SIAM J. MATH. ANAL.
Vol. 28, No. 3, pp. 669–703, May 1997

010

EXPONENTIAL ASYMPTOTICS IN A SINGULAR LIMIT FOR n-LEVEL SCATTERING SYSTEMS*

ALAIN JOYE†

Abstract. The singular limit $\varepsilon \to 0$ of the S-matrix associated with the equation $i\varepsilon d\psi(t)/dt = H(t)\psi(t)$ is considered, where the analytic generator $H(t) \in M_n(\mathbf{C})$ is such that its spectrum is real and nondegenerate for all $t \in \mathbf{R}$. Sufficient conditions allowing us to compute asymptotic formulas for the exponentially small off-diagonal elements of the S-matrix as $\varepsilon \to 0$ are made explicit and a wide class of generators for which these conditions are verified is defined. These generators are obtained by means of generators whose spectrum exhibits eigenvalue crossings which are perturbed in such a way that these crossings turn into avoided crossings. The exponentially small asymptotic formulas which are derived are shown to be valid up to exponentially small relative error by means of a joint application of the complex Wentzel–Kramers–Brillouin (WKB) method together with superasymptotic renormalization. This paper concludes with the application of these results to the study of quantum adiabatic transitions in the time-dependent Schrödinger equation and of the semiclassical scattering properties of the multichannel stationary Schrödinger equation. The results presented here are a generalization to n-level systems, $n \geq 2$, of results previously known for two-level systems only.

Key words. singular perturbations, semiclassical and adiabatic approximations, n-level S-matrix, turning-point theory

AMS subject classifications. 34E20, 34L25, 81Q20

PII. S0036141095288847

1. Introduction. Several problems of mathematical physics lead to the study of the scattering properties of linear ordinary differential equations in a singular limit

$$i\varepsilon\psi'(t) = H(t)\psi(t), \quad t \in \mathbf{R}, \quad \varepsilon \to 0, \tag{1.1}$$

where the prime denotes the derivative with respect to t, $\psi(t) \in \mathbf{C}^n$, and $H(t) \in M_n(\mathbf{C})$ for all t. A system described by such an equation will be called an n-level system. Let us mention, for example, the study of the adiabatic limit of the time-dependent Schrödinger equation or the semiclassical limit of the one-dimensional multichannel stationary Schrödinger equation at energies above the potential barriers, to which we will return below. When the generator $H(t)$ is well behaved at $+\infty$ and $-\infty$, the scattering properties of the problem can be described by means of a matrix naturally associated with equation (1.1), the so-called S-matrix. This matrix relates the behavior of the solution $\psi(t)$ as $t \to -\infty$ to that of $\psi(t)$ as $t \to +\infty$. Assuming that the spectrum $\sigma(t)$ of $H(t)$ is real and nondegenerate,

$$\sigma(t) = \{e_1(t) < e_2(t) < \cdots < e_n(t)\} \in \mathbf{R}, \tag{1.2}$$

the S-matrix is essentially given by the identity matrix

$$S = \mathrm{diag}(s_{11}(\varepsilon), s_{22}(\varepsilon), \ldots, s_{nn}(\varepsilon)) + \mathcal{O}(\varepsilon^\infty), \quad \text{where } s_{jj}(\varepsilon) = 1 + \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0, \tag{1.3}$$

* Received by the editors July 10, 1995; accepted for publication (in revised form) February 16, 1996.

http://www.siam.org/journals/sima/28-3/28884.html

† Centre de Physique Théorique, CNRS Marseille, Luminy Case 907, F-13288 Marseille cedex 9, France and Phymat, Université de Toulon et du Var, B.P. 132, 83957 La Garde cedex, France (joye@cpt.univ-mrs.fr).

provided $H(t)$ is C^∞; see, e.g., [F1], [F2], and [W]. Moreover, if $H(t)$ is assumed to be analytic, it was proven in various situations that the off-diagonal elements s_{jk} of S are exponentially decreasing [FF], [W], [F1], [F2], [JKP], [JP4]:

$$s_{jk} = \mathcal{O}\left(\mathrm{e}^{-\kappa/\varepsilon}\right), \quad \forall j \neq k, \tag{1.4}$$

as $\varepsilon \to 0$. See also [JP1], [N], [M], and [Sj] for corresponding results in infinite-dimensional spaces. Since the physical information is often contained in these off-diagonal elements, it is of interest to be able to give an asymptotic formula for s_{jk} rather than a mere estimate.

For two-level systems (or systems reducible to this case (see [JP2], [J], and [MN])), the situation is now reasonably well understood, at least under generic circumstances. Indeed, a rigorous study of the S-matrix associated with (1.1) when $n = 2$ under the hypotheses loosely stated above is provided in the recent paper [JP4]. The treatment presented unifies, in particular, earlier results obtained for either the time-dependent adiabatic Schrödinger equation (see, e.g., [JP3] and the references therein) or the study of the above barrier reflexion in the semiclassical limit (see, e.g., [FF] and [O]). Further references are provided in [JP4]. As an intermediate result, the asymptotic formula

$$s_{jk} = g_{jk}\mathrm{e}^{-\Gamma_{jk}/\varepsilon}\,(1 + \mathcal{O}(\varepsilon)), \quad \varepsilon \to 0, \tag{1.5}$$

for $j \neq k \in \{1, 2\}$ with $g_{jk} \in \mathbf{C}$ and $\mathrm{Re}\Gamma_{jk} > 0$ is proven in [JP4]. As is well known, to get an asymptotic formula for s_{jk}, one has to consider (1.1) in the complex plane, in particular in the vicinity of the degeneracy points of the analytic continuations of eigenvalues $e_1(z)$ and $e_2(z)$. Provided the level lines of the multivalued function

$$\mathrm{Im}\int_0^z e_1(z') - e_2(z')dz' = \mathrm{cst}, \tag{1.6}$$

called Stokes lines, naturally associated with (1.1) behave properly in the complex plane, the so-called complex Wentzel–Kramers–Brillouin (WKB) method allows to prove (1.5). More importantly, however, it is also shown in [JP4] how to improve (1.5) to an asymptotic formula accurate up to an exponentially small relative error:

$$s_{jk} = g^*_{jk}(\varepsilon)\mathrm{e}^{-\Gamma^*_{jk}(\varepsilon)/\varepsilon}(1 + \mathcal{O}(\mathrm{e}^{-\kappa/\varepsilon})), \quad \varepsilon \to 0, \tag{1.7}$$

with $g^*_{jk}(\varepsilon) = g_{jk} + \mathcal{O}(\varepsilon)$ and $\Gamma^*_{jk}(\varepsilon) = \Gamma_{jk} + \mathcal{O}(\varepsilon^2)$. This is achieved by using a complex WKB analysis jointly with the recently developed superasymptotic theory [Be], [N], [JP2]. Note that when given a generator, the principal difficulty in justifying formulas (1.5) and (1.7) is the verification that the corresponding Stokes lines (1.6) display the proper behavior *globally* in the complex plane, which may or may not be the case [JKP]. However, this condition is always satisfied when the complex eigenvalue degeneracy is close to the real axis, as shown in [J]. See also [MN] and [R] for recent related results.

For n-level systems, with $n \geq 3$, the situation is by no means as well understood. There are some results obtained with particular generators. In [D], [CH1], [CH2], and [BE], certain elements of the S-matrix are computed if $H(t) = H^*(t)$ depends linearly on t, $H(t) = A + tB$ for some particular matrices A and B. The choices of A and B are such that all components of the solution $\psi(t)$ can be deduced from the first one and an exact integral representation of this first component can be obtained. The integral representation is analyzed by standard asymptotic techniques, and this leads to results which are valid for any $\varepsilon > 0$, as in the case for the classical Landau–Zener generator. The study of the three-level problem when $H(t) = H^*(t) \in M_3(\mathbf{R})$ is tackled in the closing section of the very interesting paper [HP]. A nonrigorous and essentially local discussion of the behavior of the level lines of $\text{Im} \int_0^z e_j(z') - e_k(z')dz'$, $j \neq k = 1, 2, 3$, is provided, and it justifies in very favorable cases an asymptotic formula for some elements of the S-matrix. See also the review [So], where a nonrigorous study of (1.1) is made close to a complex degeneracy point of a group of eigenvalues by means of an exact solution to a model equation. However, no asymptotic formula for s_{jk}, $j \neq k$, can be found in the literature for general n-level systems, $n \geq 3$. This is due to the fact that the direct generalization of the method used successfully for two-level systems may lead to seemingly inextricable difficulties for $n = 3$. Indeed, with three eigenvalues, one has to consider three sets of level lines $\text{Im} \int_0^z e_j(z') - e_k(z')dz'$ to deal with (1.1) in the complex plane, and the conditions that they have to fulfill in order for the limit $\varepsilon \to 0$ to be controlled may be incompatible for a given generator; see [F1], [F2], and [HP]. It should be mentioned, however, that there are specific examples in which this difficult problem can be mastered. Such a result was recently obtained in the semiclassical study [Ba] of a particular problem of resonances for which similar considerations in the complex plane are required.

The goal of this paper is to provide some general insight into the asymptotic computation of the S-matrix associated with n-level systems, $n \geq 3$, based on a generalization of the techniques which proved to be successful for two-level systems. The content of this paper is twofold. On one hand, we set up a general framework in which asymptotic formulas for the exponentially small off-diagonal coefficients can be proven. On the other hand, we actually prove such formulas for a wide class of n-level systems. In the first part of the paper, we give our definition of the S-matrix associated with equation (1.1) and make explicit the symmetries it inherits from the symmetries of $H(t)$ for $t \in \mathbf{R}$ (Proposition 2.1). We then turn to the determination of the analyticity properties of the eigenvalues and eigenvectors of $H(z)$, $z \in \mathbf{C}$, which are at the root of the asymptotic formulas that we derive later (Lemma 3.1). The next step is the formulation of sufficient conditions adapted to the scattering situation that we consider, under which a complex WKB analysis allows us to prove a formula like (1.5) (Proposition 4.1). The conditions stated are similar but not identical to those given in [JKP] or [HP]. As a final step, we show how to improve the asymptotic formula (1.5) to (1.7) by means of superasymptotic machinery (Proposition 5.2 and Lemma 5.2). We then turn to the second part of the paper, where we show that a wide class of generators fits into our framework and satisfies our conditions. These generators are obtained by perturbation of generators whose eigenvalues display degeneracies on the real axis (in the spirit of [J]). We prove that for these generators, in the absence of any symmetry of the generator $H(t)$, at least one element per column in the S-matrix can be asymptotically computed (Theorem 6.1). This is the main technical section of the paper. The major advantage of this construction is that it is sufficient to look at the behavior of the eigenvalues on the real axis to check if the conditions are satisfied. The closing section contains an application of our general results to the study of quantum adiabatic transitions in the time-dependent Schrödinger equation and of the semiclassical scattering properties of the multichannel stationary Schrödinger equation. In particular, we make explicit use of the symmetries of the S-matrix to increase the number of its elements for which an asymptotic formula holds. In the latter application, further specific symmetry properties of the S-matrix are derived (Lemma 7.1).

2. Definition and properties of the *S*-matrix. We consider the evolution equation

$$i\varepsilon\psi'(t) = H(t)\psi(t), \quad t \in \mathbf{R}, \quad \varepsilon \to 0, \tag{2.1}$$

where the prime denotes the derivative with respect to t, $\psi(t) \in \mathbf{C}^n$, and $H(t) \in M_n(\mathbf{C})$ for all t. We make some assumptions on the generator $H(t)$. The first is the usual analyticity condition in this context.

H1. *There exists a strip*

$$S_\alpha = \{z \in \mathbf{C} | \, |\mathrm{Im} z| \leq \alpha\}, \quad \alpha > 0, \tag{2.2}$$

such that $H(z)$ is analytic for all $z \in S_\alpha$.

Since we are studying scattering properties, we need sufficient decay at infinity.

H2. *There exist two nondegenerate matrices $H(+), H(-) \in M_n(\mathbf{C})$ and $a > 0$ such that*

$$\lim_{t \to \pm\infty} |t|^{1+a} \sup_{|s| \leq \alpha} \|H(t+is) - H(\pm)\| < \infty. \tag{2.3}$$

We finally give a condition which has to do with the physics behind the problem.

H3. *For $t \in \mathbf{R}$, the spectrum of $H(t)$, denoted by $\sigma(t)$, is real and nondegenerate*

$$\sigma(t) = \{e_1(t) < e_2(t) < \cdots < e_n(t)\} \subset \mathbf{R}, \tag{2.4}$$

and there exists $g > 0$ such that

$$\inf_{\substack{j \neq k \\ t \in \mathbf{R}}} |e_j(t) - e_k(t)| \geq g. \tag{2.5}$$

As a consequence of H3, for each $t \in \mathbf{R}$, there exists a complete set of projectors $P_j(t) = P_j^2(t) \in M_n(\mathbf{C})$, $j = 1, 2, \ldots, n$, such that

$$\sum_{j=1}^{n} P_j(t) \equiv \mathbf{I}, \tag{2.6}$$

$$H(t) = \sum_{j=1}^{n} e_j(t) P_j(t), \tag{2.7}$$

and there exists a basis of $\mathbf{C}^n$ of eigenvectors of $H(t)$. We determine these eigenvectors $\varphi_j(t)$, $j = 1, 2, \ldots, n$, uniquely (up to a constant) by requiring them to satisfy

$$H(t)\varphi_j(t) = e_j(t)\varphi_j(t), \tag{2.8}$$

$$P_j(t)\varphi_j'(t) \equiv 0, \quad j = 1, 2, \ldots, n. \tag{2.9}$$

Explicitly, if $\psi_j(t)$, $j = 1, 2, \ldots, n$, form a complete set of differentiable eigenvectors of $H(t)$, the eigenvectors

$$\varphi_j(t) = \mathrm{e}^{-\int_0^t \xi_j(t')dt'} \psi_j(t) \quad \text{s.t. } \varphi_j(0) = \psi_j(0) \tag{2.10}$$

with

$$\xi_j(t) = \frac{\langle \psi_j(t) | P_j(t) \psi_j'(t) \rangle}{\|\psi_j(t)\|^2}, \quad j = 1, \ldots, n, \tag{2.11}$$

verify (2.9). The fact that this choice leads to an analytic set of eigenvectors close to the real axis will be proven below. We expand the solution $\psi(t)$ along the basis just constructed, thus defining the unknown coefficients $c_j(t)$, $j = 1, 2, \ldots, n$, to be determined,

$$\psi(t) = \sum_{j=1}^{n} c_j(t) \mathrm{e}^{-i\int_0^t e_j(t')dt'/\varepsilon} \varphi_j(t). \tag{2.12}$$

The phases $\mathrm{e}^{-i\int_0^t e_j(t')dt'/\varepsilon}$ (see H3) are introduced for convenience. By inserting (2.12) into (2.1), we get the following differential equation for the $c_j(t)$'s:

$$c_j'(t) = \sum_{k=1}^{n} a_{jk}(t) \mathrm{e}^{i\Delta_{jk}(t)/\varepsilon} c_k(t), \tag{2.13}$$

where

$$\Delta_{jk}(t) = \int_0^t (e_j(t') - e_k(t'))dt' \tag{2.14}$$

and

$$a_{jk}(t) = -\frac{\langle \varphi_j(t) | P_j(t) \varphi_k'(t) \rangle}{\|\varphi_j(t)\|^2}. \tag{2.15}$$

Here $\langle \cdot | \cdot \rangle$ denotes the usual scalar product in $\mathbf{C}^n$. Our choice (2.9) implies $a_{jj}(t) \equiv 0$. It is also shown below that the $a_{jk}(t)$'s are analytic functions in a neighborhood of the real axis and that hypothesis H2 implies that they satisfy the estimate

$$\lim_{t \to \pm\infty} \sup_{j \neq k} |t|^{1+a} \, |a_{jk}(t)| < \infty. \tag{2.16}$$

As a consequence of this last property and of the fact that the eigenvalues are real by assumption, the following limits exist:

$$\lim_{t \to \pm\infty} c_j(t) = c_j(\pm\infty). \tag{2.17}$$

We are now able to define the associated S-matrix, $S \in M_n(\mathbf{C})$, by the identity

$$S \begin{pmatrix} c_1(-\infty) \\ c_2(-\infty) \\ \vdots \\ c_n(-\infty) \end{pmatrix} = \begin{pmatrix} c_1(+\infty) \\ c_2(+\infty) \\ \vdots \\ c_n(+\infty) \end{pmatrix}. \tag{2.18}$$

Such a relation makes sense because of the linearity of equation (2.13). It is a well-known result that under our general hypotheses, the S-matrix satisfies

$$S = \mathbf{I} + \mathcal{O}(\varepsilon). \tag{2.19}$$

Note that the jth column of the S-matrix is given by the solution of (2.13) at $t = \infty$ subjected to the initial conditions $c_k(-\infty) = \delta_{jk}$, $k = 1, 2, \ldots, n$.

