to the Riemann sum (6) is denoted by

$$T = \int_0^\infty H_s \, da^i_j(s).$$

One can compute the action of T on a "good" vector f of its domain and obtain explicit formulas which are not worth developping here (cf [At1] for more details). For general operator processes $(H_t)_{t\geq 0}$ (still adapted but not a step process anymore) and for a general f, these explicit formulas can be extended and they are kept as a definition for the domain and for the action of the operator

$$T = \int_0^\infty H_s \, da^i_j(s).$$

The maximal domain and the explicit action of the above operator can be described but also are not worth developing here (cf [A-L]). The main point with these quantum stochastic integrals is that, when composed, they satisfy a Ito-type integration by part formula. This formula can be summarized as follows, without taking care at all of domain constraints. Let

$$T = \int_0^\infty H_s \, da^i_j(s), \;\; S = \int_0^\infty K_s \, da^k_l(s).$$

For every $t \in \mathbb{R}^+$ put

$$T_t = \int_0^\infty H_s \mathbb{1}_{[0,t]}(s)\, da^i_j(s)$$

and the same for S_t. We then have

$$TS = \int_0^\infty H_s S_s\, da^i_j(s) + \int_0^\infty T_s K_s\, da^k_l(s) + \int_0^\infty H_s K_s\, \widehat{\delta}_{il}\, da^k_j(s), \qquad (7)$$

where

$$\widehat{\delta}_{il} = \begin{cases} \delta_{il} & \text{if } (i,l) \neq (0,0) \\ 0 & \text{if } (i,l) = (0,0). \end{cases}$$

The last term appearing in this Ito-type formula is often summarized by saying that the quantum noises satisfy the formal formula:

$$da^i_j(s) da^k_l(s) = \widehat{\delta}_{il}\, da^k_j(s).$$

III.3 Embedding and approximation by the Toy Fock space

We now describe the way the chain and its basic operators can be realized as a subspace of the Fock space and a projection of the quantum noises. The subspace associated with the atom chain is attached to the choice of some partition of $\mathbb{R}^+$ in such a way that the expected properties are satisfied:

– the associated subspaces increase when the partition refines and they constitute an approximation of Φ when the diameter of the partition goes to 0,

– the associated basic operators are restrictions of the others when the partition increases and they constitute an approximation of the quantum noises when the diameter of the partition goes to 0.

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ be a partition of $\mathbb{R}^+$ and $\delta(\mathcal{S}) = \sup_i |t_{i+1} - t_i|$ be the diameter of $\mathcal{S}$. For $\mathcal{S}$ fixed, define $\Phi_n = \Phi_{[t_{n-1}, t_n]}$, $n \in \mathbb{N}^*$. We clearly have that Φ is naturally isomorphic to the countable tensor product $\otimes_{n \in \mathbb{N}^*} \Phi_n$ (which is again understood to be defined with respect to the stabilizing sequence $(\Omega)_{n \in \mathbb{N}}$).

For all $n \in \mathbb{N}^*$, define for $i, j \in \{1, \ldots, N\}$

$$\begin{aligned}
e_i(n) &= \frac{\chi^i_{t_n} - \chi^i_{t_{n-1}}}{\sqrt{t_n - t_{n-1}}} \in \Phi_n\ , \\
a^i_0(n) &= \frac{a^i_0(t_n) - a^i_0(t_{n-1})}{\sqrt{t_n - t_{n-1}}} \circ P_{1]}, \\
a^i_j(n) &= P_{1]} \circ \left(a^i_j(t_n) - a^i_j(t_{n-1})\right) \circ P_{1]}, \\
a^0_i(n) &= P_{1]} \circ \frac{a^0_i(t_n) - a^0_i(t_{n-1})}{\sqrt{t_n - t_{n-1}}}\ , \\
a^0_0(n) &= P_{0]},
\end{aligned}$$

where for $i = 0, 1$ and $P_{i]}$ is the orthogonal projection onto $L^2(\mathcal{P}_i)$. The above definitions are understood to be valid on Φ_n only, the corresponding operator acting as the identity operator I on the others Φ_m's.

For every $\sigma \in \mathcal{P} = \mathcal{P}_{I\!N^*,N}$, define e_σ from the $e_i(n)$'s in the same way as for $T\Phi$:

$$e_\sigma = \Omega \otimes \ldots \otimes \Omega \otimes e_{i_1}(n_1) \otimes \Omega \otimes \ldots \otimes \Omega \otimes e_{i_2}(n_2) \otimes \ldots$$

in $\otimes_{n \in I\!N^*} \mathcal{H}_n$. Define $T\Phi(\mathcal{S})$ to be the space of $f \in \Phi$ which are of the form

$$f = \sum_{\sigma \in \mathcal{P}} f(\sigma) e_\sigma$$

(note that the condition $\|f\|^2 = \sum_{\sigma \in \mathcal{P}} |f(\sigma)|^2 < \infty$ is automatically satisfied). The space $T\Phi(\mathcal{S})$ can be clearly and naturally identified to the spin chain $T\Phi$. The space $T\Phi(\mathcal{S})$ is a closed subspace of Φ. We denote by $P_{\mathcal{S}}$ the operator of orthogonal projection from Φ onto $T\Phi(\mathcal{S})$.

The main point is that the above operators $a^i_j(n)$ act on $T\Phi(\mathcal{S})$ in the same way as the the basic operators of $T\Phi$ (cf [AP1], Proposition 8).

Proposition 3. – *We have, for all $i, j = 1, \ldots, N$*

$$\begin{cases} a^i_0(n)\, e_j(n) = \delta_{ij}\Omega \\ a^i_0\, \Omega = 0 \end{cases}$$

$$\begin{cases} a^i_j(n)\, e_k(n) = \delta_{ik} e_j(n) \\ a^i_j\, \Omega = 0 \end{cases}$$

$$\begin{cases} a^0_i(n)\, e_j(n) = 0 \\ a^0_i(n)\, \Omega = e_i(n) \end{cases}$$

$$\begin{cases} a^0_0(n)\, e_k(n) = 0 \\ a^0_0\, \Omega = \Omega. \end{cases}$$

Thus the action of the operators a^i_j on the $e_i(n)$ is exactly the same as the action of the corresponding operators on the spin chain of section II; the operators $a^i_j(n)$ act on $T\Phi(\mathcal{S})$ exactly in the same way as the corresponding operators do on $T\Phi$. We have completely embedded the toy Fock space structure into the Fock space.

We are now going to see that the Fock space Φ and its basic operators $a^i_j(t)$, $i, j \in \{0, 1, \ldots, n\}$ can be approached by the toy Fock spaces $T\Phi(\mathcal{S})$ and their basic operators $a^i_j(n)$. We are given a sequence $(\mathcal{S}_n)_{n \in I\!N}$ of partitions which are getting finer and finer and whose diameter $\delta(\mathcal{S}_n)$ tends to 0 when n tends to $+\infty$. Let $T\Phi(n) = T\Phi(\mathcal{S}_n)$ and $P_n = P_{\mathcal{S}_n}$, for all $n \in I\!N$. We then have the following convergence result (see [AP1], Theorem 10), where the reader needs to recall the domain $\mathcal{D}$ introduced in section III.2.

