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Spectral Analysis of Unitary Band Matrices

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Abstract: This paper is devoted to the spectral properties of a class of unitary operators with a matrix representation displaying a band structure. Such band matrices appear as monodromy operators in the study of certain quantum dynamical systems. These doubly infinite matrices essentially depend on an infinite sequence of phases which govern their spectral properties. We prove the spectrum is purely singular for random phases and purely absolutely continuous in case they provide the doubly infinite matrix with a periodic structure in the diagonal direction. We also study some properties of the singular spectrum of such matrices considered as infinite in one direction only.

1. Introduction

The dynamical stability of quantum systems governed by a time periodic Hamiltonian is often characterized in terms of the spectral properties of the corresponding monodromy operator, a unitary operator defined as the evolution generated by the Hamiltonian over a period. A first rough classification consists in determining whether or not the spectrum of the monodromy operator contains an absolutely continuous (a.c.) component. The presence of absolutely continuous spectrum is a signature of unstable quantum systems, whereas a purely singular spectrum is a characteristic of quantum stability.

For smooth Hamiltonians, these spectral properties can be obtained through the study of an associated self-adjoint operator, the so-called Floquet or quasi-energy operator [Ho1], [Y]. In case the Hamiltonian is singular, e.g. when it corresponds to a kicked system, one is often lead to consider the monodromy operator directly [Co2]. In both situations, one is typically confronted with a problem where a dense pure point operator is perturbed either by the addition of a self-adjoint operator in the first case, or by a multiplicative unitary perturbation in the second case. A more or less detailed spectral analysis can thus be performed provided a perturbative framework of some sort is available, or in case disorder is present. See e.g. [Be, DS, DLSV, GY, Ho2, Ho3, N, J] for the smooth case, and also the review [Co2, Co1, dO, ADE, Bo] for the kicked case.

The dynamical quantum systems we address here are characterized by a monodromy operator given by a product of two pure point unitaries, neither of which can be considered a perturbation of the other. However, the spectral analysis can be carried over under certain circumstances due to the fact the monodromy operator has a band structure in some basis. The motivation of the construction of such operators is borrowed from the work [BB] which we briefly recall below.

As noted by these authors, this structure allows us to adapt the techniques developed in the study of one dimensional discrete Schrödinger operators to the unitary framework in order to obtain results about the spectrum of such monodromy operators.

Let us briefly summerize the paper. In Sec. 2, we define explicitly the class of unitary operators on the integer lattices $\mathbb{Z}$ and $\mathbb{N}$ that we shall study and discuss their relationship to [BB]. These operators depend on transmission and reflection amplitudes at each lattice point. Some simple perturbative results for essential and absolutely continuous spectra are obtained in Sec. 3. Here, the moduli of the transmission and reflection amplitudes may vary from point to point, but in the remainder of the paper these moduli are assumed constant on the lattice. In Secs. 4 and 5, we consider the random case, in which the phases are independent and randomly distributed on the circle, and we prove that the spectrum is purely singular. To do this, we first establish a version of the Ishii-Pastur Theorem according to which the absolutely continuous part of the spectrum is almost surely supported on the closure of the set where the Lyapunov exponent vanishes and then prove that the Lyapunov exponent is everywhere positive. In Sec. 6, we consider the coherent case, in which the phases are eventually periodic. We identify the absolutely continuous spectrum, and show that the singular continuous spectrum is absent. Finally, in Sec. 7, we give an example in which the phases are almost periodic and the spectrum is purely singular continuous.

2. Construction of the Monodromy Operator

We consider a class of monodromy operators whose construction is motivated by the study of a model of electronic transport in a ring threaded by a linear time dependent magnetic flux, as discussed in [BB], and references therein. Neglecting the curvature of the ring, the instantaneous Hamiltonian of the one-body Schrödinger operator corresponds to that of a one dimensional Schrödinger operator with a periodic potential describing the material of the ring and time dependent boundary conditions of Floquet type. With a choice of linear flux, the time plays the role of the quasi-momentum. Therefore, as a function of time, the Hamiltonian is periodic and its instantaneous spectrum is given by the band structure corresponding to the potential. Under some adiabaticity condition, the evolution operator is assumed to couple states by adjacent pairs of states only by means of the Landau-Zener mechanism. The concerned states are those whose corresponding eigenvalues become close to one another. Thus, a given state with index k say, is coupled once to the one with index $k-1$ and once with the one index $k+1$. This yields the band structure of the evolution operator over a period in the basis of eigenvectors at time zero, say. We refer the reader to this paper for physical background and further description of the regime in which the model holds. Let us now define our monodromy operator following the main lines of the construction sketched above.

Our separable Hilbert space is $l^2(\mathbb{Z})$ and we denote the canonical basis by $\{\varphi_k\}_{k\in\mathbb{Z}}$. In order to make contact with the above model, we shall also state results for $l^2(\mathbb{N}^*)$.

The most general 2×2 unitary matrix depends on 4 parameters and can be written as

$$S = \mathrm{e}^{-i\theta} \begin{pmatrix} r\mathrm{e}^{-i\alpha} & it\mathrm{e}^{i\gamma} \\ it\mathrm{e}^{-i\gamma} & r\mathrm{e}^{i\alpha} \end{pmatrix}, \tag{2.1}$$

where α, γ, θ belong to the torus $\mathbb{T}$ and the real parameters t, r, also called reflection and transition coefficients, are linked by $r^2 + t^2 = 1$.

We introduce an infinite set of such matrices $\{S_k\}_{k\in\mathbb{Z}}$, where S_k depends on the phases $\alpha_k, \gamma_k, \theta_k$, and the reflection and transition coefficients t_k, r_k. They are the building blocks of our monodromy operator in $l^2(\mathbb{Z})$.

