Spectral Properties of Non-Unitary Band Matrices

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Abstract

We consider families of random non-unitary contraction operators defined as deformations of CMV matrices which appear naturally in the study of random quantum walks on trees or lattices. We establish several deterministic and almost sure results about the location and nature of the spectrum of such non-normal operators as a function of their parameters. We relate these results to the analysis of certain random quantum walks, the dynamics of which can be studied by means of iterates of such random non-unitary contraction operators.

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1 Introduction

The spectral theory of self-adjoint and unitary operators is a well established topic in mathematics with a rich structure revealed by numerous important results, and which has found many applications, particularly in mathematical physics. See for example the textbooks [Ka, RS, DS, D4, Ku] selected from the abundant literature on the topic. By contrast, the general spectral theory of operators enjoying less symmetry, that is non-normal operators, is more vast, technically more involved and less well understood. However, the spectral theory of non self-adjoint operators has been the object of many works, in various setups of regimes, as can be seen from the works [GoKr, SFBK, D1, D2, TE, D3, Sj, CL, CCL, CD] and references therein. In particular, several analyses of non self-adjoint operators focus on tri-diagonal operators, when expressed in a certain basis, see [D1, D2, CL, CD]. Since Jacobi matrices provide generic models of self-adjoint operators, it is quite natural to deal with non self-adjoint tri-diagonal matrices which are deformations of Jacobi matrices. Moreover, certain models of this sort are physically relevant, see e.g. [HN, GoKh, FZ].

In this paper, we introduce and analyze the spectral properties of another set of non-normal operators possessing a band structure in a certain basis, which share similarities with the tri-diagonal non-self-adjoint operators mentioned above. Our operators have a five-diagonal structure and are obtained as deformations of certain unitary operators called

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CMV matrices, see [Si] for a detailed account. The role played by CMV matrices for unitary operators is similar to that played by Jacobi matrices for self-adjoint operators: they provide generic models of unitary operators; hence we call our models non-unitary operators. The non-unitary operators considered in this paper arise naturally in the study of random quantum walks on certain infinite graphs, which provide unitary dynamical systems of interest for physics, computer science and probability theory, see for example the reviews [Ke, Ko, V-A, J3]. In particular, random quantum walks defined on $\mathbb{Z}$ are given by special cases of CMV matrices. The study of the spectral properties of random unitary operators and quantum walks defined on trees or lattices, see e.g. [BHJ, HJS, JM, ASW, J2, HJ], may lead to the analysis of certain autocorrelation functions. We show in Section 2 below that in certain cases, the analysis of these autocorrelation functions reduces to the study of iterates of our non-unitary operators, which provides a direct link between spectral properties of non-unitary operators and random quantum walks. Moreover, the structure of our non-unitary operators allows us to determine the spectral nature of the corresponding random quantum walks they are related to.

While the non-unitary operators we study correspond to deformations of random CMV matrices of a special type, and consequently are rather sparse, we show in Section 4.4 that due to certain symmetries they possess, our main results also apply to deformations of random unitary CMV type matrices of a much more general form. Those random unitary operators appear as models in condensed matter physics and can be considered as natural unitary analogs of Anderson type models, see [BB, BHJ, HJS]. The corresponding non-unitary deformations they give rise to are thus of a quite general form, displaying generically non zero elements at all entries of the familiar 5-diagonal structure CMV type matrices possess. In that sense, our spectral analysis applies to non-unitary deformations of typical random CMV type matrices addressed in the literature, which corresponds in this richer framework to the analyses of the non self-adjoint Anderson or Feinberg-Zee models addressed e.g. in [D1, D2, CD].

1.1 Main results

The non-unitary operators T_ω addressed here are random operators on the Hilbert space $l^2(\mathbb{Z})$ with the following structure: In the canonical basis of $l^2(\mathbb{Z})$, denoted by $\{e_j\}_{j\in\mathbb{Z}}$, T_ω is defined as the infinite matrix

$$T_\omega = \begin{pmatrix} \ddots & e^{i\omega_{2j-1}}\gamma & e^{i\omega_{2j-1}}\delta & & & \\ & 0 & 0 & & & \\ & 0 & 0 & e^{i\omega_{2j+1}}\gamma & e^{i\omega_{2j+1}}\delta & \\ & e^{i\omega_{2j+2}}\alpha & e^{i\omega_{2j+2}}\beta & 0 & 0 & \\ & & & 0 & 0 & \\ & & & e^{i\omega_{2j+4}}\alpha & e^{i\omega_{2j+4}}\beta & \ddots \end{pmatrix}, \tag{1}$$

where the dots mark the main diagonal and the first column is the image of the vector e_{2j}. The phases $\{e^{i\omega_j}\}_{j\in\mathbb{Z}}$ are iid random variables and the deterministic coefficients, when arranged in a matrix $C_0 \in M_2(\mathbb{C})$, are constrained by the requirement that C_0 be a projection

on $\mathbb{C}^2$ of a unitary matrix on $\mathbb{C}^3$:

$$C_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ s.t. } \tilde{C} = \begin{pmatrix} \alpha & r & \beta \\ q & g & s \\ \gamma & t & \delta \end{pmatrix} \in U(3), \text{ with } 0 \leq g \leq 1. \tag{2}$$

When C_0 itself is unitary, which corresponds to $g = 1$, T_ω is a unitary random CMV matrix describing a random quantum walk, the spectral properties of which are known, see [JM, ASW]. In general, however, C_0 is a contraction, and T_ω is a non-normal contraction, i.e. a non-unitary operator. We note here that, in general, T_ω is not a seminormal operator, i.e. $[T_\omega^*, T_\omega]$ is not definite, see [C]. Non-unitary operators T_ω constrained by condition (2) appear as a natural objects in the study of the spectral properties of random quantum walks defined on the lattice $\mathbb{Z}^2$ or on $\mathcal{T}_4$, the homogeneous tree of coordination number 4, as explained in Section 3. This provides us with an independent motivation to focus on the characterization (2) here, although other choices of deformations of CMV matrices are obviously possible. Actually, Section 4.4 shows that our spectral results extend to operators of the form $\widetilde{T}_\omega$ defined in the same basis as that used for (1) by the random infinite matrix

$$\widetilde{T}_\omega = \tag{3}$$

$$\begin{pmatrix}
\ddots & e^{i(\omega_{4j-1}+\omega_{4j-3})}\gamma\delta & e^{i(\omega_{4j-1}+\omega_{4j-3})}\delta^2 & & & \\
& e^{i(\omega_{4j-1}+\omega_{4j})}\gamma\beta & e^{i(\omega_{4j-1}+\omega_{4j})}\delta\beta & & & \\
& e^{i(\omega_{4j+1}+\omega_{4j+2})}\gamma\alpha & e^{i(\omega_{4j+1}+\omega_{4j+2})}\gamma\beta & e^{i(\omega_{4j+3}+\omega_{4j+1})}\gamma\delta & e^{i(\omega_{4j+3}+\omega_{4j+1})}\delta^2 & \\
& e^{i(\omega_{4j+2}+\omega_{4j+4})}\alpha^2 & e^{i(\omega_{4j+2}+\omega_{4j+4})}\alpha\beta & e^{i(\omega_{4j+3}+\omega_{4j+4})}\gamma\beta & e^{i(\omega_{4j+3}+\omega_{4j+4})}\delta\beta & \\
& & & e^{i(\omega_{4j+5}+\omega_{4j+6})}\gamma\alpha & e^{i(\omega_{4j+1}+\omega_{4j+2})}\gamma\beta & \\
& & & e^{i(\omega_{4j+6}+\omega_{4j+8})}\alpha^2 & e^{i(\omega_{4j+6}+\omega_{4j+8})}\alpha\beta & \ddots
\end{pmatrix},$$

with entries characterised by (2). When $g = 1$, the CMV type random operator $\widetilde{T}_\omega$ is unitary. The extension of our spectral analysis to the non-unitary deformation $\widetilde{T}_\omega$ is provided by the identity $\sigma(\widetilde{T}_\omega) = \sigma(T_\omega^2)$ and the spectral mapping theorem.

Our main spectral results about T_ω read as follows. After dealing with some special cases and with the translation invariant situation where $e^{i\omega_j} = 1$, $j \in \mathbb{Z}$, we show in Theorem 4.6 that the polar decomposition of $T_\omega = V_\omega K$ has the following structure: the isometric part V_ω is actually unitary and has the same matrix structure as T_ω, i.e. V_ω a one dimensional random quantum walk. Moreover, the self-adjoint part K is deterministic with spectrum consisting in two infinitely degenerate eigenvalues $\{g, 1\}$ only. One consequence of this fact is that T_ω is a completely non-unitary contraction operator for $g < 1$, so that the random quantum walk operator it comes from has no singular spectrum, see Proposition 4.14. This special structure also allows us to get informations on the spectrum of T_ω in terms of properties on $\sigma(V_\omega)$ and $\sigma(K)$, by applying a general result stated as Theorem 5.1 and Corollary 5.3. This result determines parts of the resolvent set of a bounded operator of the form $T = AB$ with A, B bounded, invertible and normal, in terms of the spectra of A and B. A direct consequence is that the disc of radius $g > 0$ centered at 0 is always contained in the resolvent set of $T_\omega = V_\omega K$ and, when V_ω contains a gap in its spectrum, other non-trivial explicitly determined sets also belong to $\rho(T_\omega)$, see Lemmas 5.5 and 5.7.

Then, we take advantage of the fact that the two spectral projectors of K induce a natural bloc structure for T_ω which suggests the use of the Schur-Feshbach map. It turns

out the blocs of the decomposition of V_ω are tridiagonal operators. This fact allows us to provide conditions on the parameter $g \in]0,1[$ in Theorem 5.13 which ensure that the spectrum of T_ω is contained in a centered ring with inner radius g and outer radius strictly smaller than one. It also allows us to show in Lemma 4.12 that the circles of radii 1 and g cannot support any eigenvalues of T_ω. These results are deterministic, but we further show that they hold for any realization of the random phases $\{e^{i\omega_j}\}_{j\in\mathbb{Z}}$. Finally, we take a closer look at the case $g = 0$, the farthest to the unitary case, in some sense. Assuming the random phases are uniformly distributed and making use of ergodicity, we show that the almost sure spectrum of T_ω consists in the origin and a centered ring whose inner and outer radii we determine. Also, in case the peripheral spectrum of T_ω coincides with the unit circle, we get that it contains no eigenvalue, whereas the spectrum of V_ω is pure point, and that of the corresponding random quantum walk operator is absolutely continuous, see Proposition 6.3.

The rest of the paper is organized as follows. Section 2 provides a short summary of the relevant informations needed to make connection between the non-unitary operators T_ω considered in this paper and random quantum walks on $\mathcal{T}_4$ and $\mathbb{Z}^2$. The link is made explicit in Section 3. The spectral properties of non-unitary operators is developed in the following two sections, together with the consequences which can be drawn for the random quantum walks they are related to and the explicit link between T_ω and $\widetilde{T}_\omega$. The last section is devoted to the case $g = 0$.

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2 Random Quantum Walks on $\mathbb{Z}^2$ and $\mathcal{T}_4$

We provide here the basics on simple random quantum walks defined on the lattice $\mathbb{Z}^2$ and the homogeneous tree $\mathcal{T}_4$, of coordination number 4. Such quantum walks naturally depend on a $U(4)$-matrix valued parameter C which drives the walk and monitors the effects of the disorder at the same time. In the next section, we focus on certain families of matrix valued parameters of interest which directly lead to the non-unitary operators T_ω considered in this paper. We also explain the consequences of our analysis of T_ω for the corresponding random quantum walks.

For more about random quantum walks and their spectral properties, we refer the reader to the reviews [Ko, V-A, J3] and papers [BHJ, HJS, JM, J2, HJ] and references therein.

We describe random quantum walks on the graph $\mathcal{T}_4$ only according to [HJ], and will simply mention the occasional changes necessary to deal with the lattice case, as in [J2].

2.1 Random quantum walks on $\mathcal{T}_4$

Let $\mathcal{T}_4$ be a homogeneous tree of degree 4, that we will consider as the tree of the free group generated by $A_4 = \{a, b, a^{-1}, b^{-1}\}$, with $aa^{-1} = a^{-1}a = e = bb^{-1} = b^{-1}b$, e being

the identity element of the group; see Figure (1). We choose a vertex of $\mathcal{T}_4$ to be the root of the tree, denoted by e. Each vertex $x = x_1 x_2 \dots x_n$, $n \in \mathbb{N}$ of $\mathcal{T}_4$ is a reduced word of finitely many letters from the alphabet A_4 and an edge of $\mathcal{T}_4$ is a pair of vertices (x, y) such that $xy^{-1} \in A_4$. The number of nearest neighbors of any vertex is thus 4 and any pair of vertices x and y can be joined by a unique set of edges, or path in $\mathcal{T}_4$. We identify

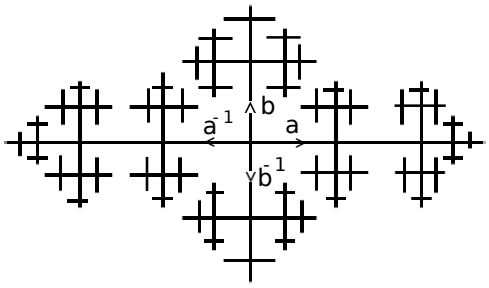

Figure 1: construction of $\mathcal{T}_4$

$\mathcal{T}_4$ with its set of vertices, and define the configuration Hilbert space of the walker by $l^2(\mathcal{T}_4) = \left\{\psi = \sum_{x\in\mathcal{T}_4} \psi_x |x\rangle \text{ s.t. } \psi_x \in \mathbb{C},\ \sum_{x\in\mathcal{T}_4} |\psi_x|^2 < \infty\right\}$, where $|x\rangle$ denotes the element of the canonical basis of $l^2(\mathcal{T}_4)$ which sits at vertex x. The coin Hilbert space (or spin Hilbert space) of the quantum walker on $\mathcal{T}_4$ is $\mathbb{C}^4$. The elements of the ordered canonical basis of $\mathbb{C}^4$ are labelled by the letters of the alphabet A_4 as $\{|a\rangle, |b\rangle, |a^{-1}\rangle, |b^{-1}\rangle\}$. The total Hilbert space is

$$\mathcal{K} = l^2(\mathcal{T}_4) \otimes \mathbb{C}^4 \quad \text{with canonical basis} \quad \left\{x \otimes \tau \equiv |x\rangle \otimes |\tau\rangle, \quad x \in \mathcal{T}_4, \tau \in A_4\right\}. \tag{4}$$

The quantum walk on the tree is characterized by the dynamics defined as the composition of a unitary update of the coin (or spin) variables in $\mathbb{C}^4$ followed by a coin (or spin) state dependent shift on the tree. Let $C \in U(4)$, $U(4)$ denoting the set of 4×4 unitary matrices on $\mathbb{C}^4$. The unitary update operator given by $\mathbb{I} \otimes C$ acts on the canonical basis of $\mathcal{K}$ as

$$(\mathbb{I} \otimes C)x \otimes \tau = |x\rangle \otimes C|\tau\rangle = \sum_{\tau' \in A_4} C_{\tau'\tau}\, x \otimes \tau', \tag{5}$$

where $\{C_{\tau'\tau}\}_{(\tau',\tau)\in A_4^2}$ denote the matrix elements of C. The coin state dependent shift S on $\mathcal{K}$ is defined by

$$S = \sum\nolimits_{\tau\in A_4} S_\tau \otimes |\tau\rangle\langle\tau|, \tag{6}$$

where for all $\tau \in A_4$ the unitary operator S_τ is a shift that acts on $l^2(\mathcal{T}_4)$ as $S_\tau|x\rangle = |x\tau\rangle, \forall x \in \mathcal{T}_4$, with $S_\tau^{-1} = S_\tau^* = S_{\tau^{-1}}$. A quantum walk on $\mathcal{T}_4$ is then defined as the one step unitary evolution operator on $\mathcal{K} = l^2(\mathcal{T}_4) \otimes \mathbb{C}^4$ given by

$$U(C) = S(\mathbb{I} \otimes C) = \sum_{\substack{\tau\in A_4 \\ x\in\mathcal{T}_4}} |x\tau\rangle\langle x| \otimes |\tau\rangle\langle\tau|C, \tag{7}$$

where $C \in U(4)$ is a parameter. A random quantum walk is defined via the following natural generalization. Let $\mathcal{C} = \{C(x) \in U(4)\}_{x\in\mathcal{T}_4}$ be a family of coin matrices indexed by the vertices $x \in \mathcal{T}_4$. A quantum walk with site dependent coin matrices is defined by

$$U(\mathcal{C}) \;=\; \sum_{\tau\in A_4, x\in\mathcal{T}_4} |x\tau\rangle\langle x| \otimes |\tau\rangle\langle\tau|C(x). \tag{8}$$