Theorem 4. –

i) The orthogonal projectors P_n converge strongly to the identity operator I on Φ. That is, any $f \in \Phi$ can be approached in Φ by a sequence $(f_n)_{n \in I\!N}$ such that $f_n \in T\Phi(n)$ for all $n \in I\!N$.

ii) If $\mathcal{S}_n = \{0 = t_0^n < t_1^n < \cdots < t_k^n < \cdots\}$*, then for all* $t \in \mathbb{R}^+$*, all* $i, j = 1, \ldots, n$ *the operators*

$$\sum_{k; t_k^n \le t} a_j^i(k),$$

$$\sum_{k; t_k^n \le t} \sqrt{t_k^n - t_{k-1}^n}\, a_0^i(k),$$

$$\sum_{k; t_k^n \le t} \sqrt{t_k^n - t_{k-1}^n}\, a_i^0(k)$$

$$\text{and} \sum_{k; t_k^n \le t} (t_k^n - t_{k-1}^n)\, a_0^0(k)$$

converge strongly on $\mathcal{D}$ *to* $a_j^i(t)$*,* $a_0^i(t)$*,* $a_i^0(t)$ *and* $a_0^0(t)$ *respectively.*

We have fulfilled our duties: not only the space $T\Phi(\mathcal{S})$ recreates $T\Phi$ and its basic operators as a subspace of Φ and a projection of its quantum noises, but, when $\delta(\mathcal{S})$ tends to 0, this realisation constitutes an approximation of the space Φ and of its quantum noises.

III.4 Quantum Langevin equations

In this article what we call quantum Langevin equation is actually a restricted version of what is usually understood in the physical literature (cf [G-Z]); by this we mean that we study here the so-called quantum stochastic differential equations as defined by Hudson and Parthasarathy and heavily studied by further authors ([H-P], [Fag]). This type of quantum noise perturbation of the Schrödinger equation is exactly the type of equation which we will get as the continuous limit of our Hamiltonian description of repeated quantum interactions.

Quantum stochastic differential equations are operator-valued equations on $\mathcal{H}_S \otimes \Phi$ of the form

$$dU_t = \sum_{i,j=0}^{N} L_j^i U_t \, da_j^i(t),$$

with initial condition $U_0 = I$. The above equation has to be understood as an integral equation

$$U_t = I + \int_0^t \sum_{i,j=0}^{N} L_j^i U_t \, da_j^i(t),$$

the operators L_j^i being bounded operators on $\mathcal{H}_S$ alone which are ampliated to $\mathcal{H}_S \otimes \Phi$.

The main motivation and application of that kind of equation is that it gives an account of the interaction of the small system $\mathcal{H}_S$ with the bath Φ in terms of quantum noise perturbation of a Schrödinger-like equation. Indeed, the first term of the equation

$$dU_t = L_0^0 U_t \, dt + \ldots$$

describes the induced dynamics on the small system, all the other terms are quantum noises terms. One of the main application of these equations is that they give explicit constructions of unitary dilations of semigroups of completely positive maps of $\mathcal{B}(\mathcal{H}_S)$ (see [H-P] and also section VII of this article).

Let us here only recall one of the main existence, uniqueness and boundedness theorem connected to quantum Langevin equations. The literature is huge about those equations; we refer to [Par] for the result we mention here. In the following, by coherent vectors we mean elements of the space $\mathcal{E}$ generated by the $u \otimes \varepsilon(f)$, with $u \in \mathcal{H}_S$, $f \in L^2(\mathbb{R}^+; \mathbb{C}^n)$ and

$$[\varepsilon(f)](\sigma) = \prod_{(s,i)\in\sigma} f_i(s),$$

the usual coherent vectors of the Fock space Φ.

Theorem 5. – *If all the operators L^i_j are bounded on $\mathcal{H}_S$ then the quantum stochastic differential equation*

$$U_t = I + \sum_{i,j=0}^{N} \int_0^t L^i_j U_s \, da^i_j(s)$$

admits a unique solution defined on the space of coherent vectors.

The solution $(U_t)_{t\geq 0}$ is made of unitary operators if and only if there exist on $\mathcal{H}_S$, a self-adjoint operator H, operators L_i, $i = 1, \ldots, N$ and operators S^i_j, $i, j = 1, \ldots, N$ such that the matrix $(S^i_j)_{i,j=1,\ldots,N}$ is unitary and the coefficients L^i_j are of the form

$$L^0_0 = -(iH + \frac{1}{2}\sum_{k=1}^{N} L^*_k L_k)$$

$$L^0_j = L_j$$

$$L^i_0 = -\sum_{k=1}^{N} L^*_k S^k_i$$

$$L^i_j = S^i_j - \delta_{ij} I.$$

■

III.5 Convergence theorems

We are finally able to state the main result of [AP1] which shows the convergence of repeated interactions models to quantum stochastic differential equations.

Let τ be a parameter in $\mathbb{R}^+$, which is thought of as representing a small time interval. Let $U(\tau)$ be a unitary operator on $\mathcal{H}_S \otimes \mathbb{C}^{N+1}$, with coefficients $U^i_j(\tau)$ as a matrix of operators on $\mathcal{H}_S$ (this operator has to be though of as corresponding to the unitary operator U of section II). Let $V_k(\tau)$ be the associated repeated interaction operator:

$$V_{k+1}(\tau) = U_{k+1}(\tau) V_k(\tau)$$

with the same notation as in section II. In the following we will drop dependency in τ and write simply U, or V_k. Besides, we denote

$$\varepsilon_{ij} = \frac{1}{2}(\delta_{0i} + \delta_{0j})$$

for all i, j in $\{0, \ldots, N\}$. That is, for $i, j \geq 1$

$$\varepsilon_{i0} = \varepsilon_{0j} = \frac{1}{2}, \quad \varepsilon_{ij} = 0, \quad \varepsilon_{00} = 1.$$

Note that from now on we take the embeding of $T\Phi$ in Φ for granted and we consider, without mentionning it, all the repeated quantum interactions to happen in $T\Phi(\tau)$, the subspace of Φ associated to the partition $\mathcal{S} = \{t_i = i\tau; i \in \mathbb{N}\}$. The main result of [AP1] (Theorem 13 in this reference) is the following.

Theorem 6. – *Assume that there exist bounded operators L^i_j, $i, j \in \{0, \ldots, n\}$ on $\mathcal{H}_S$ such that*

$$\lim_{\tau \to 0} \frac{U^i_j(\tau) - \delta_{ij} I}{\tau^{\varepsilon_{ij}}} = L^i_j$$

for all $i, j = 0, \ldots, n$. Then, for almost all t the operators $V_{[t/\tau]}$ converge strongly, when $\tau \to 0$, to V_t, the unitary solution of the quantum stochastic differential equation

$$dV_t = \sum_{i,j=0}^{n} L^i_j V_t \, da^i_j(t)$$

with initial condition $V_0 = I$.

IV. The G.N.S. representation of the heat bath

In order to apply Theorem 6 to our repeated interaction model, we need the state of the photon to be a vector state (i.e. a pure state) instead of a density matrix. This is easily performed by considering the so-called G.N.S. representation (or cyclic representation) of the photon system.

This representation can be described in the following way. Consider the space $\mathcal{H} = \mathcal{L}(\mathbb{C}^{N+1})$ of endomorphisms of $\mathbb{C}^{N+1}$. Consider a given density matrix ρ_β on $\mathcal{H}$, which is supposed to be in diagonal form $\rho_\beta = \mathrm{diag}(\beta_0, \ldots, \beta_N)$, where all the β_i are strictly positive. The space $\mathcal{H} = \mathcal{L}(\mathbb{C}^{N+1})$ is made into a Hilbert space when equipped with the scalar product :

$$\langle A, B \rangle = \mathrm{tr}(\rho_\beta \, A^* B),$$

for all $A, B \in \mathcal{H}$. The associated norm on $\mathcal{H}$ is denoted by $||\cdot||$. This Hilbert space is $(N+1)^2$-dimensional and we shall describe one of its orthonormal basis as follows. We denote by X^0_0 the identity endomorphism. Then, for $i = 1, \ldots, N$, we put X^i_i to be the diagonal matrices with diagonal coefficients $\{\lambda^1_i, \ldots, \lambda^N_i\}$ such that

$$\langle X^i_i, X^j_j \rangle = \delta_{ij}$$

for all $i, j = 0, 1, \dots N$. Such a family clearly exists for its diagonal elements are obtained by extending the vector $(1, \dots, 1) \in \mathbb{C}^{N+1}$ into an orthonormal basis of $\mathbb{C}^{N+1}$ equiped with the scalar product

$$\sum_{i=0}^{N} \beta_i \, \overline{x}_i y_i.$$

For $i \neq j \in \{0, \dots, N\}$ we put X^i_j to be the element of $\mathcal{H}$ given by

$$X^i_j = \frac{1}{\sqrt{\beta_i}} \, a^i_j.$$

It is then a straightforward computation to check that $\{X^i_j; i, j = 0, \dots N\}$ forms an orthonormal basis of $\mathcal{H}$.