Let P_j be the orthogonal projector on the span of φ_j, φ_{j+1} in $l^2(\mathbb{Z})$. We introduce U_e, U_o, two 2×2 block diagonal unitary operators on $l^2(\mathbb{Z})$ defined by

$$\begin{aligned} U_e &= \sum_{k\in\mathbb{Z}} P_{2k} S_{2k} P_{2k}, \\ U_o &= \sum_{k\in\mathbb{Z}} P_{2k+1} S_{2k+1} P_{2k+1}, \end{aligned} \tag{2.2}$$

or, in matrix representation in the canonical basis,

$$U_e = \begin{pmatrix} \ddots & & & & \\ & S_{-2} & & & \\ & & S_0 & & \\ & & & S_2 & \\ & & & & \ddots \end{pmatrix} \tag{2.3}$$

and similarly for U_o, with S_{2k+1} in place of S_{2k}. Note that the 2×2 blocks in U_e are shifted by one with respect to those of U_o along the diagonal.

We now define the monodromy operator U, object of our investigations, by

$$U = U_o U_e, \tag{2.4}$$

such that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} U\varphi_{2k} &= ir_{2k}t_{2k-1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k-1})}\mathrm{e}^{-i(\alpha_{2k}-\gamma_{2k-1})}\varphi_{2k-1} \\ &\quad + r_{2k}r_{2k-1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k-1})}\mathrm{e}^{-i(\alpha_{2k}-\alpha_{2k-1})}\varphi_{2k} \\ &\quad + ir_{2k+1}t_{2k}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k+1})}\mathrm{e}^{-i(\gamma_{2k}+\alpha_{2k+1})}\varphi_{2k+1} \\ &\quad - t_{2k}t_{2k+1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k+1})}\mathrm{e}^{-i(\gamma_{2k}+\gamma_{2k+1})}\varphi_{2k+2}, \\ U\varphi_{2k+1} &= -t_{2k}t_{2k-1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k-1})}\mathrm{e}^{i(\gamma_{2k}+\gamma_{2k-1})}\varphi_{2k-1} \\ &\quad + it_{2k}r_{2k-1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k-1})}\mathrm{e}^{i(\gamma_{2k}+\alpha_{2k-1})}\varphi_{2k} \\ &\quad + r_{2k}r_{2k+1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k+1})}\mathrm{e}^{i(\alpha_{2k}-\alpha_{2k+1})}\varphi_{2k+1} \\ &\quad + ir_{2k}t_{2k+1}\mathrm{e}^{-i(\theta_{2k}+\theta_{2k+1})}\mathrm{e}^{i(\alpha_{2k}-\gamma_{2k+1})}\varphi_{2k+2}. \end{aligned} \tag{2.5}$$

In matrix form, without expliciting the elements, we have the structure

$$U = \begin{pmatrix} \ddots & & & & & & & & & \\ * & * & * & * & & & & & & \\ * & * & * & * & & & & & & \\ & & * & * & * & * & & & & \\ & & * & * & * & * & & & & \\ & & & & * & * & * & * & & \\ & & & & * & * & * & * & & \\ & & & & & & * & * & * & * \\ & & & & & & * & * & * & * \\ & & & & & & & & & \ddots \end{pmatrix}. \tag{2.6}$$

In the regime considered in [BB], the transition coefficients t_k, and all elements of the scattering matrices S_k can be computed from the band functions of the periodic background potential. In particular, the transition coefficients may admit a limit as $k \to \infty$.

In this paper, we briefly show how to get information on the spectral properties of the monodromy operators defined on $l^2(\mathbb{N}^*)$ from those of operators defined on $l^2(\mathbb{Z})$. Also, we briefly demonstrate how spectral properties of U when the t_k's have limits $t_\pm$ as $k \to \pm\infty$ can be related to those of the limiting operator with constant (in k) transition coefficients t. Then we focus on the case of constant transition and reflection coefficients $t_k = t \in]0, 1[$ i.e. $r_k = r \in]0, 1[$, for all $k \in \mathbb{Z}$, which is the main object of our analysis. This corresponds to a regime of the original model in which the sole behavior of the scattering phases $\theta_k, \gamma_k, \alpha_k$ determine the spectral properties of U. It is argued in [BB] on the basis of numerical computations that in case these phases have a coherent behavior as functions of k, if they are periodic say, U has an a.c. component in its spectrum, whereas U should be singular if some phases are random. Following their arguments, we are aiming at a rigorous version of similar statements in our setting.

3. First Properties

At this point, we have slightly generalized the construction proposed by [BB] in order to define our monodromy operator, a unitary pentadiagonal band matrix. Before going further in the analysis, one can ask whether simpler unitary band matrices could provide interesting models, spectrally speaking, as is the case in the self-adjoint setting where the discrete Schrödinger operators are tridiagonal, though non-trivial. The next lemma answers this question negatively, validating our model from another point of view. Its proof can be found in the Appendix.

Lemma 3.1. *If U is unitary and tridiagonal, then U is either unitarily equivalent to a (bilateral) shift operator, or it is an infinite direct sum of* 2×2 *and* 1×1 *unitary matrices.*

On the other hand, it is straightforward to construct unitary band matrices with larger width starting with general unitary finite size matrices, following the same steps as above.

Perturbative results. In the physical context alluded to above, the natural Hilbert space is $l^2(\mathbb{N}^*)$, with $\mathbb{N}^*$ the set of positive integers, and the definition of the unitary monodromy operator, say U^+, is

$$\begin{aligned} &U^+\varphi_1 = r_1 \mathrm{e}^{-i(\theta_0+\theta_1)}\mathrm{e}^{-i\alpha_1}\varphi_1 + it_1\mathrm{e}^{-i(\theta_0+\theta_1)}\mathrm{e}^{-i\gamma_1}\varphi_2, \\ &U^+\varphi_k, \ \ k > 1, \end{aligned} \tag{3.1}$$

as in (2.5). We shall also define U^- on $l^2(-\mathbb{N}^*)$ in a similar fashion. Consider U^e on $l^2(\mathbb{Z})$ defined by (2.5) with even matrix elements

$$\{t_{-k}, \theta_{-k}, \alpha_{-k}, \gamma_{-k}\} = \{t_k, \theta_k, \alpha_k, \gamma_k\} \quad \forall k \in \mathbb{N}. \tag{3.2}$$

Theorem 3.1. *Let U^+ and U^e be as above and let $U^+_{a.c.}$ and $U^e_{a.c.}$ denote their restriction to their respective absolutely continuous subspaces. Then*

$$\sigma_{ess}(U^+) = \sigma_{ess}(U^e), \quad \text{and} \quad U^+_{a.c.} \oplus U^+_{a.c.} \simeq U^e_{a.c.},$$

where $\simeq$ means unitary equivalence.

