General Adiabatic Evolution with a Gap Condition

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Abstract

We consider the adiabatic regime of two parameters evolution semigroups generated by linear operators that are analytic in time and satisfy the following gap condition for all times: the spectrum of the generator consists in finitely many isolated eigenvalues of finite algebraic multiplicity, away from the rest of the spectrum. The restriction of the generator to the spectral subspace corresponding to the distinguished eigenvalues is not assumed to be diagonalizable.

The presence of eigenilpotents in the spectral decomposition of the generator typically leads to solutions which grow exponentially fast in some inverse power of the adiabaticity parameter, even for real spectrum. In turn, this forbids the evolution to follow all instantaneous eigenprojectors of the generator in the adiabatic limit. Making use of superadiabatic renormalization, we construct a different set of time-dependent projectors, close to the instantaneous eigeprojectors of the generator in the adiabatic limit, and an approximation of the evolution semigroup which intertwines exactly between the values of these projectors at the initial and final times. Hence, the evolution semigroup follows the constructed set of projectors in the adiabatic regime, modulo error terms we control.

1 Introduction

Singular perturbations of differential equations play an important role in various areas of mathematics and mathematical physics. Such perturbations typically appear when one considers problems that display several different time and/or length scales. In particular, the semiclassical analysis of quantum phenomena and the study of evolution equations in the adiabatic regime lead to singularly perturbed linear differential equations which are the object of many recent works. See for example the monographs [14], [11], [13], [29], [40]. The description of certain non conservative phenomena with distinct time scales also gives rise to non-autonoumous linear evolution equations, which are more general than those stemming from conservative systems, and whose adiabatic regime is of physical relevance, see e.g. [32], [33], [41], [35], [36], [37], [2], [3], [1].

The present paper is devoted to the study of general linear evolution equations in the adiabatic limit under some mild spectral conditions on the generator. The chosen set up is sufficiently general to cover most applications where the time dependent generator is characterized by a gap condition on its spectrum. Let us describe informally our result, the precise Theorem being formulated in Section 2 below.

We consider a general linear evolution equation in a Banach space $\mathcal{B}$ of the form

$$i\varepsilon\partial_t U(t,s) = H(t)U(t,s), \quad U(s,s) = \mathbb{I}, \quad s \le t \in [0,1] \tag{1.1}$$

in the adiabatic limit $\varepsilon \to 0^+$, for a time-dependent generator $H(t)$. This equation describes a rescaled non-autonomous evolution generated by a slowly varying linear operator $H(t)$. The evolution operator $U(t,s)$ evidently depends on ε, even though this is not emphasized in the notation.

The generator $H(t)$ is assumed to depend analytically on time and to have for any fixed t a spectrum $\sigma(H(t))$ divided into two disjoint parts, $\sigma(H(t)) = \sigma(t) \cup \sigma_0(t)$, where $\sigma(t)$ consists in a finite number of complex eigenvalues $\sigma(t) = \{\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)\}$ which remain isolated from one another as t varies in $[0,1]$. Moreover, the spectral projector of $H(t)$ associated with $\sigma(t)$, denoted by $P(t)$, is assumed to be finite dimensional. The part of $H(t)$ which corresponds to the spectral projector $P_0(t)$ associated with $\sigma_0(t)$ can be unbounded, bounded or zero. In the first case we need to assume $H(t)$ generates a *bona fide* evolution operator.

This spectral assumption, or gap condition, is familiar in the quantum adiabatic context where $\mathcal{B}$ is a Hilbert space on which $H(t)$ is further assumed to be self-adjoint, see [10], [25], [30], [7], [1], for example. Note that it is still possible to study the quantum adiabatic limit by altering the gap condition in different ways, as shown in [6], [18], [11], [5], [15], [39], [3], [4].

By contrast to previous studies of similar general problems [12], [32], [28], [23], [1], we do not assume that the restriction of $H(t)$ to the spectral subspace $P(t)\mathcal{B}$ is diagonalizable. Such situations take place in the study of open quantum systems by means of phenomenological time-dependent master equations, [35], [36], [41], [37]. We come back to the approach of [35] below.

Therefore, for the part $H(t)P(t)$ of the generator, we have a complete spectral decomposition

$$H(t)P(t) = \sum_{j=1}^{n} \lambda_j(t)P_j(t) + D_j(t), \tag{1.2}$$

where the $P_j(t)$'s are eigenprojectors and the $D_j(t)$'s are eigenilpotents associated to the eigenvalue $\lambda_j(t)$ that satisfy

$$\sum_{j=1}^{n} P_j(t) = P(t), \quad P_j(t)P_k(t) = \delta_{jk}P_j(t), \quad \text{and } D_j(t) = P_j(t)D_j(t)P_j(t). \tag{1.3}$$

In case $\mathcal{B}$ is a Hilbert space on which $H(t)$ is self-adjoint or if $H(t)$ is diagonalizable with real simple isolated eigenvalues only, the evolution $U(t,s)$ *follows the instantaneous eigenprojectors* $P_j(t)$ in the adiabatic regime in the sense that

$$U(t,s)P_j(s) = P_j(t)U(t,s) + O(\varepsilon), \quad \text{as} \quad \varepsilon \to 0, \tag{1.4}$$

as shown in [10], [25], [30], [7], [1], and [12], [28], [23], for example. In other words, transitions between different spectral subspaces are suppressed as $\varepsilon \to 0$, while the restriction of the evolution within these subspaces dominates the error term. This relation remains true for certain eigenprojectors if the eigenvalues are allowed to have negative imaginary parts, [32], [1]. This fact is also well-known and crucial in the study of the Stokes phenomenon appearing in singularly perturbed differential equations [14]: under analyticity assumptions, one considers certain paths in the complex t-plane, called canonical of dissipative paths, along which an equivalent of (1.4) is true in order to get bounds on, or to compute exponentially small quantities in $1/\varepsilon$ stemming from singularities in the complex t-plane. Such methods are used in [20], [21], [23] and [19], to bound or to compute exponentially small transitions in the adiabatic limit when the relevant eigenvalues are real on the real axis.

However, when eigenilpotent are present in the spectral decomposition (1.2), the relation (1.4) cannot hold in general *for all* eigenprojectors $P_j(t)$, even for real-valued eigenvalues $\lambda_j(t)$. Indeed, assuming the eigenvalues are real for the discussion, the solution $U(t,s)$ generically grows like e^{D/ε^β}, for some positive D and $0<\beta<1$, due to the presence of eigenilpotents. Hence, the error term in (1.4) becomes of order $\varepsilon e^{D/\varepsilon^\beta}$, which is still smaller than $\|U(t,s)\|$. However, the *transition amplitudes* between spectral subspaces $P_j(t)U(t,0)P_k(0)$ are typically *exponentially increasing* as $\varepsilon \to 0$, rather than vanishing as ε. An explicit example of this fact is provided at the end of the Introduction. We come back to this mechanism below.

In this context, our main result reads as follows. We construct a different set of time-dependent projectors $P_j^{q^*(\varepsilon)}(t)$ which approximates the eigenprojectors $P_j(t)$ in the adiabatic regime $\varepsilon \to 0$. And we show that the evolution $U(t,s)$ can be approximated up to an error which vanishes as $\varepsilon \to 0$ by a simpler evolution, $V^{q^*(\varepsilon)}(t,s)$, which exactly follows the constructed approximations $P_j^{q^*(\varepsilon)}(t)$ of the instantaneous eigenprojectors. In other words, we restore the expected adiabatic behaviour, *i.e.* suppression of certain transitions, by trading the instantaneous eigenprojectors for other nearby projectors in the limit $\varepsilon \to 0$.

When the eigenvalues are complex valued, the "dynamical phases" $e^{-i\int_s^t \lambda_j(u)\,du/\varepsilon}$ contribute other factors which may be exponentially growing, depending on the imaginary parts of the eigenvalues and which further need to be taken care of as in [32] or [1]. In case $H(t)$ is unbounded, we assume the part $H(t)P_0(t)$ generates a semigroup bounded by $|e^{-i\int_s^t \lambda_0(u)\,du/\varepsilon}|$, for some function $\lambda_0(t)$.

More precisely, for all $j=1,\cdots,n$ and for any $0\le t\le 1$, we construct perturbatively a set of projectors close to the spectral projectors of $H(t)$, see Section 5,

$$P_j^{q^\star(\varepsilon)}(t) = P_j(t) + O(\varepsilon), \quad \text{and } P_0^{q^\star(\varepsilon)}(t) \equiv \mathbb{I} - \sum_{j=1}^n P_j^{q^\star(\varepsilon)}(t). \tag{1.5}$$

Let $W^{q^\star(\varepsilon)}(t)$ be the intertwining operator naturally associated with the projectors $P_k^{q^\star(\varepsilon)}(t)$, $k=0,\cdots,n$ introduced by Kato [25], such that

$$W^{q^\star(\varepsilon)}(t)P_k^{q^\star(\varepsilon)}(0) = P_k^{q^\star(\varepsilon)}(t)W^{q^\star(\varepsilon)}(t), \quad k=0,\cdots,n. \tag{1.6}$$

The approximation is then defined by

$$V^{q^\star(\varepsilon)}(t,0) = W^{q^\star(\varepsilon)}(t)\Phi^{q^\star(\varepsilon)}(t,0) \tag{1.7}$$

where $\Phi^{q^\star(\varepsilon)}(t,s)$ commutes with all the $P_k^{q^\star(\varepsilon)}(0)$, $k=0,\cdots,n$ for any t and satisfies a certain singularly perturbed linear differential equation, see (6.17) below, which describes the effective evolution *within* the fixed subspaces $P_k^{q^\star(\varepsilon)}(0)\mathcal{B}$. Therefore, the following exact intertwining relation holds

$$V^{q^\star(\varepsilon)}(t,0)P_k^{q^\star(\varepsilon)}(0) = P_k^{q^\star(\varepsilon)}(t)V^{q^\star(\varepsilon)}(t,0), \quad k=0,\cdots,n. \tag{1.8}$$

Introducing $\omega(t) = \max_{k=0,\cdots,n} \Im\lambda_k(t)$ to control the norm of the "dynamical phases", we prove the existence of $\kappa > 0$ such that for any $0 \le t \le 1$

$$U(t,0) = V^{q^\star(\varepsilon)}(t,0) + O(te^{-\kappa/\varepsilon} e^{\int_0^t \omega(u)\, du/\varepsilon}), \tag{1.9}$$

where $\|V^{q^\star(\varepsilon)}(t,0)\| = O(e^{\int_0^t \omega(u)\, du/\varepsilon} e^{D/\varepsilon^\beta})$, for some $D \ge 0$ and $0 < \beta < 1$. Note that the first term always dominates the exponentially smaller error term, and moreover, that this error term tends to zero as $\varepsilon \to 0$ for times up to $T > 0$, of order one, such that $\int_0^T \omega(u)\, du = \kappa$. The latter property ensures that the transition amplitudes $\|P_j^{q^\star(\varepsilon)}(t)U(t,0)P_k^{q^\star(\varepsilon)}(0)\|$, $j \ne k$, vanish in the adiabatic limit, provided $0 \le t \le T$.