We now have the usual G.N.S. representation π of $\mathcal{L}(\mathbb{C}^{N+1})$ into $\mathcal{L}(\mathcal{H})$ given by

$$\pi(A)B = AB,$$

for all $A \in \mathcal{L}(\mathbb{C}^{N+1})$, $B \in \mathcal{H}$. In the framework of that representation, note that X^0_0 is then the vector state on $\mathcal{H}$ which represents the state ρ_β on $\mathbb{C}^{N+1}$: indeed, for all $A \in \mathcal{L}(\mathbb{C}^{n+1})$ we have

$$\langle X^0_0, \pi(A) X^0_0 \rangle = \operatorname{tr}(\rho_\beta \, A).$$

That is, in the orthonormal basis we have choosen, X^0_0 is the only important vector that could not be choosen to be different. The rest of the choice for our orthonormal basis is just convenient for the computations, but the final result does not depend on it.

Now, any operator K on $\mathcal{H}_S \otimes \mathbb{C}^{N+1}$ is transformed by π into an operator on $\mathcal{H}_S \otimes \mathcal{H}$. That is, $\pi(K)$ is a $(N+1)^2 \times (N+1)^2$-matrix with coefficients $K^{i,j}_{k,l}$ in $\mathcal{L}(\mathcal{H}_S)$. These coefficients are given by

$$K^{i,j}_{k,l} = \operatorname{tr}_{\mathcal{H}}(\rho_\beta \, (X^k_l)^* K X^i_j)$$

where $\operatorname{tr}_{\mathcal{H}}(H)$ denotes the partial trace of H along $\mathcal{H}$, that is, this is the operator on $\mathcal{H}_S$ given by the sum of the diagonal coefficients of H as a $\mathcal{L}(\mathcal{H}_S)$-valued $(N+1)^2 \times (N+1)^2$-matrix.

When taking the continuous limit on the space $\mathcal{H}_S \otimes \otimes_{\mathbb{N}^*} \mathcal{H}$, we end up into the space

$$\mathcal{H}_S \otimes \bigotimes_{\mathbb{R}^+} \mathcal{H},$$

that is,

$$\mathcal{H}_S \otimes \Gamma_s \left(L^2(\mathbb{R}^+; \mathbb{C}^{(N+1)^2-1}) \right).$$

The associated quantum noises, following the same basis, are thus denoted by $da^{i,j}_{k,l}(t)$, $i, j, k, l = 0, \dots, N$.

V. The limit quantum Langevin equation

We are now in conditions to apply Theorem 6. We consider the repeated interaction model described in section II, with its associated operators H, U_k, V_k. Taking the G.N.S. representation of all that we end up in the space $\mathcal{H}_S\otimes\otimes_{\mathbb{N}^*}\mathcal{H}$, which we embbed inside a continuous tensor product $\mathcal{H}_S\otimes\otimes_{\mathbb{R}^+}\mathcal{H}$, as explained in section III.3. The main result of this article is then the following.

Theorem 7. – *In the continuous limit $\tau\to 0$, the repeated interaction dynamics $V_{[t/\tau]}$ converges strongly on $\mathcal{H}_S\otimes\Phi$, for all t, to the (unitary) solution of the quantum Langevin equation*

$$\begin{aligned} dU_t = &-\left[iH_S + i\sum_{i=0}^N \beta_i\gamma_i\, I + \frac{1}{2}\sum_{i=1}^N(\beta_0 V_i^*V_i + \beta_i V_iV_i^*)\right]U_t\, dt \\ &- i\sum_{i=1}^N\Big[\sqrt{\beta_i}\, V_i\, U_t\, da^{i,0}_{0,0}(t) + \sqrt{\beta_0}\, V_i^*\, U_t\, da^{0,i}_{0,0}(t) \\ &+\sqrt{\beta_i}\, V_i^*\, U_t\, da^{0,0}_{i,0}(t) + \sqrt{\beta_0}\, V_i\, U_t\, da^{0,0}_{0,i}(t)\Big]\,. \end{aligned} \tag{8}$$

Proof

First, we represent the operators H and U on $\mathcal{H}_S\otimes\mathbb{C}^{N+1}$ as $(N+1)\times(N+1)$-matrices with coefficients in $\mathcal{L}(\mathcal{H}_S)$, following the orthonormal basis $\{e_0, e_1, \ldots, e_N\}$. We get

$$H(\tau) = \begin{pmatrix} H_S+\gamma_0 I & \frac{1}{\sqrt{\tau}}V_1^* & \frac{1}{\sqrt{\tau}}V_2^* & \ldots & \frac{1}{\sqrt{\tau}}V_N^* \\ \frac{1}{\sqrt{\tau}}V_1 & H_S+\gamma_1 I & 0 & \ldots & 0 \\ \frac{1}{\sqrt{\tau}}V_2 & 0 & H_S+\gamma_2 I & \ldots & 0 \\ \vdots & \vdots & \ldots & \ddots & \vdots \\ \frac{1}{\sqrt{\tau}}V_N & 0 & 0 & \ldots & H_S+\gamma_N I \end{pmatrix}.$$

We need now to compute the associated unitary operator $U=e^{-i\tau H}$ in the same framework. But as we wish to apply Theorem 6, we do not need to wonder about the exact expression of U, but only the expansion of its coefficients in powers of τ up to the pertinent orders given by Theorem 6. We get that U is represented by the matrix

$$\begin{pmatrix} I-i\tau H_S-i\tau\gamma_0 I & -i\sqrt{\tau}\,V_1^*+o(\tau^{3/2}) & \dots & -i\sqrt{\tau}\,V_N^*+o(\tau^{3/2}) \\ -\frac{1}{2}\tau\sum_{i=1}^N V_i^*V_i+o(\tau^2) & & & \\ -i\sqrt{\tau}\,V_1+o(\tau^{3/2}) & I-i\tau H_S-i\tau\gamma_1 I & \dots & -\frac{1}{2}V_1V_N^*+o(\tau^2) \\ & -\frac{1}{2}\tau V_1V_1^*+o(\tau^2) & & \\ \vdots & \vdots & \ddots & \vdots \\ -i\sqrt{\tau}\,V_N+o(\tau^{3/2}) & -\frac{1}{2}\tau V_NV_1^*+o(\tau^2) & \dots & I-i\tau H_S-i\tau\gamma_N I \\ & & & -\frac{1}{2}V_NV_N^*+o(\tau^2) \end{pmatrix}.$$

This is for the expression of U as an operator on $\mathcal{H}_S\otimes\mathbb{C}^{N+1}$. Now our aim is to compute the coefficients of the matrix of $\pi(U)$ in the G.N.S. representation, that is, as a $(N+1)^2\times(N+1)^2$-matrix with coefficients in $\mathcal{L}(\mathcal{H}_S)$.