Proof. We can write on $l^2(-\mathbb{N}^*) \oplus \mathbb{C} \oplus l^2(\mathbb{N}^*)$,

$$\begin{aligned} U^e &= \begin{pmatrix} CU^+C^{-1} & & \\ & 1 & \\ & & U^+ \end{pmatrix} + F \\ &= \begin{pmatrix} C & & \\ & 1 & \\ & & \mathbb{I} \end{pmatrix} \begin{pmatrix} U^+ & & \\ & 1 & \\ & & U^+ \end{pmatrix} \begin{pmatrix} C^{-1} & & \\ & 1 & \\ & & \mathbb{I} \end{pmatrix} + F, \end{aligned} \tag{3.3}$$

where absent elements denote zeros, $\mathbb{I}$ is the identity, C is the operator

$$\begin{aligned} C : l^2(\mathbb{N}^*) &\to l^2(-\mathbb{N}^*) \\ \varphi_k &\mapsto \varphi_{-k} \end{aligned} \tag{3.4}$$

and F is a finite rank operator. Noting that $\sigma(CU^+C^{-1}) = \sigma(U^+)$, we get the result by Weyl's and Birman-Krein's theorems on invariance of essential, resp. absolutely continuous, spectrum, under compact, resp. trace class, perturbation. □

Let us now consider the situation where the transition coefficients of the operator U defined by (2.5) satisfy

$$\lim_{k\to\pm\infty} t_k = t_\pm \Longleftrightarrow \lim_{k\to\pm\infty} r_k = r_\pm. \tag{3.5}$$

We measure the convergence by means of the quantities $\delta^\pm$ defined by

$$\delta^+(j) = \max\{r_j r_{j-1} - r_+^2\,,\ t_j t_{j-1} - t_+^2\,,\ t_j r_{j\pm 1} - r_+ t_+\}\,, \quad j \in \mathbb{N}, \tag{3.6}$$

and similarly for δ^-. Let $U^\pm(t_\pm)$ be defined on $l^2(\pm\mathbb{N}^*)$ by (3.1) with $t_k = t_\pm$ and $r_k = r_\pm$, for all $k \in \pm\mathbb{N}^*$.

Theorem 3.2. *Assume (3.5) and let U and $U^{\pm}(t_{\pm})$ be as above. Then*

$$\sigma_{ess}(U) = \sigma_{ess}(U^{+}(t_{+})) \cup \sigma_{ess}(U^{-}(t_{-})).$$

If, furthermore, there exists $\epsilon > 1/2$ such that $\sup_{j\in\mathbb{N}} \delta^{\pm}(j) j^{2\epsilon} < \infty$,

$$U_{a.c.} \simeq U^{+}_{a.c.}(t_{+}) \oplus U^{-}_{a.c.}(t_{-}).$$

Proof. Let us introduce the asymptotic unitary operator $U_{-,+}$ by

$$U_{-,+} = \begin{pmatrix} U^{-}(t_{-}) & & \\ & 1 & \\ & & U^{+}(t_{+}) \end{pmatrix}. \tag{3.7}$$

The difference between the actual and asymptotic operators is given by the operator

$$\Delta = U - U_{-,+} \tag{3.8}$$

whose matrix elements $\Delta(j,k) = \langle \varphi_j | \Delta \varphi_k \rangle$ satisfy for $|k| > 1$,

$$|\Delta(j,k)| \leq \{\delta^{\pm}(j+1)\} \text{ if } |j-k| \leq 2 0 \text{ otherwise}. \tag{3.9}$$

Therefore, approximating Δ by a finite matrix Δ_N, we can use the Schur condition, [K], P. 143, to estimate the norm of the difference $\Delta - \Delta_N$ and get $\|\Delta - \Delta_N\| \to 0$ as $N \to \infty$. This, in turn, shows that Δ is compact and that the essential spectra of U and $U_{-,+}$ coincide and yields the first assertion. The second is proven following arguments used in [Ho2]. Let $\epsilon > 1/2$ and set $\langle j \rangle = (1+j^2)^{1/2}$. We define $\Lambda = \text{diag}\ \{\langle j \rangle^{\epsilon}\}$ in the basis $\{\varphi_k\}_{k\in\mathbb{Z}}$. As $\Delta = \Lambda^{-1}(\Lambda\Delta\Lambda)\Lambda^{-1}$, where Λ^{-1} is Hilbert-Schmidt, Δ will be trace class as soon as $\Lambda\Delta\Lambda$ is bounded. Its non-zero matrix elements are

$$(\Lambda\Delta\Lambda)(j,k) = (\langle j \rangle \langle k \rangle)^{\epsilon} \Delta(j,k), \quad k = j, j \pm 1, j \pm 2, \tag{3.10}$$

so that we get boundedness as above from the Schur condition and the estimate (3.9).
□

Remarks. i) An analogous statement is obviously true for operators defined on $l^2(\mathbb{N}^*)$.
ii) The condition $\sup_{j\in\mathbb{N}} \delta^{\pm}(j) j^{2\epsilon} < \infty$ for some $\epsilon > 1/2$, actually is necessary as well to have Δ trace-class. Indeed, in case $t_{\pm} = t \iff r_{\pm} = r$ with $tr \neq 0$, $t \neq r$, and $t - t_j = 1/j^{\alpha}$, one checks that $\delta(j) \sim c/j^{\alpha}$, for some constant c. Assuming that Δ is trace class, we have $\sum_{j\in\mathbb{Z}} |\Delta(j,j)| < \infty$ which is equivalent to $\sum_{j\in\mathbb{Z}} 1/\langle j \rangle^{\alpha} < \infty$ and requires $\alpha > 2\epsilon$, for some $\epsilon > 1/2$.
iii) It is clear that similar perturbative results hold for more general cases where the phases have a limiting behavior as well.

