

Localization for Random Unitary Operators

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Abstract

We consider unitary analogs of 1–dimensional Anderson models on $l^2(\mathbb{Z})$ defined by the product $U_\omega = D_\omega S$ where S is a deterministic unitary and D_ω is a diagonal matrix of i.i.d. random phases. The operator S is an absolutely continuous band matrix which depends on a parameter controlling the size of its off-diagonal elements. We prove that the spectrum of U_ω is pure point almost surely for all values of the parameter of S . We provide similar results for unitary operators defined on $l^2(\mathbb{N})$ together with an application to orthogonal polynomials on the unit circle. We get almost sure localization for polynomials characterized by Verblunsky coefficients of constant modulus and correlated random phases.

1 Introduction

Unitary operators displaying a band structure with respect to a distinguished basis of $l^2(\mathbb{N})$ or $l^2(\mathbb{Z})$ appear in different contexts. For example, such operators describe the quantum dynamics of certain models in solid state physics, see e.g. [BB], [BHJ] and references therein. Unitary band matrices also appear naturally in the study of orthogonal polynomials on the unit circle with respect to a measure, as was recognized recently in [CMV]. For a detailed account on orthogonal polynomials on the unit circle, see [S1] (which is briefly surveyed in [S3]). In both situations, the spectral properties of these unitary infinite matrices play an important role.

The spectral analysis of a certain set of deterministic and random unitary operators with band structure is undertaken in [BHJ] and [J1], [J2]. The random cases studied in the first two papers concern a set of matrices on $l^2(\mathbb{Z})$ which are (up to unitary equivalence) of the following form: $U_\omega = D_\omega S$ where S is a deterministic unitary and D_ω is a diagonal matrix of random phases, $\text{diag}\{e^{-i\theta_k^\omega}\}$. The operator S is an absolutely continuous band matrix which depends on a parameter $t \in]0, 1[$ which controls the size of its off-diagonal elements, see Section 2 below.

A related class of unitary random operators has been studied in [K] with quite different methods. We will comment on this at the end of Section 2.

When the phases are i.i.d random variables, typical results obtained for discrete one-dimensional random Schrödinger operators are shown in [BHJ] and [J1] to hold in the unitary setting as well. For instance, the availability of a transfer matrix formalism to express generalized eigenvectors allows to introduce a Lyapunov exponent, to prove a unitary version of the Ishii-Pastur Theorem, and get absence of absolutely continuous spectrum [BHJ]. A density of states can be introduced and a Thouless formula is proven in [J1]. Since these operators can be naturally considered as unitary analogs of the self-adjoint Anderson model, generalizations to d -dimensions are introduced in [J2]. Their localization properties are studied by means of an adaptation to the unitary setup of the fractional moment method due to Aizenman and Molchanov [AM]. In [J2] it is shown for arbitrary dimension that localization takes place if the common distribution of the phases is absolutely continuous and the parameters of the d -dimensional deterministic unitary S are such that S is close to the identity. This is the unitary analog of the familiar large disorder regime under which localization holds for the d -dimensional Anderson model. When applied to the one-dimensional case, this yields localization only if the parameter t is sufficiently small.

One of the goals of the present paper is to prove localization for all values of the parameter t , thereby completing the analogy with the self-adjoint one dimensional Anderson model, where localization holds without an additional disorder assumption. We also complete the picture by considering products of the same sort on $l^2(\mathbb{N})$, for which we prove localization as well.

Another motivation comes from the application of such results to orthogonal polynomials on the unit circle (OPUC), with respect to a measure $d\mu$. These polynomials are characterized by a sequence of complex numbers $\{\alpha_k\}_{k \in \mathbb{N}}$, such that $|\alpha_k| < 1$, $k \in \mathbb{N}$, called the Verblunsky coefficients. These coefficients allow to construct a unitary infinite matrix on $l^2(\mathbb{N})$, the so-called CMV matrix [CMV], which is the equivalent in the OPUC setting of the Jacobi matrix for orthogonal polynomials on the real line. This matrix represents multiplication by z , $z \in S^1$, on $L^2(S^1, d\mu)$ so that its spectral measure is $d\mu$, see [S1]. When the Verblunsky coefficients $\{\alpha_k(\omega)\}_{k \in \mathbb{N}}$ are random, the fine structure of the corresponding random measure $d\mu_\omega$ is of interest. As mentioned in [J1], when $|\alpha_k(\omega)| = r$ for all

$k \in \mathbb{N}$, and only the phases of $\alpha_k(\omega)$ are random, the CMV matrix is unitarily equivalent to the product $D_\omega S$ on $l^2(\mathbb{N})$ which, modulo boundary conditions at site 0, is of the form considered on $l^2(\mathbb{Z})$ above. The point now is that the random phases of the coefficients $\alpha_k(\omega)$ are correlated if the phases of the diagonal matrix D_ω are independent.

Therefore we get as a corollary of our general analysis that localization takes place for random OPUC with certain types of correlated Verblunsky coefficients. Let us mention here that previous localization results for CMV-matrices provided in [GT], [T], [S1], [S2], and [Su] essentially consider independent Verblunsky coefficients.

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2 The Model and Main Result

We introduce in this section the set of infinite random unitary matrices on $l^2(\mathbb{Z})$ we will be interested in. We focus on the properties related to the localization result we are to prove. For further details and generalizations, we refer the reader to [BHJ], [J1] and [J2].

Let us denote by $|k\rangle$ the unit vector at site $k \in \mathbb{Z}$, so that $\{|k\rangle\}_{k \in \mathbb{Z}}$ forms an orthonormal basis of $l^2(\mathbb{Z})$. We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is identified with $\{\mathbb{T}^{\mathbb{Z}}\}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ being the torus, and $\mathbb{P} = \otimes_{k \in \mathbb{Z}} \mathbb{P}_k$, where $\mathbb{P}_k = \nu$ for any $k \in \mathbb{Z}$ and ν is a fixed probability measure on \mathbb{T} , and \mathcal{F} the σ -algebra generated by the cylinders. We introduce a set of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_k : \Omega \rightarrow \mathbb{T}, \quad \text{s.t.} \quad \theta_k^\omega = \omega_k, \quad k \in \mathbb{Z}. \quad (2.1)$$

These random variables $\{\theta_k\}_{k \in \mathbb{Z}}$ are thus i.i.d. on \mathbb{T} .