In general, the S-matrix defined above has no particular properties besides that of being invertible. However, when the generator $H(t)$ satisfies some symmetry properties, the same is true for S. Since such properties are important in applications, we show below that if $H(t)$ is self-adjoint with respect to some indefinite scalar product, then S is unitary with respect to another indefinite scalar product. Let $J \in M_n(\mathbf{C})$ be an invertible self-adjoint matrix. We define an indefinite metric on $\mathbf{C}^n$ by means of the indefinite scalar product

$$(\cdot, \cdot)_J = \langle \cdot | J \cdot \rangle. \tag{2.20}$$

It is easy to check that the adjoint $A^{\#}$ of a matrix A with respect to the $(\cdot, \cdot)_J$ scalar product is given by

$$A^{\#} = J^{-1} A^* J. \tag{2.21}$$

PROPOSITION 2.1. *Let $H(t)$ satisfy* H1 *and* H2 *and possess n distinct eigenvalues $\forall t \in \mathbf{R}$. Furthermore, assume that $H(t)$ is self-adjoint with respect to the scalar product $(\cdot, \cdot)_J$,*

$$H(t) = H^{\#}(t) = J^{-1} H^*(t) J, \quad \forall t \in \mathbf{R}, \tag{2.22}$$

and the eigenvectors $\varphi_j(0)$ of $H(0)$ satisfy

$$(\varphi_j(0), \varphi_j(0))_J = \rho_j, \quad \rho_j \in \{-1, 1\}, \quad \forall j = 1, \ldots, n. \tag{2.23}$$

Then the eigenvalues of $H(t)$ are real $\forall t \in \mathbf{R}$ and the S-matrix is unitary with respect to the scalar product $(\cdot, \cdot)_R$, where $R = R^* = R^{-1}$ is the real diagonal matrix $R =$ $\text{diag}(\rho_1, \rho_2, \ldots, \rho_n)$,

$$S^{\#} = R S^* R = S^{-1}. \tag{2.24}$$

Remark. The condition $(\varphi_j(0), \varphi_j(0))_J = \pm 1$ can always be satisfied by suitable renormalization provided $(\varphi_j(0), \varphi_j(0))_J \neq 0$.

The main interest of this proposition is that when the S-matrix possesses symmetries, some of its elements can be deduced from resulting identities without resorting to their actual computations.

A simple proof of Proposition 2.1 that makes use of notions discussed in the next section can be found in Appendix A. Proposition 2.1 can actually be used for the two main applications that we deal with in section 7. Note that in specific cases, further symmetry properties can be derived for the S-matrix; see section 7.

3. Analyticity properties. The generator $H(z)$ is analytic in S_α; hence the solution of the linear equation (2.1) $\psi(z)$ is analytic in S_α as well. However, the eigenvalues and eigenprojectors of $H(z)$ may have singularities in S_α. Let us recall some basic properties, the proofs of which can be found in [K]. The eigenvalues and eigenprojectors of a matrix analytic in a region of the complex plane have analytic continuations in that region with possible singularities located at points z_0, called exceptional points. In a neighborhood free of exceptional points, the eigenvalues are given by branches of analytic functions and their multiplicities are constant. One eigenvalue can therefore be analytically continued until it coincides at z_0 with one or several other eigenvalues. The set of such points defines the set of exceptional points. The eigenvalues may possess branching points at an exceptional z_0, where they are continuous, whereas the eigenprojectors are also multivalued but diverge as $z \to z_0$. Hence by hypothesis H3, the n distinct eigenvalues $e_j(t)$ defined on the real axis are

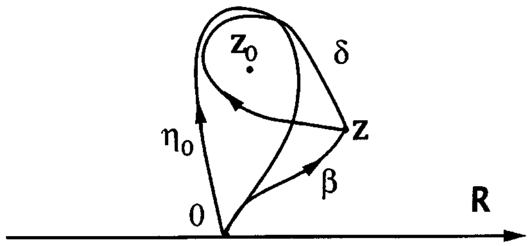

FIG. 1. *The paths β, δ, and η_0 in $S_\alpha \backslash \Omega$.*

analytic on the real axis and possess multivalued analytic continuations in S_α, with possible branching points at the set of degeneracies Ω, given by

$$\Omega = \{z_0 | \; e_j(z_0) = e_k(z_0) \quad \text{for some } k \text{ and } j \text{ and some analytic continuation}\}. \tag{3.1}$$

By assumption H2, Ω is finite, and by H3, $\Omega \cap \mathbf{R} = \emptyset$ and $\Omega = \overline{\Omega}$ due to Schwarz's principle. Similarly, the eigenprojectors $P_j(t)$ defined on the real axis are analytic on the real axis and possess multivalued analytic continuations in S_α with possible singularities at Ω. To see more precisely what happens to these multivalued functions when we turn around a point $z_0 \in \Omega$, we consider the construction described in Figure 1. Let f be a multivalued analytic function in $S_\alpha \backslash \Omega$. We denote by $f(z)$ the analytic continuation of the restriction of f around 0 along some path $\beta \in S_\alpha \backslash \Omega$ from 0 to z. Then we perform the analytic continuation of $f(z)$ along a negatively oriented loop δ based at z around a unique point $z_0 \in \Omega$, and we denote by $\widetilde{f}(z)$ the function that we get when we come back to the starting point. (If δ is positively oriented, the construction is similar.) For later purposes, we define η_0 as the negatively oriented loop homotopic to the loop based at the origin encircling z_0 obtained by following β from 0 to z, δ from z back to z, and β in the reverse sense from z back to the origin. We will keep this notation in the rest of this section. It follows from the discussion above that if we perform the analytic continuation of the set of eigenvalues $\{e_j(z)\}_{j=1}^n$, along a negatively oriented loop around $z_0 \in \Omega$, we get the set $\{\widetilde{e}_j(z)\}_{j=1}^n$ with

$$\widetilde{e}_j(z) = e_{\sigma_0(j)}(z), \quad j = 1, \dots, n, \tag{3.2}$$

where

$$\sigma_0 : \; \{1, 2, \dots, n\} \to \{1, 2, \dots, n\} \tag{3.3}$$

is a permutation that depends on η_0. Similarly, and with the same notations, we get for the analytic continuations of the projectors around z_0

$$\widetilde{P}_j(z) = P_{\sigma_0(j)}(z), \quad j = 1, \dots, n. \tag{3.4}$$

Let us consider now the eigenvectors $\varphi_j(t)$. We define $W(t)$ as the solution of

$$\begin{aligned} W'(t) &= \sum_{j=1}^{n} P_j'(t) P_j(t) W(t) \\ &\equiv K(t) W(t), \qquad W(0) = \mathbf{I}, \end{aligned} \tag{3.5}$$

where $t \in \mathbf{R}$. It is well known [K], [Kr] that $W(t)$ satisfies the intertwining identity

$$W(t) P_j(0) = P_j(t) W(t), \quad j = 1, 2, \dots, n, \quad \forall t \in \mathbf{R}, \tag{3.6}$$

so that if $\{\varphi_j(0)\}_{j=1}^n$ denotes a set of eigenvectors of $H(0)$, the vectors defined by

$$\varphi_j(t) = W(t)\varphi_j(0) \tag{3.7}$$

are eigenvectors of $H(t)$. Moreover, using the identity $Q(t)Q'(t)Q(t) \equiv 0$, which is true for any differentiable projector, it is easily checked that condition (2.9) is satisfied by these vectors. The generator $K(t)$ is analytic on the real axis and can be analytically continued in $S_\alpha\backslash\Omega$. Actually, $K(z)$ is single valued in $S_\alpha\backslash\Omega$. Indeed, let us consider the analytic continuation of $K(z)$ around $z_0 \in \Omega$. We get from (3.4) that

$$\widetilde{P}_j'(z) = P'_{\sigma_0(j)}(z) \tag{3.8}$$

so that

$$\begin{aligned}\widetilde{K}(z) &= \sum_{j=1}^n \widetilde{P}_j'(z)\widetilde{P}_j(z) = \sum_{j=1}^n P'_{\sigma_0(j)}(z)P_{\sigma_0(j)}(z)\\ &= \sum_{k=1}^n P_k'(z)P_k(z) = K(z).\end{aligned} \tag{3.9}$$

Consequently, $W(t)$ can be analytically continued in $S_\alpha\backslash\Omega$, where it is multivalued and satisfies both (3.5) and (3.6) with $z \in S_\alpha\backslash\Omega$ in place of $t \in \mathbf{R}$. Moreover, the relation between the analytic continuation $W(z)$ from 0 to some point $z \in S_\alpha\backslash\Omega$ and the analytic continuation $\widetilde{W}(z)$ is given by a monodromy matrix $W(\eta_0)$ such that

$$\widetilde{W}(z) = W(z)W(\eta_0), \tag{3.10}$$

where η_0 is the negatively oriented loop based at the origin which encircles only $z_0 \in \Omega$ (see Figure 1). Note also that the analytic continuation $W(z)$ is invertible in $S_\alpha\backslash\Omega$ and $W^{-1}(z)$ satisfies

$$W^{-1\prime}(z) = -W^{-1}(z)K(z), \qquad W^{-1}(0) = \mathbf{I}. \tag{3.11}$$

As a consequence, the eigenvectors (3.7) possess multivalued analytic extensions in $S_\alpha\backslash\Omega$. Indeed, it is easily seen that the analytic continuation of $\varphi_j(z)$ along a negatively oriented loop around $z_0 \in \Omega$, $\widetilde{\varphi}_j(z)$, is proportional to $\varphi_{\sigma_0(j)}(z)$. Hence we introduce the quantity $\theta_j(\eta_0) \in \mathbf{C}$ by the definition

$$\widetilde{\varphi}_j(z) = \mathrm{e}^{-i\theta_j(\eta_0)}\varphi_{\sigma_0(j)}(z), \quad j = 1, 2, \ldots, n. \tag{3.12}$$

Note that this is equivalent to $W(\eta_0)\varphi_j(0) = \mathrm{e}^{-i\theta_j(\eta_0)}\varphi_{\sigma_0(j)}(0)$ (see (3.10)). Let us consider the couplings (2.15). Using the definition (3.7), the invertibility of $W(t)$, and the identity (3.6), it is not difficult to see that we can rewrite

$$a_{jk}(t) = -\frac{\langle\varphi_j(0)|P_j(0)W(t)^{-1}K(t)W(t)\varphi_k(0)\rangle}{\|\varphi_j(0)\|^2}, \quad t \in \mathbf{R}, \tag{3.13}$$

which is analytic on the real axis and can be analytically continued in $S_\alpha\backslash\Omega$, where it is multivalued. Thus the same is true for the coefficients $c_j(t)$ which satisfy the linear differential equation (2.13), and their analytic continuations satisfy the same equation with $z \in S_\alpha\backslash\Omega$ in place of $t \in \mathbf{R}$. We now come to the main identity of this section regarding the coefficients $c_j(z)$. Let us denote by $c_j(z)$ the analytic continuation of

$c_j(0)$ from 0 to some $z \in S_\alpha \backslash \Omega$. We perform the analytic continuation of $c_j(z)$ along a negatively oriented loop around $z_0 \in \Omega$ and denote by $\widetilde{c}_j(z)$ the function that we get when we come back at the starting point z.

LEMMA 3.1. *For any $j = 1, \dots, n$, we have*

$$\widetilde{c}_j(z) e^{-i\int_{\eta_0} e_j(u)du/\varepsilon} e^{-i\theta_j(\eta_0)} = c_{\sigma_0(j)}(z) \tag{3.14}$$

where η_0, $\theta_j(\eta_0)$ and $\sigma_0(j)$ are defined as above.

Proof. It follows from hypothesis H1 that $\psi(z)$ is analytic in S_α so that

$$\begin{aligned} \sum_{j=1}^n c_j(z) \mathrm{e}^{-i\int_0^z e_j(u)du/\varepsilon} \varphi_j(z) &= \sum_{j=1}^n \widetilde{c}_j(z) \mathrm{e}^{-i\int_0^z \widetilde{e_j(u)} du/\varepsilon} \widetilde{\varphi}_j(z) \\ &= \sum_{j=1}^n \widetilde{c}_j(z) \mathrm{e}^{-i\int_{\eta_0} e_j(u)du/\varepsilon} \mathrm{e}^{-i\int_0^z e_{\sigma_0(j)}(u)du/\varepsilon} \mathrm{e}^{-i\theta_j(\eta_0)} \varphi_{\sigma_0(j)}(z). \end{aligned} \tag{3.15}$$

We conclude by the fact that $\{\varphi_j(z)\}_{j=1}^n$ is a basis. ◻

Remark. It is straightforward to generalize the study of the analytic continuations around one singular point of the functions given above to the case where the analytic continuations are performed around several singular points since Ω is finite. The loop η_0 can be rewritten as a finite succession of individual loops encircling only one point of Ω so that the permutation σ_0 is given by the composition of a finite number of individual permutations. Thus the factors $\mathrm{e}^{-i\theta_j(\eta_0)}$ in (3.12) should be replaced by a product of such factors, each associated with one individual loop, and the same is true for the factors $\exp(-i\int_{\eta_0} e_j(z)dz/\varepsilon)$ in Lemma 3.1. This process is performed in the proof of Theorem 6.1.

4. Complex WKB analysis.

This section is devoted to basic estimates on the coefficients $c_j(z)$ in certain domains extending to infinity in both the positive and negative directions inside the strip S_α. We first consider what happens in neighborhoods of $\pm\infty$. It follows from assumption H2 by a direct application of the Cauchy formula that (possibly by reducing α by an arbitrarily small amount)

$$\lim_{t\to\pm\infty} \sup_{|s|\le\alpha} |t|^{1+a} \|H'(t+is)\| < \infty. \tag{4.1}$$

Hence the same is true for the single-valued matrix $K(z)$:

$$\lim_{t\to\pm\infty} \sup_{|s|\le\alpha} |t|^{1+a} \|K(t+is)\| < \infty. \tag{4.2}$$

Let $0 < T \in \mathbf{R}$ be such that

$$\min_{z\in\Omega} \mathrm{Re} z > -T \quad \text{and} \quad \max_{z\in\Omega} \mathrm{Re} z < +T. \tag{4.3}$$

All quantities encountered so far are analytic in $S_\alpha \cap \{z||\mathrm{Re}z| > T\}$, and we denote with a "$\sim$" any analytic continuation in that set. As noticed earlier,

$$\widetilde{W}'(z) = K(z)\widetilde{W}(z), \quad z \in S_\alpha \cap \{z||\mathrm{Re}z| > T\} \tag{4.4}$$

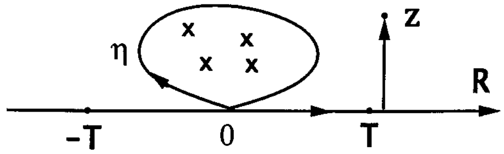

FIG. 2. *The path of integration for* $\widetilde{\Delta}_{jk}(z)$ *(the x's denote points of* Ω*).*

so that it follows from (4.2) that the limits

$$\lim_{t\to\pm\infty} \widetilde{W}(t+is) = \widetilde{W}(\pm\infty) \tag{4.5}$$

exist uniformly in $s \in \,]-\alpha, \alpha[$. Consequently (see (3.13)),

$$\lim_{t\to\pm\infty} |t|^{1+a} \sup_{|s|\le\alpha} |\widetilde{a}_{jk}(t+is)| < \infty, \quad \forall j,k \in \{1,\dots,n\}. \tag{4.6}$$

Finally, for $|t| > T$, we can write

$$\begin{aligned} \mathrm{Im}\widetilde{\Delta}_{jk}(t+is) &= \mathrm{Im}\left(\int_\eta e_j(z)dz - \int_\eta e_k(z)dz\right) \\ &\quad + \int_0^s \mathrm{Re}(e_{\sigma(j)}(t+is') - e_{\sigma(k)}(t+is'))ds', \end{aligned} \tag{4.7}$$

where this equation is obtained by deforming the path of integration from 0 to $z = t + is$ into a loop η based at the origin, which may encircle points of Ω, followed by the real axis from 0 to Rez and a vertical path from Rez to z (see Figure 2) and σ is the corresponding permutation. Hence we have

$$\sup_{z\in S_\alpha\cap\{z||\mathrm{Re}z|>T\}} \mathrm{Im}\widetilde{\Delta}_{jk}(z) < \infty, \tag{4.8}$$

which together with (4.6) yields the existence of the limits

$$\lim_{t\to\pm\infty} \widetilde{c}_j(t+is) = \widetilde{c}_j(\pm\infty) \tag{4.9}$$

uniformly in $s \in \,]-\alpha, \alpha[$. We now define the domains in which useful estimates can be obtained.