The case $t_{+} = t_{-} = 0$ is of particular interest and allows a stronger result.

Theorem 3.3. *Consider U on $l^2(\mathbb{Z})$ defined by (2.5) and U^{+} on $l^2(\mathbb{N}^*)$ defined by (3.1). If* $\liminf_{k\to\pm\infty} t_k = 0$, *then*

$$\sigma_{a.c.}(U) = \sigma_{a.c.}(U^{+}) = \emptyset. \tag{3.11}$$

Proof. We consider U only, the proof for U^+ being similar. Let U_n be equal to U with $t_n = 0$ and

$$F_n = U - U_n. \tag{3.12}$$

The matrix of U_n is separated into two disjoint blocks and F_n is a rank four operator with $\|F_n\| \leq ct_n$. The hypothesis insures the existence of a subsequence $t_{n(k)}$ going to zero as fast as we wish, say as $\langle k\rangle^{-2}$, when $k \to \pm\infty$. We set

$$G = \sum_{k\in\mathbb{Z}} F_{n(k)} \quad \text{and} \quad \tilde{U} = U - G. \tag{3.13}$$

By construction, we have for some constant $\tilde{c}$,

$$\|G\|_1 \leq 4 \sum_{k\in\mathbb{Z}} \|F_{n(k)}\| \leq \sum_{k\in\mathbb{Z}} \frac{\tilde{c}}{\langle k\rangle^2} < \infty, \tag{3.14}$$

and $\tilde{U}$ is pure point, hence the result. □

Remarks. i) In case there exists a subsequence $\{t_{n(k)}\}$ such that

$$\lim_{k\to+\infty} t_{n(k)} = t_+ \quad \text{and} \quad \lim_{k\to-\infty} t_{n(k)} = t_-, \tag{3.15}$$

a similar construction is valid and we get an approximation of the form

$$U = \tilde{U}_{-,+} + G_{-,+}, \tag{3.16}$$

where $\tilde{U}_{-,+}$ contains an infinite number of t_- and t_+ in its matrix representation and $G_{-,+}$ is trace class. However we do not know the spectral properties of such $\tilde{U}_{-,+}$'s.
ii) If $U^+(0)$ defined as in Theorem 3.2 is such that its pure point spectrum possesses a finite number of accumulation points only, then, if $\lim t_j \to 0$, $\sigma(U^+)$ is pure point with finitely many accumulation points as well. This will be true in case the phases have a coherent behavior, see Sect. 6.

Motivated by the previous theorems, we now address the spectral properties of the limiting operators.

Constant reflection and transition coefficients. From now, $t_k = t$, $r_k = r$, $\forall k \in \mathbb{Z}$. We first note that the extreme cases where $rt = 0$ are spectrally trivial.

Proposition 3.1. *In case $t = 0$ i.e. $r = 1$, U is pure point and if $t = 1$ i.e. $r = 0$, U is purely absolutely continuous.*

Proof. The first case is trivial. In the second case, we observe that U is reduced by the supplementary subspaces L^+, respectively L^-, generated by the vectors in the canonical basis with even indices, respectively odd indices. Moreover $U|_{L^\pm}$ is unitarily equivalent to the shift operator, hence the result. □

Remark. As a typical corollary, we get the following spectral properties for monodromy operators U^+ defined on $l^2(\mathbb{N}^*)$ according to (3.1):

$$\sigma_{a.c.}(U^+) = S^1 \text{ if } 1 - t_j \sim 1/j^\alpha, \quad \alpha > 1. \tag{3.17}$$

The rest of the paper is devoted to studying the limiting operator when $t_k = t \in]0, 1[$ i.e. $r_k = r \in]0, 1[$, for all $k \in \mathbb{Z}$.

All phases in the definition of U do not play the same role, as the following lemma shows. On the one hand it justifies the choice made in [BB] where the phases γ_k are taken equal to zero and, on the other hand, it will be very useful below.

Lemma 3.2. *If we denote the matrix (2.6) by* $M(\{\theta_k\}, \{\alpha_k\}, \{\gamma_k\})$*, and identify* U *to it, we have for any sequences* $\{\theta_k\}, \{\alpha_k\}, \{\gamma_k\}$, $k \in \mathbb{Z}$,

$$U \equiv M(\{\theta_k\}, \{\alpha_k\}, \{\gamma_k\}) \simeq M(\{\theta_k\}, \{\alpha_k\}, \{0\}).$$

Remarks. i) As a corollary, we can replace the sequence $\{\gamma_k\}$, $k \in \mathbb{Z}$ in the definition of U by any other sequence $\{\gamma'_k\}$.
ii) The same statement is true for U^+ defined on $l^2(\mathbb{N}^*)$ by (3.1).