As we already discussed in section IV, the coefficients of $\pi(U)$ are obtained by computing the quantities

$$U_{k,l}^{i,j}=\langle X_l^k,\pi(U)X_j^i\rangle=\mathrm{tr}_{\mathcal{H}}(\rho_\beta\,(X_l^k)^*UX_j^i).$$

In order to get $U_{0,0}^{0,0}$ we have to compute

$$\mathrm{tr}_{\mathcal{H}}(\rho_\beta\,X_0^0UX_0^0)=\mathrm{tr}_{\mathcal{H}}(\rho_\beta\,U).$$

This gives, using $\sum_{i=0}^N\beta_i=1$

$$\begin{aligned} U_{0,0}^{0,0}=\beta_0&\left(I-i\tau H_S-i\tau\gamma_0 I-\frac{1}{2}\tau\sum_{i=1}^N V_i^*V_i+o(\tau^2)\right)+\\ &+\beta_1\left(I-i\tau H_S-i\tau\gamma_1 I-\frac{1}{2}\tau V_1V_1^*+o(\tau^2)\right)+\dots\\ &+\beta_N\left(I-i\tau H_S-i\tau\gamma_N I-\frac{1}{2}\tau V_NV_N^*+o(\tau^2)\right)\end{aligned}$$

$$U_{0,0}^{0,0}=I-i\tau H_S-i\tau\sum_{i=1}^N\beta_i\gamma_i\,I-\frac{1}{2}\tau\sum_{i=0}^N(\beta_0V_i^*V_i+\beta_iV_iV_i^*)+o(\tau^2). \qquad (9)$$

We now compute the $U_{0,0}^{i,j}$ terms, for $i\neq j$. That is, we compute

$$U_{0,0}^{i,j}=\mathrm{tr}_{\mathcal{H}}(\rho_\beta\,UX_j^i).$$

This trace is equal to

$$\frac{1}{\sqrt{\beta_i}}\langle e_i,\rho_\beta\,Ue_j\rangle=\sqrt{\beta_i}\,\langle e_i,Ue_j\rangle.$$

Then two distinct cases appear. If $j=0$ we have

$$Ue_0=\begin{pmatrix} I-i\tau H_S-i\tau\gamma_0 I-\frac{1}{2}\tau\sum_{i=1}^N V_i^*V_i+o(\tau^2) \\ -i\sqrt{\tau}\,V_1+o(\tau^{3/2}) \\ \vdots \\ -i\sqrt{\tau}\,V_N+o(\tau^{3/2}) \end{pmatrix}$$

and thus

$$U^{i,0}_{0,0} = -i\sqrt{\tau}\sqrt{\beta_i}\, V_i + o(\tau^{3/2}). \tag{10}$$

But when $j \neq 0$ we have

$$Ue_j = \begin{pmatrix} -i\sqrt{\tau}\, V_j^* + o(\tau^{3/2}) \\ -\frac{1}{2}\tau V_1 V_j^* + o(\tau^2) \\ \vdots \\ I - i\tau H_S - i\tau\gamma_j I - \frac{1}{2}\tau V_j V_j^* + o(\tau^2) \\ \vdots \\ -\frac{1}{2}\tau V_N V_j^* + o(\tau^2) \end{pmatrix}.$$

Now, if $i = 0$ we get

$$U^{0,j}_{0,0} = -i\sqrt{\beta_0}\sqrt{\tau}\, V_j^* + o(\tau^{3/2}), \tag{11}$$

if $i \neq 0$ and $i \neq j$ we get

$$U^{i,j}_{0,0} = -\frac{1}{2}\sqrt{\beta_i}\,\tau\, V_i V_j^* + o(\tau^2). \tag{12}$$

A similar computation gives

$$U^{0,0}_{0,j} = -i\sqrt{\tau}\sqrt{\beta_0}\, V_j + o(\tau^{3/2}), \tag{13}$$

$$U^{0,0}_{i,0} = -i\sqrt{\beta_i}\sqrt{\tau}\, V_i^* + o(\tau^{3/2}) \tag{14}$$

and, still for $i \neq j$ and $i, j \neq 0$

$$U^{0,0}_{i,j} = -\frac{1}{2}\sqrt{\beta_i}\,\tau\, V_j V_i^* + o(\tau^2). \tag{15}$$

Now, consider $i \neq j$ and $k \neq l$, we have

$$\begin{aligned} U^{i,j}_{k,l} &= \mathrm{tr}_{\mathcal{H}}(\rho_\beta\, (X^k_l)^* U X^i_j) = \sum_p \langle X^k_l \rho_\beta\, e_p, U X^i_j\, e_p\rangle \\ &= \sum_p \beta_p \frac{1}{\sqrt{\beta_k \beta_i}} \langle a^k_l e_p, U a^i_j e_p\rangle \\ &= \delta_{ik}\langle e_l, U e_j\rangle. \end{aligned}$$

That is, if $l = 0$ and $j \neq 0$

$$U^{i,j}_{k,0} = \delta_{ik}(-i\sqrt{\tau} V_j^* + o(\tau^{3/2})) \tag{16}$$

and for $l = j = 0$

$$U^{i,0}_{k,0} = \delta_{ik}(I - i\tau H_S - i\tau\gamma_0 I - \frac{1}{2}\sum_{i=1}^N V_i^* V_i + o(\tau^2)). \tag{17}$$

If $l \neq 0$ and $l \neq j$ with $j \neq 0$ we have

$$U^{i,j}_{k,l} = \delta_{ik}(-\frac{1}{2}\tau\, V_l V_j^* + o(\tau^2)), \tag{18}$$

if $l = j \neq 0$

$$U^{i,j}_{k,j} = \delta_{ik}(I - i\tau H_S - i\tau\gamma_l\, I - \frac{1}{2}\tau\, V_l V_l^* + o(\tau^2)) \tag{19}$$

and finally if $l \neq 0$ and $j = 0$ then

$$U_{k,l}^{i,0} = \delta_{ik}(-i\sqrt{\tau} V_l + o(\tau^{3/2})). \tag{20}$$

We now study the $U_{j,j}^{i,i}$ terms. Recall that the X_i^i are diagonal matrices whose coefficients $(\lambda_i^0, \dots \lambda_i^N)$, $i = 0, \dots, N$ form an orthonormal basis of $\mathbb{C}^{N+1}$ equipped with the scalar product $\sum_i \beta_i \overline{x_i} y_i$. We get

$$\begin{aligned}
U_{0,0}^{i,i} = \operatorname{tr}(\rho_\beta U X_i^i) &= \sum_{k=0}^{N} \langle e_k, \rho_\beta U X_i^i e_k \rangle \\
&= \sum_{k=0}^{N} \beta_k \lambda_i^k \langle e_k, U e_k \rangle \\
&= \sum_{k=1}^{N} \beta_k \lambda_i^k (I - i\tau H_S - i\tau\gamma_k I - \frac{1}{2}\tau\, V_k V_k^* + o(\tau^2)) \\
&\quad + \beta_0 \lambda_i^0 (I - i\tau H_S - i\tau\gamma_0 I - \frac{1}{2} \sum_{i=1}^{N} V_i^* V_i + o(\tau^2)) \\
&= \sum_{k=1}^{N} \beta_k \lambda_i^k (-i\tau\gamma_k I - \frac{1}{2}\tau\, V_k V_k^* + o(\tau^2)) \\
&\quad + \beta_0 \lambda_i^0 (-i\tau\gamma_0 I - \frac{1}{2} \sum_{i=1}^{N} V_i^* V_i + o(\tau^2))
\end{aligned} \tag{21}$$

where we have used $\sum_k \beta_k \lambda_i^k = 0$. In the same way,

$$\begin{aligned}
U_{i,i}^{0,0} = &\sum_{k=1}^{N} \beta_k \overline{\lambda_i^k} (-i\tau\gamma_k\, I - \frac{1}{2}\tau\, V_k V_k^* + o(\tau^2)) \\
&+ \beta_0 \overline{\lambda_i^0} (-i\tau\gamma_0 I - \frac{1}{2} \sum_{i=1}^{N} V_i^* V_i + o(\tau^2)).
\end{aligned} \tag{22}$$