In case $\mathcal{B}$ is a Hilbert space and $H(t)$ is self-adjoint, both the evolution and its approximation are unitary and D can be chosen equal to zero. The intertwining identity (1.8) and (1.9) show that the transitions between the different subspaces $P_j^{q^\star(\varepsilon)}(0)\mathcal{B}$ are exponentially small in ε, while the transitions between the spectral subspaces of H are of order ε. Constructions leading to approximations $V^{q^\star(\varepsilon)}$ of this type with exponentially small error term go under the name *superadiabtic renormalization*, according to the terminology coined by Berry [9], in this quantum adiabatic context. The first general rigorous construction of this type appears in [31], but we shall use that of [22]. The statement (1.9) is thus very similar to the Adiabatic Theorem of quantum mechanics [25], [30], [7], [1]... and, more precisely, to the subsequent exponentially accurate versions in an analytic context provided in [21], [31], [22], [24], [16], [17]... or variants therof. However, while the improvement of the error term in (1.9) from $O(\varepsilon)$ to $O(e^{-\kappa/\varepsilon})$ by considering $P_j^{q^\star(\varepsilon)}$ in place of P_j in the adiabatic context is just that, *improvement*, in case there are non-zero nilpotents in the decomposition (1.2), it becomes *necessary* to consider $P_j^{q^\star(\varepsilon)}$ and achieve exponential accuracy to get a result on the vanishing of transition amplitudes between certain subspaces.

This can be understood as follows. As $\Phi_\varepsilon(t,0)$ commutes with all the $P_k^{q^\star(\varepsilon)}(0)$, $k = 0, \cdots, n$ for any t we can write

$$\Phi^{q^\star(\varepsilon)}(t,0) = \sum_{k=0}^n P_k^{q^\star(\varepsilon)}(0)\Phi^{q^\star(\varepsilon)}(t,0)P_k^{q^\star(\varepsilon)}(0) \equiv \sum_{k=0}^n \Phi_k^{q^\star(\varepsilon)}(t,0). \tag{1.10}$$

The operator $\Phi_j^{q^\star(\varepsilon)}(t,0)$ describing the evolution within the fixed subspaces $P_j^{q^\star(\varepsilon)}(0)\mathcal{B}$ satisfies for $j \ge 1$,

$$\begin{aligned} &i\varepsilon\partial_t \Phi_j^{q^\star(\varepsilon)}(t,0) = (\lambda_j(t)P_j^{q^\star(\varepsilon)}(0) + \widetilde{D}_j(t,\varepsilon) + O(\varepsilon))\Phi_j^{q^\star(\varepsilon)}(t,0), \\ &\Phi_j^{q^\star(\varepsilon)}(0,0) = P_j^{q^\star(\varepsilon)}(0), \end{aligned} \tag{1.11}$$

where $\widetilde{D}_j(t,\varepsilon)$ denotes the nilpotent $\widetilde{D}_j(t,\varepsilon) = W^{q^\star(\varepsilon)^{-1}}(t)D_j(t)W^{q^\star(\varepsilon)}(t)$. We can write

$$\Phi_j^{q^\star(\varepsilon)}(t,0) = e^{-\frac{i}{\varepsilon}\int_0^t \lambda_j(u)du}\Psi_j^{q^\star(\varepsilon)}(t,0), \tag{1.12}$$

where the operator $\Psi_j^{q^\star(\varepsilon)}$ is essentially generated by a nilpotent. Such adiabatic evolutions generated by perturbations of analytic nilpotents are studied in Section 4. We show that $\Psi_j^{q^\star(\varepsilon)}$ typically grows when $\varepsilon \to 0$ as

$$\Psi_j^{q^\star(\varepsilon)}(t,0) \simeq e^{c/\varepsilon^{\beta_j}}, \quad \text{with} \quad 0 < \beta_j < 1, \tag{1.13}$$

wheras $\Psi_j^{q^\star(\varepsilon)}(t,0)$ remains bounded as $\varepsilon \to 0$ iff $D_j(t) \equiv 0$. The growth in e^{1/ε^β}, $0 < \beta < 1$, of adiabatic evolutions generated by certain nilpotents is already present the works [42] and [38]. Hence, to compensate the exponential growth in $1/\varepsilon^{\beta_j}$ of the $\Psi_j^{q^\star(\varepsilon)}(t,0)$'s which induces transitions between the *instantaneous eigenspaces* of the same order, see the example below, it is necessary to push the estimates to exponential order, see (1.9), by trading the P_j's for the $P_j^{q^\star(\varepsilon)}$. This requires analyticity of the data, see Section 5. Analyticity is also essential in Section 4 where the properties of nilpotent generators and the adiabatic evolutions they generate are studied.

Let us finally comment on the paper [35]. It addresses, at a theoretical physics level, the evolution of master equations describing open quantum systems in which the components of the Lindblad generator are slowly varying functions of time. Mathematically, this corresponds to a particular case of problem (1.1) with a generator containing nilpotents in its decomposition (1.2) and for which $\omega(t) \equiv 0$. The authors argue under certain *implicit* conditions on the evolution, that it is possible to approximate $U(t,0)$ by some operator $V^\varepsilon(t,0)$ which satisfies the intertwining relation (1.8) with the *instantaneous projectors* $P_j(t)$ in place of the approximate projectors $P_j^{q^\star(\varepsilon)}(t) = P_j(t) + O(\varepsilon)$. The authors recognize that such a statement cannot be true generically, and we confirm their conclusion. The statement does hold, however, under the hypotheses of [1], that is when the nilpotent part of the generator in the corresponding subspace $P_j(t)\mathcal{B}$ is absent, together with an *a priori* bound on the evolution (see also remark iii) at the end of the Section). It also holds when the considered spectral subspace $P_j(t)\mathcal{B}$ is always decoupled from the others, an example of this sort is indeed provided in [35]. Otherwise the error term becomes too large due to the growth (1.13).

The paper is organized as follows. We close the introduction by the example alluded to above and then provide the precise hypotheses and the mathematical statement corresponding to our main result. The rest of the paper is devoted to the proof of it. The main steps consists in Section 4 which studies adiabatic evolutions generated by (perturbations of) analytic nilpotents. The iterative scheme providing the adiabatic renormalization of [22] is shortly recalled in Section 5. The approximations and its properties are presented in Section 6.

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1.1 About the effect of nilpotents

We consider here an explicitely solvable model defined by simple generator with two real valued distinct eigenvalues possessing a nilpotent in its spectral decomposition. We show that this nilpotent induces exponentially increasing transitions (in $1/\varepsilon^\beta$, $\beta < 1$) between the instantaneous eigenspaces, thereby underlying the necessity to use superadiabatic renormalization to achieve our result. We also identify the approximated projectors $P_j^{q^*(\varepsilon)}$ that the evolution follows.

Let H be a constant 3×3 matrix in canonical basis $\{e_1, e_2, e_3\}$ defined by

$$H = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.14}$$

and let L be another constant 3×3 matrix defined by

$$L = \begin{pmatrix} 0 & 0 & -k \\ -k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.15}$$

where the non-zero scalars a, k will be chosen later on. We set

$$S(t) := e^{-itL}, \quad H(t) := S(t)HS^{-1}(t), \tag{1.16}$$

and consider the adiabatic evolution $U(t,0)$ defined for any $t \in [0,1]$ by

$$i\varepsilon U'(t,0) = H(t)U(t,0), \quad U(0,0) = \mathbb{I}. \tag{1.17}$$

The spectrum of $H(t)$ is $\{0,1\}$ and its decomposition reads

$$H(t) = S(t)(0P_0 + D_0 + 1P_1)S^{-1}(t) \equiv 0P_0(t) + D_0(t) + 1P_1(t). \tag{1.18}$$

where $P_0 = e_1\langle e_1| + e_2\langle e_2|$, $P_1 = e_3\langle e_3|$ and $D_0 = a\ e_1\langle e_2|$. Here $\{\langle e_j|\}_{j=1,2,3}$ denotes the adjoint basis of $\{e_j\}_{j=1,2,3}$.