Proof. Let V be the unitary operator defined by

$$V\varphi_k = \mathrm{e}^{i\zeta_k}\varphi_k, \quad k \in \mathbb{Z}. \tag{3.18}$$

One checks easily that the operator $V^{-1}UV$ has the form $M(\{\theta_k\}, \{\alpha_k\}, \{0\})$ in the canonical basis provided for all $j \in \mathbb{Z}$,

$$\zeta_j - \zeta_{j-1} = -\gamma_{j-1}. \tag{3.19}$$

This is realized by taking, for example, $\zeta_0 = 0$ and

$$\zeta_k = -\sum_{j=0}^{k-1} \gamma_j, \quad \zeta_{-k} = \sum_{j=-1}^{-k} \gamma_j, \quad k \in \mathbb{N}^*. \tag{3.20}$$

□

Generalized eigenvectors. Without making use yet of the freedom we have in the sequence $\{\gamma_k\}$, $k \in \mathbb{Z}$, we now turn to the eigenvalue equation

$$\begin{aligned} U\psi &= \mathrm{e}^{i\lambda}\psi, \\ \psi &= \sum_{k\in\mathbb{Z}} c_k\varphi_k, \ \ c_k \in \mathbb{C}, \quad \lambda \in \mathbb{C}. \end{aligned} \tag{3.21}$$

One sees from the structure (2.6) of the operator U, that if ψ satisfies (3.21), a linear relation between the coefficients (c_{2k}, c_{2k+1}) and (c_{2k-2}, c_{2k-1}) of the form

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = T(k) \begin{pmatrix} c_{2k-2} \\ c_{2k-1} \end{pmatrix} \tag{3.22}$$

must exist, provided some 2×2 matrix is invertible. Using the definition (2.5), straightforward computations show that the matrix $T(k)$ has elements

$$\begin{aligned}
T(k)_{11} &= -\mathrm{e}^{-i(\lambda+\gamma_{2k-1}+\gamma_{2k-2}+\theta_{2k-1}+\theta_{2k-2})},\\
T(k)_{12} &= i\frac{r}{t}\left(\mathrm{e}^{-i(\lambda+\gamma_{2k-1}-\alpha_{2k-2}+\theta_{2k-1}+\theta_{2k-2})} - \mathrm{e}^{-i(\gamma_{2k-1}-\alpha_{2k-1})}\right),\\
T(k)_{21} &= i\frac{r}{t}\left(\mathrm{e}^{-i(\theta_{2k-2}-\theta_{2k}+\gamma_{2k}+\gamma_{2k-1}+\gamma_{2k-2}+\alpha_{2k-1})}\right.\\
&\qquad \left. -\,\mathrm{e}^{-i(\lambda+\theta_{2k-2}+\theta_{2k-1}+\gamma_{2k}+\gamma_{2k-1}+\gamma_{2k-2}+\alpha_{2k})}\right), \qquad (3.23)\\
T(k)_{22} &= -\frac{1}{t^2}\mathrm{e}^{i(\lambda+\theta_{2k}+\theta_{2k-1}-\gamma_{2k}-\gamma_{2k-1})}\\
&\quad +\frac{r^2}{t^2}\mathrm{e}^{-i(\gamma_{2k}+\gamma_{2k-1})}\left(\mathrm{e}^{i(\theta_{2k}-\theta_{2k-2}+\alpha_{2k-2}-\alpha_{2k-1})}+\mathrm{e}^{-i(\alpha_{2k}-\alpha_{2k-1})}\right)\\
&\quad -\frac{r^2}{t^2}\mathrm{e}^{-i(\lambda+\theta_{2k-2}+\theta_{2k-1}+\gamma_{2k}+\gamma_{2k-1}+\alpha_{2k}-\alpha_{2k-2})},
\end{aligned}$$

provided $t \neq 0$. We also compute

$$\det T(k) = \mathrm{e}^{-i(\theta_{2k-2}-\theta_{2k}+\gamma_{2k}+2\gamma_{2k-1}+\gamma_{2k-2})} \qquad (3.24)$$

so that $|\det T(k)| = 1$.

Therefore, once the coefficients (c_0, c_1) are given, we compute for any $k \in \mathbb{N}^*$,

$$\begin{aligned}
\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} &= T(k)\cdots T(2)T(1)\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \equiv \Phi(k)\begin{pmatrix} c_0 \\ c_1 \end{pmatrix},\\
\begin{pmatrix} c_{-2k} \\ c_{-2k+1} \end{pmatrix} &= T(-k+1)^{-1}\cdots T(-1)^{-1}T(0)^{-1}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \equiv \Phi(-k)\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.
\end{aligned} \qquad (3.25)$$

The multiplicity of possible eigenvalues is therefore bounded by two.

4. Random Setting

Apart from the fact that our transfer matrix is complex valued instead of the usual real valued setting suiting the discrete Schrödinger case, we will see that here also one Lyapunov exponent is enough to describe the spectral properties of U, when the phases are random and the transfer matrices $T(k)$ are independent and identically distributed.

Making use of Lemma 3.2, let us introduce a probabilistic space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is identified with $\{\mathbb{T}^{\mathbb{Z}}\}$, $\mathbb{T}$ being the torus, and $\mathbb{P} = \otimes_{k\in\mathbb{Z}}\mathbb{P}_0$, where $\mathbb{P}_0$ is the uniform distribution on $\mathbb{T}$ with $\mathcal{F}$ the σ-algebra generated by the cylinders. We introduce the set of random vectors on $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\begin{aligned}
\Omega &\rightarrow \mathbb{T}^2\\
\omega &\mapsto (\theta_k, \alpha_k) \;\; k \in \mathbb{Z},
\end{aligned} \qquad (4.1)$$

$$\text{with} \quad \theta_k(\omega) = \omega_{2k}\,, \quad \alpha_k(\omega) = \omega_{2k+1}\,.$$

The random vectors $\{\beta_k\}_{k\in\mathbb{Z}}$ are i.i.d. and uniformly distributed on $\mathbb{T}^2$.

We denote by U_ω the random unitary operator corresponding to the random infinite matrix (2.6),

$$U_\omega = M(\{\theta_k(\omega)\}, \{\alpha_k(\omega)\}, \{0\}). \qquad (4.2)$$

Introducing the shift operator S on Ω by

$$S(\omega)_k = \omega_{k+2}, \quad k \in \mathbb{Z}, \tag{4.3}$$

we get an ergodic set $\{S^j\}_{j\in\mathbb{Z}}$ of translations. With the unitary operator V_j defined on the canonical basis of $l^2(\mathbb{Z})$ by

$$V_j \varphi_k = \varphi_{k-2j}, \quad \forall k \in \mathbb{Z}, \tag{4.4}$$

we observe that for any $j \in \mathbb{Z}$,

$$U_{S^j\omega} = V_j U_\omega V_j^*. \tag{4.5}$$

Therefore, our random operator U_ω is an ergodic operator. The spectral projectors $E_\Delta(\omega)$ of U_ω, where Δ is a Borel set of $\mathbb{T}$, define a weakly measurable projector valued family of operators on Ω and the spectrum of U_ω is deterministic, see [CL]. However, we shall not make use of these properties below.