Consider $\Omega = \mathbb{T}^{\mathcal{T}_4\times A_4}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the torus, as a probability space with σ algebra generated by the cylinder sets and measure $\mathbb{P} = \otimes_{\substack{x\in\mathcal{T}_4\\ \tau\in A_4}} d\nu$ where $d\nu(\theta) = l(\theta)d\theta$, $l \in L^\infty(\mathbb{T})$, is a probability measure on $\mathbb{T}$. Let $\{\omega_x^\tau\}_{x\in\mathcal{T}_4,\tau\in A_4}$ be a set of i.i.d. random variables on the torus $\mathbb{T}$ with common distribution $d\nu$. We will note $\Omega \ni \omega = \{\omega_x^\tau\}_{x\in\mathcal{T}_4,\tau\in A_4}$. Our random quantum walks are constructed by means of the following families of site dependent random coin matrices: Let $\mathcal{C}_\omega = \{C_\omega(x) \in U(4)\}_{x\in\mathcal{T}_4}$ be the collection of random coin matrices depending on a fixed matrix $C \in U(4)$, where, for each $x \in \mathcal{T}_4$, $C_\omega(x)$ is defined by its matrix elements $C_\omega(x)_{\tau\tau'} = e^{i\omega_{x\tau}^\tau} C_{\tau\tau'}$, $\tau, \tau' \in A_4^2$. The site dependence appears in the random phases only of the matrices $C_\omega(x)$, which have a fixed skeleton $C \in U(4)$. We consider random quantum walks defined by the operator

$$U_\omega(C) := U(\mathcal{C}_\omega) \text{ on } \mathcal{K} = l^2(\mathcal{T}_4) \otimes \mathbb{C}^4 \tag{9}$$

depending on $C \in U(4)$. Defining a random diagonal unitary operator on $\mathcal{K}$ by

$$\mathbb{D}_\omega x \otimes \tau = e^{i\omega_x^\tau} x \otimes \tau, \quad \forall (x, \tau) \in \mathcal{T}_4 \times A_4, \tag{10}$$

we get that $U_\omega(C)$ is manifestly unitary thanks to the identity

$$U_\omega(C) = \mathbb{D}_\omega U(C) \;\text{ on } \mathcal{K}. \tag{11}$$

2.2 Random quantum walks on $\mathbb{Z}^2$

The definition of a random quantum walk of the same type on $\mathbb{Z}^2$ instead of $\mathcal{T}_4$ is the same, *mutatis mutandis*: the sites $x \in \mathcal{T}_4$ are replaced by $x \in \mathbb{Z}^2$ so that the configuration space $l^2(\mathcal{T}_4)$ is replaced by $l^2(\mathbb{Z}^2)$ but the coin space remains $\mathbb{C}^4$ in the definition of $\mathcal{K}$. Thus the update operator $\mathbb{I}\otimes C$ is the same on $l^2(\mathbb{Z}^2)\otimes\mathbb{C}^4$ and on $l^2(\mathcal{T}_4)\otimes\mathbb{C}^4$. Only the definition of the shifts S_τ in $S = \sum_{\tau\in A_4} S_\tau \otimes |\tau\rangle\langle\tau|$, see (6), needs to be slightly changed. We associate the letters τ of the alphabet A_4 with the canonical basis vectors $\{e_1, e_2\}$ of $\mathbb{R}^2$ as follows $a \leftrightarrow e_1$, $a^{-1} \leftrightarrow -e_1$, $b \leftrightarrow e_2$, $b^{-1} \leftrightarrow -e_2$ and define the action of S_τ on $l^2(\mathbb{Z}^2)$ accordingly: for any $x = (x_1, x_2) \in \mathbb{Z}^2$, $S_a|x\rangle = |x + e_1\rangle$, $S_{a^{-1}}|x\rangle = |x - e_1\rangle$, $S_b|x\rangle = |x + e_2\rangle$, $S_{b^{-1}}|x\rangle = |x - e_2\rangle$. The random quantum walk is then defined by $U_\omega(C)$, as in (11).

Remark 2.1 *All the results concerning $U_\omega(C)$ proven below for random quantum walks defined on $\mathcal{T}_4$ hold for walks defined on $\mathbb{Z}^2$ as well, with the adaptations given above.*

2.3 Spectral Criteria

The main issue about random quantum walks concerns the long time behavior of the discrete random unitary dynamical system on the Hilbert space $\mathcal{K}$ they give rise to by iteration of

$U_\omega(C)$. The resulting dynamics is related to the spectral properties of $U_\omega(C)$ studied in the papers [HJS, JM, ASW, J2, HJ] on $\mathbb{Z}^d$ and $\mathcal{T}_d$, as a function of $d \in \mathbb{N}$ and of the unitary matrix valued parameter C. We recall here well known spectral criteria which make a direct link between random quantum walks $U_\omega(C)$ on $\mathcal{T}_4$ and $\mathbb{Z}^2$ and T_ω defined in (1).

For a unitary operator U on a separable Hilbert space $\mathcal{H}$, the spectral measure $d\mu_\phi$ on the torus $\mathbb{T}$ associated with a normalized vector $\phi \in \mathcal{H}$ decomposes as $d\mu_\phi = d\mu_\phi^p + d\mu_\phi^{ac} + d\mu_\phi^{sc}$ into its pure point, absolutely continuous and singular continuous components. The corresponding orthogonal spectral subspaces are denoted by $\mathcal{H}^\#(U)$, with $\# \in \{p, ac, sc\}$. Then, see *e.g.* [RS], Wiener or RAGE Theorem relates the autocorrelation function $n \mapsto \langle \phi | U^n \phi \rangle$ to the spectral properties of U:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} |\langle \phi | U^n \phi \rangle|^2 = \sum_{\theta \in \mathbb{T}} (\mu_\phi^p \{\theta\})^2, \tag{12}$$

whereas the absolutely continuous spectral subspace of U, $\mathcal{H}^{ac}(U)$, is given by

$$\mathcal{H}^{ac}(U) = \overline{\Big\{ \phi \mid \sum_{n\in\mathbb{N}} |\langle \phi | U^n \phi \rangle|^2 < \infty \Big\}}. \tag{13}$$

For example, consider $U(C)$ on $\mathcal{K}$ given by (7). For any $C \in U(4)$, $\langle x \otimes \tau | U(C)^{2n+1} x \otimes \tau \rangle = 0$ for any $n \in \mathbb{Z}$ and $x \otimes \tau \in \mathcal{K}$, because $U(C)$ is off-diagonal. Moreover, if $C = \mathbb{I}$, $S = U(\mathbb{I})$ further satisfies $\langle x \otimes \tau | S^{2n} x \otimes \tau \rangle = \delta_{0,n}$, for all $x \otimes \tau \in \mathcal{K}$, so that $d\mu_{x\otimes\tau} = \frac{d\theta}{2\pi}$ and $\sigma(S) = \sigma_{ac}(S) = \mathbb{S}$, the whole unit circle. The same holds for $U(\mathbb{I}) = S$ defined on $\mathbb{Z}^2$.

3 Quantum Walks and Non-Unitary Operators

We consider here random quantum walks on $\mathcal{T}_4$ characterized by coin matrices C with a diagonal element of modulus one. As explained below, the non-trivial part of the dynamics they give rise to induces a systematic drift in one space direction. In other words, the dynamics induces a leakage of the wave vectors in one direction that is associated with a purely absolutely continuous part of spectrum of the corresponding evolution operator. We approach this spectral question by analysing the restriction of $U_\omega(C)$ to a one-dimensional subspace that defines the random contractions T_ω we study in this paper. The consequences for such quantum walks of our results about the contractions T_ω, namely the proof that the evolution operator is purely absolutely continuous for all realisations of the disorder, are spelled out in Lemma 3.3 and Proposition 4.14. Finally, we note that from the perspective of the determination of the spectral phase diagram for random quantum walks on $\mathcal{T}_4$, the corresponding set of coins matrices is not covered by the work [HJ].

Without loss, we assume that the coin matrix C with a diagonal element of modulus one takes the following form in the ordered basis $\{|a\rangle, |b\rangle, |a^{-1}\rangle, |b^{-1}\rangle\}$,

$$C = \begin{pmatrix} \alpha & r & \beta & 0 \\ q & g & s & 0 \\ \gamma & t & \delta & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix} \equiv \begin{pmatrix} \tilde{C} & \mathbf{0} \\ \mathbf{0}^T & e^{i\theta} \end{pmatrix} \in U(4), \quad \text{where } \tilde{C} = \begin{pmatrix} \alpha & r & \beta \\ q & g & s \\ \gamma & t & \delta \end{pmatrix} \in U(3), \tag{14}$$

with $\theta \in \mathbb{T}$ and $1 \geq g \geq 0$. The assumption $g \geq 0$ always holds at the price of a multiplication of C, and thus of $U_\omega(C)$, by a global phase which does not affect the spectral properties. By construction, $U_\omega(C)$ admits $\mathcal{K}^{b^{-1}}$, the subspace characterized by a coin variable equal to $|b^{-1}\rangle$, as an invariant subspace on which it acts as the shift $S_{b^{-1}}$, up to phases. Hence

$$\sigma\left(U_\omega(C)|_{\mathcal{K}^{b^{-1}}}\right) = \sigma_{ac}\left(U_\omega(C)|_{\mathcal{K}^{b^{-1}}}\right) = \mathbb{S}. \tag{15}$$

Let $\mathcal{K}^\perp$ be the complementary invariant subspace

$$\mathcal{K}^\perp = \overline{\text{span}}\Big\{x \otimes \tau \mid x \in \mathcal{T}_4, \tau \in \{a, b, a^{-1}\}\Big\}, \tag{16}$$

where the notation $\overline{\text{span}}$ means the closure of the span of vectors considered. On $\mathcal{K}^\perp$ the action of $U_\omega(C)$ on the quantum walker makes it move horizontally back and forth, but it only makes it go up vertically, see Figure (1). In a sense, the dynamics induces a leakage of the vectors in the direction corresponding to the coin state $|b\rangle$. In order to assess that $U_\omega(C)|_{\mathcal{K}^\perp}$ has purely absolutely continuous spectrum, an application of criterion (13) leads us to consider $\langle\psi|U_\omega(C)^n\psi\rangle$, $n \geq 0$, with normalized vector $\psi \in \mathcal{K}^\perp$. Note that by construction, for all $x \in \mathcal{T}_4$, all $\tau \in \{a, b, a^{-1}\}$

$$\langle x \otimes b|U_\omega(C)^n x \otimes \tau\rangle = \delta_{n,0}\delta_{b,\tau}, \quad \forall\, n \in \mathbb{N}, \forall\, x \in \mathcal{T}_4. \tag{17}$$

In particular, all spectral measures $d\mu_{x\otimes b}(\theta) = \frac{d\theta}{2\pi}$ on $\mathbb{T}$ and $\sigma\left(U_\omega(C)|_{\mathcal{K}^\perp}\right) = \mathbb{S}$ as well. We thus have,

$$\mathcal{H}_b = \overline{\text{span}}\Big\{x \otimes \tau \mid x \in \mathcal{T}_4, \tau \in \{b, b^{-1}\}\Big\} \subset \mathcal{H}^{ac}(U_\omega(C)). \tag{18}$$

3.1 Reduction to One Space Dimension

To this end we introduce the horizontal subspace associated with the direction a

$$\mathcal{H}_0 = \overline{\text{span}}\Big\{x \otimes \tau \mid x = a^m \in \mathcal{T}_4, m \in \mathbb{Z}, \tau \in \{a, a^{-1}\}\Big\} \subset \mathcal{K}^\perp \subset \mathcal{K}, \tag{19}$$

and $P_0 : \mathcal{K} \to \mathcal{K}$, the orthogonal projector onto $\mathcal{H}_0$. All vectors in this subspace live on the horizontal one dimensional lattice passing through the root of $\mathcal{T}_4$. We can actually consider vectors on any other horizontal one dimensional lattice by attaching $\mathcal{H}_0$ to any other vertex. To study $P_0U_\omega(C)^nP_0$, $n \geq 0$ we first note the following simple lemma which allows us to focus on the restriction of $U_\omega(C)$ to $\mathcal{H}_0$.

Lemma 3.1 *Let $T_\omega : \mathcal{H}_0 \to \mathcal{H}_0$ be defined by $T_\omega = P_0U_\omega(C)P_0|_{\mathcal{H}_0}$ and $T = T_\omega|_{\omega=(\cdots,0,0,0,\cdots)}$. Then, T_ω is a contraction,*

$$T_\omega = \mathbb{D}_\omega^0 T, \ \textit{where} \ \ \mathbb{D}_\omega^0 = \text{diag}\ (e^{i\omega_x^\tau}), \tag{20}$$

is the restriction of (10) to $\mathcal{H}_0$, and, for any $n \in \mathbb{N}$, $P_0U_\omega(C)^nP_0|_{\mathcal{H}_0} = T_\omega^n$.

Proof: First, we have $\|T_\omega\| = \|P_0U_\omega(C)P_0\| \leq 1$ and $[\mathbb{D}_\omega, P_0] = 0$ proves the second statement. Set $Q_0 = \mathbb{I} - P_0$ and let us show that for all $k \geq 1$, $P_0U_\omega(C)^kQ_0U_\omega(C)P_0 = 0$. Indeed, for any basis vector $x \otimes \tau$ of $\mathcal{H}_0$, $Q_0U_\omega(C)x \otimes \tau$ is proportional to $xb \otimes b$, where $xb \neq a^m$, for all $m \in \mathbb{Z}$. Consequently, $P_0U_\omega(C)^k xb \otimes b = 0$, for any $k \geq 1$, which yields the result. ∎

Remarks 3.2 *i) The contraction T can be written according to (7) as*

$$T = S(\mathbb{I} \otimes C_0) = S_a \otimes |a\rangle\langle a|C_0 + S_{a^{-1}} \otimes |a^{-1}\rangle\langle a^{-1}|C_0, \tag{21}$$

where $C_0 = \Pi_0 C \Pi_0|_{\Pi_0 \mathbb{C}^4}$ with $\Pi_0 = |a\rangle\langle a| + |a^{-1}\rangle\langle a^{-1}|$ is a contraction which takes the form

$$C_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \textit{in the ordered basis } \{|a\rangle, |a^{-1}\rangle\}. \tag{22}$$

We will say that C_0 characterizes the operator T.
ii) Such an operator, or its higher dimensional analogs, define contractive quantum walks.

Since T_ω is not normal in general, the inequalities spr $(T_\omega) \leq \|T_\omega\| \leq 1$ are not necessarily saturated. Actually, we prove below, Corollary 4.7, that $\|T_\omega\| = 1$, so that we need to extract spectral informations about T_ω in order to get decay as $n \to \infty$ of the autocorrelation function $|\langle\psi|U_\omega(C)^n\psi\rangle|$, $\psi \in \mathcal{H}_0$. Hence,

Lemma 3.3 *With the notations above,* spr $(T_\omega) < 1 \;\Rightarrow\; U_\omega(C)$ *is purely ac,* $\forall \omega \in \Omega$.