DEFINITION. *Let* $j \in \{1,\dots,n\}$ *be fixed. A* dissipative domain for the index j, $D_j \subset S_\alpha\backslash\Omega$, *is such that*

$$\sup_{z\in D_j} \mathrm{Re}z = \infty, \qquad \inf_{z\in D_j} \mathrm{Re}z = -\infty \tag{4.10}$$

and is defined by the property that for any $z \in D_j$ *and any* $k \in \{1,\dots,n\}$*, there exists a path* $\gamma^k \subset D_j$ *parameterized by* $u \in \,]-\infty, t]$ *which links* $-\infty$ *to* z,

$$\lim_{u\to-\infty} \mathrm{Re}\gamma^k(u) = -\infty, \qquad \gamma^k(t) = z, \tag{4.11}$$

with

$$\sup_{z\in D_j} \sup_{u\in]-\infty,t]} \left|\frac{d}{du}\gamma^k(u)\right| < \infty, \tag{4.12}$$

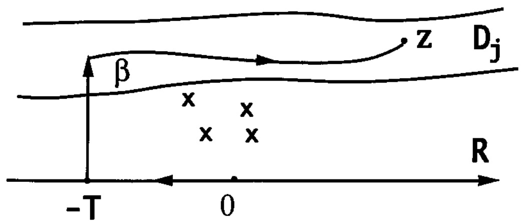

FIG. 3. *The path β along which the analytic continuation of $\Delta_{jk}(t)$ in D_j is taken.*

and satisfies the monotonicity condition

$$\mathrm{Im}\widetilde{\Delta}_{jk}(\gamma^k(u)) \text{ is a nondecreasing function of } u \in\,]-\infty, t]. \tag{4.13}$$

Such a path is a dissipative path for $\{jk\}$. *Here $\widetilde{\Delta}_{jk}(z)$ is the analytic continuation of*

$$\Delta_{jk}(t) = \int_0^t (e_j(t') - e_k(t'))dt', \quad t \in \mathbf{R}, \tag{4.14}$$

in D_j along a path β described in Figure 3 *going from* 0 *to $-T \in \mathbf{R}$ along the real axis and then vertically up or down until it reaches D_j, where $T > 0$ is chosen as in* (4.3).

Let $\widetilde{c}_k(z)$, $k = 1, 2, \ldots, n$, $z \in D_j$, be the analytic continuations of $c_k(t)$ along the same path β which are solutions of the analytic continuation of (2.13) in D_j along β:

$$\widetilde{c}_k'(z) = \sum_{l=1}^{n} \widetilde{a}_{kl}(z) \mathrm{e}^{i\widetilde{\Delta}_{kl}(z)/\varepsilon} \widetilde{c}_l(z). \tag{4.15}$$

We take as initial conditions in D_j

$$\lim_{\mathrm{Re}\, z \to -\infty} \widetilde{c}_k(z) = \lim_{t \to -\infty} c_k(t) = \delta_{jk}, \quad k = 1, \ldots, n, \tag{4.16}$$

and we define

$$x_k(z) = \widetilde{c}_k(z) \mathrm{e}^{i\widetilde{\Delta}_{jk}(z)/\varepsilon}, \qquad z \in D_j, \quad k = 1, \ldots, n. \tag{4.17}$$

LEMMA 4.1. *In a dissipative domain for the index j, we get the estimates*

$$\sup_{z \in D_j} |x_j(z) - 1| = \mathcal{O}(\varepsilon), \tag{4.18}$$

$$\sup_{z \in D_j} |x_k(z)| = \mathcal{O}(\varepsilon), \quad \forall k \neq j. \tag{4.19}$$

Remark. The real axis is a dissipative domain for all indices. In this case, we have $\widetilde{c}_j(t) \equiv c_j(t)$. Hence we get from the application of the lemma for all indices successively that $S = \mathbf{I} + \mathcal{O}(\varepsilon)$.

The estimates we are looking for are then just a direct corollary.

PROPOSITION 4.1. *Assume that there exists a dissipative domain D_j for the index j. Let η_j be a loop based at the origin which encircles all of the degeneracies between the real axis and D_j and let σ_j be the permutation of labels associated with η_j, in the spirit of the remark ending the previous section. The loop η_j is negatively (respectively,* *positively) oriented if D_j is above (respectively, below) the real axis. Then the solution of* (2.13) *subjected to the initial conditions $c_k(-\infty) = \delta_{jk}$ satisfies*

$$c_{\sigma_j(j)}(+\infty) = \mathrm{e}^{-i\theta_j(\eta_j)}\mathrm{e}^{-i\int_{\eta_j} e_j(z)dz/\varepsilon}\left(1 + \mathcal{O}(\varepsilon)\right), \tag{4.20}$$

$$c_{\sigma_j(k)}(+\infty) = \mathcal{O}\big(\varepsilon \mathrm{e}^{\mathrm{Im}\int_{\eta_j} e_j(z)dz/\varepsilon + h_j(e_{\sigma_j(j)}(+\infty) - e_{\sigma_j(k)}(+\infty))/\varepsilon}\big), \tag{4.21}$$

with $h_j \in [H_j^-, H_j^+]$, where $H_j^{\pm}$ is the maximum (respectively, minimum) imaginary part of the points at $+\infty$ in D_j:

$$H^+ = \limsup_{t\to+\infty} \sup_{s|t+is\in D_j} s, \qquad H^- = \liminf_{t\to+\infty} \inf_{s|t+is\in D_j} s. \tag{4.22}$$

Thus we see that it is possible to get the (exponentially small) asymptotic behavior of the element $s_{\sigma_j(j),j}$ of the S-matrix, provided there exists a dissipative domain for the index j. The difficult part of the problem is, of course, to prove the existence of such domains D_j, which do not necessarily exist, and to have enough of them to compute the asymptotic of the whole S-matrix. This task is the equivalent for n-level systems of studying the global behavior of the Stokes lines for two-level systems. We postpone this aspect of the problem until the next section. Note that we also get from this result an exponential bound on the elements $s_{\sigma_j(k),j}$ of the S-matrix, $k \neq j$, which may or may not be useful. If η_j encircles no point of Ω, we cannot get the asymptotic behavior of $s_{\sigma_j(j),j}$ but only get the exponential bounds. Since our main concern is asymptotic behaviors, we call the corresponding dissipative domain trivial.

Remark. In contrast with the two-level case (see [JP4]) we have to work with dissipative domains instead of working with one dissipative path for all indices. Indeed, it is not difficult to convince oneself with specific three-level cases that such a dissipative path may not exist, even when the eigenvalue degeneracies are close to the real axis. In return, we prove below the existence of dissipative domains in this situation.

Proof of Proposition 4.1. The asymptotic relation is a direct consequence of Lemma 3.1, (4.9), (4.17), and the first part of Lemma 4.1. The estimate is a consequence of the same equations, the second estimate of Lemma 4.1, and the identity, for $t > T$,

$$\begin{aligned}\mathrm{Im}\widetilde{\Delta}_{jk}(t+is) = \mathrm{Im}\left(\int_{\eta_j} e_j(z)dz - \int_{\eta_j} e_k(z)dz\right)\\ + \int_0^s \mathrm{Re}(e_{\sigma_j(j)}(t+is') - e_{\sigma_j(k)}(t+is'))ds'.\end{aligned} \tag{4.23}$$

The path of integration from 0 to z for $\widetilde{\Delta}_{jk}(z)$ is deformed into the loop η_j followed by the real axis from 0 to Rez and a vertical path from Rez to z. It remains to take the limit $t \to +\infty$. □

Proof of Lemma 4.1. We rewrite equations (4.15) and (4.16) as an integral equation and perform an integration by parts on the exponentials:

$$\begin{aligned}\widetilde{c}_k(z) = \delta_{jk} - i\varepsilon\sum_{l=1}^n \frac{\widetilde{a}_{kl}(z)}{\widetilde{e}_k(z) - \widetilde{e}_l(z)}\mathrm{e}^{i\widetilde{\Delta}_{kl}(z)/\varepsilon}\widetilde{c}_l(z)\\ + i\varepsilon\sum_{l=1}^n \int_{-\infty}^z \left(\frac{\widetilde{a}_{kl}(z')}{\widetilde{e}_k(z') - \widetilde{e}_l(z')}\right)' \mathrm{e}^{i\widetilde{\Delta}_{kl}(z')/\varepsilon}\widetilde{c}_l(z')dz'\end{aligned}$$

$$+ i\varepsilon \sum_{l,m=1}^{n} \int_{-\infty}^{z} \frac{\widetilde{a}_{kl}(z')\widetilde{a}_{lm}(z')}{\widetilde{e}_k(z') - \widetilde{e}_l(z')} \mathrm{e}^{i\widetilde{\Delta}_{km}(z')/\varepsilon} \widetilde{c}_m(z')dz'. \tag{4.24}$$

Since all eigenvalues are distinct in $S_\alpha \backslash \Omega$, the denominators are always different from 0. In terms of the functions x_k, we get

$$\begin{aligned} x_k(z) = \delta_{jk} - i\varepsilon \sum_{l=1}^{n} \frac{\widetilde{a}_{kl}(z)}{\widetilde{e}_k(z) - \widetilde{e}_l(z)} x_l(z) \\ + i\varepsilon \sum_{l=1}^{n} \int_{-\infty}^{z} \left(\frac{\widetilde{a}_{kl}(z')}{\widetilde{e}_k(z') - \widetilde{e}_l(z')} \right)' \mathrm{e}^{i(\widetilde{\Delta}_{jk}(z) - \widetilde{\Delta}_{jk}(z'))/\varepsilon} x_l(z')dz' \\ + i\varepsilon \sum_{l,m=1}^{n} \int_{-\infty}^{z} \frac{\widetilde{a}_{kl}(z')\widetilde{a}_{lm}(z')}{\widetilde{e}_k(z') - \widetilde{e}_l(z')} \mathrm{e}^{i(\widetilde{\Delta}_{jk}(z) - \widetilde{\Delta}_{jk}(z'))/\varepsilon} x_m(z')dz'. \end{aligned} \tag{4.25}$$

We introduce the quantity

$$|||x|||_j = \sup_{\substack{z \in D_j \\ l=1,\ldots,n}} |x_l(z)| \tag{4.26}$$

and consider for each k equation (4.25) along the dissipative path $\gamma^k(u)$ described in the definition of D_j such that

$$\left| \mathrm{e}^{i(\widetilde{\Delta}_{jk}(\gamma^k(t)) - \widetilde{\Delta}_{jk}(\gamma^k(u)))/\varepsilon} \right| \leq 1 \tag{4.27}$$

when $u \leq t$ along that path. Due to the integrability of the $\widetilde{a}_{kl}(z)$ at infinity and the uniform boundedness of $d\gamma^k(u)/du$, we get the estimate $|x_k(z) - \delta_{kj}| \leq \varepsilon |||x|||_j A$ for some constant A uniform in $z \in D_j$; hence $|||x|||_j \leq 1 + \varepsilon |||x|||_j A$. Consequently, for ε small enough, $|||x|||_j \leq 2$ and the result follows. □

5. Superasymptotic improvement. All of the results above can be improved substantially by using the so-called superasymptotic renormalization method [Be], [N], [JP2]. The joint use of complex WKB analysis and superasymptotic renormalization is very powerful, as demonstrated recently in [JP4] for two-level systems, and, roughly speaking, it allows us to replace all remainders $\mathcal{O}(\varepsilon)$ by $\mathcal{O}(\mathrm{e}^{-\kappa/\varepsilon})$, where $\kappa > 0$. We briefly show how to achieve this improvement in the case of n-level systems.

Let $H(z)$ satisfy H1, H2, and H3 in S_α, and let

$$\widehat{S}_\alpha = S_\alpha \backslash \cup_{r=1,\ldots,p} (J_r \cup \overline{J_r}), \tag{5.1}$$

where each J_r is an open domain containing only one point of Ω in the open upper half-plane. Hence any analytic continuation $e_j(z)$ of $e_j(t)$, $t \in \mathbf{R}$, in $\widehat{S}_\alpha$ is isolated in the spectrum of $H(z)$ so that $e_j(z)$ is analytic and multivalued in $\widehat{S}_\alpha$, and the same is true for the corresponding analytic continuation $P_j(z)$ of $P_j(t)$, $t \in \mathbf{R}$. Let σ_r be the permutation associated with the loop ζ_r based at the origin which encircles J_r once such that

$$\widetilde{e}_j(z) = e_{\sigma_r(j)}(z), \tag{5.2}$$

with the convention of section 3. The matrix $K(z)$ is analytic and single valued in $\widehat{S}_\alpha$. Consider the single-valued analytic matrix

$$H_1(z, \varepsilon) = H(z) - i\varepsilon K(z), \quad z \in \widehat{S}_\alpha. \tag{5.3}$$

For ε small enough, the spectrum of $H_1(z,\varepsilon)$ is nondegenerate $\forall z \in \widehat{S}_\alpha$ so that its eigenvalues $e_j^1(z,\varepsilon)$ and eigenprojectors $P_j^1(z,\varepsilon)$ are multivalued analytic functions in $\widehat{S}_\alpha$. Moreover, for ε small enough, the analytic continuations of $e_j^1(z,\varepsilon)$ and $P_j^1(z,\varepsilon)$ around J_r satisfy $\widetilde{e}_j^1(z) = e^1_{\sigma_r(j)}(z)$ and $\widetilde{P}_j^1(z) = P^1_{\sigma_r(j)}(z)$, as can be easily deduced from (5.2) by perturbation theory. Consequently, the matrix

$$K_1(z,\varepsilon) = \sum_{j=1}^m P_j^{1\prime}(z,\varepsilon)P_j^1(z,\varepsilon) \tag{5.4}$$

is analytic and single valued in $\widehat{S}_\alpha$. Defining the single-valued matrix

$$H_2(z,\varepsilon) = H(z) - i\varepsilon K_1(z,\varepsilon), \quad z \in \widehat{S}_\alpha, \tag{5.5}$$

we can repeat the argument for ε small enough. By induction, we set for any $q \in \mathbf{N}$

$$H_q(z,\varepsilon) = H(z) - i\varepsilon K_{q-1}(z,\varepsilon), \tag{5.6}$$

$$K_{q-1}(z,\varepsilon) = \sum_{j=1}^m P_j^{q-1\prime}(z,\varepsilon)P_j^{q-1}(z,\varepsilon), \quad z \in \widehat{S}_\alpha, \tag{5.7}$$

for ε is small enough. We have

$$H_q(z,\varepsilon) = \sum_{j=1}^m e_j^q(z,\varepsilon)P_j^q(z,\varepsilon), \tag{5.8}$$

where the eigenvalues and eigenprojections are multivalued in $\widehat{S}_\alpha$ and satisfy

$$\widetilde{e}_j^q(z,\varepsilon) = e^q_{\sigma_r(j)}(z,\varepsilon), \tag{5.9}$$

$$\widetilde{P}_j^q(z,\varepsilon) = P^q_{\sigma_r(j)}(z,\varepsilon), \quad j = 1,\dots,n, \tag{5.10}$$

with the notations of (5.2). We quote from [JP4] and [JP2] the main proposition regarding this construction.