As it stands, the transfer matrix $T(k)$ depending on the random vectors β_{2k}, β_{2k-1}, β_{2k-2} seems to be correlated with $T(k+1)$ and $T(k-1)$. Using the same Lemma 3.2, we can replace the sequence $\{0\}$ in (4.2) by $\{(-1)^{k+1}\alpha_k\}$, so that we consider explicitly $M(\{\theta_k\}, \{\alpha_k\}, \{(-1)^{k+1}\alpha_k\})$ and the corresponding transfer matrices. Thus, in terms of the new variable, with $\lambda \in \mathbb{R}$,

$$\eta_k(\lambda) = \theta_k + \theta_{k-1} + \alpha_k - \alpha_{k-1} + \lambda, \tag{4.6}$$

the transfer matrix can be written as

$$T(k) \equiv T(\eta_{2k}(\lambda), \eta_{2k-1}(\lambda)) \tag{4.7}$$

with $\forall k \in \mathbb{Z}$,

$$\begin{aligned}
T(k)_{11} &= -\mathrm{e}^{\{-i\eta_{2k-1}(\lambda)\}},\\
T(k)_{12} &= i\tfrac{r}{t}\left(\mathrm{e}^{-i\eta_{2k-1}(\lambda)} - 1\right),\\
T(k)_{21} &= i\tfrac{r}{t}\left(\mathrm{e}^{i(\eta_{2k}(\lambda)-\eta_{2k-1}(\lambda))} - \mathrm{e}^{-i\eta_{2k-1}(\lambda)}\right),\\
T(k)_{22} &= -\tfrac{1}{t^2}\,\mathrm{e}^{i\eta_{2k}(\lambda)} + \tfrac{r^2}{t^2}\left(\mathrm{e}^{i(\eta_{2k}(\lambda)-\eta_{2k-1}(\lambda))} + 1 - \mathrm{e}^{-i\eta_{2k-1}(\lambda)}\right).
\end{aligned} \tag{4.8}$$

Therefore, introducing the set of random vectors

$$\delta_k = (\eta_{2k}(\lambda), \eta_{2k-1}(\lambda)) \in \mathbb{T}^2, \quad k \in \mathbb{Z}, \tag{4.9}$$

we observe that the set of random transfer matrices $\{T(k)\}_{k\in\mathbb{Z}}$ will be independent provided the set of random vectors $\{\delta_k\}_{k\in\mathbb{Z}}$ are independent.

Using properties of the characteristic functions of random vectors

$$\Phi_\beta(n_1, n_2) = \mathbb{E}\left(e^{-i(n_1\beta_1(\omega)+n_2\beta_2(\omega))}\right), \quad n_1, n_2 \in \mathbb{Z}, \tag{4.10}$$

we get the following lemma.

Lemma 4.1. *If the vector $\{\beta_k\}_{k\in\mathbb{Z}}$ are i.i.d and uniform, the random vectors $\{\delta_k\}_{k\in\mathbb{Z}}$ are also i.i.d. and uniformly distributed on $\mathbb{T}^2$. In turn, the set of transfer matrices $\{T(k)\}_{k\in\mathbb{Z}}$ are i.i.d. random matrices in $Gl_2(\mathbb{C})$.*

We can now state our main result in the random setting:

Theorem 4.1. *Let U_ω be defined by its matrix elements (2.5) with $t \in (0, 1)$. Assuming the phases $\{\alpha_k\}_{k\in\mathbb{Z}}$ and $\{\theta_k\}_{k\in\mathbb{Z}}$ are i.i.d. and uniform on $\mathbb{T}$, we have almost surely*

$$\sigma_{a.c.}(U_\omega) = \emptyset.$$

The next section is devoted to the proof of this theorem.

Remarks. i) The same result for U_ω^+ defined by (3.1) holds by Theorem 3.2.
ii) In case the phases $\alpha_k \in \mathbb{T}$ are deterministic and of the form $\alpha_k = ak + b$, $a, b \in \mathbb{R}$, whereas the θ_k's are i.i.d. and uniform, the conclusions of the above lemma and theorem still hold. The same is true if the θ_k's are deterministic and constant whereas the α_k's are i.i.d. and uniform.
iii) To motivate our hypotheses on the uniform distribution of the phase vectors β_k, we recall the

Lemma 4.2. *If X_k, $k \in \mathbb{Z}$, is a set of i.i.d. random variables on $\mathbb{T}$ with support not reduced to a point, then the random variables $Y_k^\pm = X_k \pm X_{k-1}$, $k \in \mathbb{Z}$ are independent if and only if the X_k are uniformly distributed.*

A proof of Lemmas 4.1 and 4.2 can be found in the Appendix.

5. Lyapunov Exponents

As the map (4.6) is measurable, we can realize our transfer matrices as an i.i.d. random process on the same probabilistic space $(\Omega, \mathcal{F}, \mathbb{P})$ in such a way that

$$T(k, \omega) = T(\omega_{2k}, \omega_{2k-1}), \quad k \in \mathbb{Z}, \;\; \forall\omega \in \mathbb{T}^{\mathbb{Z}}, \tag{5.1}$$

with the C^∞ map $T : \mathbb{T}^2 \to Gl_2(\mathbb{C})$ defined in (4.7). Therefore,

$$T(k + 1, \omega) = T(k, S(\omega)), \quad \forall k \in \mathbb{Z}. \tag{5.2}$$

The set of translations $\{S^j\}_{j\in\mathbb{Z}}$ is ergodic and we can write for all $k \in \mathbb{N}^*$,

$$\begin{aligned}\Phi(k, \omega) &= T(k, \omega)T(k - 1, \omega) \cdots T(1, \omega)\\ &= T(1, S^k(\omega))T(1, S^{k-1}(\omega)) \cdots T(1, \omega).\end{aligned} \tag{5.3}$$

Similarly, we set $\Phi(0, \omega) = I\!I_2$ and

$$\Phi(-k, \omega) = T^{-1}(0, S^{-k+1}(\omega))T^{-1}(0, S^{-k+2}(\omega)) \cdots T^{-1}(0, \omega). \tag{5.4}$$

Therefore $\{\Phi(k, \omega)\}_{k\in\mathbb{N}}$ defines a random ergodic linear dynamical system over $Gl_2(\mathbb{C})$ generated by the map $T(1, \cdot)$ and $\{\Phi(-k, \omega)\}_{k\in\mathbb{N}}$ defines another one generated by $T^{-1}(0, \cdot)$.