Proof: If the spectral radius of T_ω satisfies spr $(T_\omega) < 1$, then, for any $\epsilon > 0$ s.t. $|\ln(\text{spr } (T_\omega))| - \epsilon > 0$, $\|T_\omega^n\| \leq (\text{spr } (T_\omega)e^\epsilon)^n$, if n is large enough. Thus, for any normalized $\psi \in \mathcal{H}_0$, we have $|\langle\psi|U_\omega^n\psi\rangle| = |\langle\psi|T_\omega^n\psi\rangle| \leq e^{-n(|\ln(\text{spr } (T_\omega))| - \epsilon)}$, if n is large enough. Thus $\mathcal{H}_0 \subset \mathcal{H}^{ac}(U_\omega)$. Since $\mathcal{H}_0$ can be attached to any vertex of the tree, we get the result. ∎

Remark 3.4 *We show below in Proposition 4.14 that a finer analysis of the structure of T_ω implies that $U_\omega(C)$ is purely ac for all ω, if $g < 1$.*

4 One-Dimensional Contractive Quantum Walk

We turn to the analysis of the random contractive quantum walk defined by (20) and (21) with parameters

$$C_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ s.t. } \tilde{C} = \begin{pmatrix} \alpha & r & \beta \\ q & g & s \\ \gamma & t & \delta \end{pmatrix} \in U(3) \text{ and } 0 \leq g \leq 1. \tag{23}$$

We view this problem as a question of independent interest in the spectral analysis of non self-adjoint or, more adequately in the present context, non-unitary operators.

We start by the following simple property relating C_0 to $\tilde{C}$.

Lemma 4.1 *Let $C_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a contraction on $\mathbb{C}^2$ which is not unitary. Then, there exists $\tilde{C} \in U(3)$ such that (23) holds.*

Proof: By exchanging the basis vectors, we can look for $\tilde{C}$ in the bloc form $\tilde{C} = \begin{pmatrix} C_0 & u \\ \bar{v}^T & g \end{pmatrix}$, where u, v denote vectors in $\mathbb{C}^2$ and $g \in [0, 1]$. Imposing that $\tilde{C} \in U(3)$, we get,

$$\begin{aligned} C_0^* C_0 &= \mathbb{I}_{\mathbb{C}^2} - |v\rangle\langle v|, \quad \|v\|^2 = 1 - g^2, \quad C_0 v = -gu \\ C_0 C_0^* &= \mathbb{I}_{\mathbb{C}^2} - |u\rangle\langle u|, \quad \|u\|^2 = 1 - g^2, \quad C_0^* u = -gv. \end{aligned} \tag{24}$$

It follows that $\sigma(C_0^*C_0) = \{1, g^2\}$, which determines $0 \leq g < 1$ and the norm of the corresponding eigenvector v of $C_0^*C_0$. If $g \neq 0$, then $u = -C_0 v/g$. In case $g = 0$, u is a normalized eigenvector of Ker C_0^*. ∎

Identifying the subspace $\mathcal{H}_0$ with $l^2(\mathbb{Z})$, we get a representation of T_ω by a 5-diagonal doubly infinite matrix. Let $\{e_j\}_{j\in\mathbb{Z}}$, resp. $\{a^m \otimes \tau\}_{m\in\mathbb{Z}}^{\tau\in\{a,a^{-1}\}}$, be the canonical orthonormal basis of $l^2(\mathbb{Z})$, resp. $\mathcal{H}_0$. We map the latter to the former according to the rule

$$e_{2j} = a^j \otimes a, \quad e_{2j+1} = a^j \otimes a^{-1}, \quad j \in \mathbb{Z} \tag{25}$$

and relabel the random phases ω_x^τ accordingly, so that we can identify T_ω with the matrix

$$T_\omega = \mathbb{D}_\omega^0 T = \begin{pmatrix} \ddots & e^{i\omega_{2j-1}\gamma} & e^{i\omega_{2j-1}\delta} & & & \\ & 0 & 0 & & & \\ & 0 & 0 & e^{i\omega_{2j+1}\gamma} & e^{i\omega_{2j+1}\delta} & \\ & e^{i\omega_{2j+2}\alpha} & e^{i\omega_{2j+2}\beta} & 0 & 0 & \\ & & & 0 & 0 & \\ & & & e^{i\omega_{2j+4}\alpha} & e^{i\omega_{2j+4}\beta} & \ddots \end{pmatrix}, \tag{26}$$

where the dots mark the main diagonal and the first column is the image of the vector e_{2j}. We note three special cases which allow for a complete description of the spectrum of T_ω.

Lemma 4.2 *If $\alpha = \delta = 0$, the subspaces* span $\{e_{2j+1}, e_{2j+2}\}$ *reduce T_ω. We have*

$$T_\omega = \oplus_{j\in\mathbb{Z}} T_\omega^{(j)}, \text{ where } T_\omega^{(j)} = \begin{pmatrix} 0 & \gamma e^{i\omega_{2j+1}} \\ \beta e^{i\omega_{2j+2}} & 0 \end{pmatrix}, \ j \in \mathbb{Z}, \tag{27}$$

$\sigma(T_\omega) = \cup_{j\in\mathbb{Z}}\{\pm g^{1/2} e^{i\theta/2} e^{i(\omega_{2j+1}+\omega_{2j+2})/2}\}$, *and* $g = \min(|\beta|, |\gamma|)$, $\theta = \arg(\beta\gamma)$.

If $\beta = \gamma = 0$, the subspaces $\mathcal{H}_+ = \overline{span}\{e_{2j}\}_{j\in\mathbb{Z}}$ and $\mathcal{H}_- = \overline{span}\{e_{2j+1}\}_{j\in\mathbb{Z}}$ reduce T_ω. We have, with $S_\pm$ the standard shifts on $\mathcal{H}_\pm$,

$$T_\omega = T_\omega^{(+)} \oplus T_\omega^{(-)}, \tag{28}$$

where, $T_\omega^{(+)} = T_\omega|_{\mathcal{H}_+}$ is unitarily equivalent to $|\alpha|S_+$, similarly $T_\omega^{(+)} = T_\omega|_{\mathcal{H}_+}$ is unitarily equivalent to $|\delta|S_-$. $\sigma(T_\omega) = \mathbb{S} \cup g\mathbb{S}$, and $g = \min(|\alpha|, |\delta|)$.

If $g = 1$, T_ω is unitary with $\sigma_c(T_\omega) = \emptyset$, almost surely, unless $C_0 \in U(2)$ is diagonal, in which case $\sigma(T_\omega) = \sigma_{ac}(T_\omega) = \mathbb{S}$.

Proof: The decompositions of T_ω under the assumptions made is straightforward. The only point is the determination of the spectral radius when the coefficients are constrained by (23). We consider $\alpha = \delta = 0$ only, the other case being similar. In such a case (23) implies $\bar{q}s = 0$ so that either $q = t = 0$ or $s = r = 0$. In which case $|\gamma| = 1$, or $|\beta| = 1$. In the first case, $g^2 + |r|^2 = 1 = |r|^2 + |\beta|^2$, so that $g = |\beta| = \min(|\beta|, |\gamma|)$. The case $|\beta| = 1$ is similar. Finally, the case $g = 1$ implies that C_0 is unitary, so that T_ω is a one dimensional random quantum walk, and [JM] applies to yield the result. ∎

Remark 4.3 *Quantum walks of the general form (11) can be defined on $\mathbb{Z}^d$ or $\mathcal{T}_{2d}$, with $d \in \mathbb{N}$, using the obvious extension to higher dimensions, see [HJ]. When reduced to a one*

dimensional lattice of the form $\mathcal{H}_0$, they give rise to a contractive quantum walk which has the form of a CMV type matrix of the kind (26). In general, $U(\mathcal{C})$ is not a dilation of the corresponding contractive quantum walk. However, if the quantum walk $U_\omega(C)$ defined on $\mathcal{T}_{2d}$, say, with coin matrix $C \in U(2d)$ having similar properties as for $d = 2$, this property is still true: let us denote the coin states basis by $\{|a_j\rangle, |a_j^{-1}\rangle\}_{j=1,\dots,d}$ and assume $C|a_j^{-1}\rangle = e^{-i\theta_j}|a_j^{-1}\rangle$, for $j = 2, \dots, d$. Consider the subspace $\mathcal{H}_0$ associated with the direction a_1 and P_0 the corresponding orthogonal projection onto $\mathcal{H}_0$; then $U_\omega(C)$ is a dilation of the contraction $T_\omega = P_0 U_\omega(C) P_0$, i.e. Lemma 3.1 holds.

4.1 Translation invariant case

The deterministic, translation invariant case characterized by $\mathbb{D}_\omega = \mathbb{I}$, i.e. $T_\omega = T$, is best tackled by Fourier methods. We map $l^2(\mathbb{Z})$ unitarily onto $L^2(\mathbb{T}; \mathbb{C}^2)$ via the identification

$$\psi = \sum_{j\in\mathbb{Z}} c_j |j\rangle \in l^2(\mathbb{Z}) \;\leftrightarrow\; f(x) = \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix} \in L^2(\mathbb{T}; \mathbb{C}^2), \tag{29}$$

where $f_+(x) = \sum_j c_{2j} e^{i2jx}$, $f_-(x) = \sum_j c_{2j+1} e^{i(2j+1)x}$, $x \in \mathbb{T}$. Then T is unitarily equivalent on $L^2(\mathbb{T}; \mathbb{C}^2)$ to the multiplication operator by the analytic matrix valued function

$$T \simeq T(x) = \begin{pmatrix} \alpha e^{i2x} & \beta e^{ix} \\ \gamma e^{-ix} & \delta e^{-i2x} \end{pmatrix}. \tag{30}$$

The following criteria for more symmetries hold true.

Lemma 4.4 *i) T is self-adjoint $\Leftrightarrow$ $C_0 = \begin{pmatrix} 0 & e^{i\nu} \\ e^{-i\nu} & 0 \end{pmatrix}$, $\nu \in \mathbb{R}$. This implies $g = 1$, T is unitary and $\sigma(T) = \{-1, 1\}$.*
ii) T_ω is unitary $\Leftrightarrow$ $|\det C_0| = \left|\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right| = 1$.

Proof: We have T is sef-adjoint if and only if $T(x)$ is self-adjoint for all $x \in \mathbb{T}$, which together with (23) readily implies the first statement. The second statement is a consequence of the general simple lemma

Lemma 4.5 *Let $W \in M_d(\mathbb{C})$ be a contraction. Then, W is unitary $\Leftrightarrow$ $|\det(W)| = 1$.*

Indeed, T_ω is unitary if and only if T is unitary, which is true, see (21) if and only if C_0 is unitary, and the lemma applies to the last matrix valued contraction.
Proof: The direct implication is trivial. Assume $|\det(W)| = 1$ and consider the spectral decomposition

$$W = \sum_{k=1}^{m} \lambda_k P_k + D_k, \tag{31}$$

where $\sigma(W) = \{\lambda_k\}_{1\le k\le m}$, and $\{P_k\}_{1\le k\le m}$, resp. $\{D_k\}_{1\le k\le m}$, are the eigenprojectors, resp. eigennilpotents of W. Since W is a contraction the condition on the determinant implies $|\lambda_k| = 1$, $k = 1, 2, \dots, m$. Moreover, $\|W^n\| \le 1$ for all $n \ge 0$, so that all eigennilpotents

are equal to zero, since

$$W^n = \sum_{k=1}^{m} \lambda_k^n P_k + \sum_{r=0}^{K} D_k^r \lambda_k^{n-r} \binom{n}{r}, \quad n \geq K, \tag{32}$$

where K is the maximal index of nilpotency of the $D_k's$. Eventually, the general property $\|P_k\| \geq 1$ together with $\sigma(W) \subset \mathbb{S}$ imply that $\|P_k\| = 1$ for W to be a contraction, so that $P_k = P_k^*$ for all $k = 1, 2, \ldots, m$. ■

As T is unitarily equivalent to a multiplication operator, its spectrum is readily obtained in the generic case. For all $x \in \mathbb{T}$, consider the eigenvalues of $T(x)$

$$\lambda_\pm(x) = \frac{1}{2}\left(\alpha e^{i2x} + \delta e^{-i2x} \pm \{(\alpha e^{i2x} + \delta e^{-i2x})^2 - 4(\alpha\delta - \beta\gamma)\}^{1/2}\right). \tag{33}$$

Assume that $\mathbb{T} \cap Z = \emptyset$, where $Z = \{x \in \mathbb{C} \mid \lambda_-(x) = \lambda_+(x)\}$ is the finite set of exceptional points $T(x)$, see [Ka]. Then, with $P_\pm(x)$ the eigenprojectors of the diagonalizable matrix $T(x)$, we get that $(T - z)^{-1}$ is given for $z \in \rho(T)$ by the multiplication operator $R_z(x) = \frac{P_-(x)}{\lambda_-(x)-z} + \frac{P_+(x)}{\lambda_+(x)-z}$, on $L^2(\mathbb{T}; \mathbb{C}^2)$ and $\sigma(T) = \text{Ran } \lambda_- \cup \text{Ran } \lambda_+$.

4.2 Polar decomposition of T_ω

In case the contractive quantum walk T_ω is random, we cannot use Fourier transform methods to determine $\text{spr}(T_\omega)$ but, instead, we resort to the properties of its polar decomposition. Let us come back to the general case (26) and consider the unique decomposition $T_\omega = V_\omega K_\omega$, where K_ω is a non negative operator on $l^2(\mathbb{Z})$ and V_ω is an isometry on $l^2(\mathbb{Z})$. We note that due to (20), K_ω is independent of the randomness since $T_\omega^* T_\omega = T^* T = K^2$.

Theorem 4.6 *The contraction T_ω defined on $l^2(\mathbb{Z})$ by (26) with the constraint (23) admits the polar decomposition $T_\omega = V_\omega K$, where $0 \leq K \leq \mathbb{I}$ is given by*

$$K = P_1 + gP_2, \quad \textit{with} \quad \sigma(K) = \sigma_{ess}(K) = \{1, g\} \quad \textit{and} \quad \|K\| = 1, \tag{34}$$

and with infinite dimensional spectral projectors P_j, $j = 1, 2$ given in (39) below. The isometry V_ω is unitary on $l^2(\mathbb{Z})$ and takes the form $V_\omega = \mathbb{D}_\omega^0 V$, with

$$V = \frac{1}{1+g} \begin{pmatrix} \ddots & \gamma(1+g) - qt & \delta(1+g) - st & & & \\ & 0 & 0 & & & \\ & 0 & 0 & \gamma(1+g) - qt & \delta(1+g) - st & \\ & \alpha(1+g) - qr & \beta(1+g) - sr & 0 & 0 & \\ & & & 0 & 0 & \\ & & & \alpha(1+g) - qr & \beta(1+g) - sr & \ddots \end{pmatrix}, \tag{35}$$

where the dots mark the main diagonal and the first column is the image of the vector e_{2j}.

Corollary 4.7 *for all $\omega \in \Omega$, T_ω satisfies: $\|T_\omega\| = 1$ and T_ω is unitary $\Leftrightarrow g = 1$.*

Remarks 4.8 *i) Condition (23) implies* $g = \left|\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right|$.
ii) The unitary operator V *corresponds to a one-dimensional quantum walk with unitary coin matrix* $\frac{1}{1+g}\begin{pmatrix} \alpha(1+g)-qr & \beta(1+g)-sr \\ \gamma(1+g)-qt & \delta(1+g)-st \end{pmatrix}$, *according to Remark 3.2.*
iii) The random quantum walk V_ω *displays dynamical localization for all values of the parameters in (23), unless the coin matrix is diagonal, in which case it is absolutely continuous, see [JM].*
iv) When $g = 1$, *the original random quantum walk characterized by (14) decouples into one-dimensional problems the solutions of which are known, [JM]. Thus, we assume* $0 \leq g < 1$.
v) We have $0 \in \sigma(K)$ *iff* $0 \in \sigma(T)$, *and* Ker $K =$ Ker T, *since* V *is unitary.*

The proof of Theorem 4.6 entails explicit computations of K and V_ω which are detailed in the next two propostions.