We consider unitary operators of the form

$$U_\omega = D_\omega S, \quad \text{with} \quad D_\omega = \text{diag} \{e^{-i\theta_k^\omega}\} \quad (2.2)$$

and

$$S = \begin{pmatrix} \ddots & & & & & & \\ & rt & -t^2 & & & & \\ & r^2 & -rt & & & & \\ & rt & r^2 & rt & -t^2 & & \\ & -t^2 & -tr & r^2 & -rt & & \\ & & & rt & r^2 & & \\ & & & -t^2 & -tr & \ddots & \end{pmatrix}, \quad (2.3)$$

where the translation along the diagonal is fixed by $\langle 2k - 2 | S | 2k \rangle = -t^2$, $k \in \mathbb{Z}$. The real parameters t and r are linked by $r^2 + t^2 = 1$ to ensure unitarity. Due to unitary equivalence it suffices to consider $0 \leq t, r \leq 1$. Thus S is determined by t . We shall sometimes write $S(t)$ to emphasize this dependence. The spectrum of $S(t)$ is purely absolutely continuous and is given by

$$\sigma(S(t)) = \Sigma(t) = \{e^{\pm i \arccos(1 - t^2(1 + \cos(y)))}, y \in \mathbb{T}\}. \quad (2.4)$$

For this and other properties of S , relations between U_ω and the physical model alluded to in Section 1 or links with orthogonal polynomials, see [J1] and Section 6.

Remarks:

i) In this definition, S plays the role of the free Laplacian in the self-adjoint Anderson model. Thus U_ω may be considered as a “unitary Anderson-type model”.

ii) Note that the band structure (2.3) is inherited by U_ω . There is no interesting tridiagonal unitary analog of S (and thus U_ω). In fact, it has been shown in Lemma 3.1 of [BHJ] that a tridiagonal unitary operator is either an infinite direct sum of 1×1 and 2×2 blocks or unitarily equivalent (via a multiplication operator) to the standard bilateral shift. In both cases the spectral theory of U_ω would be trivial.

iii) As $t \rightarrow 0$, $S(t)$ tends to the identity operator, whereas as $t \rightarrow 1$, $S(t)$ tends to a direct sum of shift operators. Accordingly, the spectrum of U_ω is pure point if $t = 0$ and purely absolutely continuous if $t = 1$, [BHJ].

Even with the given band structure, our choice of S as the unitary analog of the Laplacian is somewhat arbitrary. One could generalize S as follows: Choose two arbitrary unitary 2×2 -matrices B_1 and B_2 . Let U_e be the unitary operator in $l^2(\mathbb{Z})$ found as the direct sum of identical B_1 -blocks with blocks starting at even indices. Similarly, construct U_o with identical B_2 -blocks, but blocks starting at odd indices. Now let $S' = U_e U_o$. Any such S' is unitary, has the same block structure as S , and is 2-periodic (and thus has trivial spectral theory via Fourier analysis, see Proposition 6.1 [BHJ]). S is found for the particular choice

$$B_1 = \begin{pmatrix} r & t \\ -t & r \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} r & -t \\ t & r \end{pmatrix}. \quad (2.5)$$

Another choice of interest for S' is the one corresponding to

$$B_1 = e^{-i\beta_1} \begin{pmatrix} r e^{-i\alpha_1} & i t e^{i\gamma_1} \\ i t e^{-i\gamma_1} & r e^{i\alpha_1} \end{pmatrix} \quad \text{and} \quad B_2 = e^{-i\beta_2} \begin{pmatrix} r e^{-i\alpha_2} & i t e^{i\gamma_2} \\ i t e^{-i\gamma_2} & r e^{i\alpha_2} \end{pmatrix}, \quad (2.6)$$

which is the most general case except for the fact that the real parameter t is the same in both matrices. Without spelling out the details, it is not difficult to show on the basis of Lemma 3.2 and Section 4 in [BHJ], that the product $D_\omega S'$ is unitarily equivalent (via a multiplication operator again) to $D_\omega \bar{S}$, where

$$\bar{S} = e^{-i\sigma} \begin{pmatrix} \ddots & r t e^{-i\delta} & -t^2 e^{-i\delta} & & & & \\ & r^2 e^{-i\delta} & -r t e^{-i\delta} & & & & \\ & r t e^{i\delta} & r^2 e^{i\delta} & r t e^{-i\delta} & -t^2 e^{-i\delta} & & \\ & -t^2 e^{i\delta} & -t r e^{i\delta} & r^2 e^{-i\delta} & -r t e^{-i\delta} & & \\ & & & r t e^{i\delta} & r^2 e^{i\delta} & & \\ & & & -t^2 e^{i\delta} & -t r e^{i\delta} & \ddots & \end{pmatrix}, \quad (2.7)$$

with $\delta = \alpha_1 - \alpha_2$ and $\sigma = \beta_1 + \beta_2$. Thus our model (2.3) is covered by (2.6) if $\delta = 0$ or $\delta = \pi$, up to a trivial global phase. It is likely that much of our work can be extended to at least this subclass of the operators S' , but we haven't tried to do so.

One of our motivations for including only the single real parameter t in our choice of S , other than simplicity, is that it takes the role of a disorder parameter in the unitary Anderson-type model (2.2). For this consider Remark iii) above and the (multi-dimensional) localization result of [J2]. When restricted to the present one dimensional

setting, it says the following: If the i.i.d. phases θ_k have an absolutely continuous distribution with bounded density, i.e. $d\nu(\theta) = \tau(\theta)d\theta$ on \mathbb{T} , where $0 \leq \tau(\theta) \in L^\infty(\mathbb{T})$, then there exists $t_0 > 0$ such that $t < t_0$ implies that $\sigma(U_\omega)$ is pure point almost surely. Thus t near 0 corresponds to large disorder, t near 1 to small disorder.

In the one-dimensional context, one expects to have localization as soon as $t < 1$ and ν has an absolutely continuous component. Indeed, we show the

Theorem 2.1 *Let U_ω be defined by (2.1), (2.2) and (2.3). If the distribution $d\nu$ of the i.i.d. phases possesses a non-trivial absolutely continuous component and $\text{supp } \nu$ has non-empty interior, then U_ω is pure point almost surely, with exponentially decaying eigenfunctions.*

Here $\text{supp } \nu$ refers to the topological support of ν , i.e. the set of all λ such that $\nu(\lambda - \varepsilon, \lambda + \varepsilon) > 0$ for all $\varepsilon > 0$. The requirement that the $\text{supp } \nu$ contains an open set stems from the method used here to prove positivity of Lyapunov exponents, see Proposition 3.1 and its proof in Section 7 below. This assumption can be dropped, as follows from the work in preparation [HS]. For example, it will be shown there that the Lyapunov exponents are strictly positive if the support of ν contains at least three points.

We need that ν has an a.c. component due to the use of a spectral averaging argument in Section 4 below. One may expect that Theorem 2.1 carries over to singularly distributed ν , as was proven for discrete and continuous one-dimensional Anderson models in [CKM] and [DSS], respectively. But this would require to develop a completely different and much more involved approach.