Finally, using $\sum_{k=0}^{N} \beta_k\, \overline{\lambda_j^k} \lambda_i^k = \delta_{ij}$, we obtain

$$\begin{aligned}
U_{j,j}^{i,i} &= \sum_{k=0}^{N} \beta_k\, \overline{\lambda_j^k} \lambda_i^k \langle e_k, U e_k \rangle \\
&= \delta_{ij}(I - i\tau H_S) + \sum_{k=1}^{N} \beta_k\, \overline{\lambda_j^k} \lambda_i^k (-i\tau\gamma_k I - \frac{1}{2}\tau\, V_k V_k^* + o(\tau^2)) \\
&\quad + \beta_0 \overline{\lambda_j^0} \lambda_i^0 (-i\tau\gamma_0 - \frac{1}{2}\tau \sum_{j=1}^{N} V_j^* V_j + o(\tau^2)).
\end{aligned} \tag{23}$$

The last terms to be considered are those of type $U_{k,l}^{i,i}$ (and conversely $U_{i,i}^{k,l}$), with

$k \neq l$ and $i \neq 0$. We have

$$U^{i,i}_{k,l} = \mathrm{tr}(\rho_\beta\,(X^k_l)^* U X^i_i) = \sum_p \langle e_p, \rho_\beta\,(X^k_l)^* U X^i_i e_p\rangle$$
$$= \sum_p \beta_p\,\lambda^p_i \langle X^k_l e_p, U e_p\rangle$$
$$= \beta_k\,\lambda^k_i \frac{1}{\sqrt{\beta_k}}\langle e_l, U e_k\rangle$$

which gives

$$\sqrt{\beta_0}\,\lambda^0_i(-i\sqrt{\tau}V_l + o(\tau^{3/2})) \tag{24}$$

or

$$\sqrt{\beta_k}\lambda^k_i(-\frac{1}{2}\tau V_l V^*_k + o(\tau^2)) \tag{25}$$

depending on $k = 0$ or not. The case of $U^{k,l}_{i,i}$ is similar and needs not be explicited for anyway it will not contribute to the continuous limit.

We can now apply Theorem 6. Following the rules of Theorem 6, we need to check that there exists bounded operators $L^{i,j}_{k,l}$ on $\mathcal{H}_S$ such that

$$s - \lim_{\tau\to 0} \frac{U^{i,j}_{k,l} - \delta_{(i,j),(k,l)}\, I}{\tau^{\varepsilon^{i,j}_{k,l}}} = L^{i,j}_{k,l}\,,$$

where $\varepsilon^{0,0}_{0,0} = 1$, $\varepsilon^{0,0}_{k,l} = \varepsilon^{k,l}_{0,0} = 1/2$ and the others $\varepsilon^{i,j}_{k,l}$ are equal to 0.

Equality (9) shows that

$$L^{0,0}_{0,0} = -iH_S - i\sum_{i=1}^N \beta_i\gamma_i I - \frac{1}{2}\sum_i(\beta_0 V^*_i V_i + \beta_i V_i V^*_i).$$

By (10) we have

$$L^{i,0}_{0,0} = -i\sqrt{\beta_i}V_i$$

and in the same way

$$L^{0,i}_{0,0} = -i\sqrt{\beta_0}V^*_i$$
$$L^{0,0}_{i,0} - i\sqrt{\beta_i}V^*_i$$
$$L^{0,0}_{0,i} = -i\sqrt{\beta_0}V_i$$

by (11), (14) and (13) respectively.

The other terms $U^{i,j}_{0,0}$ and $U^{0,0}_{i,j}$ appear to be of order τ in (12) and (15), while only their $\sqrt{\tau}$ part contributes to the limit. As a consequence $L^{i,j}_{0,0} = L^{0,0}_{i,j} = 0$.

Terms of the form $U^{i,j}_{k,l}$ (equations (16) to (20)) contribute in the limit via the order 1 terms in $U^{i,j}_{k,l} - \delta_{(i,j),(k,l)}\, I$, that is 0 in anycase (the I term in (17) and (19) does not contribute as it indeed appears only when $(i,j) = (k,l)$.

The same holds for $U^{i,i}_{0,0}$ and $U^{0,0}_{i,i}$ which gives $L^{i,i}_{0,0} = L^{0,0}_{i,i} = 0$ (equations (21) and (22)).

The terms $U^{i,i}_{j,j}$ contribute in the limit via the order 1 terms of $U^{i,i}_{j,j} - \delta_{ij} I$. Following (23) we get a null contribution in all cases.

Finally, equality (24) and (25) show that the last coefficients also vanish in the limit.

This exactly gives the announced quantum Langevin equation. ■

Remark: One can only be impressed (at least that was the case of the authors when performing the computations) by the kind of "mathematical miracle" occuring here: all the $(N+1)^2$ terms fit perfectly in the type of conditions of Theorem 6. The number of cancellation one may hope for happens exactly.

Equation (8) takes a much more useful form if one regroups correctly the different terms. Indeed, put

$$A_i^0(t) = \sqrt{\frac{\beta_0}{\beta_0-\beta_i}}\, a_{0,i}^{0,0}(t) + \sqrt{\frac{\beta_i}{\beta_0-\beta_i}}\, a_{0,0}^{i,0}(t) \tag{26}$$

$$A_0^i(t) = \sqrt{\frac{\beta_0}{\beta_0-\beta_i}}\, a_{0,0}^{0,i}(t) + \sqrt{\frac{\beta_i}{\beta_0-\beta_i}}\, a_{i,0}^{0,0}(t) \tag{27}$$

and

$$W_i = -i\sqrt{\beta_0-\beta_i}\, V_i$$

then the equation (8) simply writes

$$\begin{aligned} dU_t = &-\left[iH_S + i\sum_{i=0}^N \beta_i\gamma_i\, I + \frac{1}{2}\sum_{i=1}^N \left(\frac{\beta_0}{\beta_0-\beta_i} W_i^* W_i + \frac{\beta_i}{\beta_0-\beta_i} W_i W_i^*\right)\right] U_t\, dt \\ &+ \sum_i \left(W_i U_t\, dA_i^0(t) - W_i^* U_t\, dA_0^i(t)\right). \end{aligned} \tag{28}$$

VI. Thermal quantum noises and their properties

In this section we concentrate on the particular quantum noises (26) and (27) that appeared above. We shall show that they are natural candidates for being qualified as "thermal quantum noises". We also show that the form of equation (25) is the generic one for unitary solutions, in the thermal case.

The situation that has appeared in the previous section can be summarized as follows.

We consider the quantities β, γ_i and thus β_i as being fixed.

First of all, there is no need to consider a Fock space over $L^2(\mathbb{R}^+; \mathbb{C}^{(N+1)^2-1})$ anymore, for most of the quantum noises $a_{k,l}^{i,j}(t)$ do not play any role in equation (8). More economical is to consider a double Fock space:

$$\widetilde{\Phi} = \Gamma_s(L^2(\mathbb{R}^+, \mathbb{C}^N)) \otimes \Gamma_s(L^2(\mathbb{R}^+, \mathbb{C}^N)).$$

Each of the copies of the Fock space accomodates the quantum noises

$$a_j^i(t) \otimes I \quad \text{and} \quad I \otimes a_j^i(t)$$

respectively, which we shall denote more simply by

$$a_j^i(t) \quad \text{and} \quad b_j^i(t)$$

respectively.