Proposition 4.9 *Assume* $0 \leq g < 1$. *The two-dimensional orthogonal subspaces* $\mathcal{H}^{(k)} = span\{e_{2k}, e_{2k+1}\}$ *reduce the operator* $K = (T^*T)^{1/2}$ *which takes the form*

$$K = \bigoplus_{k\in\mathbb{Z}} \kappa_k \quad with\ respect\ to \quad \mathcal{H}_0 = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}^{(k)}. \tag{36}$$

The bloc κ_k *acts in the ordered basis* $\{e_{2k}, e_{2k+1}\}$ *as*

$$\kappa_k = \frac{1}{|q|^2+|s|^2}\begin{pmatrix} g|q|^2+|s|^2 & \bar{q}s(g-1) \\ q\bar{s}(g-1) & g|s|^2+|q|^2 \end{pmatrix}, \quad \forall k \in \mathbb{Z}, \tag{37}$$

see (23). The spectral decomposition of κ_k *reads*

$$\kappa_k = Q_1^{(k)} + gQ_2^{(k)}, \quad where \quad Q_1^{(k)} = \frac{1}{|q|^2+|s|^2}\begin{pmatrix} |s|^2 & -\bar{q}s \\ -q\bar{s} & |q|^2 \end{pmatrix} = \mathbb{I}_2 - Q_2^{(k)}. \tag{38}$$

We deduce the spectral decomposition of K given in Theorem 4.6 immediately:

$$\sigma(K) = \{1, g\}, \quad K = P_1 + gP_2, \quad \text{where} \quad P_j = \bigoplus_{k\in\mathbb{Z}} Q_j^{(k)}, \ j = 1, 2. \tag{39}$$

Proof: A straightforward computation based on definition (26) yields

$$K^2 = \bigoplus_{k\in\mathbb{Z}} \begin{pmatrix} |\alpha|^2+|\gamma|^2 & \delta\bar{\gamma}+\beta\bar{\alpha} \\ \gamma\bar{\delta}+\alpha\bar{\beta} & |\beta|^2+|\delta|^2 \end{pmatrix} \equiv \bigoplus_{k\in\mathbb{Z}} \kappa_k^2 \tag{40}$$

with the decomposition of $\mathcal{H}_0$ given by (36). Condition (23) allows us to rewrite the blocs κ_k^2 of this decomposition as

$$\kappa_k^2 = \begin{pmatrix} 1-|q|^2 & -s\bar{q} \\ -q\bar{s} & 1-|s|^2 \end{pmatrix}, \quad \text{with} \ \begin{cases} \det \kappa_k^2 = 1-(|q|^2+|s|^2) = g^2 \\ \mathrm{tr}\kappa_k^2 = 2-(|q|^2+|s|^2) = 1+g^2. \end{cases} \tag{41}$$

Hence, $\sigma(\kappa_k^2) = \{1, g\}$ with corresponding normalized eigenvectors

$$v_1^{(k)} = \frac{1}{\sqrt{|q|^2+|s|^2}}\begin{pmatrix} s \\ -q \end{pmatrix}, \quad v_2^{(k)} = \frac{1}{\sqrt{|q|^2+|s|^2}}\begin{pmatrix} \bar{q} \\ \bar{s} \end{pmatrix}. \tag{42}$$

Explicit computations yield the spectral projectors $Q_1^{(k)} = |v_1^{(k)}\rangle\langle v_1^{(k)}|$ and $Q_2^{(k)} = \mathbb{I}_2 - Q_2^{(k)}$, and, in turn, $\kappa_k = (\kappa_k^2)^{1/2}$. The spectral decomposition of K follows immediately. ∎

We now turn to the computation of the isometry $V_\omega = \mathbb{D}_\omega^0 V$. Recall that translation invariant operators with the same band structure matrix as T are characterized by a 2×2 matrix, in the same way as T is characterized by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, see Remark 3.2.

Proposition 4.10 *For $1 > g > 0$, $V = TK^{-1}$ where $K^{-1} = \bigoplus_{k\in\mathbb{Z}} \kappa_k^{-1}$ and*

$$\kappa_k^{-1} = \frac{1}{g(1+g)} \begin{pmatrix} 1 - |s|^2 + g & s\bar{q} \\ q\bar{s} & 1 - |q|^2 + g \end{pmatrix}. \tag{43}$$

The operator V has the same band structure as T and is characterized by the unitary matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \kappa_k^{-1} = \frac{1}{1+g} \begin{pmatrix} \alpha(1+g) - qr & \beta(1+g) - sr \\ \gamma(1+g) - qt & \delta(1+g) - st \end{pmatrix}. \tag{44}$$

Remark 4.11 *The unitary operator V is well defined in the limit $g \to 0$, with the constraint (23), even though K^{-1} is not.*

Proof: The first statement is a consequence of Proposition 4.9 and of the spectral theorem. The invariance of the subspaces $\mathrm{span}\{e_{2k}, e_{2k+1}\}$ under K^{-1} and the matrix structure of T imply that V has the same structure as T. It is a matter of computation to check statement (44), systematically using constraint (23) to simplify the factor g in the denominator. ∎

4.3 Structure of the Contraction T_ω

Recall that a contraction is said to be completely non-unitary, cnu for short, if it possesses no non-trivial closed invariant subspace on which it is unitary, see *e.g.* [SFBK].

Lemma 4.12 *Let $0 \leq g < 1$. Then, for all $\omega \in \Omega$, the operator T_ω is either cnu or it is unitarily equivalent to the direct sum of a shift and of g times a shift. Consequently,*

$$\sigma_p(T_\omega) \cap \mathbb{S} = \emptyset, \quad \textit{and for } 0 < g < 1, \;\; \sigma_p(T_\omega) \cap g\mathbb{S} = \emptyset. \tag{45}$$

Proof: Assume there is a closed subspace $\mathfrak{h}_0$ such that $T_\omega|_{\mathfrak{h}_0}$ is unitary. For $\psi \in \mathfrak{h}_0$, we have $\|T_\omega\psi\| = \|\psi\|$. This implies with $T_\omega = V_\omega(P_1 + gP_2)$, that

$$(\mathbb{I} - T_\omega^* T_\omega)^{1/2}\psi = \sqrt{1-g^2}P_2\psi = 0. \tag{46}$$

Hence, $\mathfrak{h}_0 \subset P_1\mathcal{H}_0$, and, $\mathfrak{h}_0$ being invariant under T_ω, $\mathfrak{h}_o \subset \mathrm{Ker}\, P_2V_\omega P_1$. The operator $P_2V_\omega P_1$ is studied in Lemmas 5.15 and 5.20 below, where it is shown that $\mathrm{Ker}\, P_2V_\omega P_1 \neq \{0\} \Leftrightarrow P_2V_\omega P_1 = 0$ and that this is equivalent to

$$\tilde{C} \in \left\{ \begin{pmatrix} \alpha & r & 0 \\ q & g & 0 \\ 0 & 0 & \delta \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & g & s \\ 0 & t & \delta \end{pmatrix} \right\} \subset U(3). \tag{47}$$

Hence if (47) doesn't hold, T_ω is cnu, whereas in case (47) holds, Lemma 4.2 finishes the proof of the first statement. The fact that eigenvalues cannot sit on the unit circle is thus immediate, whereas, for $g > 0$, a similar argument applied to the contraction $(gT_\omega^{-1})^* = V_\omega(gP_1 + P_2)$ yields the last statement. ∎

Remark 4.13 *The operator T_ω is cnu if and only if $0 \leq g < 1$, $|\alpha| < 1$ and $|\delta| < 1$. Moreover, in case (47) holds, the corresponding random quantum walk operator $U_\omega(C)$ is purely ac by a general argument, see eq. (66) §5.4 of [HJ].*

The fact that T_ω is completely non-unitary has immediate consequences on the spectrum of $U_\omega(C)$. In particular, the following result extends the description of the spectral diagram discussed in paragraph 5.6 of [HJ].

Proposition 4.14 *If $0 \leq g < 1$, then $\sigma(U_\omega(C)) = \sigma_{ac}(U_\omega(C))$, for all $\omega \in \Omega$.*

Proof: We drop the dependence on ω and C in the notation for this proof, for simplicity. By Lemma 4.12, we can assume T is completely non-unitary. Let P_{sing} be the spectral projection onto the subspace $\mathcal{H}^{sing} = \mathcal{H}^{pp}(U) \cap \mathcal{H}^{sc}(U)$ and recall that P_0 is the orthogonal projection onto $\mathcal{H}_0$. We first show that the subspace $\mathcal{H}_0 \cap \mathcal{H}^{sing}$ reduces the operator U. Let $\psi \in \mathcal{H}_0 \cap \mathcal{H}^{sing}$,

$$U\psi = UP_{sing}\psi = P_{sing}U\psi = P_{sing}\big(P_0U\psi + (\mathbb{I} - P_0)U\psi\big), \tag{48}$$

where $(\mathbb{I} - P_0)U\psi \in \mathcal{H}_b$, see (18). Using $P_{sing}\mathcal{H}_b = 0$, we get that $U\psi = P_{sing}P_0U\psi$. But then $\|U\psi\| \leq \|P_0U\psi\| \leq \|U\psi\|$ implies $U\psi = P_0U\psi = P_0P_{sing}U\psi$ as well. Hence $\mathcal{H}_0 \cap \mathcal{H}^{sing}$ is invariant under U. By a similar argument, this subspace is invariant under U^* as well. Consequently, $\mathcal{H}^{sing}$ reduces $T = P_0U|_{\mathcal{H}_0}$, which shows that $\mathcal{H}^{sing} \cap \mathcal{H}_0 = \{0\}$ since T is cnu and $g < 1$. Repeating the argument with $\mathcal{H}_0$ replaced by the horizontal subspace attached to $x \in \mathcal{T}_4$ arbitrary eventually yields $\mathcal{H}^{sing} = \{0\}$. ∎

Remark 4.15 *In view of Lemma 4.1, one sees that Lemma 4.12 and Proposition 4.14 carry over to the cases described in Remark 4.3, in case T_ω is cnu..*

4.4 Extensions to Further Contractive Quantum Walks

We make use of a symmetry of the contractive quantum walk $T_\omega = \mathbb{D}^0_\omega T$ with T given by (21) in order relate it to $\widetilde{T}_\omega$ given by (3). Let

$$\begin{aligned} \mathcal{H}_{\mathbf{e}} &= \overline{\text{span}}\{a^m \otimes \tau, \quad m \in 2\mathbb{Z}, \tau \in \{\pm 1\}\}, \\ \mathcal{H}_{\mathbf{o}} &= \overline{\text{span}}\{a^m \otimes \tau, \quad m \in 2\mathbb{Z}+1, \tau \in \{\pm 1\}\} \end{aligned} \tag{49}$$

denote the supplementary subspaces of $\mathcal{H}_0$ consisting in even and odd sites only in configuration space. The definition (21) of T makes it clear that $T\mathcal{H}_{\mathbf{e}} \subset \mathcal{H}_{\mathbf{o}}$ and $T\mathcal{H}_{\mathbf{o}} \subset \mathcal{H}_{\mathbf{e}}$, and since $\mathbb{D}^0_\omega$ is diagonal, the same is true for T_ω. Therefore $\mathcal{H}_{\mathbf{e}}$ is invariant under T^2_ω and by Lemma 2 in [CD], $\sigma(T^2_\omega) \setminus \{0\} = \sigma(T^2_\omega|_{\mathcal{H}_{\mathbf{e}}}) \setminus \{0\}$. Actually we have

Proposition 4.16 *For all $0 \leq g \leq 1$, and with definitions (1) and (3),*

$$\widetilde{T}_\omega \simeq T^2_\omega|_{\mathcal{H}_{\mathbf{e}}} \;\Rightarrow\; \sigma(\widetilde{T}_\omega) = \sigma(T^2_\omega). \tag{50}$$

Moreover,

$$\widetilde{T}_\omega = \bigoplus_{k\in\mathbb{Z}} S_\omega(2k+1) \bigoplus_{k\in\mathbb{Z}} S_\omega(2k) \tag{51}$$

where, for all $k \in \mathbb{Z}$, we have in the basis $\{e_{2k}, e_{2k+1}\}$, resp. $\{e_{2k+1}, e_{2k+2}\}$

$$S_\omega(2k) = \mathrm{diag}(e^{i\omega_{4k-1}}, e^{i\omega_{4k+2}}) \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}, \text{resp. } S_\omega(2k+1) = \mathrm{diag}(e^{i\omega_{4k+1}}, e^{i\omega_{4k+4}}) \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}. \tag{52}$$

Proof: With the convention (25), $\mathcal{H}_{\mathbf{e}}$ is spanned by $\{e_{4k}, e_{4k+1},\ k \in \mathbb{Z}\}$. Relabelling these basis vectors according to $e_{4k} \mapsto e_{2k}$, $e_{4k+1} \mapsto e_{2k+1}$, explicit computations yield $\widetilde{T}_\omega \simeq T_\omega^2|_{\mathcal{H}_{\mathbf{e}}}$, as well as (51). Observe that $g \neq 0$ iff $\widetilde{T}_\omega$ and T_ω are boundedly invertible and that if $g = 0$, we have $0 \in \sigma(\widetilde{T}_\omega) \cap \sigma(T_\omega^2)$. This yields isospectrality of T_ω^2 and $\widetilde{T}_\omega$. ∎

Remarks 4.17 *i) The restriction $T_\omega^2|_{\mathcal{H}_{\mathbf{o}}}$ has an explicit form similar to $\widetilde{T}_\omega$ given by the composition (51) in the reversed order.*
ii) In particular, we deduce from the above that $\widetilde{T}_\omega$ is unitary iff $g = 1$, and that it is pure point for $\beta\gamma \neq 0$, whereas it is absolutely continuous if $\beta = \gamma = 0$, [JM].
iii) All the spectral results we derive for T_ω hold for $\widetilde{T}_\omega$ via the spectral mapping theorem.

5 Spectral Analysis of T_ω

We use the following notations: $\sigma_p(A)$ denotes the set of eigenvalues of a bounded operator A on $\mathcal{H}$ and $\sigma_{app}(A)$ denotes its approximate point spectrum. By definition, $\lambda \in \sigma_{app}(A)$ if and only if there exists a sequence of normalized vectors $\{\varphi_n\}_{n\in\mathbb{N}}$ such that $A\varphi_n - \lambda\varphi_n \to 0$, as $n \to \infty$. Recall that $\sigma_p(A) \subset \sigma_{app}(A)$ and $\sigma(A) = \sigma_{app}(A) \cup \overline{\sigma_p(A^*)}$, where $\overline{X} = \{\bar{x},\ |\ x \in X\}$, for any $X \subset \mathbb{C}$. Also, $\sigma_{app}(A)$ is a nonempty closed set of $\mathbb{C}$ such that $\partial\sigma(A) \subset \sigma_{app}(A)$ and one has the disjoint union $\sigma(A) = \sigma_{app}(A) \cup \overline{\sigma_{p_1}(A^*)}$, where $\sigma_{p_1}(A^*) = \{\lambda \in \mathbb{C} \mid \text{s.t. Ker } (A^* - \lambda) \neq \{0\} \text{ and } \mathrm{Ran}(A^* - \lambda) = \mathcal{H}\}$ is open in $\mathbb{C}$, see [Ku].

The starting point of analysis of the contraction T_ω is Theorem 4.4 showing that T_ω admits a polar decomposition the components of which are bounded normal operators. We are thus naturally lead to the study of spectral properties of products of such operators. The only general result we are aware of in this direction, [W], provides estimates on the position of the spectrum of such products in terms of the numerical ranges of the components, which is however not strong enough for our purpose. We will use instead

Theorem 5.1 *Let $T = AB$, where A, B are bounded normal operators on $\mathcal{H}_0$ and let $B_c(r)$ denote the open disc of radius $r > 0$ and center $c \in \mathbb{C}$. Then,*

$$\begin{aligned} B^{-1} \in \mathcal{B}(\mathcal{H}_0) \ &\Rightarrow \bigcup_{\tau\in\rho(A)} \bigcap_{b\in\sigma(B)} B_{\tau b}(|b|\ \mathrm{dist}(\tau, \sigma(A))) \subset \rho(AB), \\ A^{-1} \in \mathcal{B}(\mathcal{H}_0) \ &\Rightarrow \bigcup_{\tau\in\rho(B)} \bigcap_{a\in\sigma(A)} B_{\tau a}(|a|\ \mathrm{dist}(\tau, \sigma(B))) \subset \rho(AB). \end{aligned} \tag{53}$$

Proof: Under our assumption on τ, and since B is invertible, we have

$$T - z = (A - \tau)B + \tau B - z = (A - \tau)\left(\mathbb{I} + (A - \tau)^{-1}(\tau B - z)B^{-1}\right)B, \tag{54}$$

which shows that $T-z$ is boundedly invertible if $\|(A-\tau)^{-1}(\tau B-z)B^{-1}\| < 1$, thanks to Neumann's series. By the spectral theorem for normal operators applied to the continuous function $x \to \frac{|\tau x - z|}{|x|}$ defined on the compact set $\sigma(B)$, and using $\|(A-\tau)^{-1}\| = 1/\operatorname{dist}(\tau, \sigma(A))$, this condition is met if

$$\max_{b\in\sigma(B)} \frac{|z-\tau b|}{|b|} < \operatorname{dist}(\tau, \sigma(A)). \tag{55}$$

Therefore, given $\tau \in \rho(A)$, if $z \in \bigcap_{b\in\sigma(B)} B_{\tau b}(b \operatorname{dist}(\tau, \sigma(A)))$, then $z \in \rho(AB)$. Taking the union over $\tau \in \rho(A)$ yields (53). The second inclusion is proven analogously, using A invertible and identity for $\tau \in \rho(B)$

$$T - z = A(B-\tau) + \tau A - z = A\left(\mathbb{I} + A^{-1}(\tau A - z)(B-\tau)^{-1}\right)(B-\tau). \tag{56}$$

■

Remark 5.2 *In case A and B have bounded inverses, we get for $\tau = 0$ that $B_0(r_{AB}) \subset \rho(AB)$, where $r_{AB} = \operatorname{dist}(0, \sigma(A))\operatorname{dist}(o, \sigma(B)) > 0$.*

Applied to our case $T = VK$ with $\sigma(K) = \{g, 1\}$, $0 < g < 1$, (53) simplifies and yields more specific estimates on $\rho(T)$ as a function of the spectrum of the unitary operator V.