We further show in Section 5 that our main result is also true for unitary matrices with a similar structure defined on $l^2(\mathbb{N})$, see Theorem 5.1. This extension allows for an application of our analysis to random OPUC which is described in Section 6.

It is interesting to compare our result with the one obtained in [K]. The latter considers unitary random operators $\tilde{U}_\omega = D_\omega \tilde{S}$ quite similar to ours. D_ω is given as in (2.1), (2.2) with purely absolutely continuous random variables θ_k , while $\tilde{S} = \exp(i\kappa\Delta)$ is a deterministic unitary operator with a selfadjoint Toeplitz matrix Δ whose off-diagonal entries decrease super-exponentially in the distance from the diagonal. It is proven in [K] that for small values of the real parameter κ (i.e. in the large disorder regime) the operator U_ω almost surely has pure point spectrum with eigenfunctions decaying faster than any inverse power. While \tilde{S} has rapidly decaying entries away from the diagonal, it is not a band matrix, which excludes the use of most of the tools available for the model considered here. Instead, [K] adapts a method developed in [KMP] for selfadjoint random operators to unitary models which is very robust but gives somewhat weaker results. In particular, it does not allow to prove localization at small disorder, as expected for the one-dimensional case and proven here for our model.

3 Properties of the Model

In order to prove Theorem 2.1, we need to collect some facts about the unitary operator U_ω that are proven in [BHJ] and [J1].

Our definition (2.2) leads to ergodicity of the unitary operator U_ω . Indeed, introducing the shift operator W on Ω by

$$W(\omega)_k = \omega_{k+2}, \quad k \in \mathbb{Z}, \tag{3.1}$$

we get an ergodic set $\{W^j\}_{j \in \mathbb{Z}}$ of translations. With the unitary operators V_j defined on the canonical basis of $l^2(\mathbb{Z})$ by

$$V_j|k\rangle = |k - 2j\rangle, \forall k \in \mathbb{Z}, \quad (3.2)$$

we observe that for any $j \in \mathbb{Z}$

$$U_{W^j \omega} = V_j U_\omega V_j^*. \quad (3.3)$$

Therefore, our random operator U_ω is an ergodic unitary operator. The general theory of ergodic operators, as for example presented in [CL], chapter V, for the self-adjoint case, carries over to the unitary setting. In particular, it follows that the spectrum of U_ω is almost surely deterministic, i.e. there is a subset Σ of the unit circle such that $\sigma(U_\omega) = \Sigma$ for almost every ω . The same holds for the absolutely continuous, singular continuous and pure point parts of the spectrum: There are Σ_{ac} , Σ_{sc} and Σ_{pp} such that almost surely $\sigma_{ac}(U_\omega) = \Sigma_{ac}$, $\sigma_{sc}(U_\omega) = \Sigma_{sc}$ and $\sigma_{pp}(U_\omega) = \Sigma_{pp}$. Moreover, as shown in [J1], we can characterize Σ in terms of the support of $d\nu$ and of the spectrum $\Sigma(t)$ of $S(t)$.

Theorem 3.1 *Under the above hypotheses, the almost sure spectrum of U_ω consists in the set*

$$\Sigma = \exp(i \operatorname{supp} \nu) \Sigma(t) = \{e^{i\alpha} \Sigma(t) \mid \alpha \in \operatorname{supp} \nu\}. \quad (3.4)$$

Let us proceed by recalling some facts concerning the generalized eigenvectors and the associated Lyapunov exponent. Following [BB], [BHJ] and [J1], we study the generalized eigenvectors of U_ω by means of 2×2 transfer matrices. This is possible due to the band structure of U_ω . Our unitary operators differ from those considered in the works above by a unitary transform, so that the formulas differ a little.

Consider

$$\begin{aligned} U_\omega \psi &= e^{i\alpha} \psi, \\ \psi &= \sum_{k \in \mathbb{Z}} c_k |k\rangle, \quad c_k \in \mathbb{C}, \quad \alpha \in \mathbb{C}. \end{aligned} \quad (3.5)$$

This equation is equivalent to the relations for all $k \in \mathbb{Z}$,

$$\begin{pmatrix} c_{2(k+1)-1} \\ c_{2(k+1)} \end{pmatrix} = T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)) \begin{pmatrix} c_{2k-1} \\ c_{2k} \end{pmatrix}, \quad (3.6)$$

where the function $T : \mathbb{T}^2 \rightarrow M_2(\mathbb{C})$ is defined by

$$\begin{aligned} T(\theta, \eta)_{11} &= -e^{-i\eta} \\ T(\theta, \eta)_{12} &= \frac{r}{t} \left(e^{i(\theta-\eta)} - e^{-i\eta} \right) \\ T(\theta, \eta)_{21} &= \frac{r}{t} \left(1 - e^{-i\eta} \right) \\ T(\theta, \eta)_{22} &= -\frac{1}{t^2} e^{i\theta} + \frac{r^2}{t^2} \left((e^{i(\theta-\eta)} + 1) - e^{-i\eta} \right), \end{aligned} \quad (3.7)$$

and the phases by

$$\theta_k^\omega(\alpha) = \theta_k^\omega + \alpha. \quad (3.8)$$

Note that $\det T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)) = e^{i(\theta_{2k}^\omega - \theta_{2k+1}^\omega)}$ is independent of α and of modulus one.

Introducing the notation

$$T(k, \omega) \equiv T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)), \quad (3.9)$$

we compute for any $k \in \mathbb{N}$, assuming (c_{-1}, c_0) known,

$$\begin{aligned} \begin{pmatrix} c_{2k-1} \\ c_{2k} \end{pmatrix} &= T(k-1, \omega) \cdots T(1, \omega) T(0, \omega) \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix} \equiv \Phi(k, \omega) \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix} \\ \begin{pmatrix} c_{-2k+1} \\ c_{-2k+2} \end{pmatrix} &= T(-k, \omega)^{-1} \cdots T(-2, \omega)^{-1} T(-1, \omega)^{-1} \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix} \equiv \Phi(-k, \omega) \begin{pmatrix} c_{-1} \\ c_0 \end{pmatrix}, \end{aligned} \quad (3.10)$$

with $\Phi(0, \omega) = \mathbb{I}$. The dynamical system defined that way is ergodic,

$$\begin{aligned} \Phi(k, \omega) &= T(0, W^{k-1}(\omega)) \cdots T(0, W(\omega)) T(0, \omega) \\ \Phi(-k, \omega) &= T(-1, W^{-k+1}(\omega))^{-1} \cdots T(-1, W^{-1}(\omega))^{-1} T(-1, \omega)^{-1} \end{aligned} \quad (3.11)$$

and the determinant of the transfer matrices is of modulus one. As shown in [BHJ], it follows that for any $\alpha \in \mathbb{C}$, the Lyapunov exponent

$$\gamma_\omega(e^{i\alpha}) = \lim_{k \rightarrow \pm\infty} \frac{1}{|k|} \ln \|\Phi(k, \omega)\| \quad (3.12)$$

almost surely exists, has the same value for $k \rightarrow \infty$ and $k \rightarrow -\infty$, and takes the deterministic value

$$\gamma(e^{i\alpha}) = \lim_{k \rightarrow \pm\infty} \frac{1}{|k|} \mathbb{E}(\ln \|\Phi(k, \omega)\|). \quad (3.13)$$

We can allow spectral parameters of the form $e^{i\alpha} = z \in \mathbb{C} \setminus \{0\}$, and get from classical arguments, see *e.g.* [CFKS], that γ is a subharmonic function of z . For other properties of $\gamma(e^{i\alpha})$ and, in particular for its relations with the density of states, see [J1].