The operator $\Omega(t) := S^{-1}(t)U(t,0)$ satisfies

$$i\varepsilon\Omega'(t) = (H - \varepsilon L)\Omega(t), \quad \Rightarrow \Omega(t) = e^{-it(H-\varepsilon L)/\varepsilon}. \tag{1.19}$$

The matrix $H - \varepsilon L$ is now diagonalizable and its spectrum is

$$\{1, +\sqrt{\varepsilon ak}, -\sqrt{\varepsilon ak}\} \equiv \{1, \lambda_+(\varepsilon), -\lambda_+(\varepsilon)\} \equiv \{1, \lambda_+(\varepsilon), \lambda_-(\varepsilon)\}, \tag{1.20}$$

where $\sqrt{\cdot}$ denotes any branch of the square root function. The corresponding spectral projectors are denoted by $P_1(\varepsilon), P_+(\varepsilon)$ and $P_-(\varepsilon)$ and they are given by

$$P_1(\varepsilon) = \begin{pmatrix} 0 & 0 & \frac{\varepsilon k}{1-\varepsilon ak} \\ 0 & 0 & \frac{\varepsilon^2 k^2}{1-\varepsilon ak} \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.21}$$

$$P_+(\varepsilon) = \begin{pmatrix} \frac{\lambda_+(\varepsilon)}{\lambda_+(\varepsilon)-\lambda_-(\varepsilon)} & \frac{a}{\lambda_+(\varepsilon)-\lambda_-(\varepsilon)} & \frac{\lambda_+(\varepsilon)\varepsilon k}{(\lambda_+(\varepsilon)-\lambda_-(\varepsilon))(\lambda_+(\varepsilon)-1)} \\ \frac{\varepsilon k}{\lambda_+(\varepsilon)-\lambda_-(\varepsilon)} & \frac{\lambda_+(\varepsilon)}{\lambda_+(\varepsilon)-\lambda_-(\varepsilon)} & \frac{\varepsilon^2 k^2}{(\lambda_+(\varepsilon)-\lambda_-(\varepsilon))(\lambda_+(\varepsilon)-1)} \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.22}$$

and $P_-(\varepsilon)$ has the same expression as $P_+(\varepsilon)$ with indices $+$ and $-$ exchanged. Note that $P_\pm(\varepsilon) \simeq \pm a/\sqrt{\varepsilon ak}$ as $\varepsilon \to 0$. Whereas the projectors

$$P_1(\varepsilon) = P_1 + O(\varepsilon) \tag{1.23}$$
$$P_0(\varepsilon) = P_+(\varepsilon) + P_-(\varepsilon) = P_0 + O(\varepsilon) \tag{1.24}$$

admit expansions in powers of ε. Hence

$$U(t,0) = S(t)(e^{-it/\varepsilon}P_1(\varepsilon) + e^{-it\lambda_+(\varepsilon)/\varepsilon}P_+(\varepsilon) + e^{-it\lambda_-(\varepsilon)/\varepsilon}P_-(\varepsilon)), \tag{1.25}$$

so that, as $\varepsilon \to 0$,

$$\|U(t,0)\| \simeq \left| \frac{a}{2\sqrt{\varepsilon ak}}(e^{-it\sqrt{\varepsilon ak}/\varepsilon} - e^{it\sqrt{\varepsilon ak}/\varepsilon}) \right| \tag{1.26}$$

which diverges, whatever the nonzero value of ak is.

We now choose $ak < 0$ and $\lambda_\pm(\varepsilon) = \pm i\sqrt{\varepsilon|ak|} \in i\mathbb{R}$, for definiteness. Since $P_k(t)U(t,0)P_j(0) = S(t)P_k\Omega(t)P_j$, $j,k \in \{0,1\}$, where $S(t)$ is independent of ε, it is enough to compute $P_k\Omega(t)P_j$ to get the behaviour in ε of the transitions between the corresponding instantaneous subspaces. We get for $t > 0$ and $P_1 = e_3\langle e_3|$,

$$\begin{aligned} P_0\Omega(t)P_1 &= e^{t\sqrt{|ak|}/\sqrt{\varepsilon}} \begin{pmatrix} -\frac{\varepsilon k}{2} \\ \frac{i\varepsilon^{3/2}k^2}{2\sqrt{|ak|}} \\ 0 \end{pmatrix} \langle e_3|(1+O(\varepsilon^{1/2})), \quad \text{as } \varepsilon \to 0 & (1.27) \\ P_1\Omega(t)P_0 &\equiv 0 & (1.28) \end{aligned}$$

The first formula thus implies that the evolution $U(t,0)$ does not follow the instantaneous eigenprojector $P_1(t)$, whereas the second formula simply reflects the non-generic fact that P_0 is invariant under $H - \varepsilon L$ in our example, see the remarks below.

The model being explicitly solvable, we can readily identify the approximated projectors the evolution follows. Setting for $j = 0, 1$

$$P_j^*(t,\varepsilon) := S(t)P_j(\varepsilon)S^{-1}(t) = P_j(t) + O(\varepsilon), \tag{1.29}$$

we compute by means of (1.25)

$$U(t,0)P_j^*(0,\varepsilon) = P_j^*(t,\varepsilon)U(t,0). \tag{1.30}$$

Thus the evolution $U(t,0)$ exactly follows the projectors (1.29) whereas the transition from $P_1(0)$ to $P_0(t)$ are exponentially large in $1/\sqrt{\varepsilon}$.

Remarks:
i) If the product $ak \in \mathbb{C} \setminus \mathbb{R}^+$, a similar result holds. We took $ak < 0$ for simplicity. If the product ak is positive, the transition does vanish in the limit $\varepsilon \to 0$. This is due to the fact that the spectral projector P_0 corresponding to the unperturbed eigenvalue 0 of H is of dimension 2. For the natural generalization of this example with $\dim P_0 = d$, $d > 2$, the following holds. Generically, the splitting of the unique eigenvalue zero of the nilpotent P_0H by a perturbation of order ε yields d perturbed eigenvalues $\lambda_j(\varepsilon) \simeq \alpha\varepsilon^{1/d}e^{j2i\pi/d}$, $j = 0, \cdots, d-1$, $\alpha \in \mathbb{C}$, see [26]. Hence, one of them has a non vanishing imaginary part that produces exponentially growing contributions as $\varepsilon \to 0$.
ii) As already mentioned, this example is non-generic in the sense that P_0 is invariant under $\Omega(t)$, see (1.28). The choice of non-generic L (1.15) was made to keep the formulas simple. However, as should be clear from the analysis, a generic choice for L implies an exponential increase as $\varepsilon \to 0$ for both $P_1\Omega(t)P_0$ and $P_0\Omega(t)P_1$, when $ak \in \mathbb{C} \setminus \mathbb{R}^+$.
iii) The real unperturbed eigenvalues 0 and 1 can be replaced by any different complex numbers λ_0 and λ_1 without difficulty. The main consequence is that the exponents in (1.25) have to be changed according to $\lambda_\pm(\varepsilon) \mapsto \lambda_0 + \lambda_\pm(\varepsilon)$ and $1 \mapsto \lambda_1$. One can assume without loss that $\Im\lambda_j \leq 0$, $j = 0, 1$. Observe that if λ_0 is real and $\Im\lambda_1 < 0$, conclusions similar to (1.27) can be drawn. In case $\Im\lambda_0 < 0$ and λ_1 is real, the transition $P_0\Omega(t)P_1$ is of order ε, $\varepsilon \to 0$. This is a case where the results of [1] apply, since the evolution (1.25) becomes uniformly bounded in ε due to the exponential decay stemming from $\Im\lambda_0 < 0$.

2 Main Result

Let us specify here our hypotheses and state our result.

Let $a > 0$ and $S_a = \{z \in \mathbb{C} \mid \text{dist}(z, [0,1]) < a\}$.

H1:

Let $\{H(z)\}_{z\in\bar{S}_a}$ be a family of closed operators densely defined on a common domain $\mathcal{D}$ of a Banach space $\mathcal{B}$ and for any $\varphi \in \mathcal{D}$, the map $z \mapsto H(z)\varphi$ is analytic in S_a.

As a consequence, the resolvent $R(z,\lambda) = (H(z) - \lambda)^{-1}$ is locally analytic in z for $\lambda \in \rho(H(z))$, where $\rho(H(z))$ denotes the resolvent set of $H(z)$.

H2:

For $t \in [0,1]$, the spectrum of $H(t)$ is of the form $\sigma(H(t)) = \sigma(t) \cup \sigma_0(t)$, and there exists $G > 0$ such that

$$\inf_{t\in[0,1]} \text{dist}(\sigma(t), \sigma_0(t)) \geq G.$$

Moreover, $\sigma(t) = \{\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)\}$ where $\lambda_j(t)$, $j = 1, \cdots, n$, $n < \infty$, are eigenvalues of constant multiplicity $m_j < \infty$ such that

$$\inf_{\substack{t\in[0,1] \\ j\neq k}} \text{dist}(\lambda_j(t), \lambda_k(t)) \geq G.$$

Let $\Gamma_j \in \rho(H(t))$ be a loop encircling $\lambda_j(t)$ only. The finite dimensional spectral projectors corresponding to the eigenvalues $\lambda_j(t)$ are given by

$$P_j(t) = -\frac{1}{2\pi i}\int_{\Gamma_j} R(t,\lambda)d\lambda \quad \text{and we set} \quad P_0(t) = \mathbb{I} - \sum_{j=1}^{n} P_j(t) \equiv \mathbb{I} - P(t). \tag{2.1}$$

The loop Γ_j can be chosen locally independent of t. It is a classical perturbative fact, see [26], that **H2** also holds for the spectrum of $H(z)$ with $z \in S_a$, provided a is small enough, and that the eigenvalues are analytic functions in S_a. By this we mean that the $\inf_{t\in[0,1]}$ can be replaced by $\inf_{t\in S_a}$ in **H2**. Hence, (2.1) also holds for $z \in S_a$ and $z \mapsto P_k(z)$ is analytic in S_a, for $k = 0, \dots, n$. Consequently, the eigenilpotents given by $D_j(z) = (H(z) - \lambda_j(z))P_j(z)$ are analytic in S_a as well.

We now state a technical hypothesis needed to deal with evolution operators generated by unbounded generators. In case one works with bounded operators only, this hypothesis is not necessary.

H3:

Let $H_0(t) = P_0(t)H(t)P_0(t)$. There exists a C^1 complex valued function $t \mapsto \lambda_0(t)$ such that for all $t \in [0,1]$, $H_0(t) + \lambda_0(t)$ generates a contraction semigroup and $0 \in \rho((H_0(t) + \lambda_0(t))$.