We are now formally in good shape to apply Oseledec's and Furstenberg's Theorems to define and study the Lyapunov exponents. However, the last result is stated for real valued matrices, and, in particular, irreducibility properties of groups of matrices are a delicate matter. Therefore, we first want to map our problem to a problem involving matrices in $Gl_4(\mathbb{R})$. This is done very conveniently using the method described in [MT], which we apply to our setting. We will denote by $\langle\cdot|\cdot\rangle$ the scalar product on $\mathbb{R}^4$ or $\mathbb{C}^2$ and we introduce

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.5}$$

We define a sub-algebra of $\mathcal{A}_4(\mathbb{R})$ of $\mathcal{M}_4(\mathbb{R})$ by

$$\mathcal{A}_4(\mathbb{R}) = \left\{ \begin{pmatrix} a_1 I + a_2 J & b_1 I + b_2 J \\ c_1 I + c_2 J & d_1 I + d_2 J \end{pmatrix}, a_j, b_j, c_j, d_j \in \mathbb{R}, \ j = 1, 2. \right\}. \tag{5.6}$$

The topology on $\mathcal{M}_2(\mathbb{C})$, $\mathcal{M}_4(\mathbb{R})$ is generated by the spectral norm

$$\|A\| = \sqrt{\sum_{\lambda \in \sigma(|A|)} |\lambda|^2} \tag{5.7}$$

and that of $\mathcal{A}_4(\mathbb{R})$ is the induced topology. Let ρ be the mapping $\rho : \mathbb{C}^2 \to \mathbb{R}^4$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \Re(x) \\ -\Im(x) \\ \Re(y) \\ -\Im(y) \end{pmatrix}, \tag{5.8}$$

and $\tau : \mathcal{M}_2(\mathbb{C}) \to \mathcal{A}_4(\mathbb{R})$ be defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \Re(a)I + \Im(a)J & \Re(b)I + \Im(b)J \\ \Re(c)I + \Im(c)J & \Re(d)I + \Im(d)J \end{pmatrix}. \tag{5.9}$$

The following properties are readily checked:

Lemma 5.1. *For any $u, v \in \mathbb{C}^2$, and any $\alpha \in \mathbb{C}$,*

$$\rho(u + v) = \rho(u) + \rho(v), \tag{5.10}$$

$$\rho(\alpha u) = \Re(\alpha)\rho(u) + \Im(\alpha)\rho(iu). \tag{5.11}$$

For any $A, B \in \mathcal{M}_2(\mathbb{C})$, and $\alpha \in \mathbb{R}$,

$$\begin{aligned} &\tau(A + B) = \tau(A) + \tau(B), && \tau(AB) = \tau(A)\tau(B), \\ &\tau(\alpha A) = \alpha\tau(A), && \tau(A^*) = \tau(A)^*, \\ &\tau(A^{-1}) = \tau(A)^{-1}. \end{aligned} \tag{5.12}$$

The last formula means that if $A \in \mathcal{M}_2(\mathbb{C})$ is invertible, $\tau(A)$ is also invertible and the formula is true. Finally, for all $u \in \mathbb{C}^2$, $\forall T \in \mathcal{M}_2(\mathbb{C})$,

$$\rho(Tu) = \tau(T)\rho(u). \tag{5.13}$$

We also note the following lemma for future reference.

Lemma 5.2. *If $A \in \mathcal{M}_2(\mathbb{C})$ and $|\det(A)| = 1$, then $|\det(\tau(A))| = 1$. If A is self adjoint with eigenvalues γ_1 and γ_2 , then $\tau(A)$ is real symmetric with eigenvalues γ_1 and γ_2 of multiplicity two.*

More general results of the same sort in higher dimension can be found in [MT].

Remarks. i) Let us note as a consequence of Lemma 5.2 that the mappings ρ and τ are homeomorphisms and $\forall u \in \mathbb{C}^2$, $\forall A \in \mathcal{M}_2(\mathbb{C})$,

$$\|\rho(u)\| = \|u\|, \quad \|\tau(A)\| = \sqrt{2}\|A\|. \tag{5.14}$$

ii) The mapping ρ does not transport scalar product but it does preserve the norm. Note that we have for all $\forall u, v \in \mathbb{C}^2$, and all $T \in \mathcal{M}_2(\mathbb{C})$,

$$\langle \rho(iu)|\rho(u)\rangle = 0, \tag{5.15}$$

$$\langle \rho(u)|\tau(T)\rho(v)\rangle = \langle \rho(iu)|\tau(T)\rho(iv)\rangle = \Re(\langle u|Tv\rangle), \tag{5.16}$$

$$\langle \rho(iu)|\tau(T)\rho(v)\rangle = -\langle \rho(u)|\tau(T)\rho(iv)\rangle = \Im(\langle u|Tv\rangle). \tag{5.17}$$

Therefore, if u and v are orthogonal in $\mathbb{C}^2$, $\rho(u)$ and $\rho(v)$ are also orthogonal.