The link between the behaviour at infinity of the generalized eigenvectors and the spectrum of U_ω is provided by Sh'nol's Theorem. This is a deterministic fact which, as proven in [BHJ], carries over the the unitary operators considered here (here $E_\omega(\cdot)$ is the spectral resolution of U_ω , which we consider to be supported on \mathbb{T}):

Theorem 3.2 $\sigma(U_\omega)$ is the closure of the set

$$S_\omega = \{\alpha \in \mathbb{T}; U_\omega \phi = e^{i\alpha} \phi \text{ has a non-trivial polynomially bounded solution}\} \quad (3.14)$$

and $E_\omega(\mathbb{T} \setminus S_\omega) = 0$.

A version of the Ishii-Pastur theorem suited to unitary matrices with a band structure follows as a corollary to these arguments.

Theorem 3.3 Let U_ω be defined by (2.2), (2.1) and (2.3) and $\gamma(e^{i\alpha})$ by (3.12). Then

$$\Sigma_{ac} \subseteq \overline{\{e^{i\alpha} \in S^1; \gamma(e^{i\alpha}) = 0\}}^{ess}. \quad (3.15)$$

The positivity of the Lyapunov exponent is assessed in [BHJ] by means of Fürstenberg's Theorem. The situation considered in [BHJ] actually concerns phases which are uniformly distributed on \mathbb{T} and this is therefore easily obtained. In the present case, the support of $d\nu$ is more arbitrary and a more detailed investigation is necessary. The argument leading to the following result is presented in Section 7.

Proposition 3.1 *Assume the support ν has non-empty interior. Then, the Lyapunov exponent $\gamma(e^{i\alpha})$ associated with the ergodic linear dynamical system (3.10) is strictly positive for any $\alpha \in \mathbb{T}$.*

As a direct corollary, we get that

$$\Sigma_{ac} = \emptyset. \quad (3.16)$$

4 Proof of Theorem 2.1:

We adapt an argument which, in various forms, has been used extensively in localization proofs for various types of one-dimensional random Schrödinger operators. It combines positivity of the Lyapunov exponent with polynomial boundedness of generalized eigenfunctions (Theorem 3.2) and spectral averaging. While implicit in the literature even earlier, this strategy was first explicitly spelled out in [SW].

Let $\overline{\Omega} = \mathbb{T}^{\mathbb{Z} \setminus \{-1, 0\}}$, $\overline{\mathbb{P}} = \otimes_{k \in \mathbb{Z} \setminus \{-1, 0\}} \nu$ and $\overline{\omega} = (\dots, \overline{\omega}_{-3}, \overline{\omega}_{-2}, \overline{\omega}_1, \overline{\omega}_2, \dots)$. We will write $(\overline{\omega}, \theta_{-1}, \theta_0)$ for $(\dots, \overline{\omega}_{-3}, \overline{\omega}_{-2}, \theta_{-1}, \theta_0, \overline{\omega}_1, \overline{\omega}_2, \dots)$.

By construction $\gamma_{(\overline{\omega}, \theta_{-1}, \theta_0)}(e^{i\alpha})$ is independent of (θ_{-1}, θ_0) . It follows from Proposition 3.1 that for any $\alpha \in \mathbb{T}$, there exists $\overline{\Omega}(\alpha) \subset \overline{\Omega}$ with $\overline{\mathbb{P}}(\overline{\Omega}(\alpha)) = 1$ such that

$$\gamma_{(\overline{\omega}, \theta_{-1}, \theta_0)}(e^{i\alpha}) = \gamma(e^{i\alpha}) > 0 \quad (4.1)$$

for all (θ_{-1}, θ_0) and all $\overline{\omega} \in \overline{\Omega}(\alpha)$. Hence, by Fubini applied to $\overline{\mathbb{P}} \times |\cdot|$, we get the existence of $\overline{\Omega}_0 \in \overline{\Omega}$ with $\overline{\mathbb{P}}(\overline{\Omega}_0) = 1$ such that for every $\overline{\omega} \in \overline{\Omega}_0$ there is $A_{\overline{\omega}} \in \mathbb{T}$ with $|A_{\overline{\omega}}| = 0$ and

$$\gamma_{(\overline{\omega}, \theta_{-1}, \theta_0)}(e^{i\alpha}) > 0 \quad \text{for all } (\theta_{-1}, \theta_0) \text{ and all } \alpha \in A_{\overline{\omega}}^C. \quad (4.2)$$

Here $|\cdot|$ denotes Lebesgue-measure on \mathbb{T} .

Let us show that $A_{\overline{\omega}}^C$ is a support of the spectral resolution of $U_{(\overline{\omega}, \theta_{-1}, \theta_0)}$ for Lebesgue almost every (θ_{-1}, θ_0) .

We introduce the spectral measures μ_{ω}^j associated with $U_{\omega} = \int_{\mathbb{T}} e^{i\alpha} dE_{\omega}(\alpha)$ defined for all $j \in \mathbb{Z}$ and all Borel sets $\Delta \in \mathbb{T}$ by

$$\mu_{\omega}^j(\Delta) = \langle j | E_{\omega}(\Delta) | j \rangle. \quad (4.3)$$

By construction, the variation of a random phase at one site is described by a rank one perturbation. More precisely, dropping the subscript ω temporarily, we define \hat{D} by taking $\theta_0 = 0$ in the definition of D :

$$\hat{D} = e^{i\theta_0|0\rangle\langle 0|} D = D + |0\rangle\langle 0|(1 - e^{-i\theta_0}), \quad (4.4)$$

so that, with the obvious notations,

$$\hat{U} = \hat{D}S = e^{i\theta_0|0\rangle\langle 0|} U. \quad (4.5)$$

As for rank one perturbations of self-adjoint operators, a spectral averaging formula holds in the unitary case. In particular, see [C] and [B], for any $f \in L^1(\mathbb{T})$,

$$\int_{\mathbb{T}} d\theta_0 \int_{\mathbb{T}} f(\alpha) d\mu_{(\overline{\omega}, \theta_{-1}, \theta_0)}^0(\alpha) = \int_{\mathbb{T}} f(\alpha) d\alpha. \quad (4.6)$$

By applying this to the characteristic function of $A_{\overline{\omega}}$ we get

$$0 = |A_{\overline{\omega}}| = \int_{\mathbb{T}} \mu_{(\overline{\omega}, \theta_{-1}, \theta_0)}^0(A_{\overline{\omega}}) d\theta_0, \quad (4.7)$$

implying that $\mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^0(A_{\bar{\omega}}) = 0$ for every θ_{-1} and Lebesgue-a.e. θ_0 . Similarly, we get $\mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^{-1}(A_{\bar{\omega}}) = 0$ for every θ_0 and Lebesgue-a.e. θ_{-1} .