In other words, **H3** says that the solution $T(s) = e^{-i\lambda_0(t)s}e^{-iH_0(t)s}$ to the strong equation on $\mathcal{D}$ $i\partial_s T(s) = (H_0(t) + \lambda_0(t))T(s)$ satisfies $\|T(s)\| \leq 1$, for all $s \geq 0$. By Hille-Yoshida's Theorem, **H3** is equivalent to the following spectral condition for any $t \in [0,1]$,

$$[0,\infty) \subset \rho(-iH_0(t) - i\lambda_0(t)) \quad \text{and} \quad \|(iH_0(t) + i\lambda_0(t) + \lambda)^{-1}\| \leq \frac{1}{\lambda}, \;\; \forall \lambda > 0. \tag{2.2}$$

This hypothesis implies that the equation

$$i\varepsilon\partial_t U_0(t,s)\varphi = H_0(t)U_0(t,s)\varphi, \quad U_0(s,s)\varphi = \varphi, \quad s \le t \in [0,1], \quad \forall \varphi \in \mathcal{D}, \tag{2.3}$$

defines a unique *strongly continuous two-parameter evolution operator* $U_0(t,s)$. It means that $U_0(t,s)$ is uniformly bounded, strongly continuous in the triangle $0 \le s \le t \le 1$ and satisfies the relation $U_0(t,s)U_0(s,r) = U_0(t,r)$ for any $0 \le r \le s \le t \le 1$. Moreover, $U_0(t,s)$ maps $\mathcal{D}$ into $\mathcal{D}$, also satisfies

$$i\varepsilon\partial_s U_0(t,s)\varphi = -U_0(t,s)H_0(s)\varphi, \quad \forall \varphi \in \mathcal{D}, \tag{2.4}$$

and is such that $H_0(t)U_0(t,s)(H_0(s)+\lambda_0)^{-1}$ is bounded and continous in the triangle $0 \le s \le t \le 1$. Moreover, see [34], Thm X.70., the following bound holds

$$\|U_0(t,s)\| \le e^{\int_s^t \Im\lambda_0(u)\, du/\varepsilon}, \quad \forall s \le t \in [0,1]. \tag{2.5}$$

Since $H(t) = H_0(t) + P(t)H(t)P(t)$ where $P(t)H(t)P(t)$ is bounded and analytic in t, Hypothesis **H3** also implies existence and uniqueness of a *bona fide* evolution operator $U(t,s)$ associated with the equation

$$i\varepsilon\partial_t U(t,s)\varphi = H(t)U(t,s)\varphi, \quad U(s,s)\varphi = \varphi, \quad s \le t \in [0,1], \quad \forall \varphi \in \mathcal{D}, \tag{2.6}$$

see [27], Thm 3.6, 3.7 and 3.11.

Theorem 2.1 *Assume* **H1**, **H2** *and* **H3** *and consider $U(t,0)$ defined by (2.6). For $k = 0,\cdots,n$, let $P_k^{q^\star(\varepsilon)}(t) = P_k(t) + O(\varepsilon)$ be defined by (5.7), (5.11) and $V^{q^\star(\varepsilon)}(t,0) = W^{q^\star(\varepsilon)}(t)\Phi^{q^\star(\varepsilon)}(t,0)$ given by (6.1), (6.10), (6.12) and (5.11). Define $\omega(t) = \max_{k=0,\cdots,n}\Im\lambda_k(t)$. Then, there exists a constant $\kappa > 0$ such that for any $0 \le t \le 1$*

$$\begin{aligned} e^{-\int_0^t \omega(u)\,du/\varepsilon} U(t,0)P_k^{q^*(\varepsilon)}(0) &= e^{-\int_0^t \omega(u)\,du/\varepsilon} V^{q^*(\varepsilon)}(t,0)P_k^{q^*(\varepsilon)}(0) \\ &\quad + O(te^{-\kappa/\varepsilon} \sup_{0\le s\le t} \|e^{-\int_0^s \omega(u)\,du/\varepsilon} V^{q^*(\varepsilon)}(s,0)P_k^{q^*(\varepsilon)}(0)\|), \end{aligned}$$

with

$$V^{q^\star(\varepsilon)}(t,0)P_k^{q^\star(\varepsilon)}(0) = P_k^{q^\star(\varepsilon)}(t)V^{q^\star(\varepsilon)}(t,0), \quad k = 0,\cdots,n.$$

Moreover, for all $k \ge 0$ there exists $0 \le \beta_k < 1$, $c_k > 0$, and $d_k \ge 0$, with $d_0 = 0$, such that

$$\|V^{q^\star(\varepsilon)}(t,0)P_k^{q^\star(\varepsilon)}(0)\| \le c_k e^{d_k/\varepsilon^{\beta_k}} e^{\Im \int_0^t \lambda_k(u)du/\varepsilon},$$

with $d_j = 0$, if and only if $D_j(t) \equiv 0$, $j \in \{1,\cdots,n\}$, in (1.2).

As a direct

Corollary 2.1 *Under the hypotheses of Theorem 2.1, there exists $\kappa > 0$, $0 < \beta < 1$, and $D \ge 0$ such that*

$$U(t,0) = V^{q^*(\varepsilon)}(t,0) + O(te^{-\kappa/\varepsilon} e^{\int_0^t \omega(u)\,du/\varepsilon}),$$

where $V^{q^*(\varepsilon)}(t,0) = O(e^{\int_0^t \omega(u)\,du/\varepsilon} e^{D/\varepsilon^\beta})$.

Remarks:
0) The equivalent results hold if the initial time 0 is replaced by any $0 \leq s \leq t$, *mutatis mutandis*. See Subsection 6.1.
i) As is obvious from the formulation, the natural operators to control are $e^{-\int_0^t \omega(u)\, du/\varepsilon} U(t,0)$ and $e^{-\int_0^t \omega(u)\, du/\varepsilon} V^{q^*(\varepsilon)}(t,0)$.
ii) As particular cases of Theorem 2.1, we recover the results of [12], [32], [28], [23], [1].
iii) In case κ is sufficiently large, the different components of the leading order term have amplitudes whose instantaneous exponential decay or growth rates in $1/\varepsilon$ may change with time. More precisely, assume that

$$\int_0^t \Im\lambda_k(u)\, du > \int_0^t \omega(u)\, du - \kappa, \quad \forall t \in [0,1], \quad \text{and} \quad \forall k = 0, \cdots, n. \tag{2.7}$$

This can be achieved by perturbating weakly a generator for which all λ_k are real valued, for example. Then, for any initial condition

$$\varphi = \sum_{k=0}^{n} P_k^{q^\star(\varepsilon)}(0)\varphi \equiv \sum_{k=0}^{n} \varphi_k(\varepsilon) \in \mathcal{D}, \tag{2.8}$$

we get

$$U(t,0)\varphi = \sum_{k=0}^{n} e^{-i\int_0^t \lambda_k(u)du/\varepsilon} \Psi_k^{q^\star(\varepsilon)}(t,0)\varphi_k(\varepsilon) + O(te^{-\kappa/\varepsilon} e^{\int_0^t \omega(u)\, du/\varepsilon}), \tag{2.9}$$

where the error term is exponentially smaller than the leading terms. Each term of the sum decays or grows as $\varepsilon \to 0$ with an instantaneous exponential rate given by $\Im \int_0^t \lambda_k(u)du/\varepsilon$. Depending on the functions $t \mapsto \Im \int_0^t \lambda_k(u)du/\varepsilon$, the index of the component which is the most significant may vary with time.
iv) In case all $\lambda_k(t)$ are real, $k = 0, \cdots, n$, and $H(t)$ is diagonalizable, we can take $d_k = 0$ for all $k = 0, \cdots, n$, and $\omega(t) \equiv 0$. The evolution U and its approximation $V^{q^*(\varepsilon)}$ are then uniformly bounded in ε and differ by an error of order $e^{-\kappa/\varepsilon}$. Theorem 2.1 thus generalizes Thm 2.4 in [23] in the sense that we allow permanently degenerate eigenvalues $\lambda_j(t)$, whereas they were assumed to be simple in [23].

3 Preliminary Estimates

We start by recalling a perturbation formula for evolution operators that we will use several times in the sequel.

Let $\{A(t)\}_{t\in[0,1]}$ be a densely defined family of linear operators on a common domain $\mathcal{D}$ of a Banach space $\mathcal{B}$, and assume $t \mapsto A(t)$ is strongly continuous. Let $B(t)$ be linear, bounded and strongly continuous in $t \in [0,1]$. Assume there exist two-parameter evolution operators $T(t,s)$ and $S(t,s)$ associated with the equations

$$\begin{aligned} i\partial_t T(t,s)\varphi &= A(t)T(t,s)\varphi, \quad T(s,s) = \mathbb{I}, & (3.1)\\ i\partial_t S(t,s)\varphi &= (A(t)+B(t))S(t,s)\varphi, \quad S(s,s) = \mathbb{I}, & (3.2) \end{aligned}$$

for all $\varphi \in \mathcal{D}$ and $s \leq t \in [0,1]$. Then, for any $\varphi \in \mathcal{D}$, and any $r \leq s \leq t \in [0,1]$,

$$i\partial_s(T(t,s)S(s,r)\varphi) = T(t,s)B(s)S(s,r)\varphi, \tag{3.3}$$

so that by integration on s between r and t,

$$S(t,r)\varphi = T(t,r)\varphi - i\int_r^t ds T(t,s)B(s)S(s,r)\varphi. \tag{3.4}$$

Iterating this formula, we deduce the representation

$$\begin{aligned} S(t,r) = \sum_{n\geq 0}(-i)^n \int_r^t ds_1 \int_r^{s_1} ds_2 \cdots \int_r^{s_{n-1}} ds_n \\ \times \quad T(t,s_1)B(s_1)T(s_1,s_2)B(s_2)\cdots B(s_n)T(s_n,r). \end{aligned} \tag{3.5}$$

Further assuming that $T(t,s)$ satisfies the bound

$$\|T(t,s)\| \leq Me^{\int_s^t \omega(u)du}, \tag{3.6}$$

for a constant M and a real valued integrable function $u \mapsto \omega(u)$, we get from (3.5)

$$\|S(t,s)\| \leq Me^{\int_r^t (\omega(u)+M\|B(u)\|)\,du}. \tag{3.7}$$

As a first application of (3.7), we get from (2.5) a first estimate on $U(t,s)$ that we will improve later on

$$\|U(t,s)\| \leq e^{\int_s^t (\Im\lambda_0(u)+\|P(u)H(u)P(u)\|)\,du/\varepsilon}. \tag{3.8}$$

4 Nilpotent Generators

For later purposes, we study here the adiabatic evolution generated by an analytic nilpotent, in a finite dimensional space. We assume

N1:
For any $z \in S_a$, $N(z)$ is an analytic nilpotent valued operator in a linear space $\mathcal{B}$ of finite dimension such that for a fixed integer $d \geq 0$, $N(z)^d \equiv 0$.