PROPOSITION 5.1. *Let $H(z)$ satisfy* H1, H2, *and* H3 *in S_α, and let $\widehat{S}_\alpha$ be defined as above. Then there exist constants $c > 0$ and $\varepsilon^* > 0$ and a real function $b(t)$ with $\lim_{t\to\pm\infty} |t|^{1+a}b(t) < \infty$ such that*

$$\|K_q(z,\varepsilon) - K_{q-1}(z,\varepsilon)\| \le b(\mathrm{Re} z)\varepsilon^q c^q q!, \tag{5.11}$$

$$\|K_q(z,\varepsilon)\| \le b(\mathrm{Re} z) \tag{5.12}$$

for all $z \in \widehat{S}_\alpha$, all $\varepsilon < \varepsilon^*$, and all $q \le q^*(\varepsilon) \equiv [1/\mathrm{e}c\varepsilon]$, where $[y]$ denotes the integer part of y and e *is the basis of the neperian logarithm.*

We can deduce from this that in $\widehat{S}_\alpha$

$$e_j^q(z,\varepsilon) = e_j(z) + \mathcal{O}(\varepsilon^2 b(\mathrm{Re} z)), \tag{5.13}$$

$$P_j^q(z,\varepsilon) = e_j(z) + \mathcal{O}(\varepsilon b(\mathrm{Re} z)), \quad \forall q \le q^*(\varepsilon). \tag{5.14}$$

We introduce the notation $f^{q^*(\varepsilon)} \equiv f^*$ for any quantity f^q depending on the index q, and we henceforth drop the ε in the arguments of the functions that we encounter. We define the multivalued analytic matrix $W_*(z)$ for $z \in \widehat{S}_\alpha$ by

$$W_*{}'(z) = K_*(z)W_*(z), \qquad W_*(0) = \mathbf{I}. \tag{5.15}$$

Due to the above observations and Proposition 5.1, $W_*(z)$ enjoys all of the properties that $W(z)$ does, such as

$$W_*(z)P_j^*(0) = P_j^*(z)W_*(z), \tag{5.16}$$

$$\widetilde{W}^*(z) = W_*(z)W_*(\zeta_r) \tag{5.17}$$

and, uniformly in s,

$$\lim_{t\pm\infty} W_*(t+is) = W_*(\infty). \tag{5.18}$$

Thus we define for any $z \in \widehat{S}_\alpha$ a set of eigenvectors of $H_*(z)$ by $\varphi_j^*(z) = W_*(z)\varphi_j^*(0)$, where $H_*(0)\varphi_j^*(0) = e_j^*(0)\varphi_j^*(0)$, $j = 1,\dots,n$, that satisfy

$$\widetilde{\varphi}_j^*(0) = \exp\{-i\theta_j^*(\zeta_r)\}\varphi^*_{\sigma_r(j)}(0),$$

with $\theta_j^*(\zeta_r) = \theta(\zeta_r) + \mathcal{O}(\varepsilon) \in \mathbf{C}$. Let us expand the solution of (2.1) on this multivalued set of eigenvectors as

$$\psi(z) = \sum_{j=1}^{n} c_j^*(z)\mathrm{e}^{-i\int_0^z e_j^*(z')dz'/\varepsilon}\varphi_j^*(z). \tag{5.19}$$

Since the analyticity properties of the eigenvectors and eigenvalues of $H_*(z)$ are the same as those enjoyed by the eigenvectors and eigenvalues of $H(z)$, we get, as in Lemma 3.1,

$$\widetilde{c}_j^*(z)\mathrm{e}^{-i\int_{\zeta_r} e_j^*(u)du/\varepsilon}\mathrm{e}^{-i\theta_j^*(\zeta_r)} = c^*_{\sigma_r(j)}(z), \quad \forall z \in \widehat{S}_\alpha. \tag{5.20}$$

Substituting (5.19) in (2.1), we see that in $\widehat{S}_\alpha$ the multivalued coefficients $c_j^*(z)$ satisfy the differential equation

$$c_j^{*\prime}(z) = \sum_{k=1}^{n} a_{jk}^*(z)\mathrm{e}^{i\Delta_{jk}^*(z)/\varepsilon}c_k^*(z), \tag{5.21}$$

where

$$\Delta_{jk}^*(z) = \int_0^z e_j^*(z') - e_k^*(z')dz' \tag{5.22}$$

and

$$a_{jk}^*(z) = \frac{\langle\varphi_j^*(z)(0)|P_j^*(z)(0)W_*(z)^{-1}(K_{q^*-1}(z) - K_{q^*}(z))W_*(z)\varphi_k^*(0)\rangle}{\|\varphi_j^*(0)\|^2}; \tag{5.23}$$

compare this with (3.13). The key point of this construction is that it follows from Proposition 5.1 with $q = q^*(\varepsilon)$ that

$$|a_{jk}^*(z)| \le 2b(\mathrm{Re}z)\mathrm{e}^{-\kappa/\varepsilon}, \quad \forall z \in \widehat{S}_\alpha, \tag{5.24}$$

where $\kappa = 1/ec > 0$, and it follows from (5.13) that

$$\mathrm{Im}\Delta_{jk}^*(z) = \mathrm{Im}\Delta_{jk}(z) + \mathcal{O}(\varepsilon^2) \tag{5.25}$$

uniformly in $z \in \widehat{S}_\alpha$. Thus we deduce from (5.24) that the limits

$$\lim_{t\to\pm\infty} c_j^*(t+is) = c_j^*(\pm\infty), \quad j=1,\dots,n, \tag{5.26}$$

exist for any analytic continuation in $\widehat{S}_\alpha$. Moreover, along any dissipative path $\gamma^k(u)$ for $\{jk\}$, as defined above, we get from (5.25)

$$\left|\mathrm{e}^{i(\widetilde{\Delta}^*_{jk}(\gamma^k(t))-\widetilde{\Delta}^*_{jk}(\gamma^k(u)))/\varepsilon}\right| = \mathcal{O}(1), \quad \forall u \le t, \tag{5.27}$$

so that, reproducing the proof of Lemma 4.1, we have the following result.

LEMMA 5.1. *In a dissipative domain D_j, if $\widetilde{c}_k^*(-\infty) = c_k^*(-\infty) = \delta_{kj}$, then*

$$\widetilde{c}_j^*(z) = 1 + \mathcal{O}(e^{-\kappa/\varepsilon}), \tag{5.28}$$

$$e^{i\widetilde{\Delta}_{jk}(z)\varepsilon}\widetilde{c}_k^*(z) = \mathcal{O}(e^{-\kappa/\varepsilon}), \quad \forall k \ne j, \tag{5.29}$$

uniformly in $z \in \widehat{S}_\alpha$.

This lemma yields the following improved version of our main result.

PROPOSITION 5.2. *Under the conditions of Proposition* 4.1 *and with the same notations, if $c_k^*(-\infty) = \delta_{jk}$, then*

$$c^*_{\sigma_j(j)}(+\infty) = \mathrm{e}^{-i\theta_j^*(\eta_j)}\mathrm{e}^{-i\int_{\eta_j} e_j^*(z)dz/\varepsilon}(1+\mathcal{O}(\mathrm{e}^{-\kappa/\varepsilon})), \tag{5.30}$$

$$c^*_{\sigma_j(k)}(+\infty) = \mathcal{O}\left(\mathrm{e}^{-\kappa/\varepsilon}\mathrm{e}^{\mathrm{Im}\int_{\eta_j} e_j(z)dz/\varepsilon + h_j(e_{\sigma_j(j)}(+\infty)-e_{\sigma_j(k)}(+\infty))/\varepsilon}\right). \tag{5.31}$$

Note that we may or may not replace $e_j(z)$ by $e_j^*(z)$ in the estimate without altering the result. It remains to make the link between the S-matrix and the $c_k^*(+\infty)$'s of the proposition explicit. We define $\beta_j^{*\pm}$ by the relations

$$\varphi_j^*(\pm\infty) = \mathrm{e}^{-i\beta_j^{*\pm}}\varphi_j(\pm\infty) \tag{5.32}$$

($H_*(z)$ and $H(z)$ coincide at $\pm\infty$). By comparison with (5.19) and (2.12), we deduce the following lemma.

LEMMA 5.2. *If $c_k(t)$ and $c_k^*(t)$ satisfy $c_k(-\infty) = c_k^*(-\infty) = \delta_{jk}$, then the element kj of the S-matrix is given by*

(5.33)

$$\begin{aligned} s_{kj} = c_k(+\infty) &= e^{-i(\beta_k^{*+}-\beta_j^{*-})}e^{-i\int_0^{+\infty} e_k^*(t')-e_k(t')dt'/\varepsilon}e^{-i\int_{-\infty}^0 e_j^*(t')-e_j(t')dt'/\varepsilon}c_k^*(+\infty) \\ &\equiv e^{-i\alpha^*_{kj}}c_k^*(+\infty), \end{aligned}$$

with $\beta_j^{*\pm} = \mathcal{O}(\varepsilon)$ and $\int_{\pm\infty}^0 e_j^*(t') - e_j(t')dt'/\varepsilon = \mathcal{O}(\varepsilon)$, i.e., $e^{-i\alpha^*_{kj}} = 1 + \mathcal{O}(\varepsilon)$.

Remarks. (i) Proposition 5.2 together with Lemma 5.2 are the main results of the first part of this paper.

(ii) As a direct consequence of these estimates on the real axis, we have

$$s_{jk} = \mathcal{O}(\mathrm{e}^{-\kappa/\varepsilon}), \quad \forall k \ne j, \tag{5.34}$$

and

$$s_{jj} = \mathrm{e}^{-i\alpha^*_{jj}}(1+\mathcal{O}(e^{-\kappa/\varepsilon})). \tag{5.35}$$

(iii) It should be clear from the analysis performed above that all of the results obtained hold if the generator $H(z)$ in (2.1) is replaced by

$$H(z,\varepsilon) = H_0(z) + \mathcal{O}(\varepsilon b(\mathrm{Re} z)), \tag{5.36}$$

with $b(t) = \mathcal{O}(1/t^{1+a})$, provided $H_0(z)$ satisfies the hypotheses we assumed.

6. Avoided crossings. We now come to the second part of the paper, in which we prove asymptotic formulas for the off-diagonal elements of the S-matrix by means of the general setup presented above. To start with, we define a class of n-level systems for which we can prove the existence of one nontrivial dissipative domain for all indices. They are obtained by means of systems that exhibit degeneracies of eigenvalues on the real axis, hereafter called real crossings, which we perturb in such a way that these degeneracies are lifted and turn into avoided crossings on the real axis. When the perturbation is small enough, this process moves the eigenvalue degeneracies off the real axis, but they remain close to the place where the real crossings occurred. This method was used successfully in [J] to deal with two-level systems. We do not attempt to list all of the cases in which dissipative domains can be constructed by means of this technique but rather present a wide class of examples which are relevant in the theory of quantum adiabatic transitions and in the theory of multichannel semiclassical scattering, as described below.

Let $H(t,\delta) \in M_n(\mathbf{C})$ satisfy the following assumptions.

H4. *For each fixed $\delta \in [0,d]$, the matrix $H(t,\delta)$ satisfies* H1 *in a strip S_α independent of δ and $H(z,\delta)$ and $\partial/\partial z H(z,\delta)$ are continuous as a functions of two variables $(z,\delta) \in S_\alpha \times [0,d]$. Moreover, it satisfies* H2 *uniformly in $\delta \in [0,d]$, with limiting values $H(\pm,\delta)$ which are continuous functions of $\delta \in [0,d]$.*

H5. *For each $t \in \mathbf{R}$ and each $\delta \in [0,d]$, the spectrum of $H(t,\delta)$, denoted by $\sigma(t,\delta)$, consists of n real eigenvalues*

$$\sigma(t,\delta) = \{e_1(t,\delta), e_2(t,\delta), \ldots, e_n(t,\delta)\} \subset \mathbf{R} \tag{6.1}$$

which are distinct when $\delta > 0$:

$$e_1(t,\delta) < e_2(t,\delta) < \cdots < e_n(t,\delta). \tag{6.2}$$

When $\delta = 0$, the functions $e_j(t,0)$ are analytic on the real axis and there exists a finite set of crossing points $\{t_1 \le t_2 \le \cdots \le t_p\} \in \mathbf{R}$, $p \ge 0$, such that the following hold:

(i) $\forall t < t_1$,

$$e_1(t,0) < e_2(t,0) < \cdots < e_n(t,0). \tag{6.3}$$

(ii) $\forall j < k \in \{1,2,\ldots,n\}$, *there exists at most one t_r with*

$$e_j(t_r,0) - e_k(t_r,0) = 0, \tag{6.4}$$

and if such a t_r exists, we have

$$\frac{\partial}{\partial t}\left(e_j(t_r,0) - e_k(t_r,0)\right) > 0. \tag{6.5}$$

(iii) $\forall j \in \{1,2,\ldots,n\}$, *the eigenvalue $e_j(t,0)$ crosses eigenvalues whose indices are all superior to j or all inferior to j.*

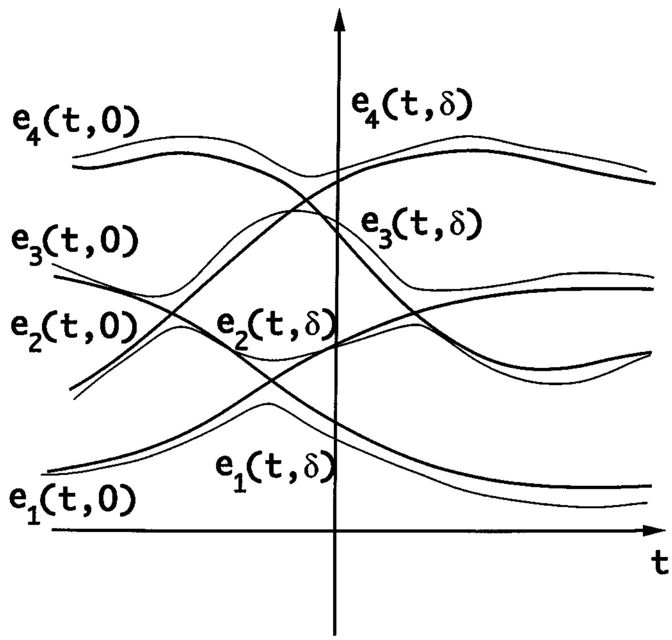

FIG. 4. *A pattern of eigenvalue crossings (bold curves) with the corresponding pattern of avoided crossings (fine curves) satisfying* H5.

Remarks. (i) The parameter δ can be understood as a coupling constant that controls the strength of the perturbation.

(ii) The eigenvalues $e_j(t, 0)$ are assumed to be analytic on the real axis, because of the degeneracies on the real axis. However, if $H(t, \delta)$ is self-adjoint for any $\delta \in [0, d]$, this is true for an indexation, as follows from a theorem of Rellich; see [K].

(iii) In Figure 4, we give an example of a pattern of crossings with the corresponding pattern of avoided crossings for which the above conditions are fulfilled.

(iv) The crossings are assumed to be generic in the sense that the derivatives of $e_j - e_k$ are nonzero at the crossing t_r.

(v) The crossing points $\{t_1, t_2, \ldots, t_p\}$ need not be distinct, which is important when the eigenvalues possess symmetries. However, for each $j = 1, \ldots, n$, the eigenvalue $e_j(t, \delta)$ experiences avoided crossings with $e_{j+1}(t, \delta)$ and/or $e_{j-1}(t, \delta)$ at a subset of distinct points $\{t_{r_1}, \ldots, t_{r_j}\} \subseteq \{t_1, t_2, \ldots, t_p\}$.

We now state the main lemma of this section regarding the analyticity properties of the perturbed levels and the existence of dissipative domains for all indices in this perturbative context.