Corollary 5.3 *Let $T = VK$ with V unitary and $0 < K = (P_1 + gP_2)$, $0 < g < 1$. Then*

$$\bigcup_{\tau\in\rho(V)} B_{\tau g}(g \operatorname{dist}(\tau, \sigma(V))) \cap B_\tau(\operatorname{dist}(\tau, \sigma(V))) \subset \rho(T), \tag{57}$$

$$\bigcup_{\tau\in\rho(K)} \bigcap_{v\in\sigma(V)} B_{\tau v}(\operatorname{dist}(\tau, \sigma(K))) \subset \rho(T). \tag{58}$$

In particular,

$$B_0(g) \subset \rho(T). \tag{59}$$

Moreover, assume the arc $(-\theta, \theta)$ belongs to $\rho(V)$, with $0 < \theta < \pi$. Then,

$$\bigcup_{\substack{\tau\in\mathbb{R}_+ \\ \alpha\in[-\theta,\theta]}} B_{e^{i\alpha}\tau}(d_{e^{i\alpha}\tau}) \cap B_{ge^{i\alpha}\tau}(g d_{e^{i\alpha}\tau}) \subset \rho(T), \tag{60}$$

$$\bigcup_{\substack{\tau\in\mathbb{R}_- \\ \alpha\in[-\pi/2,\pi/2]}} \bigcap_{e^{i\nu}\in\sigma(e^{i\alpha}V)} B_{e^{i\nu}\tau}(\delta_{e^{i\alpha}\tau}) \subset \rho(T), \quad \textit{where} \tag{61}$$

$$d_{e^{\pm i\alpha}\tau} = \operatorname{dist}(e^{\pm i\alpha}\tau, \sigma(V)) = \sqrt{\tau^2 - 2\tau\cos(\theta-\alpha) + 1} \textit{ with } \tau > 0,\ \alpha \in [0, \theta], \tag{62}$$

$$\delta_{e^{\pm i\alpha}\tau} = \operatorname{dist}(e^{\pm i\alpha}\tau, \sigma(K)) = \sqrt{\tau^2 + 2|\tau| g\cos(\alpha) + g^2} \textit{ with } \tau < 0,\ \alpha \in [0, \pi/2]. \tag{63}$$

Remarks 5.4 *i) The points $\tau \in \rho(V)$ in (57) such that $dist(\tau, \sigma(V)) = |1-\tau|$ do not yield more information than (59): $\tau < 1$ implies $\bigcap_{k\in\sigma(K)} B_{\tau k}(k(1-|\tau|)) \subset B_0(g)$ and $\tau > 1$ implies $\bigcap_{k\in\sigma(K)} B_{\tau k}(k(|\tau|-1)) \subset \mathbb{C} \setminus \overline{B_0(1)}$. This is the case when $\sigma(V) = \mathbb{S}$.*
ii) At the expense of a rotation, we can associate to any arc in $\rho(V)$ two sets (60) and (61) that belong to $\rho(T)$. The corresponding sets are both symmetrical with respect to the bisector of that arc.
iii) Lemma 4.2 or Remark 5.23 shows that (59) is optimal.

Proof: The first statements are mere rewritings of (53) and Remark 5.2 implies (59). For (60), we note that $w \in \mathbb{C}$ is such that $\mathrm{dist}(w, \sigma(V)) = |w - e^{\pm i\theta}|$ if $w = \tau e^{\pm i\alpha}$, with $\alpha \in [0, \theta]$ and $\tau \geq 0$, which establishes (62). Whereas for (61), $w = -|\tau| e^{\pm i\alpha}$ with $\alpha \in [0, \pi/2]$ satisfies $\mathrm{dist}(w, \sigma(K)) = |w - g| = ||\tau| e^{\pm i\alpha} + g|$ which yields (63). Then a change of variables allows us to express (58) as (61) under our assumptions. ∎

Without attempting to provide a complete analysis, we describe (60) and (61) in some more details and show that (61) provides less information in case $\sigma(V)$ displays one gap only. The proofs of the statements are provided in an Appendix. Let $C_c(r)$ denote the circle of center $c \in \mathbb{C}$ and radius $r > 0$ and ∂S denote the boundary of a set S. First consider (60) for $\alpha = 0$. Because the intersection of discs can be non-empty when the intersection of their boundary is empty, there is a difference between (60) and the set $D(\theta)$ such that

$$\partial D(\theta) = \bigcup_{\tau \in \mathbb{R}_+} C_\tau(d_\tau) \cap C_{g\tau}(g d_\tau), \tag{64}$$

and $D(\theta)$ contains the vertical segment between the intersection of two circles. We also set $R_\gamma(\theta) = \{z \in \mathbb{C} \,|\, \Re z > \gamma \cos(\theta)\}$.

Lemma 5.5 *With the notations above, and assuming $\alpha = 0$, the LHS of (60) is given by*

$$\bigcup_{\tau \in \mathbb{R}_+} B_\tau(d_\tau) \cap B_{g\tau}(g d_\tau) = D(\theta) \cup B_0(g) \cup R_1(\theta), \quad \text{for } \theta \in]0, \pi/2[\ , \tag{65}$$

see Fig. 2, where $\partial D(\theta)$ is given by the cubic curve

$$\begin{aligned} y^2 &= \frac{x(x^2 - x(1+g)\cos(\theta) + g)}{(1+g)\cos(\theta) - x} \ \text{with} \\ x &= -\frac{1+g}{2\tau} + (1+g)\cos(\theta) \in [0, (1+g)\cos(\theta)[, \quad \text{for } \tau \in [1/(2\cos(\theta)), \infty[. \end{aligned} \tag{66}$$

For $\pi/2 \leq \theta < \pi$,

$$\bigcup_{\tau \in \mathbb{R}_+} B_\tau(d_\tau) \cap B_{g\tau}(g d_\tau) = B_0(g) \cup R_g(\theta). \tag{67}$$

Moreover, for fixed $0 < \alpha < \theta$, assuming $0 < \theta < \pi$, we have

$$\bigcup_{\tau \in \mathbb{R}_+} B_{e^{i\alpha}\tau}(d_{e^{i\alpha}\tau}) \cap B_{ge^{i\alpha}\tau}(g d_{e^{i\alpha}\tau}) \subset \bigcup_{\tau \in \mathbb{R}_+} B_\tau(d_\tau) \cap B_{g\tau}(g d_\tau). \tag{68}$$

Remarks 5.6 *i) In particular, under our assumptions, the segment $[0, 1] \subset \rho(VK)$ if $\cos^2(\theta) < \frac{4g}{(1+g)^2} \in]0, 1[$, see Figure 2.That this condition is necessary in general can be seen on the matrix case*

$$V = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ K = \frac{1}{2}\begin{pmatrix} 1+g & 1-g \\ 1-g & 1+g \end{pmatrix} \tag{69}$$

such that $\sigma(VK) = \{\frac{1}{2}(\cos(\theta)(1+g) \pm \sqrt{\cos^2(\theta)(1+g)^2 - 4g})\} \subset \mathbb{R}_+^*$, if $\cos^2(\theta) \geq \frac{4g}{(1+g)^2}$.
ii) The points $0, ge^{i\theta}$ and $e^{i\theta}$ belong to $\partial D(\theta)$ and correspond to the values of τ given by $1/(2\cos(\theta)), (1+g)/(2\cos(\theta))$ and $(1+g)/(2g\cos(\theta))$ respectively.

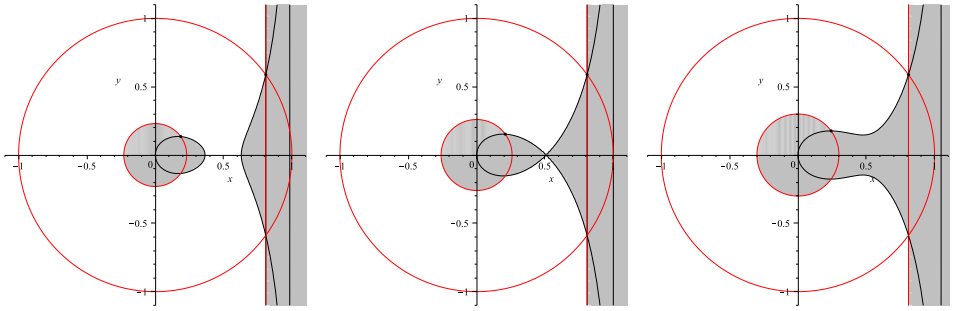

Figure 2: The sets $D \cup B_0(g) \cup R_1(\theta)$ for $0 < \theta < \pi/2$ fixed and increasing values of g. The unit circle $\mathbb{S}$ and $g\mathbb{S}$ are indicated in red, whereas the black curves denote ∂D. The vertical red line corresponds to $\partial R_1(\theta)$.

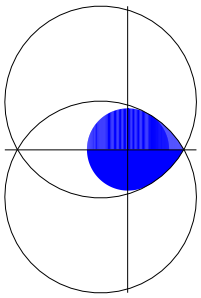

Figure 3: The set $\Gamma_{\rho,\rho'}(\theta)$.

To discuss the set (61), we need some notations. For $\rho, \rho' > 0$, we define, see Figure 3,

$$\Gamma_{\rho,\rho'}(\theta) = (B_{-e^{+i\theta}\rho}(\rho+\rho') \cap B_{-e^{-i\theta}\rho}(\rho+\rho') \cap R_{\rho'}(\theta)) \cup B_0(\rho'). \tag{70}$$

where the two discs $B_{-e^{\pm i\theta}\rho}(\rho+\rho')$ tangent to $B_0(\rho')$ at $\rho' e^{\pm i\theta}$. We prove the following in an Appendix.

Lemma 5.7 *Assume $\sigma(V) = \{e^{i\nu} \text{ s.t. } \nu \in [\theta,\pi] \cup [-\pi,-\theta]\}$, with $\theta \in]0,\pi[$. We have*

$$\bigcup_{\tau\in\mathbb{R}_-} \bigcap_{e^{i\nu}\in\sigma(V)} B_{e^{i\nu}\tau}(\delta_\tau) = B_0(g) \cup \Delta_g(\theta), \tag{71}$$

where $\Delta_g(\theta)$ denotes either the triangle defined by the points $ge^{i\theta}, ge^{-i\theta}, g/\cos(\theta)$ whenever $\theta < \pi/2$, or $\Delta_g(\theta)$ denotes the set delimited by the two non-vertical lines passing by these points and the condition $\Re z \geq g\cos(\theta)$ whenver $\theta \in [\pi/2,\pi[$. Then, for each $\alpha \in]0,\pi/2[$ fixed,

$$\bigcup_{\tau\in\mathbb{R}_-} \bigcap_{e^{i\nu}\in\sigma(V)} B_{e^{i\nu}e^{i\alpha}\tau}(\delta_{e^{i\alpha}\tau}) = e^{i\alpha} \bigcup_{|\tau|\in\mathbb{R}_+} \Gamma_{|\tau|,\delta_{\tau e^{i\alpha}}-|\tau|}(\theta). \tag{72}$$

For any $\theta \in [0,\pi[$, and all $\alpha \in]0,\pi/2[$,

$$\bigcup_{\tau\in\mathbb{R}_-} \bigcap_{e^{i\nu}\in\sigma(V)} B_{e^{i\nu}e^{i\alpha}\tau}(\delta_{e^{i\alpha}\tau}) \subset \bigcup_{\tau\in\mathbb{R}_+} B_\tau(d_\tau) \cap B_{g\tau}(gd_\tau). \tag{73}$$

Example 5.8 Let us illustrate the use of Theorem 5.1. Consider

$$\tilde{C}(\xi,\eta)=\begin{pmatrix}\cos(\eta) & \cos(\xi)\sin(\eta) & -\sin(\xi)\sin(\eta)\\ 0 & \sin(\xi) & \cos(\xi)\\ \sin(\eta) & -\cos(\xi)\cos(\eta) & \sin(\xi)\cos(\eta)\end{pmatrix}\in O(3),\quad \xi,\eta\in[0,\pi/2], \tag{74}$$

where (ξ,η) is restricted to $[0,\pi/2]^2$ for simplicity. We thus compute that

$$T,\ \text{resp. } V,\ \text{is characterized by } \begin{pmatrix}\cos(\eta) & -\sin(\eta)\sin(\xi)\\ \sin(\eta) & \cos(\eta)\sin(\xi)\end{pmatrix},\ \text{resp. } \begin{pmatrix}\cos(\eta) & -\sin(\eta)\\ \sin(\eta) & \cos(\eta)\end{pmatrix}. \tag{75}$$

Moreover, Fourier methods yield

$$\sigma(V)=\{z\in\mathbb{S}\mid \arg z\in[\eta,\pi-\eta]\cup[-\pi+\eta,-\eta]\}. \tag{76}$$

Assuming the common distribution $d\nu$ of phases has support given by

$$\text{supp } d\nu=[-\epsilon,\epsilon],\ \text{with } \epsilon<\eta, \tag{77}$$

we have thanks to the general almost sure relation $\sigma(V_\omega)=\sigma(V)e^{i\,\text{supp}(d\nu)}$ which holds for products of unitary operators of that sort, see Section 5.1 of [J1], for example,

$$\sigma(V_\omega)=\{z\in\mathbb{S}\mid \arg z\in[\eta-\epsilon,\pi-\eta+\epsilon]\cup[-\pi+\eta-\epsilon,-\eta+\epsilon]\},\ \text{a.s.} \tag{78}$$

Hence, Corollary 5.3 applies with $\theta=\eta-\epsilon$ and $g=\sin(\xi)$, and gives rise to two regions of $\rho(T_\omega)$: one described in Lemma 5.5, and its symmetric image with respect to the vertical axis. In particular, the spectrum of the corresponding T_ω is separated into two disjoint parts if

$$\cos^2(\eta-\epsilon)\le\frac{4\sin(\xi)}{(1+\sin(\xi))^2}. \tag{79}$$

Let us continue with some general links between the spectral properties of T_ω and $U_\omega(C)$.

Lemma 5.9 *Let U be unitary on $\mathcal{H}$ and P_0 be an orthogonal projector. For any $\varphi\in\mathcal{H}$*

$$UP_0\varphi = e^{i\theta}\varphi\Rightarrow\varphi=P_0\varphi \text{ and } e^{i\theta}\varphi=U\varphi=P_0UP_0\varphi, \tag{80}$$

$$P_0U\varphi = e^{i\theta}\varphi\Rightarrow\varphi=P_0\varphi \text{ and } e^{i\theta}\varphi=U\varphi=P_0UP_0\varphi. \tag{81}$$

Moreover, writing $Q_0=\mathbb{I}-P_0$, we get

$$\text{Ker } Q_0UP_0=\{0\}\Rightarrow\sigma_p(UP_0)\cap\mathbb{S}=\sigma_p(P_0U)\cap\mathbb{S}=\sigma_p(P_0UP_0)\cap\mathbb{S}=\emptyset. \tag{82}$$

Furthermore, let $T=P_0UP_0|_{P_0\mathcal{H}}$. If $e^{i\theta}\in\sigma_{app}(T)\setminus\sigma_p(T)$, then $e^{i\theta}\in\sigma_{app}(U)$.