The detailed analysis of the properties of analytic nilpotent matrices is performed in Section 5 of the book [8]. It is shown in particular that such operators have the following structure. For any nilpotent $N(z)$ satisfying **N1** in S_a, there exists a finite set of points $Z_0 \subset S_{a'}$, with $a' < a$, and, there exists a family of invertible operators $\{S(z)\}_{z\in S_{a'}\setminus Z_0}$ such that for any $z \in S_{a'} \setminus Z_0$,

$$N(z) = S^{-1}(z)NS(z) \tag{4.1}$$

with $S(z)$ and $S^{-1}(z)$ meromorphic in $S_{a'}$ and regular in $S_{a'} \setminus Z_0$. The set Z_0 where $N(z)$ is not similar to the constant nilpotent N is called the set of *weakly splitting points* of $N(z)$. At these points, the range and kernel of $N(z)$ change.

We consider $Y(t,s)$, defined as the solution to

$$\varepsilon\partial_t Y(t,s) = N(t)Y(t,s), \quad Y(s,s) = \mathbb{I}, \quad \forall s,t \in [0,1], \tag{4.2}$$

and estimate the way $Y(t,s)$ depends on ε, as $\varepsilon \to 0$. Note that we don't need to impose $s \leq t$ since we deal with bounded generators.

In case N is constant, with $N^{d-1} \neq 0$, $Y(t,s) = e^{(t-s)N/\varepsilon}$ behaves polynomially in $1/\varepsilon$, *i.e.* like $((t-s)/\varepsilon)^{d-1}$, as $\varepsilon \to 0$. When $N(t)$ is not constant, one may expect that $Y(t,s)$ explodes less fast than $e^{c/\varepsilon}$, which is the worst behavious as $\varepsilon \to 0$ for bounded generators. In such cases, however, $Y(t,s)$ grows typically faster than polynomially in $1/\varepsilon$, as the following example shows. For $N(z)$ given by

$$N(t) = \begin{pmatrix} t & -1 \\ t^2 & -t \end{pmatrix}, \tag{4.3}$$

we get that the solution $Y(t,0)$ to (4.2) reads

$$Y(t,0) = \begin{pmatrix} \cosh(\frac{t}{\sqrt{\varepsilon}}) & -\frac{1}{\sqrt{\varepsilon}}\sinh(\frac{t}{\sqrt{\varepsilon}}) \\ t\cosh(\frac{t}{\sqrt{\varepsilon}}) - \sqrt{\varepsilon}\sinh(\frac{t}{\sqrt{\varepsilon}}) & \cosh(\frac{t}{\sqrt{\varepsilon}}) - \frac{t}{\sqrt{\varepsilon}}\sinh(\frac{t}{\sqrt{\varepsilon}}) \end{pmatrix}, \tag{4.4}$$

which behaves as $e^{t/\sqrt{\varepsilon}}$, when $\varepsilon \to 0$. The growth is nevertheless slower than exponential in $1/\varepsilon$. We show that the characteristic behaviour of Y generated by an analytic nilpotent operator is similar. For later purposes, we actually consider generators given by an order ε perturbation of a nilpotent.

Proposition 4.1 *Suppose the nilpotent $N(t)$ satisfies* **N1** *and let $\{A(t)\}_{t\in[0,1]}$ be a C^0 family of operators on $\mathcal{B}$. Then, there exist $c > 0$ and $0 < \beta < 1$ such that the solution $Y(t,s)$ of*

$$\varepsilon\partial_t Y(t,s) = (N(t) + \varepsilon A(t))Y(t,s), \quad Y(s,s) = \mathbb{I}, \quad \forall s,t \in [0,1], \tag{4.5}$$

satisfies uniformly in $t, s \in [0,1]$

$$\|Y(t,s)\| \leq c e^{c/\varepsilon^\beta}.$$

Remarks:
i) Asymptotic expansions as $\varepsilon \to 0$ of solutions to such equations are derived in [42], [38], in the neighbourhood of points which are not weakly splitting points for $N(z)$.
ii) In case both 0 and 1 are not weakly splitting points and A is analytic, it is possible to take $\beta = (d-1)/d$, which is the optimal exponent, see the example. As we shall not need such improvements, we don't give a proof.
iii) The adiabatic evolution generated by an analytic nilpotent does not have to grow exponentially fast in $1/\varepsilon^\beta$, as $\varepsilon \to 0$. Consider for example (4.3) and (4.4) along the imaginary t-axis. However, such evolutions cannot be uniformly bounded in ε, as the next Lemma shows, under slightly stronger conditions.
iv) It is actually enough to assume $t \mapsto \|A(t)\|$ is uniformly bounded on $[0,1]$.

Lemma 4.1 *Assume $\{N(t)\}_{t\in[0,1]}$ is a C^1 family of nilpotents and $\{A(t)\}_{t\in[0,1]}$ is a C^0 family of operators on $\mathcal{B}$. Consider $Y(t,s)$ the solution to (4.5). Then*

$$\sup_{\substack{\varepsilon>0 \\ s\leq t}} \|Y(t,s)\| < \infty \iff N(u) \equiv 0 \quad \forall s \leq u \leq t.$$

Proof of Proposition 4.1: The proof consists in two steps. First we prove the result for generators with more structure and then, making use of the results of Section 5 in [8] on the detailed structure of analytic nilpotents, we extend it to the general case.

Lemma 4.2 *Assume $N(t) = S^{-1}(t)NS(t)$ where N satisfies $N^d = 0$ and where $\{S(t)\}_{t\in[0,1]}$ is a C^1 family of invertible operators. Let $\{A(t)\}_{t\in[0,1]}$ be a C^0 family of operators and set $B(t) = S(t)A(t)S^{-1}(t) + S'(t)S^{-1}(t)$. Then, there exists $c > 0$ such that the solution $Y(t,s)$ of (4.5) satisfies*

$$\|Y(t,s)\| \leq \|S^{-1}(t)\|\|S(s)\|\frac{c}{\varepsilon^{(d-1)/d}}e^{\int_s^t(1+c\|B(u)\|)du/\varepsilon^{(d-1)/d}}, \quad \forall s \leq t \in [0,1].$$

Remarks:
0) The constant c depends on N only.
i) If $s \geq t$, the same estimate holds with $\int_t^s \|B(u)\|du$ in the exponent.
ii) This Lemma also holds in infinite dimension.

Proof of Lemma 4.2: Let $Z(t,s) = S(t)Y(t,s)S^{-1}(s)$. This operator satisfies by construction

$$\varepsilon\partial_t Z(t,s) = (N + \varepsilon B(t))Z(t,s), \quad Z(s,s) = \mathbb{I}, \quad \forall s,t \in [0,1]. \tag{4.6}$$

Let us compare $Z(t,s)$ with

$$Z_0(t,s) = e^{N(t-s)/\varepsilon}, \quad s,t \in [0,1] \tag{4.7}$$

by means of (3.5). We get

$$\begin{aligned} Z(t,r) = \sum_{n\geq 0}\int_r^t ds_1 \int_r^{s_1} ds_2 \cdots \int_r^{s_{n-1}} ds_n \\ \times \quad Z_0(t,s_1)B(s_1)Z_0(s_1,s_2)B(s_2)\cdots B(s_n)Z_0(s_n,r). \end{aligned} \tag{4.8}$$

Consider now

$$Z_\delta(s) = e^{(N-\delta)s}, \quad \text{for} \quad \delta > 0. \tag{4.9}$$

This operator is such that there exists a $c > 0$, which depends on N only, such that

$$\|Z_\delta(s)\| \leq c/\delta^{d-1} \ \ \forall s \geq 0, \ \text{and} \ \ 0 < \delta \leq 1. \tag{4.10}$$

Indeed, on the one hand, we have for $s \geq s_0$, with s_0 large enough $\|Z_\delta(s)\| \leq Ke^{-\delta s}s^{d-1}$, where K is some constant which depends on N only. Maximizing over $s \geq 0$, we get $e^{-\delta s}s^{d-1} \leq e^{1-d}\frac{(d-1)^{d-1}}{\delta^{d-1}}$. On the other hand, for all $0 \leq s \leq s_0$, we have $\|Z_\delta(s)\| \leq e^{s_0\|N\|}$, so that if $0 < \delta \leq 1$, (4.10) holds with $c = \max(e^{s_0\|N\|}, K((d-1)/e)^{d-1})$.

Coming back to (4.8) in which we make use of the relation

$$Z_0(t,s) = Z_\delta((t-s)/\varepsilon)e^{\delta(t-s)/\varepsilon}, \tag{4.11}$$

and (4.10), we get

$$\begin{aligned} \|Z(t,r)\| &\leq e^{\delta(t-r)/\varepsilon}\sum_{n\geq 0}\int_r^t ds_1 \int_r^{s_1} ds_2 \cdots \int_r^{s_{n-1}} ds_n \\ &\quad \times\|Z_\delta((t-s_1)/\varepsilon)B(s_1)Z_\delta((s_1-s_2)/\varepsilon)B(s_2)\cdots B(s_n)Z_\delta((s_n-r)/\varepsilon)\| \\ &\leq \frac{ce^{\delta(t-r)/\varepsilon}}{\delta^{d-1}}\sum_{n\geq 0}\frac{(c\int_r^t \|B(s)\|\,ds/\delta^{d-1})^n}{n!} \\ &= \frac{ce^{\delta(t-r)/\varepsilon}}{\delta^{d-1}}e^{c\int_r^t \|B(s)\|\,ds/\delta^{d-1}}. \end{aligned} \tag{4.12}$$

The left hand side is independent of δ, which we can chose as $\delta = \varepsilon^{1/d}$, so that we eventually get

$$\|Z(t,r)\| \leq \frac{c}{\varepsilon^{(d-1)/d}} e^{\int_r^t (1+c\|B(s)\|)\, ds/\varepsilon^{(d-1)/d}}, \tag{4.13}$$

from which the result follows. ■

Let us go on with the proof of the Proposition. If $Z_0 \cap [0,1] = \emptyset$, Lemma (4.2) applies and Proposition 4.1 holds. If not, there exist a finite set of real points $\{0 \leq t_1 < t_2 < \cdots < t_m \leq 1\}$ and a finite set of integers $\{p_j\}_{j=1,\cdots,m}$ such that

$$\max(\|S(t)\|, \|S^{-1}(t)\|, \|S'(t)S^{-1}(t)\|) = O(1/(t-t_j)^{p_j}), \quad \text{as} \quad t \to t_j. \tag{4.14}$$

Since Y is an evolution operator, we can split the integration range in finitely many intervals, so that it is enough to control $Y(t,s)$ for $s \leq t \in [v,w] \subset \mathbb{R}$ where $[v,w]$ contains one singular point only. Call this singular point t_0 and the corresponding integer p_0.