LEMMA 6.1. *Let $H(t, \delta)$ satisfy* H4 *and* H5. *We can choose $\alpha > 0$ small enough so that the following assertions are true for sufficiently small $\delta > 0$:*

(i) *Let $\{t_{r_1}, \ldots, t_{r_j}\}$ be the set of avoided crossing points experienced by $e_j(t, \delta)$, $j = 1, \ldots, n$. For each j, there exists a set of distinct domains $J_r \in S_\alpha$, where $r \in \{r_1, \ldots, r_j\}$,*

$$J_r = \{z = t + is |\, 0 \leq |t - t_r| < L, \quad 0 < g < s < \alpha'\}, \tag{6.6}$$

with L small enough, $\alpha' < \alpha$, and $g > 0$ such that $e_j(-\infty, \delta)$ can be analytically *continued in*

$$S_\alpha^j = S_\alpha \setminus \cup_{r=r_1,\ldots,r_j} (J_r \cup \overline{J_r}). \tag{6.7}$$

(ii) *Let t_r be an avoided crossing point of $e_j(t,\delta)$ with $e_k(t,\delta)$, $k = j \pm 1$. Then the analytic continuation of the restriction of $e_j(t,\delta)$ around t_r along a loop based at $t_r \in \mathbf{R}$ which encircles J_r once yields $\widetilde{e}_j(t_r,\delta)$ back at t_r with*

$$\widetilde{e}_j(t_r,\delta) = e_k(t_r,\delta). \tag{6.8}$$

(iii) *For each $j = 1,\ldots,n$, there exists a dissipative domain D_j above or below the real axis in $S_\alpha \cap \{z = t + is| |s| \geq \alpha'\}$. The permutation σ_j associated with these dissipative domains* (*see Proposition* 4.1) *are all given by $\sigma_j = \sigma$, where σ is the permutation that maps the index of the kth eigenvalue $e_j(\infty,0)$ numbered from the lowest one on k for all $k \in \{1,2,\cdots,n\}$.*

Remarks. (i) In part (ii), the same result is true along a loop encircling $\overline{J_r}$.

(ii) The dissipative domains D_j of part (iii) are located above (respectively, below) all of the sets J_r (respectively, $\overline{J_r}$), $r = 1,\ldots,p$.

(iii) The main interest of this lemma is that the sufficient conditions required for the existence of dissipative domains in the complex plane can be deduced from the behavior of the eigenvalues on the *real* axis.

(iv) We emphasize that more general types of avoided crossings than those described in H5 may lead to the existence of dissipative domains for *certain* indices, but we want to obtain dissipative domains for *all* indices. For example, if part (iii) of H5 is satisfied for certain indices only, then part (iii) of Lemma 6.1 is satisfied for those indices only.

(v) Note also that there are patterns of eigenvalue crossings for which there exist no dissipative domain for some indices. For example, if $e_j(t,0)$ and $e_k(t,0)$ display two crossings, it is not difficult to see from the proof of the lemma that no dissipative domains can exist for j or k.

We postpone the proof of Lemma 6.1 to the end of this section and continue with its consequences. By applying the results of the previous section, we get the following result.

THEOREM 6.1. *Let $H(t,\delta)$ satisfy* H4 *and* H5. *If $\delta > 0$ is small enough, the elements $\sigma(j)j$ of the S-matrix, with $\sigma(j)$ defined in Lemma* 6.1, *are given in the limit $\varepsilon \to 0$ for all $j = 1,\ldots,n$ by*

$$s_{\sigma(j)j} = \prod_{k=j}^{\sigma(j)\mp 1} e^{-i\theta_k(\zeta_k)} e^{-i\int_{\zeta_k} e_k(z,\delta)dz/\varepsilon} \left(1 + \mathcal{O}(\varepsilon)\right), \quad \sigma(j) \begin{cases} > j, \\ < j, \end{cases} \tag{6.9}$$

where for $\sigma(j) > j$ (*respectively, $\sigma(j) < j$*), *ζ_k, $k = j,\ldots,\sigma(j)-1$* (*respectively, $k = j,\ldots,\sigma(j)+1$*), *denotes a negatively* (*respectively, positively*) *oriented loop based at the origin which encircles the set J_r* (*respectively, $\overline{J_r}$*) *corresponding to the avoided crossing between $e_k(t,\delta)$ and $e_{k+1}(t,\delta)$* (*respectively, $e_{k-1}(z,\delta)$*) *at t_r, $\int_{\zeta_k} e_k(z,\delta)dz$ denotes the integral along ζ_k of the analytic continuation of $e_k(0,\delta)$, and $\theta_k(\zeta_k)$ is the corresponding factor defined by* (3.12); *see Figure* 5.

More accurately, with the notations of section 5, *we have the improved formula*

$$s_{\sigma(j)j} = e^{-i\alpha^*_{\sigma(j)j}} \prod_{k=j}^{\sigma(j)\mp 1} e^{-i\theta^*_k(\zeta_k)} e^{-i\int_{\zeta_k} e^*_k(z,\delta)dz/\varepsilon} (1 + \mathcal{O}(e^{-\kappa/\varepsilon})), \quad \sigma(j) \begin{cases} > j, \\ < j. \end{cases} \tag{6.10}$$

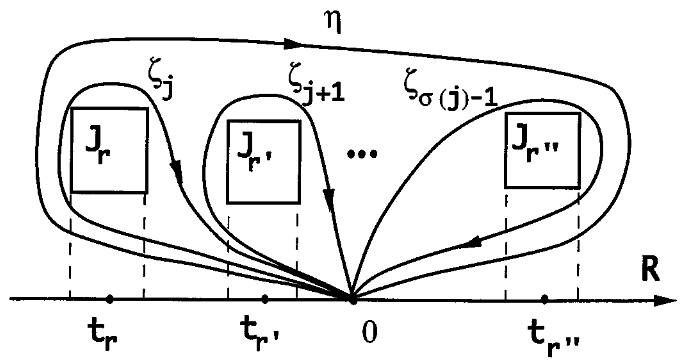

FIG. 5. *The loops* η_j *and* ζ_k, $k = j, \ldots, \sigma(j) - 1$.

The elements $\sigma(l)j$, $l \neq j$, *are estimated by*

$$(6.11)\quad s_{\sigma(l)j} = \mathcal{O}\left(\varepsilon e^{h(e_{\sigma(j)}(\infty,\delta) - e_{\sigma(l)}(\infty,\delta))/\varepsilon} \prod_{k=j}^{\sigma(j)\mp 1} e^{\mathrm{Im}\int_{\zeta_k} e_k(z,\delta)dz/\varepsilon}\right), \quad \sigma(j) \begin{cases} > j, \\ < j, \end{cases}$$

where h *is strictly positive* (*respectively, negative*) *for* $\sigma(j) > j$ (*respectively,* $\sigma(j) < j$).

Remarks. (i) Since the eigenvalues are continuous at the degeneracy points, we have that

$$(6.12)\qquad \lim_{\delta\to 0} \mathrm{Im} \int_{\zeta_k} e_k(z,\delta)dz = 0, \quad \forall k = 1, \ldots, p.$$

(ii) The remainders $\mathcal{O}(\varepsilon)$ depend on δ, but it should be possible to get estimates that are valid as both ε and δ tend to zero, in the spirit of [J], [MN], and [R].

(iii) This result shows that at least one off-diagonal element per column of the S-matrix can be computed asymptotically. However, it is often possible to get more elements by making use of the symmetries of the S-matrix. Moreover, if there exist dissipative domains that go above or below other eigenvalue degeneracies further away in the complex plane, other elements of the S-matrix can be computed.

(iv) Finally, note that all starred quantities in (6.10) depend on ε.

Proof of Theorem 6.1. The first thing to determine is whether the loops ζ_k are above or below the real axis. Since the formulas that we deduce from the complex WKB analysis are asymptotic, it suffices to choose the case that yields exponential decay of $s_{\sigma(j)j}$. It is readily checked in the proof of Lemma 6.1 below that if $\sigma(j) > j$, D_j is above the real axis and if $\sigma(j) < j$, D_j is below the real axis. Then it remains to explain how to pass from the loop η_j given in Proposition 4.1 to the set of loops ζ_k, $k = j, \ldots, \sigma(j) - 1$. We briefly deal with the case where $\sigma(j) > j$; the other case is similar. It follows from Lemma 6.1 that we can deform η_j into the set of loops ζ_k, each associated with one avoided crossing, as described in Figure 5. Thus we have

$$(6.13)\qquad \int_{\eta_j} = \sum_{k=j}^{\sigma(j)-1} \int_{\zeta_k}$$

for the decay rate and (see (3.10))

$$(6.14)\qquad W(\eta_j) = W(\zeta_{\sigma(j)-1}) \cdots W(\zeta_{j+1}) W(\zeta_j)$$

for the prefactors. Let ν_j be a negatively oriented loop based at t_r which encircles J_r as described in Lemma 6.1. Now consider the loop ζ_j associated with this avoided crossing and deform it to the path obtained by going from 0 to t_r along the real axis, from t_r to t_r along ν_j, and back from t_r to the origin along the real axis. By point (ii) of Lemma 6.1, we get

$$\widetilde{e}_j(0,\delta) = e_{j+1}(0,\delta) \tag{6.15}$$

along ζ_j, and, accordingly (see (3.12)),

$$\widetilde{\varphi}_j(0,\delta) = \mathrm{e}^{-i\theta_j(\zeta_j)}\varphi_{j+1}(0,\delta). \tag{6.16}$$

This justifies the first factor in the formula. By repeating the argument at the next avoided crossings, keeping in mind that we get $e_{j+1}(0,\delta)$ at the end of ζ_j and so on, we get the final result. The estimate on $s_{\sigma(l)j}$ is obtained by direct application of lemma 6.1. □

Proof of Lemma 6.1. In what follows, we shall denote "$\frac{\partial}{\partial t}$" by a "$'$." We must consider the analyticity properties of $\widetilde{e}_j(z,\delta)$ and define domains in which every point z can be reached from $-\infty$ by means of a path $\gamma(u)$, $u \in\ ]-\infty,t]$, $\gamma(t)=z$ such that $\mathrm{Im}\widetilde{\Delta}_{jk}(\gamma(u),\delta)$ is nondecreasing in u for certain indices $j\neq k$ when $\delta>0$ is fixed. Note that by Schwarz's principle, if $\gamma(u)$ is dissipative for $\{jk\}$, then $\overline{\gamma(u)}$ is dissipative for $\{kj\}$. When $\gamma(u)=\gamma_1(u)+i\gamma_2(u)$ is differentiable, saying that $\gamma(u)$ is dissipative for $\{jk\}$ is equivalent to

$$\begin{aligned}&\mathrm{Re}(\widetilde{e}_j(\gamma(u),\delta)-\widetilde{e}_k(\gamma(u),\delta))\dot{\gamma}_2(u)+\mathrm{Im}(\widetilde{e}_j(\gamma(u),\delta)-\widetilde{e}_k(\gamma(u),\delta))\dot{\gamma}_1(u)\geq 0,\\ &\qquad \forall u\in\ ]-\infty,t],\end{aligned} \tag{6.17}$$

where "$\dot{}$" denotes the derivative with respect to u. Moreover, if the eigenvalues are analytic in a neighborhood of the real axis, we have in that neighborhood the relation

$$\mathrm{Im}(\widetilde{e}_j(t+is,\delta)-\widetilde{e}_k(t+is,\delta)) = \int_0^s \mathrm{Re}\ (\widetilde{e}'_j(t+is',\delta)-\widetilde{e}'_k(t+is',\delta))ds', \tag{6.18}$$

which is a consequence of the Cauchy–Riemann identity. We proceed as follows. We construct dissipative domains above and below the real axis when $\delta=0$, and we show that they remain dissipative for the perturbed quantities $\widetilde{\Delta}_{jk}(z,\delta)$, provided δ is small enough. We introduce some quantities to be used in the construction. Let $C_r\subset\{1,\ldots,n\}^2$ denote the set of distinct couples of indices such that the corresponding eigenvalues experience one crossing at $t=t_r$. Similarly, $N\subset\{1,\ldots,n\}^2$ denotes the set of couples of indices such that the corresponding eigenvalues never cross.

Let $I_r=[t_r-L,t_r+L]\in\mathbf{R}$, $r=1,\ldots,p$, with L so small that

$$\min_{r\in\{1,\ldots,p\}}\ \min_{\{jk\}\in C_r,\,j<k}\ \inf_{t\in I_r}(e'_j(t,0)-e'_k(t,0))\equiv 4c>0. \tag{6.19}$$

This relation defines the constant c, and we also define b by

$$\min_{r\in\{1,\ldots,p\}}\ \min_{\{jk\}\in C_r,\,j<k}\ \inf_{t\in\mathbf{R}\setminus I_r}|e_j(t,0)-e_k(t,0)|\geq 4b>0, \tag{6.20}$$

$$\min_{\{jk\}\in N,\,j<k}\ \inf_{t\in\mathbf{R}}|e_j(t,0)-e_k(t,0)|\geq 4b>0. \tag{6.21}$$

We further introduce

$$I_r^\alpha=\{z=t+is|t\in I_r,|s|\leq\alpha\},\quad r=1,\ldots,p. \tag{6.22}$$

Then we choose α small enough so that the only points of degeneracy of eigenvalues in S_α are on the real axis and

$$\min_{r\in\{1,\dots,p\}} \min_{\{jk\}\in C_r,\, j<k} \inf_{z\in I_r^\alpha} \mathrm{Re}(e_j'(z,0)-e_k'(z,0)) > 2c > 0 \tag{6.23}$$

$$\min_{r\in\{1,\dots,p\}} \min_{\{jk\}\in C_r,\, j<k} \inf_{z\in S_\alpha\setminus I_r^\alpha} |\mathrm{Re}(e_j(z,0)-e_k(z,0))| > 2b > 0 \tag{6.24}$$

$$\min_{\{jk\}\in N,\, j<k} \inf_{z\in S_\alpha} |\mathrm{Re}(e_j(z,0)-e_k(z,0))| > 2b > 0. \tag{6.25}$$

The fact that this choice is always possible is a consequence of the analyticity of $e_j(z,0)$ close to the real axis and of the fact that we can essentially work in a compact because of hypothesis H4. Let $a(t)$ be integrable on $\mathbf{R}$ and such that

$$\frac{a(t)}{2} > \max_{j<k\in\{1,\dots,n\}} \sup_{|s|\le\alpha} \left|\mathrm{Re}(e_j'(t+is,0)-e_k'(t+is,0))\right|. \tag{6.26}$$

It follows from H4 that such functions exist.