Proof: Taking the norm of the left hand side of (80) yields $P_0\varphi=\varphi$, $Q_0UP_0\varphi=0$ and the first identities follow. For (81), $P_0U\varphi=e^{i\theta}\varphi=P_0e^{i\theta}\varphi$ gives the results directly. Now, $P_0U\varphi=e^{i\theta}\varphi\ \Leftrightarrow\ UP_0\psi=e^{i\theta}\psi$ where $\psi=U\varphi$ shows with (80) that (81) implies $Q_0UP_0\psi=0$. Similarly, $P_0UP_0\varphi=e^{i\theta}\varphi$ implies $Q_0UP_0\varphi=0$. Thus, if Ker $Q_0UP_0=\{0\}$, we get the absence of eigenvalue of modulus one for UP_0, P_0U and P_0UP_0. Finally, let $e^{i\theta}\in\sigma_{app}(T)\setminus\sigma_p(T)$ and $\varphi_n\in P_0\mathcal{H}$ s.t. $\|\varphi_n\|=1$ and $T\varphi_n-e^{i\theta}\varphi_n\to0$. By assumption, $\|U\varphi_n\|^2=\|e^{i\theta}\varphi_n+(P_0U\varphi_n-e^{i\theta}\varphi_n)\|^2+\|Q_0U\varphi_n\|^2$, where the parenthesis in the right hand side tends to zero, as $n\to\infty$. As U is unitary, we have $\lim_{n\to\infty}Q_0U\varphi_n=0$. Consequently, $e^{i\theta}\in\sigma_{app}(U)$ since $U\varphi_n-e^{i\theta}\varphi_n=T\varphi_n-e^{i\theta}\varphi_n+Q_0U\varphi_n\to0$, as $n\to\infty$. ∎

Remark 5.10 *i) The same result holds with T^* and U^* in place of T and U.*
ii) If Ker $(Q_0UP_0) = \{0\}$, $\lim_{n\to\infty} Q_0U\varphi_n = 0$ *implies that the operator* $[Q_0UP_0]^{-1}$: Ran $Q_0UP_0 \subset Q_0\mathcal{H} \to P_0\mathcal{H}$ *is not bounded.*

Let us also recall the following properties.

Lemma 5.11 *Let $T = V(P_1+gP_2)$ and $\varphi \in \mathcal{H}_0$ such that $T\varphi = \lambda\varphi$. Then for all $0 < g < 1$,*

$$\begin{aligned} |\lambda| = 1 \quad &\Rightarrow \quad \varphi = P_1\varphi \text{ and } V\varphi = P_1VP_1\varphi = \lambda\varphi, \\ |\lambda| = g \quad &\Rightarrow \quad \varphi = P_2\varphi \text{ and } V\varphi = P_2VP_2\varphi = (\lambda/g)\varphi. \end{aligned} \tag{83}$$

Consequently,

$$\begin{aligned} \text{Ker } P_2VP_1 = \{0\} \quad &\Rightarrow \quad \sigma_p(T) \cap \mathbb{S} = \emptyset, \text{ and} \\ \text{Ker } P_1VP_2 = \{0\} \quad &\Rightarrow \quad \sigma_p(T) \cap g\mathbb{S} = \emptyset. \end{aligned} \tag{84}$$

If $g = 0$,

$$\sigma(T) = \sigma(P_1VP_1|_{P_1\mathcal{H}_0}) \cup \{0\}. \tag{85}$$

Proof: All statements except the last one are consequences of the proof of Lemma 4.12. If $g = 0$, $T = VP_1$, so that Ker $T = P_2\mathcal{H}_0$. Statement (85) is a consequence of (89) and (90) in the proof of Theorem 5.13 below. ∎

Remarks 5.12 *i) Analogous statements hold when T is replaced by $(P_1 + gP_2)V = V^*TV$. In particular, the results hold for T^*.*

Next, we come back to our random setting and make further use of the structure of K to apply the Feschbach-Schur method in order to obtain conditions on the coefficients of $\tilde{C}$ (23) that ensure that for all realizations $\omega \in \Omega$, spr $(T_\omega) < \|T_\omega\| = 1$, in case $g < 1$.

Theorem 5.13 *Let $T_\omega = V_\omega(P_1 + gP_2)$, where P_j are defined in (34) and $0 \le g < 1$. Consider $P_jVP_k = V_{jk}$, $j,k \in \{1,2\}$, as operators on $P_k\mathcal{H}$. If $\|V_{11}\| < 1$, then, for all realizations $\omega \in \Omega$*

$$g < \frac{1 - \|V_{11}\|}{\|V_{21}\|\|V_{12}\| + \|V_{22}\|(1 - \|V_{11}\|)} \quad \Rightarrow \quad \text{spr } (T_\omega) < 1. \tag{86}$$

Moreover, the set $\{|z| < g\} \cup \{r(V) < |z| \le 1\} \subset \rho(T_\omega)$ for all $\omega \in \Omega$, where

$$r(V) = \frac{1}{2}\left(\|V_{11}\| + g\|V_{22}\| + \sqrt{(\|V_{11}\| - g\|V_{22}\|)^2 + 4g\|V_{21}\|\|V_{12}\|\|}\right). \tag{87}$$

Remarks 5.14 *i) The result is deterministic and holds for any operator $T = V(P_1 + gP_2)$, where V is unitary and $\{P_j\}_{j=1,2}$ are supplementary orthogonal projectors.*
ii) In case V_ω is given by Theorem 4.6, (86) yields a somehow implicit condition since the norms $\|V_{jk}\|$ depend on g, see Lemma 5.15 and Example 5.19 below.
iii) Remark 5.23 below shows that $r(V)$ is optimal.
iv) This infinite dimensional result is reminiscent of the works [WF, B], which consider matrices of the form $T_\omega = V_\omega K$ where V_ω is a unitary, Haar distributed matrix and $K > 0$ is given. It is shown under various assumptions that a density of eigenvalues of T_ω can be defined, which is supported in a deterministic ring.

Proof: It is enough to prove the second statement. We start with the deterministic case. Given $K = P_1 + gP_2$, we split $\mathcal{H}_0$ as $\mathcal{H}_0 = \mathcal{H}_1 \bigoplus \mathcal{H}_2$ where $\mathcal{H}_j = P_j\mathcal{H}_0$. Writing $T = VK$ as a bloc structure according to this decomposition, we have for any $z \in \mathbb{C}$

$$T - z\mathbb{I} = \begin{pmatrix} V_{11} - z\mathbb{I}_1 & gV_{12} \\ V_{21} & gV_{22} - z\mathbb{I}_2 \end{pmatrix}, \tag{88}$$

where $\mathbb{I}_j = P_j|_{\mathcal{H}_j}$ is the identity operator in $\mathcal{H}_j$ and $V_{jk} = P_jVP_k$ are understood as operators from $\mathcal{H}_k$ to $\mathcal{H}_j$, $j, k \in \{1, 2\}$. For any $z \in \rho(gV_{22})$, we consider the Schur complement $F(z) \in \mathcal{B}(\mathcal{H}_1)$ defined by

$$F(z) = (V_{11} - z\mathbb{I}_1) - gV_{12}(gV_{22} - z\mathbb{I}_2)^{-1}V_{21}, \tag{89}$$

such that

$$z \in \rho(T) \cap \rho(gV_{22}) \Leftrightarrow 0 \in \rho(F(z)). \tag{90}$$

As V is unitary, we have $g\|V_{22}\| \leq g < 1$, so that $F : \{|z| > g\} \to \mathcal{B}(\mathcal{H}_1)$ is well defined. If $z \in \rho(V_{11}) \cap \mathbb{S}$, we can write

$$F(z) = (V_{11} - z\mathbb{I}_1)\left(\mathbb{I}_1 - g(V_{11} - z\mathbb{I}_1)^{-1}V_{12}(gV_{22} - z\mathbb{I}_2)^{-1}V_{21}\right), \tag{91}$$

which has a bounded inverse if $g\|(V_{11} - z\mathbb{I}_1)^{-1}V_{12}(gV_{22} - z\mathbb{I}_2)^{-1}V_{21}\| < 1$. Assuming that $\|V_{11}\| < 1$, we have $\{|z| > \|V_{11}\|\} \subset \rho(V_{11})$ and for $|z| > \max(g\|V_{22}\|, \|V_{11}\|)$,

$$g\|(V_{11} - z\mathbb{I}_1)^{-1}V_{12}(gV_{22} - z\mathbb{I}_2)^{-1}V_{21}\| \leq \frac{g\|V_{12}\|\|V_{21}\|}{(|z| - \|V_{11}\|)(|z| - g\|V_{22}\|)}. \tag{92}$$

The inner radius $r(V)$ of the ring (87) is defined so that the right hand side above is strictly smaller than one and it satisfies $\max(g\|V_{22}\|, \|V_{11}\|) \leq r(V) < 1$ whenever $g < \frac{1-\|V_{11}\|}{\|V_{21}\|\|V_{12}\|+\|V_{22}\|(1-\|V_{11}\|)}$. Thus, according to (90), this implies that the ring (87) belongs to the resolvent set of T, which yields the result for T in place of T_ω.

To get the result for the random case with V replaced by V_ω, it is enough to show that

$$\|P_jV_\omega P_k\| = \|P_jVP_k\| = \|V_{jk}\|, \quad \forall\ j, k \in \{1, 2\}. \tag{93}$$

This is a consequence of the following lemma, which ends the proof of the theorem. ∎

Lemma 5.15 *Let $\{v_j^{(k)}\}_{k\in\mathbb{Z}}$ be the orthonormal basis of $\mathcal{H}_j$, $j = 1, 2$ given by (42). Then*

$$\begin{aligned}
P_1V_\omega P_1v_1^{(k)} &= \frac{1}{1-g^2}\left(-e^{i\omega_{2k-1}}\bar{q}(s\gamma - q\delta)v_1^{(k-1)} + e^{i\omega_{2k+2}}\bar{s}(s\alpha - q\beta)v_1^{(k+1)}\right) \qquad (94)\\
P_2V_\omega P_2v_2^{(k)} &= \frac{1}{1-g^2}\left(-e^{i\omega_{2k-1}}stv_2^{(k-1)} - e^{i\omega_{2k+2}}qrv_2^{(k+1)}\right)\\
P_2V_\omega P_1v_1^{(k)} &= \frac{1}{1-g^2}\left(s(s\gamma - q\delta)e^{i\omega_{2k-1}}v_2^{(k-1)} + q(s\alpha - q\beta)e^{i\omega_{2k+2}}v_2^{(k+1)}\right)\\
P_1V_\omega P_2v_2^{(k)} &= \frac{1}{1-g^2}\left(e^{i\omega_{2k-1}}\bar{q}tv_1^{(k-1)} - \bar{s}re^{i\omega_{2k+2}}v_1^{(k+1)}\right).
\end{aligned}$$

Defining coefficients $w_{\pm}^{(ij)}$ *by*

$$P_i V_\omega P_j v_j^{(k)} = e^{i\omega_{2k-1}} w_+^{(ij)} v_i^{(k-1)} + e^{i\omega_{2k+2}} w_-^{(ij)} v_i^{(k+1)}, \tag{95}$$

we have

$$\|P_j V_\omega P_k\| \;=\; |w_+^{(jk)}| + |w_-^{(jk)}| = \|V_{jk}\| \tag{96}$$

and, for all $i, j \in \{1, 2\}$,

$$\text{Ker } P_i V_\omega P_j \neq \{0\} \Leftrightarrow P_i V_\omega P_j = 0. \tag{97}$$

Let $\mathbb{D}_\eta^{(j)}$ *and* $\mathbb{D}_\xi^{(j)}$ *be defined in the orthonormal basis* $\{v_j^{(k)}\}_{k\in\mathbb{Z}}$ *of* $\mathcal{H}_j$ *by* $\mathbb{D}_\eta^{(j)} =$ diag $(e^{i\eta_k^{(j)}})$, *and* $\mathbb{D}_\xi^{(j)} =$ diag $(e^{i\xi_k^{(j)}})$, *where, for* $p \geq 1$

$$\eta_{2p}^{(j)} \;=\; \sum_{l=0}^{p} \omega_{4l} - \sum_{l=0}^{p-1} \omega_{4l+1}, \qquad \eta_{2p+1}^{(j)} = \sum_{l=0}^{p} \omega_{4l+2} - \sum_{l=0}^{p-1} \omega_{4l+3} \tag{98}$$

$$\xi_{2p}^{(j)} \;=\; \sum_{l=0}^{p-1} \omega_{4l+3} - \sum_{l=0}^{p-1} \omega_{4l+2}, \quad \xi_{2p+1}^{(j)} = \sum_{l=0}^{p} \omega_{4l+1} - \sum_{l=0}^{p} \omega_{4l}, \tag{99}$$

and, for $p \leq 0$

$$\eta_{2p}^{(j)} \;=\; -\sum_{l=p+1}^{1} \omega_{4l} + \sum_{l=p}^{1} \omega_{4l+1}, \qquad \eta_{2p+1}^{(j)} = -\sum_{l=p+1}^{1} \omega_{4l+2} + \sum_{l=p}^{0} \omega_{4l+3} \tag{100}$$

$$\xi_{2p}^{(j)} \;=\; -\sum_{l=p}^{0} \omega_{4l+3} + \sum_{l=p}^{1} \omega_{4l+2}, \quad \xi_{2p+1}^{(j)} = -\sum_{l=p+1}^{1} \omega_{4l+1} + \sum_{l=p+1}^{1} \omega_{4l}. \tag{101}$$

Then,

$$P_j V_\omega P_k = \mathbb{D}_\eta^{(j)} V_{jk} \mathbb{D}_\xi^{(k)} \simeq \mathbb{D}_\xi^{(k)} \mathbb{D}_\eta^{(j)} V_{jk}. \tag{102}$$

Proof: The expressions of $P_i V_\omega P_j$ in the bases $\{v_j^{(k)}\}_{k\in\mathbb{Z}}$ are obtained by explicit computations making use of (42),

$$e_{2k} = \frac{(\bar{s} v_1^{(k)} + q v_2^{(k)})}{\sqrt{|q|^2 + |s|^2}}, \quad e_{2k+1} = \frac{-\bar{q} v_1^{(k)} + s v_2^{(k)}}{\sqrt{|q|^2 + |s|^2}}, \tag{103}$$

and of the constraint (23). Identity (96) is established by a classical argument and (97) is a direct consequence of this identity. Relation (102) is also a matter of verification. ∎

Remark 5.16 *With* $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = g e^{i\chi}$, *see Remark 4.8, and constraint (23), we have*

$$\|V_{11}\| = \frac{|\delta - \bar{\alpha} g e^{i\chi}| + |\alpha - \bar{\delta} g e^{i\chi}|}{1 - g^2}, \tag{104}$$

where the first / second term is the modulus of the coefficient of $v_1^{(k-1)}$ / $v_1^{(k+1)}$ *in (95).*

We establish further properties of V_{jk} and $P_jV_\omega P_k$ as operators from $\mathcal{H}_k$ to $\mathcal{H}_j$, that we present in an abstract form.

Proposition 5.17 *Let W be an operator that takes a tridiagonal form in an orthonormal basis of $l^2(\mathbb{Z})$ whose sole non zero coefficients satisfy*

$$|W_{j,j+1}| = W_-, \text{ and } |W_{j,j-1}| = W_+, \ \forall j \in \mathbb{Z}. \tag{105}$$

Assume, without loss, that $W_+ \geq W_- > 0$. Then, $\|W\| = W_+ + W_-$ and

$$\begin{aligned} &\text{If } W_+ \leq 1, \qquad \{|z| < W_+ - W_-\} \subset \rho(W) \\ &\text{If } W_+ > 1, \qquad \{|z| < (W_+ - W_-)/(2W_+ - 1)\} \subset \rho(W). \end{aligned} \tag{106}$$

If W is further translation invariant, $W_{j,j+1} = w_-$, and $W_{j,j-1} = w_+, \forall j \in \mathbb{Z}$, then W is normal and spr $(W) = \|W\| = |w_+| + |w_-|$.