Assume to start with that $v < t_0 < w$. Let $\delta > 0$ be small enough and $v \leq s < t_0 < t \leq w$ so that we can write

$$Y(t,s) = Y(t, t_0+\delta) Y(t_0+\delta, t_0-\delta) Y(t_0-\delta, s). \tag{4.15}$$

The first and last terms of the right hand side can be estimates by Lemma 4.2, whereas we get for the middle term

$$\|Y(t_0+\delta, t_0-\delta)\| \leq e^{\frac{1}{\varepsilon}\int_{t_0-\delta}^{t_0+\delta} \|N(u)+\varepsilon A(u)\| du}. \tag{4.16}$$

Altogether this yields

$$\begin{aligned} \|Y(t,s)\| \quad &\leq \quad c^2 \|S^{-1}(t)\| \|S(t_0+\delta)\| \|S^{-1}(t_0-\delta)\| \|S(s)\| / \varepsilon^{2(d-1)/d} \\ &\quad \times e^{(\int_s^{t_0-\delta} + \int_{t_0+\delta}^t)(1+c\|B(u)\|)du/\varepsilon^{(d-1)/d}} e^{+\frac{1}{\varepsilon}\int_{t_0-\delta}^{t_0+\delta} \|N(u)+\varepsilon A(u)\| du}. \end{aligned} \tag{4.17}$$

By (4.14), there exists a constant c (that may change from line to line) which in dependent of ε such that the pre-exponential factors are bounded by c/δ^{2p_0}. Also, since $N(t)$ is C^1 and $A(t)$ is C^0 on $[0,1]$,

$$\int_{t_0-\delta}^{t_0+\delta} \|N(u)+\varepsilon A(u)\|\, du \leq c\delta \quad \text{and} \quad \int_{t_0+\delta}^{t} \|B(u)\|\, du \leq c/\delta^{p_0} \tag{4.18}$$

and similarly for $\int_s^{t_0-\delta} \|B(u)\|$. Hence, $Y(t,s)$ satisfies the bound

$$\|Y(t,s)\| \leq c e^{c(\frac{1}{\delta^{p_0}\varepsilon^{(d-1)/d}} + \frac{\delta}{\varepsilon})} / (\delta^{2p_0} \varepsilon^{2(d-1)/d}). \tag{4.19}$$

Choosing $\delta = \delta(\varepsilon) = \varepsilon^{\frac{1}{d(p_0+1)}}$ in order to balance the contributions in the exponent, we get with a suitable constant c

$$\|Y(t,s)\| \leq c e^{c/\varepsilon^{\frac{(p_0+1)d-1}{(p_0+1)d}}} / \varepsilon^{\frac{2(d(p_0+1)-1)}{(p_0+1)d}}. \tag{4.20}$$

Picking $\frac{(p_0+1)d-1}{(p_0+1)d} < \beta_0 < 1$, we get for yet another constant c

$$\|Y(t,s)\| \leq c e^{c/\varepsilon^{\beta_0}}. \tag{4.21}$$

A similar analysis yields the same result in case $t_0 = u$ or $t_0 = w$. As there are only finitely many weakly splitting points to take care of, taking for $\beta < 1$ the largest of the β_j, for $j = 1, \cdots, m$, we get the result. ■

Remarks:
i) The proof is valid in arbitrary dimension, assuming only (4.14) at a finite number of points.
ii) The exponents $p_i > 0$ in (4.14) need not be integers.

Proof of Lemma 4.1: Let $Y(t,s)$ be a solution to (4.5) and assume $N(u) \equiv 0$ for all $s \le u \le t$. Then $\|Y(t,s)\| \le e^{\int_s^t \| A(u)| \, du}$, which shows one implication. We prove the reverse implication by contradiction. Assume there exists $u_0 \in [s,t]$ such that the nilpotent $N(u_0) \ne 0$ and $\|Y(t,s)\| \le c$, uniformly as $\varepsilon \to 0$, for all $0 \le s \le t \le 1$. We compare $Y(t,s)$ with

$$Z_0(t,s) = e^{N(u_0)(t-s)/\varepsilon} \tag{4.22}$$

and get the following estimate from (3.4) and (3.5)

$$\|Z_0(t,s)\| \le c e^{c \int_s^t \|N(u) - N(u_0) + \varepsilon A(u)\| \, du/\varepsilon}. \tag{4.23}$$

By Taylor's formula, there exists a $\delta > 0$ such that $t - s \le \delta$ implies

$$\int_s^t \|N(u) - N(u_0) + \varepsilon A(u)\| \, du \le c\delta(\delta + \varepsilon), \tag{4.24}$$

for another constant c. Hence, if $t - s \le \delta$, with δ small enough,

$$\|Z_0(t,s)\| \le c e^{c\delta^2/\varepsilon}, \tag{4.25}$$

for some c. On the other hand, if $t - s = \delta$ and $\varepsilon \ll \delta$, we have for some c,

$$\|Z_0(t,s)\| = c(\delta/\varepsilon)^{d-1}. \tag{4.26}$$

Thus, by letting δ and ε tend to zero in such a way that $\delta^2 \ll \varepsilon \ll \delta$, we get a contradiction between (4.25) and (4.26), which finishes the proof of the statement.

5 Iterative Scheme

We present here the iterative construction which leads to the construction of $V^{q^*(\varepsilon)}(t,s)$ developed in [22], to which we refer the reader for proofs and more details. The first general construction of this kind is to be found in [31].

Assume **H1** and **H2** with $a > 0$ small enough so that **H2** holds in S_a.

By perturbation theory in $z \in S_a$, if $z_0 \in S_a$ and $\Gamma_j \in \rho(H(z_0))$, $j = 1, \cdots, n$ are simple loops encircling the eigenvalues $\lambda_j(z_0)$, there exists $r > 0$ such that for any $z \in B(z_0, r)$, where $B(z_0, r)$ is an open disc of radius r centered at z_0, $\Gamma_j \in \rho(H(z))$,

For $z \in B(z_0, r)$, we set

$$P_j(z) = -\frac{1}{2\pi i} \int_{\Gamma_j} (H(z) - \lambda)^{-1} \, d\lambda \equiv P_j^0(z), \quad P_0(z) = P_0^0(z), \tag{5.1}$$

$$K^0(z) = i \sum_{k=0}^{n} P_k^{0\prime}(z) P_k^0(z). \tag{5.2}$$

The operator K^0 is bounded, analytic and we define the closed operator

$$H^1(z) = H(z) - \varepsilon K^0(z) \quad \text{on} \quad \mathcal{D}. \tag{5.3}$$

For ε small enough, the gap hypothesis **H2** holds for all $z \in B(z_0, r)$, and we set for ε small enough

$$\begin{aligned} P_j^1(z) &= -\frac{1}{2\pi i}\int_{\Gamma_j} (H^1(z) - \lambda)^{-1}\, d\lambda, \quad P_0^1 = \mathbb{I} - \sum_{j=1}^n P_j^1(z) && (5.4) \\ K^1(z) &= i\sum_{k=0}^n P_k^{1\prime}(z) P_k^1(z). && (5.5) \end{aligned}$$

Note that H^1, P_k^1, $k = 0, \cdots, n$, and K^1 are ε-dependent and strongly analytic in $B(z_0, r)$.

We define inductively, for ε small enough, the following hierachy of operators for $q \geq 1$

$$\begin{aligned} H^q(z) &= H(z) - \varepsilon K^{q-1}(z) && (5.6) \\ P_j^q(z) &= -\frac{1}{2\pi i}\int_{\Gamma_j} (H^q(z) - \lambda)^{-1}\, d\lambda, \quad P_0^q = \mathbb{I} - \sum_{j=1}^n P_j^q(z) && (5.7) \\ K^q(z) &= i\sum_{k=0}^n P_k^{q\prime}(z) P_k^q(z). && (5.8) \end{aligned}$$

It is proven among other things in [22], see also [23], that the following holds:

Proposition 5.1 *There exists $\varepsilon_0 > 0$, $b > 0$ and $g > 0$ such that for all $q \leq q^*(\varepsilon) \equiv [g/\varepsilon]$ and all $z \in B(z_0, r)$, $K^q(z)$ is analytic in S_a, and*

$$\|K^q(z) - K^{q-1}(z)\| \leq bq! \left(\frac{\varepsilon}{eg}\right)^q \tag{5.9}$$

$$\|K^q(z)\| \leq b. \tag{5.10}$$

Remarks:
i) As a corollary, for

$$q = q^*(\varepsilon) = [g/\varepsilon], \tag{5.11}$$

we get the exponential estimate

$$\|K^{q^*(\varepsilon)}(z) - K^{q^*(\varepsilon)-1}(z)\| \leq eb\, e^{-g/\varepsilon}. \tag{5.12}$$

ii) The values of ε_0 and g which determines the exponential decay above only depend on

$$\sup_{\substack{z \in B(z_0, r) \\ \lambda \in \cup_{j=1}^n \Gamma_j}} \|(H(z) - \lambda)^{-1}\| < \infty,$$

see [22] for explicit constants.
iii) Since S_a is compact, at the expense of decreasing the value of a, we can assume that proposition 5.1 holds for any $z \in S_a$, with uniform constants g, ε_0 and b.