Therefore, for all $\bar{\omega} \in \bar{\Omega}_0$, there exists $J_{\bar{\omega}} \subset \mathbb{T}^2$ such that $|J_{\bar{\omega}}^C| = 0$ and

$$(\theta_{-1}, \theta_0) \in J_{\bar{\omega}} \Rightarrow \mu_{(\bar{\omega}, \theta_{-1}, \theta_0)}^j(A_{\bar{\omega}}) = 0, \quad j \in \{-1, 0\}. \quad (4.8)$$

Fix $\bar{\omega} \in \bar{\Omega}_0$ and $(\theta_{-1}, \theta_0) \in J_{\bar{\omega}}$ and consider $\omega = (\bar{\omega}, \theta_{-1}, \theta_0)$. Below we prove

Lemma 4.1 *The subspace $\text{Span}\{|-1\rangle, |0\rangle\}$ is cyclic for U_ω .*

Therefore, we deduce from (4.8) that $E_\omega(A_{\bar{\omega}}) = 0$. If S_ω is the set from Sh'nol's Theorem 3.2, we conclude that $S_\omega \cap A_{\bar{\omega}}^C$ is a support for $E_\omega(\cdot)$.

Let $\alpha \in S_\omega \cap A_{\bar{\omega}}^C$. By Theorem 3.2, $U_\omega \psi = e^{i\alpha} \psi$ has a non-trivial polynomially bounded solution ψ . On the other hand, by (4.2), $\gamma_\omega(e^{i\alpha}) > 0$. Thus, by Osceleddec's Theorem, every solution which is polynomially bounded at $+\infty$ necessarily has to decay exponentially, and the same holds at $-\infty$. Thus ψ decays exponentially at $+\infty$ and $-\infty$, and therefore is in $l^2(\mathbb{Z})$ and an eigenfunction of U_ω . We have shown that every $\alpha \in S_\omega \cap A_{\bar{\omega}}^C$ is an eigenvalue of U_ω . As $l^2(\mathbb{Z})$ is separable, it follows that $S_\omega \cap A_{\bar{\omega}}^C$ is countable. Therefore $E_\omega(\cdot)$ has countable support and thus U_ω is pure point spectrum, in particular

$$\sigma_{sc}(U_\omega) = \emptyset \quad \text{for every } \omega \in \Omega_0 := \{(\bar{\omega}, \theta_{-1}, \theta_0) : \bar{\omega} \in \bar{\Omega}_0, (\theta_{-1}, \theta_0) \in J_{\bar{\omega}}\}. \quad (4.9)$$

From $|J_{\bar{\omega}}^C| = 0$ and the non-triviality of the a.c. component of ν we have

$$(\nu \times \nu)(J_{\bar{\omega}}) \geq (\nu_{ac} \times \nu_{ac})(J_{\bar{\omega}}) = (\nu_{ac} \times \nu_{ac})(\mathbb{T}^2) > 0. \quad (4.10)$$

As $\bar{\mathbb{P}}(\bar{\Omega}_0) = 1$, we conclude from (4.9) and (4.10) that

$$\mathbb{P}(\sigma_{sc}(U_\omega) = \emptyset) \geq \mathbb{P}(\Omega_0) = \int_{\bar{\Omega}_0} d\bar{\mathbb{P}}(\bar{\omega}) (\nu \times \nu)(J_{\bar{\omega}}) > 0. \quad (4.11)$$

By the discussion in Section 2 we know that spectral types are almost surely deterministic, thus $\Sigma_{sc} = \emptyset$. We already know $\Sigma_{ac} = \emptyset$ from (3.16). This proves that U_ω has almost surely pure point spectrum.

We still need to show that almost surely all eigenfunctions decay exponentially. To this end, note that we actually have shown above that the event ‘‘all eigenvectors of U_ω decay at the rate of the Lyapunov exponent’’ has positive probability (as this is true for all $\omega \in \Omega_0$). For the case of ergodic one-dimensional Schrödinger operators Kotani and Simon show in Theorem A.1 of [KS] that this event has probability 1 or 0. In fact, only measurability needs to be shown as the event is invariant under the ergodic transformation W . The proof of this fact provided in [KS] carries over to our model. Let us only note that, due to Lemma 4.1, we may use $\rho_\omega = \mu_\omega^{-1} + \mu_\omega^0$ as spectral measures in their argument. This completes the proof of Theorem 2.1 up to the

Proof of Lemma 4.1: We drop the sub(super)scripts ω in this proof. We have to show that any vector $|k\rangle$, $k \in \mathbb{Z}$ can be written as a linear combination of the vectors $U^n|j\rangle$, $n \in \mathbb{Z}$, $j = -1$ and $j = 0$. We compute from (2.2) and its adjoint

$$U|-1\rangle = e^{-i\theta_{-2}rt}|-2\rangle + e^{-i\theta_{-1}r^2}|-1\rangle + e^{-i\theta_0rt}|0\rangle - e^{-i\theta_1t^2}|1\rangle \quad (4.12)$$

$$U|0\rangle = -e^{-i\theta_{-2}t^2}|-2\rangle - e^{-i\theta_{-1}rt}|-1\rangle + e^{-i\theta_0r^2}|0\rangle - e^{-i\theta_1rt}|1\rangle \quad (4.13)$$

$$U^{-1}|0\rangle = e^{i\theta_0}(rt|-1\rangle + r^2|0\rangle + rt|1\rangle - t^2|2\rangle) \quad (4.14)$$

$$U^{-1}|-1\rangle = e^{i\theta_1}(-t^2|-3\rangle - rt|-2\rangle + r^2|-1\rangle - rt|2\rangle). \quad (4.15)$$

Hence, using $t^2 + r^2 = 1$,

$$|1\rangle = \frac{e^{i\theta_1}}{t} \left(e^{-i\theta_0} r |0\rangle - (tU| - 1\rangle + rU|0\rangle) \right) \quad (4.16)$$

$$|-2\rangle = \frac{e^{i\theta_{-2}}}{t} \left(rU| - 1\rangle - tU|0\rangle - e^{-i\theta_{-1}} r | - 1\rangle \right). \quad (4.17)$$