Form the operator processes

$$A_i^0(t) = \sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, a_i^0(t) + \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, b_0^i(t) \tag{28}$$

$$A_0^i(t) = \sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, a_0^i(t) + \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, b_i^0(t). \tag{29}$$

For every $f \in L^2(\mathbb{R}^+; \mathbb{C}^n)$ with coordinates (f_i) in the basis $\{e_1, \ldots, e_n\}$ put

$$A^*(f) = \sum_{i=1}^n \int_{\mathbb{R}^+} f_i(t)\, dA_i^0(t)$$

and

$$A(f) = \sum_{i=1}^n \int_{\mathbb{R}^+} \overline{f_i(t)}\, dA_0^i(t).$$

Proposition 8. – *The operators $A(f), A^*(g)$ form a non-Fock representation of the CCR algebra over $(L^2(\mathbb{R}^+; \mathbb{C}^N))$.*

Proof

The operators $A(f)$ and $A^*(g)$ have similar properties as the usual quantum noises. In particular, they admit a quantum stochastic integration theory, which is completely identical to the usual one. This does not need to be developed here. We shall only prove that the quantum Ito formula (7) is now driven by the rules:

$$dA_0^i(t)\, dA_i^0(t) = \frac{\beta_0}{\beta_0 - \beta_i}\, dt$$

and

$$dA_i^0(t)\, dA_0^i(t) = \frac{\beta_i}{\beta_0 - \beta_i}\, dt.$$

Indeed, we have

$$dA_0^i(t)\, dA_i^0(t) =$$

$$= \left(\sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, da_0^i(t) + \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, db_i^0(t)\right)\left(\sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, da_i^0(t) + \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, db_0^i(t)\right)$$

$$= \frac{\beta_0}{\beta_0 - \beta_i}\, da_0^0(t) = \frac{\beta_0}{\beta_0 - \beta_i}\, dt$$

and

$$dA_i^0(t)\, dA_0^i(t) =$$

$$= \left(\sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, da_i^0(t) + \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, db_0^i(t)\right)\left(\sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, da_0^i(t) + \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, db_i^0(t)\right)$$

$$= \frac{\beta_i}{\beta_0 - \beta_i}\, db_0^0(t) = \frac{\beta_i}{\beta_0 - \beta_i}\, dt\,.$$

By the quantum Ito formula we get

$$\begin{aligned}[A(f), A^*(g)] &= \sum_{i=1}^N \int_{I\!R^+} \overline{f_i(t)}\, g_i(t) \, \left(\frac{\beta_0-\beta_i}{\beta_0-\beta_i}\right) dt\, I \\ &= \langle f, g\rangle I.\end{aligned}$$

In other words, the operators $A(f), A^*(g)$ form a representation of the CCR algebra over $(L^2(I\!R^+;\mathbb{C}^N))$. But this clearly a non-Fock one for the creation and annihilation operator attached to this representation do not generate the whole creation and annihilation operators of the underlying (double) Fock space. ■

Now, let us form the associated Weyl operators

$$W(f) = \exp\left(\frac{A(f)+A^*(f)}{\sqrt{2}}\right).$$

We wish to compute the statistics of $W(f)$ in the vaccum state Ω. For this purpose, we use the following notation. If H is any operator on $\mathbb{C}^{N+1}$, then it acts on $L^2(I\!R^+;\mathbb{C}^N)$ by

$$[Hf](s) = \lambda_0 + \sum_{i=1}^N \lambda_i f_i(s).$$

This has to be understood as follows: in general λ_0 is chosen to be equal to 0, thus H acts on $L^2(I\!R^+;\mathbb{C}^N)$ as a multiplication operator. In our case it is the multiplication by a constant (vector).

Theorem 9. – *We have*

$$\langle \Omega, W(f)\,\Omega\rangle = \exp\left(-\frac{1}{4}\langle f, \coth(\beta\frac{H_R}{2})f\rangle\right)$$

for all $f \in L^2(I\!R^+;\mathbb{C}^N)$.

Proof

We have

$$\begin{aligned}A(f)+A^*(f) = \sum_{i=1}^N \int_0^\infty \Bigg(&\sqrt{\frac{\beta_0}{\beta_0-\beta_i}}\, f_i(s)\, da_i^0(s) + \sqrt{\frac{\beta_i}{\beta_0-\beta_i}}\, f_i(s)\, db_0^i(s) + \\ &+ \sqrt{\frac{\beta_0}{\beta_0-\beta_i}}\, \overline{f_i(s)}\, da_0^i(s) + \sqrt{\frac{\beta_i}{\beta_0-\beta_i}}\, \overline{f_i(s)}\, db_i^0(s)\Bigg).\end{aligned}$$

If we put

$$a(f) = \sum_{i=1}^N \int_0^\infty \left(f_i(s)\, da_i^0(s) + \overline{f_i(s)}\, da_0^i(s)\right)$$

and

$$b(f) = \sum_{i=1}^N \int_0^\infty \left(f_i(s)\, db_i^0(s) + \overline{f_i(s)}\, db_0^i(s)\right)$$

then the above expression shows that

$$A(f) + A^*(f) = a(\widetilde{f}) + b(\widehat{f})$$

where

$$\widetilde{f}_i(s) = \sqrt{\frac{\beta_0}{\beta_0 - \beta_i}}\, f_i(s)$$

$$\widehat{f}_i(s) = \sqrt{\frac{\beta_i}{\beta_0 - \beta_i}}\, \overline{f_i}(s).$$

Denote by W_a and W_b the usual Weyl operators associated to the noises a and b respectively. We have clearly shown that

$$W(f) = W_a(\widetilde{f}) \otimes W_b(\widehat{f}).$$

As a consequence, using usual computations on the Weyl operators

$$\begin{aligned}
\langle \Omega, W(f)\Omega\rangle &= \exp\left(-\frac{1}{4}\left(\left|\left|\widetilde{f}\right|\right|^2 + \left|\left|\widehat{f}\right|\right|^2\right)\right) \\
&= \exp\left(-\frac{1}{4}\sum_{i=1}^{n}\left(\frac{\beta_0}{\beta_0 - \beta_i} + \frac{\beta_i}{\beta_0 - \beta_i}\right)||f_i||^2\right) \\
&= \exp\left(-\frac{1}{4}\sum_{i=1}^{n}\left(\coth(\beta\frac{\gamma_i}{2})\,||f_i||^2\right)\right) \\
&= \exp\left(-\frac{1}{4}\langle f, \coth(\beta\frac{H_R}{2}) f\rangle\right).
\end{aligned}$$

■

We recover an analogue of the usual K.M.S. state statistics for a free Boson gas at thermal equilibrium. Let us discuss that point more precisely. Usually, the Hamiltonian model for a quantum heat bath is as follows. We are given a function $\omega(s)$ (in Fourier representation actually, by this does not matter much here) and the Hamiltonian of the heat bath, on the Fock space $\Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}))$ is the differential second quantization operator $d\Gamma(\omega)$ associated to the multiplication by ω.

In our discrete model, if we take the typical interacting system $\mathcal{H}_R$ to be $\mathbb{C}^2$ but with a Hamiltonian depending on the number of the copy:

$$H_R(k) = \begin{pmatrix} 0 & 0 \\ 0 & \omega(k) \end{pmatrix}$$

then in the continuous limit, the corresponding Hamiltonian on the Fock space $\Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}))$ is indeed $d\Gamma(\omega)$ (under some continuity assumption on ω, cf [AP2]).

In the case we have described here, the situation is made a little more complicated by the fact that we considered a chain of $\mathbb{C}^{N+1}$ instead of $\mathbb{C}^2$, but a lot easier by taking a constant Hamiltonian H_R. The time-dependent case stays to be explored, no doubt it will give rise to the usual free Bose gaz statistics.