Before we go on, let us recall a few facts from perturbation theory applied to our setting, that will be needed in the sequel.

Assume $q \leq q^*(\varepsilon)$ and let $\lambda \in \cup_{j=1}^n \Gamma_j \subset \rho(H(z_0))$ and $z \in B(z_0, r)$. We can write for $\varepsilon < \varepsilon_0$

$$\begin{aligned}(H^q(z)-\lambda)^{-1} &= (H(z)-\varepsilon K^{q-1}(z)-\lambda)^{-1} \\ &= (H(z)-\lambda)^{-1}+\varepsilon(H(z)-\lambda)^{-1}K^{q-1}(z)(H^q(z)-\lambda)^{-1} \\ &= (\mathbb{I}-\varepsilon(H(z)-\lambda)^{-1}K^{q-1}(z))^{-1}(H(z)-\lambda)^{-1}.\end{aligned} \tag{5.13}$$

Hence, for any $j = 1, \cdots, n$,

$$\begin{aligned}P_j^q(z) &= P_j(z) - \frac{\varepsilon}{2\pi i}\int_{\Gamma_j}(H(z)-\lambda)^{-1}K^{q-1}(z)(H^q(z)-\lambda)^{-1}\,d\lambda \\ &= P_j(z)-\varepsilon R_j^q(z)\end{aligned} \tag{5.14}$$

is analytic in z and the remainder is of order ε, together with all its derivatives. Moreover, making use of

$$(H(z)-\lambda)^{-1} = (H(z)-\lambda_0)^{-1}(\mathbb{I}-(\lambda-\lambda_0)(H(z)-\lambda)^{-1}) \tag{5.15}$$

for λ_0 in $\rho(H(z))$, we can write

$$H(z)P_j^q(z) = H(z)P_j(z)+\varepsilon F_j^q(z) \tag{5.16}$$

where $F_j^q(z)$ given by

$$H(z)(H(z)-\lambda_0)^{-1}\int_{\Gamma_j}(\mathbb{I}-(\lambda-\lambda_0)(H(z)-\lambda)^{-1})K^{q-1}(z)(H^q(z)-\lambda)^{-1}\frac{d\lambda}{2\pi i}. \tag{5.17}$$

The identity

$$H(z)(H(z)-\lambda_0)^{-1} = \mathbb{I}+\lambda_0(H(z)-\lambda_0)^{-1}, \tag{5.18}$$

shows that $F_j^q(z)$ is uniformly bounded as $\varepsilon \to 0$ and analytic.

As a consequence, we have

Lemma 5.1 *Let F_j^q be defined by (5.17). Then*

$$H^q(z)P_j^q(z) = H(z)P_j(z)+\varepsilon(F_j^q(z)-K^{q-1}(z)P_j^q(z)) \tag{5.19}$$
$$H^q(z)P_0^q(z) = H_0(z)+\varepsilon(F_0^q(z)-K^{q-1}(z)P_0^q(z)), \tag{5.20}$$

where $F_0^q(z) = -\sum_{j=1}^n F_j^q(z)$.

6 The Approximation

Let $q \leq q^*(\varepsilon)$ and consider V^q, defined as the solution to

$$\begin{aligned}&i\varepsilon\partial_t V^q(t,s)\varphi = (H^q(t)+\varepsilon K^q(t))V^q(t,s)\varphi, \\ &\varphi \in D, \quad V^q(s,s)=\mathbb{I}, \quad 0 \leq s \leq t \leq 1.\end{aligned} \tag{6.1}$$

As $H^q = H - \varepsilon K^{q-1}$ we get that

$$H^q(t)+\varepsilon K^q(t) = H_0(t)+\sum_{j=1}^n P_j(t)H(t)P_j(t)+\varepsilon(K^q(t)-K^{q-1}(t)) \tag{6.2}$$

is a bounded, smooth perturbation of $H_0(t)$. The results of [27] guarantee the existence and uniqueness of the solution to (6.1). Moreover, as is well known [26], [27], V^q further satisfies

$$V^q(t,s)P_k^q(s) = P_k^q(t)V^q(t,s), \quad \forall k = 0, \cdots, n, \quad 0 \le s \le t \le 1. \tag{6.3}$$

In order to show by means of (3.7) that V^q, with $q = q^*(\varepsilon)$, is a good approximation of U, we need to control the behaviour of the norm of V^q as $\varepsilon \to 0$. We split V^q into components within the spectral subspaces of P_k^q. Set

$$V_k^q(t,s) = V^q(t,s)P_k^q(s) \quad \text{s.t.} \quad V^q(t,s) = \sum_{k=1}^{n} V_k^q(t,s). \tag{6.4}$$

Since the projectors $\{P_k^q(s)\}_{k=0,\cdots,n}$ have norms uniformy bounded from above and below in $s \in [0,1]$ and $\varepsilon > 0$, there exists a positive constant γ such that

$$\gamma^{-1} \max_{k=0,\cdots,n} \|V_k^q(t,s)\| \le \|V^q(t,s)\| \le \gamma \max_{k=0,\cdots,n} \|V_k^q(t,s)\|. \tag{6.5}$$

We have,

Proposition 6.1 *There exist constants $C_k > 0$, $k = 0, 1, \cdots, n$, $d_j \ge 0$ and $0 < \beta_j < 1$, $j = 1, \cdots, n$ such that for all $\varepsilon < \varepsilon_0$, and all $q \le q^*(\varepsilon)$,*

$$\|V_0^q(t,s)\| = \|V^q(t,s)P_0^q(s)\| \le C_0 e^{\Im \int_s^t \lambda_0(u)du/\varepsilon} \tag{6.6}$$

$$\|V_j^q(t,s)\| = \|V^q(t,s)P_j^q(s)\| \le C_j e^{d_j/\varepsilon^{\beta_j}} e^{\Im \int_s^t \lambda_j(u)du/\varepsilon}. \tag{6.7}$$

Moreover, (6.7) holds with $d_j = 0$ if and only if $D_j(t) \equiv 0$ in (1.2).

Proof of Proposition 6.1:
We first consider $V_0^q(t,s)$, the part of V^q corresponding to the infinite dimensional subspace P_0^q. Because of (6.3), it satisfies for $0 \le s \le t \le 1$ and any $\varphi \in D$

$$i\varepsilon\partial_t V_0^q(t,s)\varphi = ((H^q(t) + \varepsilon K^q(t))P_0^q(t))V_0^q(t,s)\varphi, \quad V_0^q(s,s) = P_0^q(s). \tag{6.8}$$

Lemma 5.1 shows that the generator of $V_0^q(t,s)$ is equal to $H_0(t)$ plus a smooth bounded perturbation of order ε. We can thus compare $V_0^q(t,s)$ and $U(t,s)P_0^q(s)$ by means of (3.7). The fact that the initial condition is $P_0^q(s)$ instead of the identity simply multiplies the estimate by $\|P_0^q(s)\|$, so that we get

$$\|V^q(t,s)P_0^q(s)\| \le \|P_0^q(s)\| e^{\Im \int_s^t \lambda_0(u)du/\varepsilon} C_0' \le e^{\Im \int_s^t \lambda_0(u)du/\varepsilon} C_0, \tag{6.9}$$

where C_0' and $C_0 = C_0' \sup_{\substack{s\in[0,1] \\ \varepsilon>0}} \|P_0^q(s)\|$ are uniform in ε.

The control of the remaining components is conveniently done by taking advantage of the intertwining relation (6.3) as follows.

Let W^q be the bounded operator satisfying the equation

$$iW^{q\prime}(t) = K^q(t)W^q(t), \quad W^q(0) = \mathbb{I}. \tag{6.10}$$

This operator enjoys a certain number of properties. As K^q is smooth and bounded, the solution is given by a convergent Dyson series, and $W^q(t)$ interwines between $P_k^q(0)$ and $P_k^q(t)$.

Morerover, W^q and its inverse map D into D, see [21]. Finally, by regular perturbation theory and Proposition 5.1, $K^q = K^0 + O(\varepsilon)$ so that

$$\sup_{\substack{t\in[0,1]\\ 0<\varepsilon<1}} \|W^q(t)^{\pm 1}\| < \infty. \tag{6.11}$$

Therefore, the bounded operator defined by

$$\Phi^q(t,s) = W^q(t)^{-1}V^q(t,s)W^q(s), \quad 0 \le s \le t \le 1 \tag{6.12}$$

satisfies by construction

$$[\Phi^q(t,s), P_k^q(0)] \equiv 0, \quad \forall\, k = 0,\cdots,n \quad \forall\, 0 \le s \le t \le 1. \tag{6.13}$$

We can thus view

$$\begin{aligned}\Phi_j^q(t,s) &= \Phi^q(t,s)P_j^q(0), \quad j = 1,\cdots,n, & (6.14)\\ \Phi_0^q(t,s) &= \Phi^q(t,s)P_0^q(0) & (6.15)\end{aligned}$$

as operators in the finite dimensional Banach spaces $P_j(0)\mathcal{B}$, for $j \ge 1$ and in the infinite dimensional Banach space $P_0(0)\mathcal{B}$. Moreover, thanks to (6.11), there exists a constant C such that, uniformly in $0 \le s \le t \le 1$ and $\varepsilon > 0$,

$$C^{-1}\|V_k^q(t,s)\| \le \|\Phi_k^q(t,s)\| \le C\|V_k^q(t,s)\|, \quad k = 0,\cdots,n. \tag{6.16}$$

The operator $\Phi_j^q(t,s)$ satisfies for any $\varphi \in D$

$$\begin{aligned} i\varepsilon\partial_t\Phi_j^q(t,s)\varphi &= W^q(t)^{-1}H^q(t)V^q(t,s)W^q(s)P_j^q(0)\Phi_j^q(t,s)\\ &= P_j^q(0)W^q(t)^{-1}H^q(t)P_j^q(t)W^q(t))P_j^q(0)\Phi_j^q(t,s)\varphi\\ &\equiv \widetilde{H}_j^q(t)\Phi_j^q(t,s)\varphi, \end{aligned} \tag{6.17}$$

where the generator $\widetilde{H}_j^q(t)$ is bounded, see Lemma 5.1. In a sense, $\Phi_j^q(t,s)$ describes the evolution within the spectral subspaces. Let us further compute with $P_j^q = (P_j^q)^2$ and (5.14)

$$\begin{aligned} H^q(t)P_j^q(t) &= P_j^q(t)(H(t)P_j(t) + \varepsilon(F_j^q(t) - K^{q-1}(t))P_j^q(t) & (6.18)\\ &= P_j^q(t)(\lambda_j(t)P_j(t) + D_j(t) + \varepsilon(F_j^q(t) - K^{q-1}(t))P_j^q(t)\\ &= \lambda_j(t)P_j^q(t) + P_j^q(t)D_j(t)P_j^q(t)\\ &\quad +\varepsilon P_j^q(t)(\lambda_j(t)R_j^q(t) + F_j^q(t) - K^{q-1}(t))P_j^q(t)\\ &\equiv P_j^q(t)(\lambda_j(t) + D_j(t))P_j^q(t) + \varepsilon J_j^q(t). \end{aligned}$$

The last term is bounded, analytic in t and of order ε. We will deal with it perturbatively.