Therefore, using (4.16) in (4.14), suitable linear combinations of $| - 1\rangle, |0\rangle, U| - 1\rangle, U|0\rangle$ and $U^{-1}|0\rangle$ yield $|2\rangle$. Similarly, $| - 3\rangle$ can be obtained as a linear combination of $| - 1\rangle, |0\rangle, U| - 1\rangle, U|0\rangle$ and $U^{-1}| - 1\rangle$ using (4.17) in (4.15). These manipulations lead us from the indices $(-1, 0)$ to the set $(1, 2)$ in one direction and $(-3, -2)$ in the other direction. Due to the shape of U , we can iterate the process to reach any vector. ■

5 Half-lattice operators

In this section we indicate how to adapt the results above to random unitary matrices similar to (2.2), but defined on $l^2(\mathbb{N}_0)$, in the same spirit as in [BHJ].

Let S^+ be the unitary defined on $l^2(\mathbb{N}_0)$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, in the canonical basis $\{|j\rangle\}_{j \in \mathbb{N}_0}$ by the matrix

$$S^+ = \begin{pmatrix} -r & rt & -t^2 & & & & \\ t & r^2 & -rt & & & & \\ & rt & r^2 & rt & -t^2 & & \\ & -t^2 & -rt & r^2 & -rt & & \\ & & & tr & r^2 & & \\ & & & -t^2 & -rt & \ddots & \end{pmatrix}, \quad (5.1)$$

where the dots mean repetition of the last 4×2 block, as in (2.3). Similar to S one may write S^+ as a product of block matrices, the first factor consisting entirely of B_1 -blocks and the second of B_2 -blocks, but with a single 1×1 block as the upper left entry, determining our choice of “boundary condition”. S^+ is found by choosing the latter to be -1 . As in the discussion of Section 2 this suggests generalizations (in particular the 1×1 block could be replaced by an arbitrary phase), but we have not explored this systematically.

Then one considers

$$U_\omega^+ = D_\omega^+ S^+ \quad \text{with} \quad D_\omega^+ = \text{diag} \{e^{-i\theta_k^\omega}\}, \quad (5.2)$$

where the random phases θ_k^ω are given by (2.1), for $k \in \mathbb{N}_0$. The generalized eigenvectors $\psi = \sum_{k \geq 0} c_k |k\rangle$ defined by

$$U_\omega^+ \psi = e^{i\alpha} \psi \quad (5.3)$$

give rise to the same dynamical system (3.10) on the coefficients c_k . Starting from $(c_1, c_2)^T$, we have

$$\begin{pmatrix} c_{2(k+1)-1} \\ c_{2(k+1)} \end{pmatrix} = T(\theta_{2k}^\omega(\alpha), \theta_{2k+1}^\omega(\alpha)) \begin{pmatrix} c_{2k-1} \\ c_{2k} \end{pmatrix}, \quad k = 1, 2, \dots \quad (5.4)$$

where the transfer matrix $T(\theta, \eta)$ is given by (3.7) and $\theta_k^\omega(\alpha) = \theta_k^\omega + \alpha$ as before. This relation must be supplemented by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_0 \begin{pmatrix} (e^{-i(\theta_1+\alpha)} + r e^{-i(\theta_1-\theta_0)})/t \\ (e^{-i(\theta_1+\alpha)} + r e^{-i(\theta_1-\theta_0)})r/t^2 - (r + e^{i(\alpha+\theta_0)})/t^2 \end{pmatrix}, \quad (5.5)$$

where c_0 is free, because of the boundary condition at component 0. The (forward) Lyapunov exponent $\gamma(e^{i\alpha})$ corresponding to (5.4) defined by (3.12) exists almost surely and the conclusions of Theorems 3.2 and 3.3 remain true for U_ω^+ .

The first difference/simplification with respect to the operator U_ω defined on the whole lattice is that U_ω^+ admits a cyclic vector

Lemma 5.1 *The vector $|0\rangle$ is cyclic for U_ω^+ .*

Proof: One first checks that $|1\rangle = \frac{e^{i\theta_1}}{t} (U_\omega^+|0\rangle + re^{-i\theta_0}|0\rangle)$, and then one concludes as in the proof of Lemma 4.1. \blacksquare

As a consequence, we get

Theorem 5.1 *Let U_ω^+ be defined by (5.1), (5.2) and (2.1). If the distribution $d\nu$ of the i.i.d. phases possesses a non-trivial absolutely continuous component and its support has non-empty interior, then U_ω^+ is pure point almost surely, with exponentially decaying eigenfunctions.*

Proof: The proof is virtually the same as that of Theorem 2.1. Due to cyclicity of $|0\rangle$ it suffices to average over the single phase θ_0 , which leads to some simplifications. \blacksquare

6 Application to OPUC

The previous extension of our result to $l^2(\mathbb{N}_0)$ was aimed to pave way for the applications of our localization results to orthogonal polynomials on the unit circle (OPUC) with respect to an infinitely supported probability measure $d\mu$. Such polynomials Φ_k are determined via the Szego recursion $\Phi_{k+1}(z) = z\Phi_k(z) - \bar{\alpha}_k\Phi_k^*(z)$, $\Phi_0 = 1$, by a sequence of complex valued coefficients $\{\alpha_k\}_{k \in \mathbb{N}_0}$, such that $|\alpha_k| < 1$, called Verblunsky coefficients, which also characterize the measure $d\mu$, see [S1]. This latter relation is encoded in a five diagonal unitary matrix C on $l^2(\mathbb{N}_0)$ representing multiplication by $z \in S^1$: The measure $d\mu$ arises as the spectral measure $\mu(\Delta) = \langle 0|E(\Delta)|0\rangle$ of the cyclic vector $|0\rangle$ of C . This matrix is the equivalent of the Jacobi matrix in the case of orthogonal polynomials with respect to a measure on the real axis, and it is called the CMV matrix, after [CMV].