Remark : The parameter β used here is supposed to be the inverse of the temperature of the heat bath (more exactly $1/kT$). If we make the temperature go to 0, that is, β goes to $+\infty$, then

$$\frac{\beta_0}{\beta_0-\beta_i}=\frac{1}{1-e^{-\beta(\gamma_i-\gamma_0)}}$$

converges to 1, for $\gamma_i-\gamma_0>0$ by hypothesis, and

$$\frac{\beta_i}{\beta_0-\beta_i}=\frac{e^{-\beta\gamma_i}}{1-e^{-\beta(\gamma_i-\gamma_0)}}$$

converges to 0. This makes all the noises b_j^i being useless and $A(g)=a(g)$, for all g. We recover the usual quantum noises, the usual Weyl operators. This means that the usual quantum noises are the 0 temperature ones.

We now describe which kind of quantum Langevin equation, driven by those thermal quantum noises gives rise to a unitary evolution.

Theorem 10. – *A Langevin equation of the form*

$$dU_t=K_0^0U_t\,dt+\sum_{i=0}^n\left(K_i^0U_t\,dA_i^0(t)+K_0^iU_t\,dA_0^i(t)\right),\qquad U_0=I,$$

where the coefficients K_j^i are all bounded operators on $\mathcal{H}_S$, always admit a unique solution on the set of coherent vectors. The solution is unitary if and only if it is of the form

$$dU_t=\left(-iH-\frac{1}{2}\sum_{i=1}^n\left(\frac{\beta_0}{\beta_0-\beta_i}W_i^*W_i+\frac{\beta_i}{\beta_0-\beta_i}W_iW_i^*\right)\right)U_t\,dt+$$
$$+\sum_{i=0}^n\left(W_iU_t\,dA_i^0(t)-W_i^*U_t\,dA_0^i(t)\right)$$

for some bounded operators W_i, $i=1,\ldots,n$, on $\mathcal{H}_S$ and a self-adjoint bounded operator H on $\mathcal{H}_S$.

Proof

The existence and uniqueness result is a simple consequence of the one quoted in Theorem 5.

For characterizing the unitarity we use algebraical (formal) computations, which are the same as for the proof of Theorem 5. The analytical part of the proof is totaly identical to the one of Theorem 5. There is no need to develop it here.

Our equation is of the general form

$$dU_t=K_0^0U_t\,dt+\sum_{i=0}^n\left(K_i^0U_t\,dA_i^0(t)+K_0^iU_t\,dA_0^i(t)\right)$$

We thus also have

$$dU_t^*=U_t^*(K_0^0)^*\,dt+\sum_{i=0}^n\left(U_t^*(K_i^0)^*\,dA_0^i(t)+U_t^*(K_0^i)^*\,dA_i^0(t)\right).$$

By the quantum Ito formula we get

$$\begin{aligned} d(U_t^* U_t) &= (dU_t^*) U_t + U_t^* \, dU_t + dU_t^* \, dU_t \\ &= U_t^* (K_0^0)^* U_t \, dt + \sum_{i=0}^n \big(U_t^* (K_i^0)^* U_t \, dA_0^i(t) + U_t^* (K_0^i)^* U_t \, dA_i^0(t) \big) + \\ &\quad + U_t^* K_0^0 U_t \, dt + \sum_{i=0}^n \big(U_t^* K_i^0 U_t \, dA_i^0(t) + U_t^* K_0^i U_t \, dA_0^i(t) \big) + \\ &\quad + \frac{\beta_0}{\beta_0 - \beta_i} \sum_{i=1}^n U_t^* (K_i^0)^* K_i^0 U_t \, dt + \frac{\beta_i}{\beta_0 - \beta_i} \sum_{i=1}^n U_t^* (K_0^i)^* K_0^i U_t \, dt \\ &= U_t^* \left((K_0^0)^* + K_0^0 + \sum_{i=1}^n \left(\frac{\beta_0}{\beta_0 - \beta_i} (K_i^0)^* K_i^0 + \frac{\beta_i}{\beta_0 - \beta_i} (K_0^i)^* K_0^i \right) \right) U_t \, dt \\ &\quad + \sum_{i=0}^n \big(U_t^* \left((K_i^0)^* + K_0^i \right) U_t \big) \; dA_0^i(t) + \sum_{i=0}^n \big(U_t^* \left((K_0^i)^* + K_i^0 \right) U_t \big) \; dA_i^0(t). \end{aligned}$$

By a similar computation we obtain

$$\begin{aligned} &d(U_t U_t^*) = \\ &= \Bigg(U_t U_t^* (K_0^0)^* + K_0^0 U_t U_t^* + \sum_{i=1}^n \Bigg(\frac{\beta_0}{\beta_0 - \beta_i} K_0^i U_t U_t^* (K_0^i)^* + \\ &\qquad\qquad + \frac{\beta_i}{\beta_0 - \beta_i} K_i^0 U_t U_t^* (K_i^0)^* \Bigg) \Bigg) \; dt \\ &\quad + \sum_{i=0}^n \big(U_t U_t^* (K_i^0)^* + K_0^i U_t U_t^* \big) \; dA_0^i(t) + + \sum_{i=0}^n \big(U_t U_t^* (K_0^i)^* + K_i^0 U_t U_t^* \big) \; dA_i^0(t). \end{aligned}$$

Asking both to be equal to 0 for every t is equivalent to the following conditions :

$$K_0^i = -(K_i^0)^*$$

$$(K_0^0)^* + K_0^0 + \sum_{i=1}^n \left(\frac{\beta_0}{\beta_0 - \beta_i} (K_i^0)^* K_i^0 + \frac{\beta_i}{\beta_0 - \beta_i} K_i^0 (K_i^0)^* \right) = 0.$$

Put

$$K = K_0^0 + \frac{1}{2} \sum_{i=1}^n \left(\frac{\beta_0}{\beta_0 - \beta_i} (K_i^0)^* K_i^0 + \frac{\beta_i}{\beta_0 - \beta_i} K_i^0 (K_i^0)^* \right),$$

the last condition above exactly says

$$K = -K^*.$$

We thus obtain the announced characterization. ■

The attentive reader has noticed that this is exactly the form of equation (28)!

VII. The Lindblad generator

Going back to the usual Langevin equations of Theorem 5, let us recall a very important theorem, which is the main point in using quantum Langevin equations in order to dilate quantum dynamical semigroups.

Theorem 11. – *Consider the unitary solution $(U_t)_{t\geq 0}$ of the quantum Langevin equation*

$$dU_t = -(iH + \frac{1}{2}\sum_{k=1}^{N} L_k^* L_k) U_t \, dt + \sum_{i=0}^{N} W_i U_t \, da_i^0(t) - \sum_{i=0}^{n} W_k^* U_t \, da_0^i(t).$$

For any bounded operator X on $\mathcal{H}_S$, the application

$$t \mapsto P_t(X) = \langle \Omega, U_t^* (X \otimes I) U_t \Omega \rangle$$

is a semigroup of completely positive maps whose Lindblad generator is

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}\sum_{i=1}^{N} (W_i^* W_i X + X W_i^* W_i - 2 W_i^* X W_i).$$

■

In our thermal case we have the following the form for the Lindblad generator.