Equations (6.18) suggests to decompose $\Phi_j^q(t,s)$, $j = 1,\cdots,n$, as

$$\Phi_j^q(t,s) = e^{-i\int_s^t \lambda_j(u)\,du/\varepsilon}\Psi_j^q(t,s) \tag{6.19}$$

where $\Psi_j^q(t,s) : P_j^q(0)\mathcal{B} \to P_j^q(0)\mathcal{B}$ satisfies

$$\begin{aligned} i\varepsilon\partial_t\Psi_j^q(t,s) &= P_j^q(0)(W^q(t)^{-1}(D_j(t) + \varepsilon J_j^q(t))W^q(t)P_j^q(0)\,\Psi_j^q(t,s), & (6.20)\\ \Psi_j^q(s,s) &= P_j^q(0), \end{aligned}$$

where, in the leading part of the generator,

$$\widetilde{D}_j(t) = W^q(t)^{-1}D_j(t)W^q(t) \tag{6.21}$$

is analytic and nilpotent with $\widetilde{D}_j(t)^{m_j} = 0$, with $m_j = \dim P_j(t)$. However, the restriction of $\widetilde{D}_j(t)$ to $P_j^q(0)\mathcal{B}$, $P_j^q(0)\widetilde{D}_j(t)P_j^q(0)$, is not nilpotent. Nevertheless, $\Psi_j^q(t,s)$ satisfies the same type of estimates an evolution generated by a perturbed analytic nilponent does:

Lemma 6.1 *Let $\Psi_j^q(t,s)$ be defined by (6.19), for $j = 1, \cdots, n$. Then, there exist $0 < \beta_j < 1$ and $d_j \geq 0$, $c_j > 0$ such that*

$$\|\Psi_j^q(t,s)\| \leq c_j e^{d_j/\varepsilon^{\beta_j}}. \tag{6.22}$$

Moreover, the estimate holds with $d_j = 0$ if and only if $D_j(t) \equiv 0$ in (1.2).

Proof of Lemma 6.1: Equations (5.14) and (1.3) allow to get rid of the projectors $P_j^q(0)$ in (6.20) up to an error of order ε,

$$P_j^q(0)\widetilde{D}_j(t)P_j^q(0) = W^q(t)^{-1}P_j^q(t)D_j(t)P_j^q(t)W^q(t) = \widetilde{D}_j(t) + \varepsilon L_j^q(t), \tag{6.23}$$

where

$$\begin{aligned} L_j^q(t) &= -W^q(t)^{-1}\Big(R_j^q(t)D_j(t)P_j(t) + P_j(t)D_j(t)R_j^q(t) \\ &\quad - \varepsilon R_j^q(t)D_j(t)R_j^q(t)\Big)W^q(t) \end{aligned} \tag{6.24}$$

is analytic and of order ε^0. Since $W^q(t)^{\pm 1}$ is analytic and uniformly bounded, the nilpotent $\widetilde{D}_j(t)$ satisfies **N1** uniformly in $\varepsilon > 0$, and (6.20) and (6.24) show that the generator of $\Psi_j^q(t,s)$ satisfies the hypotheses of Proposition 4.1, which yields the estimate. The last statement stems from Lemma 4.1 . ■

It remains to gather (6.16), (6.19) and Lemma 6.1 to end the proof of Proposition 6.1. ■

6.1 End of the Proof

Given Proposition 6.1, we can finish the proof of our main statement as follows.

Applying (3.4) to U and V^q, we get

$$U(t,r) = V^q(t,r) + i\int_r^t V^q(t,s)(K^q(s) - K^{q-1}(s))U(s,r)\,ds. \tag{6.25}$$

Let $t \mapsto \omega(t)$ be the continuous function defined by

$$\omega(t) = \max_{k=0,\cdots,n} \Im\lambda_k(t). \tag{6.26}$$

Applying (6.25) to $P_k^q(r)$ and multiplication by $e^{-\int_r^t \omega(s)\,ds/\varepsilon}$ gives with (6.4)

$$\begin{aligned} \|e^{-\int_r^t \omega(u)\,du/\varepsilon}(U(t,r) - V^q(t,r))P_k^q(r)\| &\leq \int_r^t \|e^{-\int_s^t \omega(u)\,du/\varepsilon}V^q(t,s)(K^q(s) - K^{q-1}(s))\| \\ \times \Big(\|e^{-\int_r^s \omega(u)\,du/\varepsilon}(U(s,r) - V^q(s,r))P_k^q(r)\| &+ \|e^{-\int_r^s \omega(u)\,du/\varepsilon}V_k^q(s,r)\|\Big)\,ds. \end{aligned} \tag{6.27}$$

Proposition 6.1 and the definition of $\omega(t)$ yield for any $0 \leq r \leq s \leq 1$

$$\|e^{-\int_r^s \omega(u)\,du/\varepsilon}V_k^q(s,r)\| \leq C_k e^{d_k/\varepsilon^{\beta_k}}, \quad \text{(with } d_0 = 0). \tag{6.28}$$

Further taking $q = q^*(\varepsilon)$, (5.12), (6.5) show the existence of constants $B > 0$ and $0 < \kappa < g$ such that

$$\|e^{-\int_s^t \omega(u)\,du/\varepsilon}V^{q^*(\varepsilon)}(t,s)(K^{q^*(\varepsilon)}(s) - K^{q^*(\varepsilon)-1}(s))\| \leq ebCe^{D/\varepsilon^\beta}e^{-g/\varepsilon} \leq Be^{-\kappa/\varepsilon} \tag{6.29}$$

Hence, we get using $0 \leq t-s \leq 1$,

$$\begin{aligned} &\|e^{-\int_r^t \omega(u)\,du/\varepsilon}(U(t,r)-V^{q^*(\varepsilon)}(t,r))P_k^{q^*(\varepsilon)}(r)\| \\ &\qquad \leq Be^{-\kappa/\varepsilon}\int_r^t \|e^{-\int_r^s \omega(u)\,du/\varepsilon}V_k^{q^*(\varepsilon)}(s,r)\|\,ds \\ &\qquad +Be^{-\kappa/\varepsilon}\sup_{r\leq s\leq t}\|e^{-\int_r^s \omega(u)\,du/\varepsilon}(U(s,r)-V^{q^*(\varepsilon)}(s,r))P_k^{q^*(\varepsilon)}(r)\|, \end{aligned} \tag{6.30}$$

from which we deduce that if ε is so small that $Be^{-\kappa/\varepsilon} < 1/2$,

$$\begin{aligned} &\sup_{r\leq s\leq t}\|e^{-\int_r^s \omega(u)\,du/\varepsilon}(U(s,r)-V^{q^*(\varepsilon)}(s,r))P_k^{q^*(\varepsilon)}(r)\| \\ &\qquad \leq 2Be^{-\kappa/\varepsilon}(t-r)\sup_{r\leq s\leq t}\|e^{-\int_r^s \omega(u)\,du/\varepsilon}V_k^{q^*(\varepsilon)}(s,r)\|. \end{aligned} \tag{6.31}$$

In particular, our main result follows. For ε small enough, for any $0 \leq r \leq t \leq 1$, and for all $k = 0, \cdots, n$,

$$\begin{aligned} e^{-\int_r^t \omega(u)\,du/\varepsilon}U(t,s)P_k^{q^*(\varepsilon)}(r) &= e^{-\int_r^t \omega(u)\,du/\varepsilon}V_k^{q^*(\varepsilon)}(t,r) \\ &\quad +O((t-r)e^{-\kappa/\varepsilon}\sup_{r\leq s\leq t}\|e^{-\int_r^s \omega(u)\,du/\varepsilon}V_k^{q^*(\varepsilon)}(s,r)\|). \end{aligned} \tag{6.32}$$

We chose to estimate the difference $U - V^{q^*(\varepsilon)}$ applied on the projectors, because the norms of the different components $V_k^{q^*(\varepsilon)}$ vary with k. Of course, (6.32) also holds with $P_k^{q^*(\varepsilon)}$ removed and $V^{q^*(\varepsilon)}$ in place of $V_k^{q^*(\varepsilon)}$.

Making further use of (6.28) in the error term of (6.32), we get (lowering the value of $0 < \kappa < g$)

$$U(t,r) = V^{q^*(\varepsilon)}(t,r) + O((t-r)e^{-\kappa/\varepsilon}e^{\int_r^t \omega(u)\,du/\varepsilon}), \tag{6.33}$$

where $V^{q^*(\varepsilon)}(t,r) = O(e^{\int_r^t \omega(u)\,du/\varepsilon}e^{D/\varepsilon^\beta})$, for some $0 < \beta < 1$, and $D \geq 0$. ∎

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