In case the Verblunsky coefficients all have the same modulus, i.e.

$$\alpha_k = re^{i\eta_k}, \quad k = 0, 1, \dots \quad (6.1)$$

the corresponding CMV matrix reads

$$C = \begin{pmatrix} re^{-i\eta_0} & rte^{-i\eta_1} & t^2 & & & \\ t & -r^2e^{i(\eta_0-\eta_1)} & -rte^{i\eta_0} & & & \\ & rte^{-i\eta_2} & -r^2e^{i(\eta_1-\eta_2)} & & & \\ & t^2 & -rte^{i\eta_1} & & & \\ & & & rte^{-i\eta_3} & t^2 & \\ & & & -r^2e^{i(\eta_2-\eta_3)} & -rte^{i\eta_2} & \\ & & & rte^{-i\eta_4} & -r^2e^{i(\eta_3-\eta_4)} & \\ & & & & t^2 & -rte^{i\eta_3} & \ddots \end{pmatrix}. \quad (6.2)$$

Now, changing from the canonical basis $\{|j\rangle\}_{j \in \mathbb{N}_0}$ to $\{e^{i\beta_j}|j\rangle\}_{j \in \mathbb{N}_0}$ by means of the unitary B defined by $B|j\rangle = e^{i\beta_j}|j\rangle$, $j = 0, 1, \dots$, we get

$$B^{-1}CB = \begin{pmatrix} re^{-i\eta_0} & rte^{-i\eta_1}e^{i(\beta_1-\beta_0)} & t^2e^{i(\beta_2-\beta_0)} & & & \\ te^{i(\beta_0-\beta_1)} & -r^2e^{i(\eta_0-\eta_1)} & -rte^{i\eta_0}e^{i(\beta_2-\beta_1)} & & & \\ & rte^{-i\eta_2}e^{i(\beta_1-\beta_2)} & -r^2e^{i(\eta_1-\eta_2)} & & & \\ & t^2e^{i(\beta_1-\beta_3)} & -rte^{i\eta_1}e^{i(\beta_2-\beta_3)} & & \ddots & \end{pmatrix}. \quad (6.3)$$

Then, by choosing the β_j 's suitably, the matrix (6.3) becomes the negative of a matrix of the form (5.2):

$$-U^+ = \begin{pmatrix} re^{-i\theta_0} & -re^{-i\theta_0}t & t^2e^{-i\theta_0} & & & \\ -te^{-i\theta_1} & -r^2e^{-i\theta_1} & rte^{-i\theta_1} & & & \\ & -rte^{-i\theta_2} & -r^2e^{-i\theta_2} & -rte^{-i\theta_2} & t^2e^{-i\theta_2} & \\ & t^2e^{-i\theta_3} & rte^{-i\theta_3} & -r^2e^{-i\theta_3} & rte^{-i\theta_3} & \\ & & & -tre^{-i\theta_4} & -r^2e^{-i\theta_4} & \\ & & & t^2e^{-i\theta_5} & rte^{-i\theta_5} & \ddots \end{pmatrix}. \quad (6.4)$$

Here, as seen from the diagonal elements, the phases θ_k are given in terms of the phases of the Verblunsky coefficients by (set $\eta_{-1} = 0$)

$$\theta_k = \eta_k - \eta_{k-1}, \quad k = 0, 1, 2, \dots, \quad (6.5)$$

or, equivalently

$$\eta_k = \theta_k + \theta_{k-1} + \dots + \theta_0, \quad k = 0, 1, 2, \dots \quad (6.6)$$

The terms in t^2 require

$$\begin{aligned} \beta_1 - \beta_0 &= \theta_1 + \pi \\ \beta_{2k+1} - \beta_{2k-1} &= \theta_{2k+1}, \quad k = 1, 2, \dots, \\ \beta_{2k+2} - \beta_{2k} &= -\theta_{2k}, \quad k = 0, 1, \dots, \end{aligned} \quad (6.7)$$

where β_0 is free. Explicitly, for $k \geq 0$,

$$\begin{aligned} \beta_{2k+1} &= \theta_{2k+1} + \theta_{2k-1} + \dots + \theta_1 + \beta_0 + \pi \\ \beta_{2k+2} &= -(\theta_{2k} + \theta_{2k-2} + \dots + \theta_0). \end{aligned} \quad (6.8)$$

It is straightforward to check that (6.5) and (6.7) form a consistent choice in the sense that all terms in rt in (6.3) and (6.4) agree. Assuming the θ_k 's are i.i.d. random variables, Theorem 5.1 applies to this case and yields

Proposition 6.1 *Let $\alpha_k(\omega)_{k \in \mathbb{N}_0}$ be random Verblunsky coefficients of the form*

$$\alpha_k(\omega) = re^{i\eta_k(\omega)}, \quad 0 < r < 1, \quad k = 0, 1, 2, \dots \quad (6.9)$$

whose phases are distributed on \mathbb{T} according to

$$\eta_k(\omega) \sim d\nu * d\nu * \dots * d\nu, \quad (k + 1 \text{ convolutions}) \quad (6.10)$$

and $d\nu$ is a probability measure with non-trivial a.c. component and such that its support has non-empty interior. Then, the random measure $d\mu_\omega$ on S^1 with respect to which the corresponding random polynomials $\Phi_{k,\omega}$ are orthogonal is almost surely pure point.

Remark: Other localization results for random polynomials on the unit circle, [S2], [T], [Su] are proven for independent Verblunsky coefficients. Moreover, the results of [Su] and [S2] require rotational invariance of the distribution of the Verblunsky coefficients in the unit disk. By contrast, the proposition above holds for strongly correlated random Verblunsky coefficients.

7 Proof of Proposition 3.1

We want to apply Fürstenberg's Theorem. Since the latter is stated for real valued matrices, we proceed as in [BHJ] and introduce the mapping $\tau : M_2(\mathbb{C}) \rightarrow M_4(\mathbb{R})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} \Re(a)I + \Im(a)J & \Re(b)I + \Im(b)J \\ \Re(c)I + \Im(c)J & \Re(d)I + \Im(d)J \end{pmatrix}, \quad (7.1)$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (7.2)$$

This mapping is a homeomorphism from $M_2(\mathbb{C})$ to $\tau(M_2(\mathbb{C}))$ and, in particular, a group homeomorphism from the set of matrices in $M_2(\mathbb{C})$ with determinant of modulus one to the set of matrices in $M_4(\mathbb{R})$ with determinant of modulus one.

Let $G_{\alpha,\nu}$ be the closed group generated by the matrices $T(\theta,\eta)$ from (3.7), where θ and η vary in $\alpha + \text{supp } \nu$. As discussed in [BHJ], Fürstenberg's Theorem implies that $\gamma(e^{i\alpha}) > 0$ if it can be shown that $G_{\alpha,\nu}$ is strongly-irreducible and non-compact. As $\text{supp } \nu$ has non-empty interior and thus $G_{\alpha,\nu}$ has a non-trivial connected component it suffices to show

Lemma 7.1 *The group $\tau(G_{\alpha,\nu})$ is irreducible and non-compact.*

To show the first statement, one uses arguments of the same type as those developed in [BHJ]. The set $\alpha + \text{supp } \nu$ contains a non-empty open interval \mathcal{I} and it will suffice to only work with $\theta, \eta \in \mathcal{I}$.