Theorem 12. – *Consider the unitary solution $(U_t)_{t\geq 0}$ of the thermal quantum Langevin equation*

$$dU_t = \left(-iH - \frac{1}{2}\sum_{i=1}^{N} \left(\frac{\beta_0}{\beta_0 - \beta_i} W_i^* W_i + \frac{\beta_i}{\beta_0 - \beta_i} W_i W_i^* \right) \right) U_t \, dt + \\ + \sum_{i=0}^{N} \left(W_i U_t \, dA_i^0(t) - W_i^* U_t \, dA_0^i(t) \right).$$

For any bounded operator X on $\mathcal{H}_S$, the application

$$t \mapsto P_t(X) = \langle \Omega, U_t^* (X \otimes I) U_t \Omega \rangle$$

is a semigroup of completely positive maps whose Lindblad generator is

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}\sum_{i=1}^{N} \frac{\beta_0}{\beta_0 - \beta_i} (W_i^* W_i X + X W_i^* W_i - 2 W_i^* X W_i) \\ - \frac{1}{2}\sum_{i=1}^{N} \frac{\beta_i}{\beta_0 - \beta_i} (W_i W_i^* X + X W_i W_i^* - 2 W_i X W_i^*).$$

Proof

The basic computation is the same as for Theorem 11:

– By the "thermal quantum Ito formula" (see proof of Proposition 8) one computes

$$d(U_t^* (X \otimes I) U_t) = (dU_t^*)(X \otimes I) U_t + U_t^* (X \otimes I)(dU_t) + (dU_t^*)(X \otimes I)(dU_t);$$

– The only contributing term when averaging over the vacuum state is the coefficient of dt, that is, we get the equation

$$d\langle \Omega, U_t^* (X \otimes I) U_t \Omega \rangle = \langle \Omega, U_t^* (\mathcal{L}(X) \otimes I) U_t \Omega \rangle \, dt.$$

The solution is clearly a semigroup with generator $\mathcal{L}$. ■

Let us write down in a corollary, the Lindblad generator in perspective with the initial Hamiltonian.

Corollary 13 – *If the repeated interaction model is having the following total Hamiltonian:*

$$H = H_S \otimes I + I \otimes H_R + \frac{1}{\sqrt{\tau}} \sum_{i=1}^{N} \left(V_i \otimes a_i^0 + V_i^* \otimes a_0^i \right)$$

then the associated Lindblad generator in the continuous limit is

$$\begin{aligned} \mathcal{L}(X) = i[H_S, X] &- \frac{1}{2} \sum_{i=1}^{N} \beta_0 \left(V_i^* V_i X + X V_i^* V_i - 2 V_i^* X V_i \right) \\ &- \frac{1}{2} \sum_{i=1}^{N} \beta_i \left(V_i V_i^* X + X V_i V_i^* - 2 V_i X V_i^* \right). \end{aligned}$$

VIII. Thermalization

In this section we answer a very natural question in this context. Consider a given quantum system $\mathcal{H}_S$ with a given Hamiltonian H_S. Is there a natural Lindblad generator $\mathcal{L}$ (in the Schrödinger picture) on $\mathcal{H}_S$ which admits as a unique invariant state, the state

$$\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H_S}$$

and which possesses the property of return to equilibrium for this state? By "return to equilibrium" we mean the following: for every inital state ρ_0, the evolution $e^{t\mathcal{L}}(\rho_0)$ converges to the state ρ_β in the $*$-weak sense, that is,

$$\lim_{t \to +\infty} \operatorname{tr}(e^{t\mathcal{L}}(\rho_0)\, X) = \operatorname{tr}(\rho_\beta\, X)$$

for all observable X.

We shall prove in this section that the answer to the above question is positive, at least if $\mathcal{H}_S$ is finite-dimensional. For this purpose we recall a famous result by Frigerio and Veri [F-V], in a slightly extended form due to Fagnola and Rebolledo [F-R].

Theorem 14 – *Let*

$$\mathcal{L}(\rho) = -i[H, \rho] - \frac{1}{2} \sum_{i=1}^{n} \left(L_i^* L_i \rho + \rho L_i^* L_i - 2 L_i \rho L_i^* \right)$$

be a Lindblad generator (in Schrödinger picture). If the commutants

$$\{H, L_i, L_i^*; i = 1, \dots n\}' \qquad \text{and} \qquad \{L_i, L_i^*; i = 1, \dots n\}'$$

coincide then the associated dynamics possesses the property of return to equilibrium.

We consider $\mathcal{H}_S$ a $N+1$-dimensional Hilbert space, with Hamiltonian (in diagonal form)

$$H_S = \begin{pmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_N \end{pmatrix}.$$

Consider the Gibbs state $\rho_\beta = (1/Z_\beta)\, e^{-\beta H_S}$, it is also of diagonal form with diagonal elements denoted by $\beta_0, \beta_1, \dots, \beta_N$.

We put the system $\mathcal{H}_S$ in repeated quantum interaction with a chain of copies of $\mathbb{C}^{N+1}$ with the total Hamiltonian

$$H = H_S \otimes I + I \otimes H_R + \frac{1}{\sqrt{\tau}} \sum_{i=1}^{N} \left(V_i \otimes a_i^0 + V_i^* \otimes a_0^i \right),$$

where V_i is the matrix

$$V_i = \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \\ 0 & 0 & \dots & & 0 \\ \vdots & & & & \vdots \\ 0 & & 0 & \dots & 0 \end{pmatrix}$$

with the 1 being at the i-th row and where $H_R = H_S$.

Note that this means that we put the system $\mathcal{H}_R$ in repeated quantum interaction with a chain of copies of ... itself but in the desired state. This is somehow very natural!

Theorem 15 – *In the continuous limit the above repeated interaction model admits the following Lindblab generator in the Schrödinger picture:*

$$\begin{aligned} \mathcal{L}(\rho) = -i[H_S, \rho] &- \frac{1}{2} \sum_{i=1}^{N} \beta_0 \left(V_i^* V_i \rho + \rho V_i^* V_i - 2 V_i \rho V_i^* \right) \\ &- \frac{1}{2} \sum_{i=1}^{N} \beta_i \left(V_i V_i^* \rho + \rho V_i V_i^* - 2 V_i^* \rho V_i \right). \end{aligned}$$

This Lindblad generator admits

$$\rho_\beta = \frac{1}{Z_\beta}\, e^{-\beta H_S}$$

as a unique invariant state and it converges to equilibrium.

Proof

The announced Lindblad generator is just the dual of the Lindblad generator described in Corollary 13.

Let us compute $\mathcal{L}(\rho_\beta)$. We have

$$[H_S, \rho_\beta] = 0.$$

On the other hand, the V_i's have been chosen so that

$$\rho_\beta V_i = \frac{\beta_i}{\beta_0} V_i \rho_\beta.$$

This gives

$$V_i^* \rho_\beta = \frac{\beta_i}{\beta_0} \rho_b V_i^*$$

These two relation give

$$V_i^* V_i \rho_\beta + \rho_\beta V_i^* V_i - 2V_i \rho V_i^* = 2V_i^* V_i \rho_\beta - 2\frac{\beta_0}{\beta_i} V_i V_i^* \rho_\beta$$

and

$$V_i V_i^* \rho_\beta + \rho_\beta V_i V_i^* - 2V_i^* \rho V_i = 2V_i V_i^* \rho_\beta - 2\frac{\beta_i}{\beta_0} V_i^* V_i \rho_\beta.$$

Hence we get the result: $\mathcal{L}(\rho_\beta) = 0$.

Let us consider the von Neumann algebra generated by the operators V_i, V_i^*, $i = 1 \dots N$. It is easy to see that this is the whole $\mathcal{B}(\mathcal{H}_S)$. Hence the commutant of this von Neumann algebra is trivial. As we always have the obvious inclusion

$$\{H_R, V_i, V_i^*; i = 1, \dots n\}' \subset \{V_i, V_i^*; i = 1, \dots n\}'$$

we have equality of the two commutants and Theorem 14 applies. This gives the return to equilibrium property and hence the uniqueness of the invariant state. ■

Note the important following fact: we never used the fact that ρ_β is a Gibbs state, we only used the fact that it is a function of H_R. Hence the above result is valid for any state ρ which is a function of H_R.

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