Let $r\in\{1,\dots,p\}$ and $\gamma_2(u)$ be a solution of

$$\begin{cases} \dot\gamma_2(u) = -\frac{\gamma_2(u)a(u)}{b}, & u\in\,]-\infty, t_r-L],\\ \dot\gamma_2(u)=0, & u\in\,]t_r-L, t_r+L[,\\ \dot\gamma_2(u) = +\frac{\gamma_2(u)a(u)}{b}, & u\in[t_r+L,\infty[, \end{cases} \tag{6.27}$$

with $\gamma_2(t_r)>0$. Then $\gamma_2(u)>0$ for any u since

$$\begin{cases} \gamma_2(u)=\gamma_2(t_r)\mathrm{e}^{-\int_{t_r-L}^{u} a(u')du'/b}, & u\in\,]-\infty,t_r-L],\\ \gamma_2(u)=\gamma_2(t_r), & u\in\,]t_r-L,t_r+L[,\\ \gamma_2(u)=\gamma_2(t_r)\mathrm{e}^{\int_{t_r+L}^{u} a(u')du'/b}, & u\in[t_r+L,\infty[, \end{cases} \tag{6.28}$$

and since $a(u)$ is integrable, the limits

$$\lim_{u\to\pm\infty}\gamma_2(u)=\gamma_2(\pm\infty) \tag{6.29}$$

exist. Moreover, we can always choose $\gamma_2(t_r)>0$ sufficiently small so that $\gamma^r(u)\equiv u+i\gamma_2(u)\in S_\alpha$ for any real u. Let us verify that this path is dissipative for all $\{jk\}\in C_r$, $j<k$. For $u\in\,]-\infty,t_r-L]$, using

$$\mathrm{Re}(e_j(z,0)-e_k(z,0)) < -2b<0, \quad \forall z\in S_\alpha\cap\{z|\mathrm{Re}z\le t_r-L\}, \tag{6.30}$$

$$|\mathrm{Im}(e_j(t+is,0)-e_k(t+is,0))| < |s| \sup_{s'\in[0,s]} \left|\mathrm{Re}(e_j'(t+is',0)-e_k'(t+is',0))\right| \tag{6.31}$$

(see (6.18)), and the definition (6.26), we have

$$\begin{aligned} &\mathrm{Re}(e_j(\gamma^r(u),0)-e_k(\gamma^r(u),0))\dot\gamma_2(u)+\mathrm{Im}(e_j(\gamma^r(u),0)-e_k(\gamma^r(u),0))\dot\gamma_1(u)\\ &\quad= -\mathrm{Re}(e_j(\gamma^r(u),0)-e_k(\gamma^r(u),0))\frac{\gamma_2(u)a(u)}{b} + \mathrm{Im}(e_j(\gamma^r(u),0)-e_k(\gamma^r(u),0))\\ &\quad> 2\gamma_2(u)a(u)-\gamma_2(u)a(u)/2 > \gamma_2(u)a(u)>0. \end{aligned} \tag{6.32}$$

Similarly, when $u \geq t_r + L$, using

$$\text{Re}(e_j(z,0) - e_k(z,0)) > 2b > 0, \quad \forall z \in S_\alpha \cap \{z | \text{Re} z \geq t_r + L\}, \tag{6.33}$$

we get

(6.34)
$$\begin{aligned} &\text{Re}(e_j(\gamma^r(u),0) - e_k(\gamma^r(u),0))\dot{\gamma}_2(u) + \text{Im}(e_1(\gamma^r(u),0) - e_k(\gamma^r(u),0))\dot{\gamma}_1(u) \\ &\quad = \text{Re}(e_j(\gamma^r(u),0) - e_k(\gamma^r(u),0))\frac{\gamma_2(u)a(u)}{b} + \text{Im}(e_j(\gamma^r(u),0) - e_k(\gamma^r(u),0)) \\ &\quad > 2\gamma_2(u)a(u) - \gamma_2(u)a(u)/2 > \gamma_2(u)a(u) > 0. \end{aligned}$$

Finally, for $s \in [t_r - L, t_r + L]$, we have with (6.23) that

(6.35)
$$\begin{aligned} \text{Im}(e_j(\gamma^r(u),0) - e_k(\gamma^r(u),0)) &= \int_0^{\gamma_2(u)} \text{Re}(e_j'(t'+is,0) - e_k'(t'+is,0)) \\ &\geq \gamma_2(u)2c > \gamma_2(u)c > 0. \end{aligned}$$

Thus $\gamma^r(u)$ is dissipative for all $\{jk\} \in C_r$, $j < k$. Note that the last estimate shows that it is not possible to find a dissipative path for $\{jk\} \in C_r$, $j < k$ below the real axis.

Now consider $\{jk\} \in N$, $j < k$, and let $\gamma_2^+(u)$ be a solution of

$$\dot{\gamma}_2^+(u) = -\frac{\gamma_2^+(u)a(u)}{b}, \qquad \gamma_2^+(0) > 0, \quad u \in]-\infty, +\infty[, \tag{6.36}$$

i.e.,

$$\gamma_2^+(u) = \gamma_2^+(0)\mathrm{e}^{-\int_0^u a(u')du'/b}. \tag{6.37}$$

As above, we have $\gamma_2^+(u) > 0$ for any u and we can choose $\gamma_2^+(0) > 0$ small enough so that $\gamma^+(u) \equiv u + i\gamma_2^+(u) \in S_\alpha$ for any $u \in \mathbf{R}$. Since

$$\text{Re}(e_j(z,0) - e_k(z,0)) > -2b, \quad \forall z \in S_\alpha, \tag{6.38}$$

we check by a computation analogous to (6.32) that $\gamma^+(u)$ is dissipative for $\{jk\} \in N$, $j < k$. Similarly, we can verify that if $\gamma_2^-(u)$ is the solution of

$$\dot{\gamma}_2^-(u) = \frac{\gamma_2^-(u)a(u)}{b}, \qquad \gamma_2^-(0) < 0, \quad u \in]-\infty, +\infty[, \tag{6.39}$$

with $|\gamma_2^-(0)|$ small enough, the path $\gamma^-(u) \equiv u + i\gamma_2^-(u)$ below the real axis is in S_α for any $u \in \mathbf{R}$ and is dissipative for $\{jk\} \in N$, $j < k$, as well.

Finally, the complex conjugates of these paths yield dissipative paths above and below the real axis for $\{jk\} \in N$, $j > k$.

We now define the dissipative domains by means of their borders. Let $\gamma^+(u)$ and $\gamma^-(u)$, $u \in \mathbf{R}$, be two dissipative paths in S_α defined as above with $|\gamma_2^-(0)|$ sufficiently small so that $\overline{\gamma^-}$ is below γ^+. We set

$$D = \{z = t + is | 0 < -\gamma_2^-(t) \leq s \leq \gamma_2^+(t), \quad t \in \mathbf{R}\}. \tag{6.40}$$

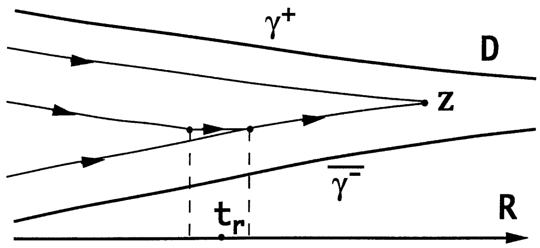

FIG. 6. *The dissipative domain D and some dissipative paths.*

Let $z \in D$, and $j \in \{1, \dots, n\}$ be fixed. By assumption H5, the set X_j of indices k such that $\{jk\} \in C_r$ for some $r \in \{1, \dots, p\}$ consists of values k that satisfy $j < k$ or it consists of values k that satisfy $j > k$. Let us assume that the first alternative takes place. Now for any $k \in \{1, \dots, n\}$, there are three cases.

(1) If $k \in X_j$, then there exists a dissipative path $\gamma^r \in D$ for $\{jk\} \in C_r$, $j < k$, constructed as above which links $-\infty$ to z. It is enough to select the initial condition $\gamma_2(t_r)$ suitably; see Figure 6.

(2) Similarly, if $j < k \notin X_j$, there exists a dissipative path $\gamma^+ \in D$ for $\{jk\}$ constructed as above which links $-\infty$ to z obtained by a suitable choice of $\gamma_2^+(0)$.

(3) Finally, if $k > j$, we can take as a dissipative path for $\{jk\}$ the path $\overline{\gamma^-} \in D$ constructed as above which links $-\infty$ to z with a suitable choice of $\gamma_2^-(0)$. Hence D is dissipative for the index j when $\delta = 0$. If j is such that the set X_j consists of points k with $k > j$, a similar argument with the complex conjugates of the above paths shows that the domain $\overline{D}$ below the real axis is dissipative for j when $\delta = 0$.

Let us show that these domains remain dissipative when $\delta > 0$ is not too large. We start by considering the analyticity properties of the perturbed eigenvalues $e_j(z, \delta)$, $\delta > 0$. Let $0 < \alpha' < \alpha$ be such that

$$I_r^{\alpha'} \cap (D \cup \overline{D}) = \emptyset, \quad \forall r = 1, \dots, p. \tag{6.41}$$

The analytic eigenvalues $e_j(z, 0)$, $j \in \{1, \dots, n\}$, are isolated in the spectrum of $H(z, 0)$ for any $z \in \widetilde{S}_\alpha$, where

$$\widetilde{S}_\alpha = S_\alpha \setminus \cup_{r=1,\dots,p} I_r^{\alpha'}. \tag{6.42}$$

For any $j = 1, \dots, n$ we get from perturbation theory [K] that the analytic continuations $\widetilde{e}_j(z, \delta)$ of $e_j(t_1 - L, \delta)$ in $\widetilde{S}_\alpha$ are all distinct in $\widetilde{S}_\alpha$, provided δ is small enough. This is due to the fact that assumption H4 implies the continuity of $H(z, \delta)$ in δ uniformly in $z \in S_\alpha$, as is easily verified. More precisely, for any fixed index j, the eigenvalue $e_j(t, \delta)$ experiences avoided crossings at the points $\{t_{r_1}, \dots, t_{r_j}\}$. We can assume without loss of generality that

$$I_k^{\alpha'} \cap I_l^{\alpha'} = \emptyset, \quad \forall k \neq l \in \{r_1, \dots, r_j\}. \tag{6.43}$$

Hence for $\delta > 0$ small enough, the analytic continuation $\widetilde{e}_j(z, \delta)$ is isolated in the spectrum of $H(z, \delta)$ uniformly in $z \in S_\alpha \setminus \cup_{r=r_1,\dots,r_j} I_r^{\alpha'}$. Since by assumption H5 there is no crossing of eigenvalues on the real axis when $\delta > 0$, there exists a $0 < g < \alpha'$ that depends on δ such that $\widetilde{e}_j(z, \delta)$ is isolated in the spectrum of $H(z, \delta)$ uniformly in $z \in S_\alpha^j$, where

$$S_\alpha^j = S_\alpha \setminus \cup_{r=r_1,\dots,r_j} (J_r \cup \overline{J_r}) \tag{6.44}$$

and

$$J_r = I_r^{\alpha'} \cap \{z|\,\mathrm{Im}z > g\}, \quad r = 1, \ldots, p. \tag{6.45}$$

Hence the singularities of $\widetilde{e}_j(z,\delta)$ are located in $\cup_{r=r_1,\ldots,r_j}(J_r \cup \overline{J_r})$, which yields the first assertion of the lemma.

Consider a path ν_r from $t_r - L$ to $t_r + L$ which goes above J_r, where t_r is an avoided crossing between $e_j(t,\delta)$ and $e_k(t,\delta)$, $k = j \pm 1$. By perturbation theory again, $e_j(t_r - L,\delta)$ and $e_k(t_r - L,\delta)$ tend to $e_{j'}(t_r - L, 0)$ and $e_{k'}(t_r - L, 0)$ as $\delta \to 0$ for some $j', k' \in 1, \ldots, n$, whereas $e_j(t_r + L,\delta)$ and $e_k(t_r + L,\delta)$ tend to $e_{k'}(t_r + L, 0)$ and $e_{j'}(t_r + L, 0)$ as $\delta \to 0$; see Figure 4. Now the analytic continuations of the restrictions of $e_j(t,\delta)$ and $e_k(t,\delta)$ around $t_r - L$ along ν_r, $\widetilde{e}_j(z,\delta)$ and $\widetilde{e}_k(z,\delta)$ tend to the analytic functions $\widetilde{e}_{j'}(z,0) = e_{j'}(z,0)$ and $\widetilde{e}_{k'}(z,0) = e_{k'}(z,0)$ as $\delta \to 0$ for all $z \in \nu_r$. Thus we deduce that for δ small enough,

$$\widetilde{e}_j(t_r + L,\delta) \equiv e_k(t_r + L,\delta) \tag{6.46}$$

since we know that $\widetilde{e}_j(t_r + L,\delta) = e_{\sigma(j)}(t_r + L,\delta)$ for some permutation σ. Hence point (iii) of the lemma follows.

Note that the analytic continuations $\widetilde{e}_j(z,\delta)$ are single valued in $\widetilde{S}_\alpha$. Indeed, the analytic continuation of $e_j(t_r - L,\delta)$ along $\overline{\nu_r}$, denoted by $\widehat{e}_j(z,\delta)$, $\forall z \in \overline{\nu_r}$, is such that

$$\widehat{e}_j(t_r + L,\delta) = \overline{\widetilde{e}_j(t_r + L,\delta)} = \widetilde{e}_j(t_r + L,\delta) = e_k(t_r + L,\delta) \tag{6.47}$$

due to Schwarz's principle. We further require δ to be sufficiently small so that the following estimates are satisfied:

$$\min_{r\in\{1,\ldots,p\}} \min_{\substack{\{jk\}\in C_r\\ j<k}} \inf_{z\in \widetilde{S}_\alpha \setminus I_r^\alpha} |\mathrm{Re}(\widetilde{e}_j(z,\delta) - \widetilde{e}_k(z,\delta))| > b > 0, \tag{6.48}$$

$$\min_{\substack{\{jk\}\in N\\ j<k}} \inf_{z\in \widetilde{S}_\alpha} |\mathrm{Re}(\widetilde{e}_j(z,\delta) - \widetilde{e}_k(z,\delta))| > b > 0, \tag{6.49}$$

$$\max_{j<k\in\{1,\ldots,n\}} \sup_{\mathrm{Im}z|\, z\in \widetilde{S}_\alpha} \left|\mathrm{Re}(\widetilde{e}'_j(z,\delta) - \widetilde{e}'_k(z,\delta))\right| < a(\mathrm{Re}z), \tag{6.50}$$

and, in the compacts $\widetilde{I}_r^\alpha = I_r^\alpha \setminus I_r^{\alpha'}$,

$$\begin{aligned} &\min_{r\in\{1,\ldots,p\}} \min_{\substack{\{jk\}\in C_r\\ j<k}} \inf_{z\in \widetilde{I}_r^\alpha} |\mathrm{Im}(\widetilde{e}_j(z,\delta) - \widetilde{e}_k(z,\delta))| \\ &> \frac{1}{2} \min_{r\in\{1,\ldots,p\}} \min_{\substack{\{jk\}\in C_r\\ j<k}} \inf_{z\in \widetilde{I}_r^\alpha} |\mathrm{Im}(\widetilde{e}_j(z,0) - \widetilde{e}_k(z,0))| > |\mathrm{Im}z|c, \end{aligned} \tag{6.51}$$

$$\begin{aligned} &\max_{r\in\{1,\ldots,p\}} \max_{j<k\in\{1,\ldots,n\}} \sup_{z\in \widetilde{I}_r^\alpha} |\mathrm{Im}(\widetilde{e}_j(z,\delta) - \widetilde{e}_k(z,\delta))| \\ &< 2 \max_{r\in\{1,\ldots,p\}} \max_{j<k\in\{1,\ldots,n\}} \sup_{z\in \widetilde{I}_r^\alpha} |\mathrm{Im}(\widetilde{e}_j(z,0) - \widetilde{e}_k(z,0))| < |\mathrm{Im}z| a(\mathrm{Re}z). \end{aligned} \tag{6.52}$$

The simultaneous requirements (6.26) and (6.50) are made possible by the continuity properties of $H'(z,\delta)$ and the uniformity in δ of the decay at $\pm\infty$ of $H(z,\delta)$ assumed in H4 together with the fact that $a(t)$ can be replaced by a multiple of $a(t)$ if necessary to satisfy both estimates. The condition on δ is given by the first inequalities in (6.51) and (6.52), whereas the second ones are just recalls.

Then it remains to check that the paths γ^r, γ^+, and γ^- defined above satisfy the dissipativity condition (6.17) for the corresponding indices. This is not difficult since the above estimates are precisely designed to preserve inequalities such as (6.32), (6.34), and (6.35). However, it should not be forgotten that in the sets $I_r^{\alpha'}$, the eigenvalues may be singular so that (6.18) cannot be used there. Therefore, when checking that a path parameterized as above by $u \in \mathbf{R}$ is dissipative, it is necessary to consider separately the case $u \in \mathbf{R}\backslash(\cup_{r=1,\ldots,p} I_r)$, where we proceed as above with (6.48), (6.49), (6.50), and (6.18), and the case $u \in \cup_{r=1,\ldots,p} I_r$, where we use use (6.51) and (6.52) instead of (6.18) as follows. If $u \in I_r$ for r such that t_r is a crossing point for $e_j(t,0)$ and $e_k(t,0)$, we take (6.51) to estimate $\mathrm{Im}(\widetilde{e}_{j'}(z,\delta) - \widetilde{e}_{k'}(z,\delta))$ for the corresponding indices j' and k', and if t_r is not a crossing point for $e_j(t,0)$ and $e_k(t,0)$, we use (6.52) to estimate $\mathrm{Im}(\widetilde{e}_{j'}(z,\delta) - \widetilde{e}_{k'}(z,\delta))$. Consequently, the domains D and $\overline{D}$ defined above keep the same dissipativity properties when $\delta > 0$ is small enough.