Remark 5.18 *The radius of both disks contained in $\rho(W)$ is smaller than one.*

Proof: The norm of W was already mentioned above. The structure of W is such that we can write $W = W^+S_+ + W^-S_-$, where the non zero matrix elements of the operator S_+/S_- lie on the diagonal immediately above/below the main diagonal, and all have modulus one; $S_\pm$ are unitarily equivalent to standard shifts. Thus, for any $|z| \neq 1$, we can write

$$\begin{aligned} W - z &= W^+(S_+ - z) + W^-S_- - z(1 - W^+) \\ &= W^+(S_+ - z)\Big(\mathbb{I} + \frac{(S_+ - z)^{-1}}{W^+}\Big(W^-S_- - z(1 - W^+)\Big)\Big). \end{aligned} \tag{107}$$

Since

$$\left\| \frac{(S_+ - z)^{-1}}{W^+}\Big(W^-S_- - z(1 - W^+)\Big) \right\| \leq \frac{W^- + |z||1 - W^+|}{W^+|1 - |z||}, \tag{108}$$

the Neumann series implies that $W - z$ admits a bounded inverse if the right hand side of (108) is bounded above by one. Considering small values of $|z|$ and dealing with the different cases for W^+, we get the result. In case W is translation invariant, we obtain by Fourier methods that W is unitarily equivalent to a scalar multiplication operator

$$W \simeq W(x) = e^{ix}w_+ + e^{-ix}w_- \ \text{ on } L^2(\mathbb{T}; \mathbb{C}). \tag{109}$$

This operator is obviously normal, which ends the proof. ∎

Hence, the translation invariant contractions $P_jVP_j|_{\mathcal{H}_j} = V_{jj}$ with tri-diagonal representations in the orthonormal basis of $\mathcal{H}_j$ given by $\{v_j^{(k)}\}_{k\in\mathbb{Z}}$, $j = 1, 2$, for $0 \leq g < 1$, with coefficients $w_\pm^{(jj)}$ defined by (95) is normal and satisfies spr $(V_{jj}) = \|V_{jj}\| = |w_+^{j(j)}| + |w_-^{(jj)}|$.

Example 5.19 Let us apply the results above to Example 5.8 where $\tilde{C}$ defined by equation (74). Recall that in this case $g = \sin(\xi)$, and $\xi, \eta \in [0, \pi/2]$. We get

$$\|V_{11}\| = \cos(\eta), \ \ \|V_{21}\| = \sin(\eta), \ \ \|V_{22}\| = \cos(\eta), \ \ \|V_{12}\| = \sin(\eta). \tag{110}$$

Thus, for $\eta, \xi \in]0, \pi/2[$ so that $g > 0$, $\|V_{11}\| < 1$ and for ξ small enough so that

$$\sin(\xi) < \frac{1 - \cos(\eta)}{\sin^2(\eta) + \cos(\eta)(1 - \cos(\eta))}, \tag{111}$$

condition (86) holds and we get

$$r(V) = \frac{1}{2}\left(\cos(\eta)(1 + \sin(\xi)) + \sqrt{\cos^2(\eta)(1 - \sin(\xi))^2 + 4\sin(\xi)\sin^2(\eta)}\right). \tag{112}$$

Actually, all corresponding operators $P_j V_\omega P_k$ in this case map the basis vector $v_k^{(n)}$ to one of $v_j^{(n\pm1)}$ only. In particular, $P_1 V_\omega P_1|_{\mathcal{H}_1}$ and $P_2 V_\omega P_2|_{\mathcal{H}_2}$ are unitarily equivalent to $\cos(\eta)S_1$ and $\cos(\eta)S_2$ respectively, where S_j is the standard shift on $P_j\mathcal{H}_j$. Hence,

$$\sigma(P_1 V_\omega P_1|_{\mathcal{H}_1}) = \cos(\eta)\mathbb{S} \quad \text{and} \quad \sigma(P_2 V_\omega P_2|_{\mathcal{H}_2}) = \cos(\eta)\mathbb{S}. \tag{113}$$

Thus, assuming a phase distribution satisfying (77) and parameters such that condition (111) holds, we have excluded the presence of spectrum of the corresponding non-unitary operator T_ω in the union of the ring of inner radius (112) and of the symmetric sets characterized by Lemma 5.5. Moreover, for suitable values of the parameters condition (79) holds as well and $\sigma(T_\omega)$ is contained in two disjoint sets separated by the real axis.

The following more specific properties hold.

Lemma 5.20 *We have*

$$\|V_{11}\| = 0 \Leftrightarrow \|V_{22}\| = 0 \Leftrightarrow \tilde{C} \in \left\{ \begin{pmatrix} 0 & 0 & \beta \\ q & g & 0 \\ \gamma & t & 0 \end{pmatrix}, \begin{pmatrix} 0 & r & \beta \\ 0 & g & s \\ \gamma & 0 & 0 \end{pmatrix} \right\} \subset U(3), \tag{114}$$

and,

$$V_{jk} = 0 \text{ for some } k \neq j \Leftrightarrow V_{jj} \text{ unitary for all } j \in \{1, 2\} \tag{115}$$

$$\Leftrightarrow V_{jj} \simeq S_j,\ S_j \text{ a shift on } \mathcal{H}_j \Leftrightarrow \tilde{C} \in \left\{ \begin{pmatrix} \alpha & r & 0 \\ q & g & 0 \\ 0 & 0 & \delta \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & g & s \\ 0 & t & \delta \end{pmatrix} \right\} \subset U(3).$$

Remarks 5.21 *i) In case V_ω is off-diagonal with respect to $\mathcal{H}_0 = \mathcal{H}_1 \bigoplus \mathcal{H}_2$, so that (114) and Lemma 4.2 hold, we saw that for all $0 \leq g < 1$ and all ω, $\sigma(T_\omega) \subset \{z \in \mathbb{C} \ ||z| = \sqrt{g}\}$. We recover this result by noting that V_ω off-diagonal implies for $z \neq 0$*

$$F(z) = -z\left(\mathbb{I}_1 - P_1 V_\omega^2 P_1 g/z^2\right), \tag{116}$$

where $P_1 V_\omega^2 P_1|_{\mathcal{H}_1}$ is unitary. Hence $F(z)$ is boundedly invertible iff $z^2 \in \sigma(gP_1 V_\omega^2 P_1|_{\mathcal{H}_1})$.
ii) In case V_ω is diagonal with respect to $\mathcal{H}_0 = \mathcal{H}_1 \bigoplus \mathcal{H}_2$, so that (115) and Lemma 4.2 hold, we saw that for all $0 \leq g < 1$ and all ω, $\sigma(T_\omega) = \mathbb{S} \cup g\mathbb{S}$.

Proof: The tridiagonal matrix representation of V_{jj} stems from (94), which yields the first statement. The last statements are obtained by discussing the conditions $w_-^{(jj)} = w_+^{(jj)} = 0$ depending on the fact that q, s are zero or not. We first note that the condition $g < 1$

forbids $q = s = 0$ or $r = t = 0$. For $\|V_{11}\| = 0$, the case $qs \neq 0$, is impossible: the expansion of $\det(\tilde{C})$ with respect to the second column and $w_- = w_+ = 0$ imply $\det(\tilde{C}) = g(\alpha\delta - \gamma\beta)$, which is of modulus 1. This implies $g = |(\alpha\delta - \gamma\beta)| = 1$ and $q = s = 0$, a contradiction. If $qs = 0$, one gets that α or δ equals 1, which with condition (14) yield the result. Similarly, $\|V_{22}\| = 0$ imply $q = t = 0$ or $s = r = 0$ and condition (14) again yields the result. The assertions regarding the off diagonal parts of V_ω are readily obtained by the same type of considerations and the fact that V_ω is unitary. ∎

5.1 Ergodicity

We briefly recall here a spectral consequences of our hypothesis on the way the randomness enters the operator T_ω. Ergodicity provides a tool to estimate from below the spectrum of T_ω, almost surely. Our setup actually enters the more general theory of pseudo-ergodic operators, as developed in [D1, D2], of which ergodic operators are special cases.

The definition (20) of $\mathbb{D}^0_\omega$ makes the operator ergodic under 2-shifts with respect to the matrix representation (26). If Σ denotes both the map from $\Omega \to \Omega$ such that $(\Sigma\omega)_j = \omega_{j+2}$, and the operator defined on $\mathcal{H}_0$ by $\Sigma e_j = e_{j+2},\ \forall j \in \mathbb{Z}$, we have

$$T_{\Sigma^k\omega} = \Sigma^{-k} T_\omega \Sigma^k,\ \forall k \in \mathbb{Z}. \tag{117}$$

Following [D1, D2] in making use of independence of the random phases and Borel-Cantelli Lemma, we get

Proposition 5.22 *Let $l \in 2\mathbb{N}$ and $\theta^{(l)} = (\theta_1, \theta_2, \cdots, \theta_l) \in (supp\ d\nu)^l \subset \mathbb{T}^l$. Set $T_{\theta^{(l)}} := \mathbb{D}^0_\omega T$, where $\omega = (\ldots, \theta^{(l)}, \theta^{(l)}, \ldots) \in \Omega$. Then,*

$$\cup_{l\in 2\mathbb{N}} \cup_{\theta^{(l)}\in\mathbb{T}^l} \sigma(T_{\theta^{(l)}}) \subset \sigma(T_\omega),\ \textit{almost surely.} \tag{118}$$

Remark 5.23 *In particular, if $d\nu(\theta) = d\theta/(2\pi)$, $\cup_{\theta\in[0,2\pi]} e^{i\theta}(\mathrm{Ran}\ \lambda_+ \cup \mathrm{Ran}\ \lambda_-) \subset \sigma(T_\omega)$, where $\lambda_\pm$ are defined in (33). This shows that statements (59) and Theorem (5.13) on the location of $\sigma(T_\omega)$ are optimal, as we argue below.*

Considering Example 5.8, one checks that when $\xi \to 0$, condition (111) holds, $\lambda_+(0) = \frac{1}{2}\left(\cos(\eta)(1+\sin(\xi)) + \sqrt{\cos^2(\eta)(1+\sin(\xi))^2 - 4\sin(\xi)}\right) > 0$ and the value $r(V)$ given in (112) becomes arbitrarily close to $\lambda_+(0)$. Also, when $\cos^2(\eta) < 4\sin(\xi)/(1+\sin(\xi))^2$ we have $|\lambda_+(0)| = g = \sin(\xi)$. Since $|\lambda_+(0)| \in \sigma(T_\omega)$ almost surely, Proposition 5.22 shows that statement (59) and Theorem (5.13) on the location of $\sigma(T_\omega)$ are optimal.

6 Special Case $g = 0$

This section is devoted to a more thorough analysis of the case $g = 0$

$$T_\omega = V_\omega P_1 \text{ corresponding to } \tilde{C} = \begin{pmatrix} \alpha & r & \beta \\ q & 0 & s \\ \gamma & t & \delta \end{pmatrix} \in U(3). \tag{119}$$

According to Lemmas 4.4 and 5.11, $T_\omega = V_\omega P_1$ is far from being unitary, Ker $T_\omega = \mathcal{H}_2$, for all $\omega \in \Omega$, and $\sigma(T_\omega) = \sigma(P_1 V_\omega P_1) \cup \{0\}$. More precisely:

Proposition 6.1 *If $g = 0$, we have for all $\omega \in \Omega$*

$$\sigma(T_\omega) \setminus \{0\} \subset \big\{ \big||\alpha| - |\delta|\big| \le |z| \le |\alpha| + |\delta| \big\}. \tag{120}$$

If $\alpha = 0$, resp. $\delta = 0$, then $P_1 V_\omega P_1|_{\mathcal{H}_1}$ is unitarily equivalent to $|\delta| S^+$, resp. $|\alpha| S^-$, and

$$\sigma(T_\omega) = \max(|\alpha|, |\delta|)\mathbb{S} \cup \{0\} \quad \textit{and} \quad \sigma_p(T_\omega) = \sigma_p(T_\omega^*) = \{0\}. \tag{121}$$

Moreover,

$$\gamma \neq qt \Leftrightarrow \beta \neq sr \quad \Rightarrow \quad V_\omega \textit{ is pure point a.s.} \tag{122}$$

$$\gamma = qt \Leftrightarrow \beta = sr \quad \Rightarrow \quad V_\omega \textit{ is purely ac, } \forall \omega \in \Omega. \tag{123}$$

Example 6.2 Let us consider an explicit parametrization of a $\tilde{C} \in O(3)$ of the kind (119)

$$\tilde{C}(\xi, \eta) = \begin{pmatrix} \cos(\xi)\sin(\eta) & \cos(\eta) & -\sin(\xi)\sin(\eta) \\ \sin(\xi) & 0 & \cos(\xi) \\ -\cos(\xi)\cos(\eta) & \sin(\eta) & \sin(\xi)\cos(\eta) \end{pmatrix} \in O(3), \ \xi, \eta \in [0, \pi/2], \tag{124}$$

where (ξ, η) is restricted to $[0, \pi/2]^2$ for simplicity. Then, $|\alpha| + |\delta| < 1$ is equivalent to $\sin(\xi + \eta) \neq 1$, i.e. $\xi + \eta \neq \pi/2$, and $\gamma = at$ is equivalent to $\cos(\xi - \eta) = 0$, i.e. $(\xi, \eta) = (\pi/2, 0)$, or $(\xi, \eta) = (0, \pi/2)$.

Proof: Remark 5.16 implies for $g = 0$ that the modulus of the coefficients of the tridiagonal operator $P_1 V_\omega P_1$ are $|\alpha|$ and $|\delta|$, so Proposition 5.17 yields the first statement. We know that $0 \in \sigma_p(T_\omega)$. Further assuming that $\alpha\delta = 0$, the same remark yields that $P_1 V_\omega P_1$ is unitarily equivalent to a shift and consequently, Lemma 5.11 yields the spectrum of T_ω. Finally, the eigenvalue equation $T_\omega \varphi = \lambda\varphi$, $\lambda \neq 0$, implies that $\varphi_1 = P_1\varphi$ satisfies $P_1 V_\omega P_1 \varphi_1 = \lambda \varphi_1$, which cannot hold for a shift. The same argument applies to T_ω^*. Then one checks on the unitary operator (35) that $\gamma = qt$ is equivalent to $\beta = sr$. In turn, this implies that V_ω is unitarily equivalent to a direct sum of two shifts. In all other cases, V_ω is pure point almost surely as shown in [JM]. ∎

From the foregoing we know that when $g = 0$, $P_1 V_\omega P_1 = \mathbb{D}_\eta^{(1)} V_{11} \mathbb{D}_\xi^{(1)}$, where

$$V_{11} = \begin{pmatrix} \ddots & \delta & & \\ \alpha & 0 & \delta & \\ & \alpha & 0 & \delta \\ & & \alpha & \ddots \end{pmatrix} \ \simeq \ e^{i(\arg\alpha - \arg\beta)/2}(e^{iy}|\alpha| + e^{-iy}|\delta|), \text{ on } \ L^2(\mathbb{T}), \tag{125}$$

and

$$e^{-i(\arg\alpha - \arg\delta)/2}\sigma(V_{11}) = E(|\alpha|, |\delta|), \tag{126}$$

where $E(|\alpha|, |\delta|)$ denotes the ellipse centered at the origin, with horizontal major axis of length $|\alpha| + |\delta|$ and vertical minor axis of length $||\alpha| - |\delta||$. When the random phases are iid and uniform, we have a complete description of the spectral properties of T_ω when $g = 0$.