Let us assume there exists a strict subspace V of \mathbb{R}^4 which is invariant under $\tau(T(\theta, \eta))$, for all $(\theta, \eta) \in \mathcal{I} \times \mathcal{I}$. The mapping $\tau(T)$ is smooth in $\mathcal{I} \times \mathcal{I}$ and it follows by approximating derivatives with finite differences that V is also left invariant by $\partial_\theta^r \partial_\eta^s \tau(T(\theta, \eta))$ for all $r, s \in \mathbb{N}_0$ and for all $(\theta, \eta) \in \mathcal{I} \times \mathcal{I}$.

In particular, we compute for $\eta = \theta \in \mathcal{I}$,

$$\tau(T(\theta, \theta)) = A_0 + A_1 \sin(\theta) + A_2 \cos(\theta), \quad (7.3)$$

$$\partial_\theta \tau(T(\theta, \eta))|_{\eta=\theta} = B_0 + B_1 \sin(\theta) + B_2 \cos(\theta), \quad (7.4)$$

with

$$A_0 = \begin{pmatrix} 0 & 0 & r/t & 0 \\ 0 & 0 & 0 & r/t \\ r/t & 0 & 2r^2/t^2 & 0 \\ 0 & r/t & 0 & 2r^2/t^2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & r/t \\ -1 & 0 & -r/t & 0 \\ 0 & r/t & 0 & -1 \\ -r/t & 0 & 1 & 0 \end{pmatrix}, \quad (7.5)$$

$A_2 = -(A_0 + \mathbb{I})$, and

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & r/t \\ 0 & 0 & -r/t & 0 \\ 0 & 0 & 0 & r^2/t^2 \\ 0 & 0 & -r^2/t^2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/t^2 & 0 \\ 0 & 0 & 0 & 1/t^2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/t^2 \\ 0 & 0 & 1/t^2 & 0 \end{pmatrix}. \quad (7.6)$$

It follows that V is invariant under A_0, A_1, A_2, B_0, B_1 and B_2 (for example by differentiating the right hand sides of (7.3) and (7.4) two more times). Since these matrices are real (anti) self-adjoint, they leave V^\perp invariant as well. Hence, V and V^\perp are generated by real eigenvectors of these matrices, if they are diagonalizable over \mathbb{R} . Note that B_1 is diagonal in the canonical basis denoted by $\{e_j\}_{j \in 1,2,3,4}$.

If V is one-dimensional, it is generated by one vector which is either in the subspace $\langle e_1, e_2 \rangle$ or in the $\langle e_3, e_4 \rangle$. As neither of these subspaces is invariant under A_1 , this is impossible. The same is true for V^\perp , so that V must be two-dimensional. V cannot coincide with any of the previously considered subspaces, by the same argument, so the only possibility left is

$$V = \langle w_1, w_2 \rangle, \quad \text{where} \quad w_1 = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix} \quad \text{and} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \quad (7.7)$$

But, to have $A_1 w_1 \in V$, there must exist $a, b \in \mathbb{R}$ such that $A_1 w_1 = a w_1 + b w_2$, i.e.

$$\begin{pmatrix} \beta \\ -\alpha \\ \beta r/t \\ -\alpha r/t \end{pmatrix} = \begin{pmatrix} a\alpha \\ a\beta \\ b\gamma \\ b\delta \end{pmatrix}. \quad (7.8)$$

The first two components imply $(1 + a^2)\beta = 0$, thus $\beta = 0$ and $\alpha = 0$, which is absurd. Hence $\tau(G_{\alpha,\nu})$ is irreducible.

Let us now turn to the second statement of Lemma 7.1. The non-compactness of $\tau(G_{\alpha,\nu})$ and $G_{\alpha,\nu}$ being equivalent, we can choose to work on $G_{\alpha,\nu}$. We will actually show a much stronger statement. For this, pick any fixed θ and η on the torus with $\theta \neq \eta$. Consider the subgroup $G(\theta, \eta)$ of $GL(2, \mathbb{C})$ generated by $T(\theta, \theta), T(\eta, \eta), T(\theta, \eta)$ and $T(\eta, \theta)$, where the latter are defined through (3.7). As $\alpha + \text{supp } \nu$ contains at least two points, non-compactness of $G_{\alpha,\nu}$ will follow from non-compactness of $G(\theta, \eta)$.

With the abbreviations $x := e^{-i\theta}$ and $z := e^{-i\eta}$ the first of them takes the form

$$T(\theta, \eta) = \begin{pmatrix} -z & \frac{r}{t}(\bar{x}z - z) \\ \frac{r}{t}(1 - z) & \frac{r^2}{t^2}(\bar{x}z + 1 - z) - \frac{1}{t^2}\bar{x} \end{pmatrix},$$

with $\det T(\theta, \eta) = \bar{x}z$, and similarly for the other three generating matrices.

Calculations (some lengthy) show that

$$C := T(\theta, \theta)T(\theta, \eta)^{-1} = \begin{pmatrix} x\bar{z} & 0 \\ \frac{r}{t}(x\bar{z} - 1) & 1 \end{pmatrix} \in G(\theta, \eta),$$

$$E := T(\eta, \theta)^{-1}T(\theta, \theta) = \begin{pmatrix} 1 & \frac{r}{t}(1 - \bar{x}z) \\ 0 & \bar{x}z \end{pmatrix} \in G(\theta, \eta),$$

$$L := CE = \begin{pmatrix} x\bar{z} & \frac{r}{t}(x\bar{z} - 1) \\ \frac{r}{t}(x\bar{z} - 1) & \bar{x}z - \frac{r^2}{t^2}|x\bar{z} - 1|^2 \end{pmatrix} \in G(\theta, \eta),$$

$$J := EC = \begin{pmatrix} x\bar{z} - \frac{r^2}{t^2}|x\bar{z} - 1|^2 & \frac{r}{t}(1 - \bar{x}z) \\ \frac{r}{t}(1 - \bar{x}z) & \bar{x}z \end{pmatrix} \in G(\theta, \eta).$$

Note that $\det L = \det J = 1$ and that $J^{-1} = L^*$. Thus we get the self-adjoint element $K := J^{-1}L$ of $G(\theta, \eta)$. In fact, K is positive definite and $\det K = 1$.

More calculation shows that

$$\begin{aligned} \operatorname{tr} K &= 1 + \frac{2r^2}{t^2} |x\bar{z} - 1|^2 + \left| x\bar{z} - \frac{r^2}{t^2} |x\bar{z} - 1|^2 \right|^2 \\ &= 2 + \frac{r^2}{t^4} |x\bar{z} - 1|^4. \end{aligned}$$

As $\theta \neq \eta$ and therefore $x\bar{z} \neq 1$ we conclude that $\operatorname{tr} K > 2$. Positivity of K implies that it has an eigenvalue strictly bigger than 1. Thus, containing all powers of K , the group $G(\theta, \eta)$ is non-compact. ■

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