Let us finally turn to the determination of the associated permutation σ. As noticed earlier, the eigenvalues $\widetilde{e}_j(z,\delta)$ are continuous in δ uniformly in $z \in \widetilde{S}_\alpha$. Hence, since the eigenvalues $e_j(z,0)$ are analytic in S_α, we have

$$\lim_{\delta\to 0} \widetilde{e}_j(\infty,\delta) = e_j(\infty,0) \quad j = 1,2,\ldots,n, \tag{6.53}$$

whereas along the real axis (see Figure 4), we have

$$\lim_{\delta\to 0} e_{\sigma(j)}(\infty,\delta) = e_j(\infty,0), \tag{6.54}$$

with σ defined in the lemma, from which the result follows. □

7. Applications. Let us consider the time-dependent Schrödinger equation in the adiabatic limit. The relevant equation is then (2.1), where $H(t) = H^*(t)$ is the time-dependent self-adjoint Hamiltonian. Thus we can take $J = \mathbf{I}$ in Proposition 2.1 to get

$$H(t) = H^*(t) = H^{\#}(t). \tag{7.1}$$

Since the norm of an eigenvector is positive, it remains to impose the gap hypothesis in H3 to fit in the framework, and we deduce that the S-matrix is unitary since $R = \mathbf{I}$. In this context, the elements of the S-matrix describe the transitions between the different levels between $t = -\infty$ and $t = +\infty$ in the adiabatic limit.

We now specify our concern a little further and consider a three-level system, i.e., $H(t) = H^*(t) \in M_3(\mathbf{C})$. We assume that $H(t)$ satisfies the hypotheses of Theorem 6.1 with an extra parameter δ, which we omit in the notation, and displays two avoided crossings at $t_1 < t_2$, as shown in Figure 7. The corresponding permutation σ is given by

$$\sigma(1) = 3, \qquad \sigma(2) = 1, \qquad \sigma(3) = 2. \tag{7.2}$$

By Theorem 6.1, we can compute asymptotically the elements s_{31}, s_{12}, s_{23}, and s_{jj}, $j = 1,2,3$. Using the unitarity of the S-matrix, we can get some more information. Introducing

$$\Gamma_j = \left|\mathrm{Im}\int_{\zeta_j} e_j(z)dz\right|, \quad j = 1,2, \tag{7.3}$$

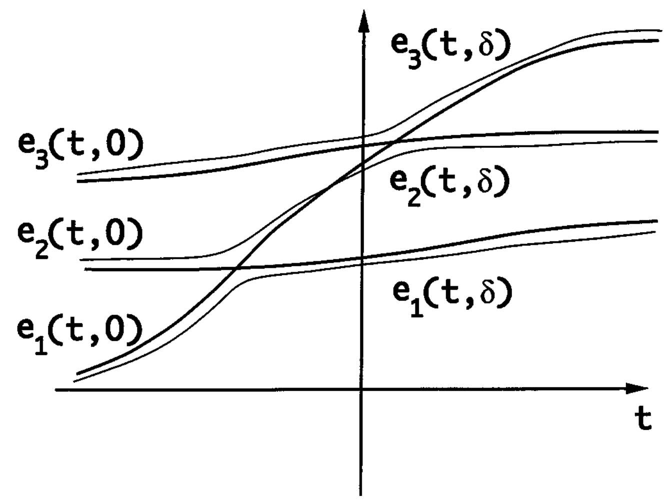

FIG. 7. *The pattern of avoided crossings in the adiabatic context.*

where ζ_j is in the upper half-plane, with the notation of section 6, it follows that

$$s_{31} = \mathcal{O}(\mathrm{e}^{-(\Gamma_1+\Gamma_2)/\varepsilon}), \qquad s_{12} = \mathcal{O}(\mathrm{e}^{-\Gamma_1/\varepsilon}), \qquad s_{23} = \mathcal{O}(\mathrm{e}^{-\Gamma_2/\varepsilon}), \tag{7.4}$$

and

$$s_{jj} = 1 + \mathcal{O}(\varepsilon), \quad j = 1, 2, 3. \tag{7.5}$$

Expressing the fact that the first and second columns as well as the second and third rows are orthogonal, we deduce

$$s_{21} = -\overline{s_{12}}\frac{s_{11}}{\overline{s_{22}}}(1 + \mathcal{O}(\mathrm{e}^{-2\Gamma_2/\varepsilon})), \tag{7.6}$$

$$s_{32} = -\overline{s_{23}}\frac{s_{33}}{\overline{s_{22}}}(1 + \mathcal{O}(\mathrm{e}^{-2\Gamma_1/\varepsilon})). \tag{7.7}$$

Finally, the estimate in Theorem 6.1 yields

$$s_{13} = \mathcal{O}(\varepsilon \mathrm{e}^{-|h|(e_2(\infty,\delta)-e_1(\infty,\delta))/\varepsilon}\mathrm{e}^{-\Gamma_2/\varepsilon}) = \mathcal{O}(\mathrm{e}^{-(\Gamma_2+\Gamma_2+K)/\varepsilon}), \tag{7.8}$$

where $K > 0$, since we have that $\Gamma_j \to 0$ as $\delta \to 0$. Hence we get

(7.9)

$$S = \begin{pmatrix} s_{11} & s_{12} & \mathcal{O}\left(\mathrm{e}^{-(\Gamma_2+\Gamma_2+K)/\varepsilon}\right) \\ -\overline{s_{12}}\frac{s_{11}}{\overline{s_{22}}}\left(1+\mathcal{O}\left(\mathrm{e}^{-2\Gamma_2/\varepsilon}\right)\right) & s_{22} & s_{23} \\ s_{31} & -\overline{s_{23}}\frac{s_{33}}{\overline{s_{22}}}\left(1+\mathcal{O}\left(\mathrm{e}^{-2\Gamma_1/\varepsilon}\right)\right) & s_{33} \end{pmatrix},$$

where all s_{jk}'s above can be computed asymptotically up to exponentially small relative error using (6.10).

The smallest asymptotically computable element s_{31} describes the transition from $e_1(-\infty, \delta)$ to $e_3(+\infty, \delta)$. The result that we obtain for this element is in agreement with the rule of thumb that claims that the transitions take place locally at the avoided crossings and can be considered as independent. Accordingly, we can only estimate the smallest element of all, s_{13}, which describes the transition from $e_3(-\infty, \delta)$ to $e_1(+\infty, \delta)$, for which the avoided crossings are not encountered in the "right order," as discussed in [HP]. It is possible, however, to get an asymptotic expression for this element in some cases. When the unperturbed levels $e_2(z, 0)$ and $e_3(z, 0)$ possess a degeneracy point in S_α and when there exists a dissipative domain for the index 3 of the unperturbed eigenvalues going above this point, one can convince oneself that s_{13} can be computed asymptotically for δ small enough using the techniques presented above.

Our second application is the study of the semiclassical scattering properties of the multichannel stationary Schrödinger equation with energy above the potential barriers. The relevant equation is then

$$-\varepsilon^2 \frac{d^2}{dt^2}\Phi(t) + V(t)\Phi(t) = E\Phi(t), \tag{7.10}$$

where $t \in \mathbf{R}$ is a space variable, $\Phi(t) \in \mathbf{C}^m$ is the wave function, $\varepsilon \to 0$ denotes Planck's constant, $V(t) = V^*(t) \in M_m(\mathbf{C})$ is the matrix of potentials, and the spectral parameter E is kept fixed and large enough so that

$$U(t) \equiv E - V(t) > 0. \tag{7.11}$$

Introducing

$$\psi(t) = \begin{pmatrix} \Phi(t) \\ i\varepsilon\Phi(t) \end{pmatrix} \in \mathbf{C}^{2m}, \tag{7.12}$$

we cast equation (7.10) into the equivalent form (2.1) for $\psi(t)$ with the generator

$$H(t) = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ U(t) & \mathbf{O} \end{pmatrix} \in M_{2m}(\mathbf{C}). \tag{7.13}$$

It is readily verified that

$$H(t) = J^{-1}H^*(t)J, \tag{7.14}$$

with

$$J = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}. \tag{7.15}$$

Concerning the spectrum of $H(t)$, we should remark that if the real and positive eigenvalues of $U(t)$, $k_j^2(t)$, $j = 1, \dots, m$ associated with the eigenvectors $u_j(t) \in \mathbf{C}^m$ are assumed to be distinct, i.e.,

$$0 < k_1^2(t) < k_2^2(t) < \cdots < k_m^2(t), \tag{7.16}$$

then the spectrum of the generator $H(t)$ given by (7.13) consists of $2m$ real distinct eigenvalues

$$-k_m(t) < -k_{m-1}(t) < \cdots < -k_1(t) < k_1(t) < k_2(t) < \cdots < k_m(t) \tag{7.17}$$

associated with the $2m$ eigenvectors

$$\chi_j^\pm(t) = \begin{pmatrix} u_j(t) \\ \pm k_j(t)u_j(t) \end{pmatrix} \in \mathbf{C}^{2m},$$
$$H(t)\chi_j^\pm(t) = \pm k_j(t)\chi_j^\pm(t). \tag{7.18}$$

We check that

$$(\chi_j^{\pm}(0), \chi_j^{\pm}(0))_J = \pm 2k_j(0)\|u_j(0)\| \neq 0, \quad j = 1, \dots, m, \tag{7.19}$$

where $\|u_j(0)\|$ is computed in $\mathbf{C}^m$, so that Proposition 2.1 applies. Before dealing with its consequences, we further make explicit the structure of S. Adopting the notation suggested by (7.17) and (7.18), we write

$$H(t) = \sum_{j=1}^{m} k_j(t) P_j^+(t) - \sum_{j=1}^{m} k_j(t) P_j^-(t), \tag{7.20}$$

$$\psi(t) = \sum_{j=1}^{m} c_j^+(t)\varphi_j^+(t) \mathrm{e}^{-i\int_0^t k_j(t')dt'/\varepsilon} + \sum_{j=1}^{m} c_j^-(t)\varphi_j^-(t) \mathrm{e}^{i\int_0^t k_j(t')dt'/\varepsilon} \tag{7.21}$$

and introduce

$$\mathbf{c}^{\pm}(t) = \begin{pmatrix} c_1^{\pm}(t) \\ c_2^{\pm}(t) \\ \vdots \\ c_m^{\pm}(t) \end{pmatrix} \in \mathbf{C}^m. \tag{7.22}$$

Hence we have the block structure

$$S \begin{pmatrix} \mathbf{c}^+(-\infty) \\ \mathbf{c}^-(-\infty) \end{pmatrix} \equiv \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \begin{pmatrix} \mathbf{c}^+(-\infty) \\ \mathbf{c}^-(-\infty) \end{pmatrix} = \begin{pmatrix} \mathbf{c}^+(+\infty) \\ \mathbf{c}^-(+\infty) \end{pmatrix}, \tag{7.23}$$

where $S_{\sigma\tau} \in M_m(\mathbf{C})$, $\sigma, \tau \in \{+, -\}$.

Let us turn to the symmetry properties of S. We get from (7.19) and Proposition 2.1 that

(7.24)

$$\begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}^{*} \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} S_{++}^* & -S_{-+}^* \\ -S_{+-}^* & S_{--}^* \end{pmatrix}.$$

In terms of the blocks $S_{\sigma\tau}$, this is equivalent to

$$S_{++}S_{++}^* - S_{+-}S_{+-}^* = \mathbf{I}, \tag{7.25}$$

$$S_{++}S_{-+}^* - S_{+-}S_{--}^* = \mathbf{O}, \tag{7.26}$$

$$S_{--}S_{--}^* - S_{-+}S_{-+}^* = \mathbf{I}. \tag{7.27}$$

The block S_{++} describes the transmission coefficients associated with a wave traveling from the right and S_{-+} describes the associated reflexion coefficients. Similarly, S_{--} and S_{+-} are related to the transmission and reflexion coefficients associated with a wave incoming from the left. It should be noted that in the case of equation (7.10), another convention is often used to define an S-matrix (see, for instance, [F1]). This gives rise to a different S-matrix with a similar interpretation. However, it is not difficult to establish a one-to-one correspondence between the two definitions. If the matrix of potentials $V(t)$ is real symmetric, we have further symmetry in the S-matrix.

LEMMA 7.1. *Let S given by* (7.23) *be the S-matrix associated with* (7.10) *under condition* (7.11). *If we further assume that $V(t) = \overline{V(t)}$, then taking $\varphi_j^{\pm}(0) \in \mathbf{R}^{2m}$, $j = 1, \dots, m$, in* (7.21), *we get*

$$S_{++} = \overline{S_{--}}, \qquad S_{+-} = \overline{S_{-+}}. \tag{7.28}$$

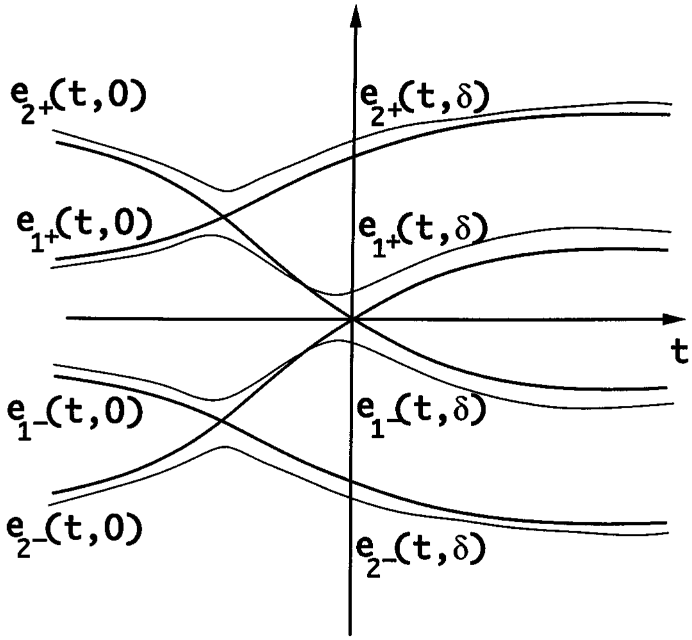

FIG. 8. *The pattern of avoided crossings in the semiclassical context.*

The corresponding results for the S-matrix defined in [F1] are derived in [MN]. The proof of this lemma can be found in Appendix B. We now consider (7.10) in the case where $U(t) = U^*(t) = \overline{U(t)} \in M_2(\mathbf{R})$, which describes a two-channel Schrödinger equation. We assume that the four-level generator $H(t)$ displays three avoided crossings at $t_1 < t_2$, two of which take place at the same point t_1 because of the symmetry of the eigenvalues, as in Figure 8. By Lemma 7.1, it is enough to consider the blocks S_{++} and S_{+-}. The transitions corresponding to elements of these blocks which we can compute asymptotically are from level 1^+ to level 2^+ and from level 2^- to level 1^+. They correspond to elements s_{21}^{++} and s_{12}^{+-}, respectively. With the notation

$$\Gamma_j = \left| \mathrm{Im} \int_{\zeta_j} k_1(z) dz \right|, \quad j = 1, 2, \tag{7.29}$$

where ζ_j is in the upper half-plane, we have the estimates

(7.30)
$$s_{21}^{++} = \mathcal{O}(e^{-\Gamma_1/\varepsilon}), \qquad s_{12}^{+-} = \mathcal{O}(e^{-(\Gamma_1+\Gamma_2)/\varepsilon}), \qquad s_{jj}^{++} = 1 + \mathcal{O}(\varepsilon), \quad j = 1, 2.$$

It follows from (7.26) and Lemma 7.1 that the matrix $S_{++}S_{+-}^T$ is symmetric. Hence

$$s_{11}^{++} s_{21}^{+-} + s_{12}^{++} s_{22}^{+-} = s_{21}^{++} s_{11}^{+-} + s_{22}^{++} s_{12}^{+-}, \tag{7.31}$$

whereas we get from (7.25) that

$$s_{11}^{++} \overline{s_{21}^{++}} + s_{12}^{++} \overline{s_{22}^{++}} = s_{11}^{+-} \overline{s_{21}^{+-}} + s_{12}^{+-} \overline{s_{22}^{+-}}. \tag{7.32}$$

The only useful estimate we get with Theorem 6.1 is

$$s_{22}^{+-} = \mathcal{O}(e^{-(\Gamma_1+\Gamma_2+K)/\varepsilon}), \quad K > 0, \tag{7.33}$$

which together with (7.30) in (7.31) yields

$$s_{21}^{+-} = s_{21}^{++} s_{11}^{+-} / s_{11}^{++} + \mathcal{O}(\mathrm{e}^{-(\Gamma_1+\Gamma_2)/\varepsilon}). \tag{7.34}$$

Thus from (7.32) and (5.34) for s_{11}^{+-},

$$s_{12}^{++} = -\overline{s_{21}^{++}}\frac{\overline{s_{11}^{++}}}{\overline{s_{22}^{++}}}(1+\mathcal{O}(\mathrm{e}^{-\kappa/\varepsilon})), \tag{7.35}$$

with

$$0 < \kappa < \min(\Gamma_1, \Gamma_2). \tag{7.36}$$

Summarizing, we have

$$S_{++} = \begin{pmatrix} s_{11}^{++} & -\overline{s_{21}^{++}}\frac{\overline{s_{11}^{++}}}{\overline{s_{22}^{++}}}\left(1+\mathcal{O}\left(\mathrm{e}^{-\kappa/\varepsilon}\right)\right) \\ s_{21}^{++} & s_{22}^{++} \end{pmatrix} \tag{7.37}$$

and

$$S_{+-} = \begin{pmatrix} \mathcal{O}\left(\mathrm{e}^{-\kappa/\varepsilon}\right) & s_{12}^{+-} \\ \mathcal{O}\left(\mathrm{e}^{-\kappa/\varepsilon}\right) & \mathcal{O}\left(\mathrm{e}^{-(\Gamma_1+\Gamma_2+K)/\varepsilon}\right) \end{pmatrix}, \tag{7.38}$$

where all elements $s_{jk}^{\sigma\tau}$ can be asymptotically computed up to exponentially small relative corrections using (6.10). We obtain no information on the first column of S_{+-} except estimate (5.34), where (7.36) necessarily holds. However, if there exists one or several other dissipative domains for certain indices, it is then possible to get asymptotic formulas for the estimated terms.