Proposition 6.3 *Assume $g = 0$ and $d\nu(\theta) = d\theta/2\pi$. Then, $T_\omega = V_\omega P_1$ satisfies*

$$\sigma(T_\omega) = \{0\} \cup \big\{ \big||\alpha| - |\delta|\big| \le |z| \le |\alpha| + |\delta| \big\}, \ \textit{a.s.} \tag{127}$$

When $|\alpha| + |\delta| = 1$, the peripheral spectra of the relevant operators coincide with $\mathbb{S}$,

$$\sigma(T_\omega) \cap \mathbb{S} = \sigma(P_1 V_\omega P_1|_{\mathcal{H}_1}) \cap \mathbb{S} = \sigma(V_\omega) = \mathbb{S}, \ \textit{a.s.} \tag{128}$$

However, the nature of the peripheral spectra of T_ω and V_ω differs for $\gamma \neq qt$,

$$\sigma_p(T_\omega) \cap \mathbb{S} = \sigma_p(T_\omega^*) \cap \mathbb{S} = \emptyset, \ \textit{whereas } \sigma_c(V_\omega) = \emptyset \ \textit{a.s.} \tag{129}$$

Remark 6.4 *This result shows in a sense that the spectral localization of V_ω does not carry over to the boundary of the spectrum of $T_\omega = V_\omega P_1$. Note that the original operator $U_\omega(C)$ is purely ac when $g < 1$, for all $\omega \in \Omega$.*

Proof: The first consequence of our assumption on the distribution of the random phases is that $P_1 V_\omega P_1 = \mathbb{D}_\omega^{(1)} V_{11}$, where the random phases of the diagonal operator $\mathbb{D}_\omega^{(1)}$ are independent and uniformly distributed, see e.g. Lemma 4.1 in [ABJ]. Hence proposition 5.22 with supp $d\nu(\cdot) = 2\pi$, together with Proposition 6.1 show that

$$\big\{\big||\alpha| - |\delta|\big| \leq |z| \leq |\alpha| + |\delta|\big\} = \bigcup_{\theta \in [0, 2\pi[} e^{i\theta} E(|\alpha|, |\delta|) = \sigma(\mathbb{D}_\omega^{(1)} V_{11}), \text{ almost surely.} \tag{130}$$

When $|\alpha| + |\delta| = 1$, the peripheral spectra equals $\mathbb{S}$ almost surely by Lemma 5.11. Finally, the nature of the peripheral spectra stems from Lemmas 5.9 and 4.12. ∎

Remark 6.5 *In case $|\alpha| = |\delta| = 1/2$, $V_{11} = \Delta_1$, the discrete Laplacian on $\mathcal{H}_1$. With $d\nu(\theta) = d\theta/2\pi$,*

$$\sigma(\mathbb{D}_\omega^{(1)} \Delta_1) = \sigma(T_\omega) = \{|z| \leq 1\}, \ \textit{almost surely}, \tag{131}$$

where $\mathbb{D}_\omega^{(1)} \Delta_1$ is a version of the random hopping model of Feinberg and Zee [FZ].

A Proof of Lemmas 5.5, 5.7, and Proposition 5.22.

Proof: [of Lemma 5.5] The determination of $\partial D(\theta)$ follows from the elimination of the parameter τ according to (66) by an explicit computation.

The relation $D(\theta) \cup B_0(g) \subset \bigcup_{\tau \in \mathbb{R}_+} B_\tau(d_\tau) \cap B_{g\tau}(g d_\tau)$ holds by construction. Let us check that $R_1(\theta)$ belongs to (60) as well. Let $(x_\tau, y_\tau) = C_\tau(d_\tau) \cap C_{g\tau}(g d_\tau)$. In order to assess the property $(x_\tau, y) \in \bigcup_{\tau' \in \mathbb{R}_+} B_{\tau'}(d_{\tau'}) \cap B_{g\tau'}(g d_{\tau'})$, for some $y \in \mathbb{R}$, we compute for any $\tau' \in \mathbb{R}$,

$$\begin{aligned} (x_\tau - \tau')^2 + y^2 &= d_{\tau'}^2 + (y^2 - y_\tau^2) + 2(\tau' - \tau)(\cos(\theta) - x_\tau) \\ (x_\tau - g\tau')^2 + y^2 &= g d_{\tau'}^2 + (y^2 - y_\tau^2) + 2g(\tau' - \tau)(g\cos(\theta) - x_\tau). \end{aligned} \tag{132}$$

Thus, for any $\tau \geq (1+g)/(2g\cos(\theta))$ so that $x_\tau > \cos(\theta)$, and any $y^2 \geq y_\tau^2$, we can take τ' large enough so that $(x_\tau, y) \in B'_\tau(d_{\tau'}) \cap B_{g\tau'}(g d_{\tau'})$.

Consider now the reverse inclusion $\bigcup_{\tau \in \mathbb{R}_+} B_\tau(d_\tau) \cap B_{g\tau}(g d_\tau) \subset D(\theta) \cup B_0(g) \cup R_1(\theta)$. By symmetry it is enough to focus on $y \geq 0$ and $x \leq \cos(\theta)$. Using (132) again, we first see that points $(x_\tau, y) \notin D(\theta) \cup B_0(g) \cup R_1(\theta)$ such that $g\cos(\theta) \leq x_\tau \leq \cos(\theta)$ and $y \geq y_\tau$

cannot belong to $B_{\tau'}(d_{\tau'}) \cap B_{g\tau'}(gd_{\tau'})$, for any τ'. Assume now $(x,y) \notin D(\theta) \cup B_0(g) \cup R_1(\theta)$ is such that $x \leq g\cos(\theta)$ and $y^2 \geq g^2 - x^2$. For any $\tau' > 0$, the relation

$$(x-\tau')^2 + y^2 = gd_{\tau'}^2 + (y^2 - (g^2 - x^2)) + 2\tau'(g\cos(\theta) - x) \geq gd_{\tau'}^2 \tag{133}$$

shows that $(x,y) \notin \bigcup_{\tau' \in \mathbb{R}_+} B_{\tau'}(d_{\tau'}) \cap B_{g\tau'}(gd_{\tau'})$, which ends the proof for $\theta < \pi/2$.

When $\pi/2 \leq \theta < \pi$, one first notes that $B_\tau(d_\tau) \cap B_{g\tau}(gd_\tau) = B_{g\tau}(gd_\tau)$. Then, any $z \in \mathbb{C}$ such that $\Re z > g\cos(\theta)$ is contained in $B_{g\tau}(gd_\tau)$ provided $\tau > 0$ is large enough.

Finally, we prove (68) assuming $0 < \alpha < \theta < \pi/2$. We first note that if $e^{i\alpha}\tau$ is such that $\Im e^{i\alpha}\tau \geq \sin(\theta)$, *i.e.* $\tau \geq \sin(\theta)/\sin(\alpha)$, then any $z \in B_{e^{i\alpha}\tau}(d_{e^{i\alpha}\tau})$, satisfies $\Re z > \cos(\theta)$, so that $\bigcup_{\tau \geq \sin(\theta)/\sin(\alpha)} B_{e^{i\alpha}\tau}(d_{e^{i\alpha}\tau}) \cap B_{ge^{i\alpha}\tau}(gd_{e^{i\alpha}\tau}) \subset R_1(\theta)$. For any $\tau < \sin(\theta)/\sin(\alpha)$, the intersection of the line passing by $e^{i\theta}$ and $e^{i\alpha}\tau$ and the real axis occurs at a point $\tau' > 0$ so that $d_{\tau'} = d_{e^{i\alpha}\tau} + |e^{i\alpha}\tau - \tau'|$. Therefore, if $z \in B_{e^{i\alpha}\tau}(d_{e^{i\alpha}\tau}) \cap B_{ge^{i\alpha}\tau}(gd_{e^{i\alpha}\tau})$, we have

$$\begin{aligned} |z-\tau'| &\leq |z - e^{i\alpha}\tau| + |e^{i\alpha}\tau - \tau'| < d_{e^{i\alpha}\tau} + |e^{i\alpha}\tau - \tau'| = d_{\tau'} \\ |z - g\tau'| &\leq |z - ge^{i\alpha}\tau| + g|e^{i\alpha}\tau - \tau'| < gd_{e^{i\alpha}\tau} + g|e^{i\alpha}\tau - \tau'| = gd_{\tau'} \end{aligned} \tag{134}$$

which shows that $z \in B_{\tau'}(\tau') \cap B_{ge^{\tau'}}(gd_{\tau'})$ and which ends the proof. A similar argument yields the result for $\pi/2 \leq \theta < \pi$. ∎

Proof: [of Lemma 5.7] We consider $0 < \theta < \pi/2$ only, the other case being similar. Let $z = \rho e^{i\beta} \in B_0(g) \cup \Delta_g(\theta)$. By symmetry and the foregoing, we can consider $0 \leq \beta \leq \pi$ only, and $\rho \geq g$. Thus, it is enough to consider $0 \leq \beta < \theta$, and $g \leq \rho < g/\cos(\theta - \beta)$. We need to show that $|\rho e^{i\beta} + |\tau| e^{i\nu}| < |\tau| + g$, for some $\tau \in \mathbb{R}^-$ and some $e^{i\nu} \in \sigma(V)$, which is equivalent to

$$2|\tau|(g - \rho\cos(\nu - \beta)) > \rho^2 - g^2 \geq 0. \tag{135}$$

Since we have $\cos(\nu - \beta) \leq \cos(\theta - \beta)$, the left hand side of (135) is bounded below by $2|\tau|(g - \rho\cos(\theta - \beta))$ which is strictly positive, so that (135) holds for $|\tau|$ large enough.

Conversely, assume $\exists\ |\tau|$ such that $\forall\ e^{i\nu} \in \sigma(V)$, we have $|z + e^{i\nu}|\tau|| < |\tau| + g$. With $z = \rho e^{i\beta}$, the geometrical properties recalled above imply that for all $\beta \notin]-\theta, \theta[$, $\rho < g$. Otherwise, the inequality is equivalent to

$$\rho^2 + 2\rho|\tau|\cos(\beta - \nu) - g(2|\tau| + g) < 0. \tag{136}$$

Therefore, denoting by $x_+(\nu)$ the positive root of (136), we must have for all allowed ν, $0 \leq \rho \leq x_+(\nu)$, where $\beta \in [-\theta, \theta]$. With $x_+(\nu) \geq x_+(\theta)$, as a consequence of $\cos(\nu - \beta) < \cos(\theta - \beta)$, we must have $0 \leq \rho \leq x_+(\theta)$, for β fixed. To get the result, one finally checks that $x_+(\theta) < g/\cos(\theta - \beta)$.

Consider now (72) and fix $\tau \leq 0$. Expression (61) with $e^{-i\alpha}V$ in place of V and the observation that $\delta_{\tau e^{i\alpha}} > |\tau|$ implies all circles $C_{e^{i\nu}\tau}(\delta_{\tau e^{i\alpha}})$ are tangent to $C_0(\delta_{\tau e^{i\alpha}} - |\tau|)$ yield (72). Note that $\Gamma_{|\tau|, \delta_{\tau e^{i\alpha}} - |\tau|}(\theta) = B_0(g)$ if $\tau = 0$.

It remains to establish (73) for $\alpha \geq 0$. We start with a few facts for $|\tau|$ fixed

$$ge^{\pm i(\theta - \alpha)} \in \partial\Gamma_{|\tau|, \delta_{\tau e^{i\alpha}} - |\tau|}(\theta) \cap C_{\tau e^{\pm i\theta}}(\delta_{\tau e^{i\alpha}}) \tag{137}$$

The point of of $e^{i\alpha}\Gamma_{|\tau|, \delta_{\tau e^{i\alpha}} - |\tau|}(\theta)$ that is most distant from the origin is $e^{i\alpha}\rho_{|\tau|} \in C_{\tau e^{i\theta}}(\delta_{\tau e^{i\alpha}}) \cap C_{\tau e^{-i\theta}}(\delta_{\tau e^{i\alpha}})$, where

$$\rho_{|\tau|} = -|\tau|\cos(\theta) + \sqrt{(g + |\tau|\cos(\alpha))^2 + |\tau|^2(\cos^2(\theta) - \cos^2(\alpha))}. \tag{138}$$

Now, if $\pi/2 > \alpha > \theta$, $\rho_{|\tau|} < g$, so that (73) is contained in $B_0(g)$. Thus we assume from now on that $\alpha \leq \theta < \pi/2$. The line tangent to $e^{i\alpha}\partial\Gamma_{|\tau|,\delta_{\tau e^{i\alpha}-|\tau|}}(\theta)$ at $ge^{i\theta}$ has equation

$$t_{|\tau|}(x) = -(x - g\cos(\theta))\frac{(g\cos(\theta) + |\tau|\cos(\theta+\alpha))}{(g\sin(\theta) + |\tau|\sin(\theta+\alpha))} + g\sin(\theta). \tag{139}$$

Note that the tangent to $e^{i\alpha}\partial\Gamma_{|\tau|,\delta_{\tau e^{i\alpha}-|\tau|}}(\theta)$ at $ge^{i(2\alpha-\theta)}$ has slope inferior to $\pi/2$. By convexity, $e^{i\alpha}\Gamma_{|\tau|,\delta_{\tau e^{i\alpha}-|\tau|}}(\theta) \subset \Delta_{|\tau|}$, where $\Delta_{|\tau|}$ is the triangle defined by the intersection point of these tangent lines, $ge^{i\theta}$ and $ge^{i(2\alpha-\theta)}$ union $B_0(\delta_{\tau e^{i\alpha}-|\tau|})$. Since the slope of the line $t_{|\tau|}$ is strictly increasing with $|\tau|$, we also have $\Delta_{|\tau|} \subset \Delta_\infty$, where the latter is set is the triangle is defined by $g\frac{\cos(\alpha)}{\cos(\theta)}e^{i\alpha}$, $ge^{i\theta}$ and $ge^{i(2\alpha-\theta)}$ union $B_0(\delta_{\tau e^{i\alpha}-|\tau|})$.To prove (73), it is enough to show that the line t_∞ does not intersects the curve (66) that defines $D(\theta)$ for $x \in]g\cos(\theta), \min(g\frac{\cos^2(\alpha)}{\cos(\theta)}, \cos(\theta))[$. With $y(x) > 0$ solution to (66), we get

$$y^2(x) - t_\infty^2(x) = \frac{(x - g\cos(\theta))(x^2 - x(\cos(\theta) + 2g\cos(\theta+\alpha)\cos(\alpha)) + g(g+1)\cos^2(\alpha))}{((1+g)\cos(\theta) - x)\sin^2(\theta+\alpha)}, \tag{140}$$

which has the sign of the second factor in the numerator, call it $p(x)$, for $x \in]g\cos(\theta), (g+1)\cos(\theta)[$. Moreover, we note that

$$p\left(g\frac{\cos^2(\alpha)}{\cos(\theta)}\right) = \frac{g^2\cos^2(\alpha)(\cos^2(\alpha) + \cos^2(\theta) - 2\cos(\theta)\cos(\alpha)\cos(\theta+\alpha))}{\cos^2(\theta)} > 0 \tag{141}$$

And since $t'_\infty(x) = -\frac{\cos(\theta+\alpha)}{\sin(\theta+\alpha)} < \tan(\theta)$, we have $t_\infty(\cos(\theta)) < \sin(\theta) = y(\cos(\theta))$, hence $p(\cos(\theta) > 0$. If the discriminant of p is negative, then $p(x)$ has no real roots, $y^2(x) - t_\infty^2(x) > 0$ and the result holds. Otherwise, denote by $x_- \leq x_+$ these roots such that $x_-x_+ = g(g+1)\cos^2(\alpha) > 0$. Hence $y^2(x) - t_\infty^2(x)$ will be positive on $]g\cos(\theta), \min(g\frac{\cos^2(\alpha)}{\cos(\theta)}, \cos(\theta))[$ if $x_- \leq x_+ < 0$, which happens if and only if $(\cos(\theta) + 2g\cos(\theta+\alpha)\cos(\alpha)) < 0$. The foregoing yields that neither $g\frac{\cos^2(\alpha)}{\cos(\theta)}$ nor $\cos(\theta)$ lies between the roots. If $g\frac{\cos^2(\alpha)}{\cos(\theta)} \leq \cos(\theta)$, we get

$$g\frac{\cos^2(\alpha)}{\cos(\theta)} < \cos(\alpha)\sqrt{g(g+1)} \leq \frac{1}{2}((\cos(\theta) + 2g\cos(\theta+\alpha)\cos(\alpha))) \leq x_+, \tag{142}$$

and the result follows. If $g\frac{\cos^2(\alpha)}{\cos(\theta)} > \cos(\theta)$, the same largument shows that $\cos(\theta) < x_+$, which ends the proof for $0 < \theta < \pi/2$. When $\pi/2 \leq \theta < \pi$, the inclusion (73) follows directly from (72) and the simple shape of $B_0(g) \cup R_g(\theta)$. ∎

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