Appendix A. Proof of Proposition 2.1. A direct consequence of the property

$$H(t) = H^{\#}(t) = J^{-1}H^*(t)J \tag{A.1}$$

is the relation $\sigma(H(t)) = \overline{\sigma(H(t))}$. Thus if $\sigma(H(0)) \subset \mathbf{R}$, then $\sigma(H(t)) \subset \mathbf{R}$ for all $t \in \mathbf{R}$ since the analytic eigenvalues are assumed to be distinct and nondegenerate for all $t \in \mathbf{R}$. Let $e_j(0)$ be the eigenvalue associated with $\varphi_j(0)$. Then due to the property $H(0) = H^{\#}(0)$,

$$(\varphi_j(0), H(0)\varphi_k(0))_J = e_k(0)(\varphi_j(0), \varphi_k(0))_J = \overline{e_j(0)}(\varphi_j(0), \varphi_k(0))_J \tag{A.2}$$

for any $j, k = 1, \dots, n$. For $j = k$, we get from the assumption $(\varphi_j(0), \varphi_j(0))_J \neq 0$ that $e_j(0) \in \mathbf{R}$, and from the fact that the eigenvalues of $H(0)$ are distinct,

$$(\varphi_j(0), \varphi_k(0))_J = 0, \quad j \neq k. \tag{A.3}$$

The resulting reality of $e_j(t)$ for all $t \in \mathbf{R}$ and $j = 1, \dots, n$ together with (A.1) yields

$$P_j(t) = J^{-1}P_j^*(t)J. \tag{A.4}$$

Hence using the fact that the P_j^*'s are projectors,

$$K(t) = \sum_{j=1}^n P_j{}'(t)P_j(t) = \sum_{j=1}^n (J^{-1}P_j^*(t)J)'J^{-1}P_j^*(t)J = J^{-1}\sum_{j=1}^n P_j^{*\prime}(t)P_j^*(t)J$$

$$= -J^{-1}\sum_{j=1}^n P_j^*(t)P_j^{*\prime}(t)J = -J^{-1}K^*(t)J. \tag{A.5}$$

Let $\Phi, \Psi \in \mathbf{C}^n$ and $W(t)$ be defined by

$$W'(t) = K(t)W(t), \qquad W(0) = \mathbf{I} \tag{A.6}$$

(see (3.5)). Then we have

$$\begin{aligned}(W(t)\Phi, W(t)\Psi)'_J &= \langle W'(t)\Phi|JW(t)\Psi\rangle + \langle W(t)\Phi|JW'(t)\Psi\rangle\\ &= \langle K(t)W(t)\Phi|JW(t)\Psi\rangle + \langle W(t)\Phi|JK(t)W(t)\Psi\rangle\\ &= \langle W(t)\Phi|J(J^{-1}K^*(t)J + K(t))W(t)\Psi\rangle \equiv 0.\end{aligned} \tag{A.7}$$

Thus in the indefinite metric, the scalar products of the eigenvectors of $H(t)$, $\varphi_j(t) = W(t)\varphi_j(0)$ (see (3.7)), are constants:

$$(\varphi_j(t), \varphi_k(t))_J \equiv (\varphi_j(0), \varphi_k(0))_J. \tag{A.8}$$

We can then normalize the $\varphi_j(0)$ in such a way that

$$(\varphi_j(t), \varphi_k(t))_J = (\varphi_j(0), \varphi_k(0))_J = \delta_{jk}\rho_j, \tag{A.9}$$

with $\rho_j \in \{+1, -1\}$. Let $\psi(t)$ and $\chi(t)$ be two solutions of (2.1). By an argument similar to the one above using (A.1), we deduce

$$(\chi(t), \psi(t))_J \equiv (\chi(0), \psi(0))_J. \tag{A.10}$$

Inserting the decompositions

$$\psi(t) = \sum_{j=1}^{n} c_j(t)\mathrm{e}^{-i\int_0^t e_j(t')dt'/\varepsilon}\varphi_j(t), \tag{A.11}$$

$$\chi(t) = \sum_{j=1}^{n} d_j(t)\mathrm{e}^{-i\int_0^t e_j(t')dt'/\varepsilon}\varphi_j(t) \tag{A.12}$$

in this last identity yields

$$\begin{aligned}\sum_{j,k=1}^{n} \overline{d}_k(t)c_j(t)(\varphi_k(t), \varphi_j(t))_J\mathrm{e}^{i\int_0^t (e_k(t')-e_j(t'))/\varepsilon dt'} &= \sum_{j}^{n} \overline{d}_j(t)\rho_j c_j(t)\\ \equiv \sum_{j=1}^{n} \overline{d}_j(0)\rho_j c_j(0) &= \sum_{j=1}^{n} \overline{d}_j(\pm\infty)\rho_j c_j(\pm\infty).\end{aligned} \tag{A.13}$$

Since the initial conditions for the coefficients,

$$c_j(-\infty) = \delta_{jk}, \qquad d_j(-\infty) = \delta_{jl}, \tag{A.14}$$

imply

$$c_j(+\infty) = s_{jk}, \qquad d_j(+\infty) = s_{jl}, \tag{A.15}$$

introducing the matrix $R = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n) \in M_n(\mathbf{C})$, we get from (A.13) that

$$R = S^*RS, \tag{A.16}$$

which is equivalent to the assertion $S^{-1} = RS^*R$. □

Appendix B. Proof of Lemma 7.1. Let $G = G^* = G^{-1}$ be given in block structure by

$$G = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix} \in M_{2m}(\mathbf{C}) \tag{B.1}$$

and $H(t)$ be given by (7.13) with $U(t) = \overline{U(t)} = U^*(t)$. Since

$$GH(t)G = -H(t), \qquad \overline{H(t)} = H(t)\,, \tag{B.2}$$

and the eigenvalues of $H(t)$ are real, it is readily verified that

$$GP_j^{\pm}(t)G = P_j^{\mp}(t), \qquad \overline{P_j^{\pm}(t)} = P_j^{\pm}(t), \quad j = 1, \dots, m. \tag{B.3}$$

Hence

$$K(t) = \sum_{\substack{j=1 \\ \tau=\pm}}^{m} P_j^{\tau\prime}(t) P_j^{\tau}(t) = \overline{K(t)} = GK(t)G, \tag{B.4}$$

from which it follows that the solution $W(t)$ of

$$W'(t) = K(t)W(t), \qquad W(0) = \mathbf{I} \tag{B.5}$$

satisfies

$$W(t) = \overline{W(t)} = GW(t)G. \tag{B.6}$$

Since the matrix of potentials $U(0)$ is real symmetric, its eigenvectors $u_j(0)$ may be chosen real so that we can assume that

$$\varphi_j^{\pm}(0) = \begin{pmatrix} u_j(0) \\ \pm k_j(0) u_j(0) \end{pmatrix} \in \mathbf{R}^{2m}. \tag{B.7}$$

Thus it follows from the above that

$$\varphi_j^{\pm}(t) = W(t)\varphi_j^{\pm}(0) \in \mathbf{R}^{2m}, \quad \forall t \in \mathbf{R}, \tag{B.8}$$

and satisfies

$$G\varphi_j^{\pm}(t) = GW(t)GG\varphi_j^{\pm}(0) = W(t)G\varphi_j^{\pm}(0) = \varphi_j^{\mp}(t). \tag{B.9}$$

Finally, the main consequence of (B.2) is that if $\psi(t)$ is a solution of

$$i\varepsilon\psi'(t) = H(t)\psi(t), \tag{B.10}$$

then $\varphi(t) = G\overline{\psi(t)}$ is another solution, as is easily verified. Thus we can write with (7.21), (B.8), and (B.9) that

$$\begin{aligned} \varphi(t) &= \sum_{j=1}^{m} d_j^{+}(t)\varphi_j^{+}(t) \mathrm{e}^{-i\int_0^t k_j(t')dt'/\varepsilon} + \sum_{j=1}^{m} d_j^{-}(t)\varphi_j^{-}(t) \mathrm{e}^{i\int_0^t k_j(t')dt'/\varepsilon} \\ &= \sum_{j=1}^{m} \overline{c_j^{+}(t)}\varphi_j^{-}(t) \mathrm{e}^{i\int_0^t k_j(t')dt'/\varepsilon} + \sum_{j=1}^{m} \overline{c_j^{-}(t)}\varphi_j^{+}(t) \mathrm{e}^{-i\int_0^t k_j(t')dt'/\varepsilon}, \end{aligned} \tag{B.11}$$

i.e.,

$$d_j^+(t) = \overline{c_j^-(t)},$$

$$d_j^-(t) = \overline{c_j^+(t)}, \quad \forall j = 1, \ldots, m, \quad \forall t \in \mathbf{R}. \tag{B.12}$$

Finally, using the definition (7.23) and the above property for $t = \pm\infty$, we get for any $\mathbf{d}^{\pm}(-\infty) \in \mathbf{C}^m$ that

(B.13)

$$\begin{pmatrix} \mathbf{d}^+(+\infty) \\ \mathbf{d}^-(+\infty) \end{pmatrix} = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \begin{pmatrix} \mathbf{d}^+(-\infty) \\ \mathbf{d}^-(-\infty) \end{pmatrix} = \begin{pmatrix} \overline{S_{--}} & \overline{S_{-+}} \\ \overline{S_{+-}} & \overline{S_{++}} \end{pmatrix} \begin{pmatrix} \mathbf{d}^+(-\infty) \\ \mathbf{d}^-(-\infty) \end{pmatrix},$$

from which the result follows. □

Acknowledgments. It is a great pleasure to thank Charles-Edouard Pfister for many enlightening and fruitful discussions which took place in Marseille and Lausanne. The hospitality of the Institut de Physique Théorique de l'EPFL, where part of this work was performed, is acknowledged.

REFERENCES

[Ba] H. Baklouti, *Asymptotique de largeurs de résonnances pour un modèle d'Effet tunnel microlocal*, thèse, Université de Paris Nord, Paris, 1995.

[Be] M. V. Berry, *Histories of adiabatic quantum transitions*, Proc. Roy. Soc. London Ser. A, 429 (1990), pp. 61–72.

[BE] S. Brundobler and V. Elser, *S-matrix for generalized Landau–Zener problem*, J. Phys. A, 26 (1993), pp. 1211–1227.

[CH1] C. E. Carroll and F. T. Hioe, *Generalization of the Landau–Zener calculation to three-level systems*, J. Phys. A, 19 (1986), pp. 1151–1161.

[CH2] C. E. Carroll and F. T. Hioe, *Transition probabilities for the three-level Landau–Zener model*, J. Phys. A, 19 (1986), pp. 2061–2073.

[D] Yu. N. Demkov, *Adiabatic perturbation of discrete spectrum states*, Soviet Phys. Dokl., 11 (1966), p. 138.

[F1] M. Fedoriuk, *Méthodes Asymptotiques pour les Equations Différentielles Ordinaires Linéaires*, Mir, Moscow, 1987.

[F2] M. V. Fedoryuk, *Analysis* I, in Encyclopaedia of Mathematical Sciences, Vol. 13, R. V. Gamkrelidze, ed., Springer-Verlag, Berlin, New York, Heidelberg, 1989.

[FF] N. Fröman and P. O. Fröman, *JWKB Approximation: Contributions to the Theory*, North–Holland, Amsterdam, 1965.

[HP] J.-T. Hwang and P. Pechukas, *The adiabatic theorem in the complex plane and the semi-classical calculation of non-adiabatic transition amplitudes*, J. Chem. Phys., 67 (1977), pp. 4640–4653.

[J] A. Joye, *Proof of the Landau–Zener formula*, Asymptotic Anal., 9 (1994), pp. 209–258.

[JKP] A. Joye, H. Kunz, and C.-E. Pfister, *Exponential decay and geometric aspect of transition probabilities in the adiabatic limit*, Ann. Phys., 208 (1991), pp. 299–332.

[JP1] A. Joye and C.-E. Pfister, *Exponentially small adiabatic invariant for the Shhrödinger equation*, Comm. Math. Phys., 140 (1991), pp. 15–41.

[JP2] A. Joye and C.-E. Pfister, *Superadiabatic evolution and adiabatic transition probability between two non-degenerate levels isolated in the spectrum*, J. Math. Phys., 34 (1993), pp. 454–479.

[JP3] A. Joye and C.-E. Pfister, *Quantum adiabatic evolution*, in On Three Levels: Micro-, Meso-, and Macro-Approaches in Physics (Leuven Conference Proceedings), M. Fannes, C. Meas, and A. Verbeure, eds., Plenum Press, New York, 1994, pp. 139–148.

[JP4] A. Joye and C.-E. Pfister, *Semi-classical asymptotics beyond all orders for simple scattering systems*, SIAM J. Math. Anal., 26 (1995), pp. 944–977.

[K] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, New York, Heidelberg, 1980.

[Kr] S. G. KREIN, *Linear Differential Equations in Banach Spaces*, Transl. Math. Monographs 29, AMS, Providence, RI, 1971.

[M] A. MARTINEZ, *Precise exponential estimates in adiabatic theory*, J. Math. Phys., 35 (1994), pp. 3889–3915.

[MN] P. A. MARTIN AND G. NENCIU, *Semi-classical inelastic S-matrix for one-dimensional N-states systems*, Rev. Math. Phys., 7 (1995), pp. 193–242.

[N] G. NENCIU, *Linear adiabatic theory and applications: Exponential estimates*, Comm. Math. Phys., 152 (1993), pp. 479–496.

[O] F. W. J. OLVER, *General connection formulae for Liouville–Green approximations in the complex plane*, Philos. Trans. Roy. Soc. London Ser. A, 289 (1978), pp. 501–548.

[R] T. RAMOND, *Semiclassical study of quantum scattering on the line*, preprint, Université de Paris Nord, Paris, 1994.

[Sj] J. SJÖSTRAND, *Projecteurs adiabatiques du point de vue pseudodifférentiel*, C. R. Acad. Sci. Paris Sér. I Math., 22 (1993), pp. 217–220.

[So] E. A. SOLOV'EV, *Nonadiabatic transitions in atomic collisions*, Soviet Phys. Uspekhi, 32 (1989), pp. 228–250.

[W] W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations*, Wiley–Interscience, New York, 1965.