Existence of the Lyapunov Exponents. Using this operator $\tau : Gl_2(\mathbb{C}) \to Gl_4(\mathbb{R})$, we can now consider the random ergodic linear dynamical system over $Gl_4(\mathbb{R})$ defined from $\{\Phi(k,\omega)\}_{k\in\mathbb{N}}$ by

$$\Psi(k,\omega) = \tau(\Phi(k,\omega)) \tag{5.18}$$

generated by the map $\tau(T(1,\cdot)) : \Omega \to Gl_4(\mathbb{R})$. We will work similarly if $-k \in \mathbb{N}$.

We now apply Oseledec's Theorem according to [A], Thm. 3.4.11, specialized to our setting.

Proposition 5.1. *Let the random ergodic dynamical system generated by the map $\tau(T(1,\cdot)) : \Omega \to Gl_4(\mathbb{R})$. Then, on an invariant set $\Omega_0 \subset \Omega$ of $\mathbb{P}$-measure one, the following limit exists*

$$\lim_{n\to\infty} (\Psi(n,\omega)^*\Psi(n,\omega))^{1/2n} = \Lambda(\omega). \tag{5.19}$$

The matrix $\Lambda(\omega)$ possesses at most 2 distinct eigenvalues of multiplicities 2, denoted by

$$\mathrm{e}^{\gamma_1} \geq \mathrm{e}^{\gamma_2} \equiv \mathrm{e}^{-\gamma_1} > 0, \tag{5.20}$$

associated with at most two eigenspaces $\mathcal{E}_1(\omega)$, $\mathcal{E}_2(\omega)$. The Lyapunov exponents $\gamma_1 \geq \gamma_2$ are constant almost surely.
If $\gamma_1 > 0$, there exists a filtration of $\mathbb{R}^4$,
$\{0\} \subset \mathcal{V}(\omega) \subset \mathbb{R}^4$ such that

$$\mathcal{V}(\omega) = \mathcal{E}_2(\omega), \quad \textit{and}\ \mathbb{R}^4 = \mathcal{E}_2(\omega) \oplus \mathcal{E}_1(\omega), \tag{5.21}$$

and $u \in \mathcal{V}(\omega)$ iff

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Psi(n,\omega)u\| = \gamma_2 = -\gamma_1 < 0, \tag{5.22}$$

and $u \in \mathbb{R}^4 \setminus \mathcal{V}(\omega)$

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Psi(n,\omega)u\| = \gamma_1 > 0. \tag{5.23}$$

Moreover, there exists a splitting

$$\mathbb{R}^4 = E_2(\omega) \oplus E_1(\omega) \tag{5.24}$$

such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Psi(n,\omega)u\| = \gamma_j \Leftrightarrow u \in E_j(\omega) \setminus \{0\}. \tag{5.25}$$

Proof. We need to check the hypotheses of the Ergodic Multiplicative (Oseledec's) Theorem, see e.g. [A], Thm. 3.4.11, in order to get the existence of the limit. All norms being equivalent, considering the maximum modulus of the matrix elements, we get the existence of a finite constant depending only on $0 < t < 1$ such that $C(t)^{-1} \le \|T(1,\omega)\| \le C(t)$. As $|\det(T(1,\omega))| = 1$, the same bound is true for $T(1,\omega)^{-1}$. The properties of τ finally yield

$$(\ln^+ \|\tau(T(1,\cdot))\| + \ln^+ \|\tau(T(1,\cdot)^{-1})\|) \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \tag{5.26}$$

where $\ln^+(x) = \max\{\ln(x), 0\}$, $x > 0$, which ensures the existence of the limit. The statements about the number of Lyapunov exponents, their relations and multiplicities are shown as follows. For any n, the 2×2 matrix $\Phi(n,\omega)^*\Phi(n,\omega)$ is positive, of determinant one so that it either possesses two distinct eigenvalues $\sigma_1(n,\omega) > \sigma_2(n,\omega) = 1/\sigma_1(n,\omega) > 0$ (of multiplicity one), or it is the identity matrix. Therefore, $\Psi(n,\omega)^*\Psi(n,\omega) = \tau(\Phi(n,\omega)^*\Phi(n,\omega))$ has two distinct eigenvalues $\sigma_1(n,\omega) > \sigma_2(n,\omega) = 1/\sigma_1(n,\omega) > 0$ of multiplicity two, or it is the identity matrix in $\mathbb{R}^4$. The determinant being continuous, the limit $\Lambda(\omega)$ is also positive of determinant equal to one. By continuity of τ and τ^{-1}, $\Lambda(\omega)$ also belongs to $\mathcal{A}_4(\mathbb{R})$ and there exists $\kappa(\omega) \in \mathcal{M}_2(\mathbb{C})$ such that $\tau(\kappa(\omega)) = \Lambda(\omega)$. Moreover, the relation $\kappa(\omega) = \lim_{n\to\infty} \Phi(n,\omega)^*\Phi(n,\omega)$ shows that $\kappa(\omega)$ is also positive of determinant one, which proves that the multiplicities of the eigenvalues of $\Lambda(\omega)$ is two or it is the identity matrix. □

Corollary 5.1. *Under the same hypotheses as above, there exists almost surely a subspace $\mathcal{V}_0(\omega)$ of $\mathbb{C}^2$ of complex dimension 1 such that*

$$\begin{aligned} &\forall u \in \mathcal{V}_0(\omega) \setminus \{0\}, \quad \lim_{n\to+\infty} \tfrac{1}{n} \ln \|\Phi(n,\omega)u\| = -\gamma_1 < 0, \\ &\forall u \in \mathbb{C}^2 \setminus \mathcal{V}_0(\omega), \quad \lim_{n\to+\infty} \tfrac{1}{n} \ln \|\Phi(n,\omega)u\| = \gamma_1 > 0. \end{aligned} \tag{5.27}$$

Also, there exists a splitting $\mathbb{C}^2 = E_2^0(\omega) \oplus E_1^0(\omega)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)u\| = \gamma_j \Leftrightarrow u \in E_j^0(\omega) \setminus \{0\}. \tag{5.28}$$

Proof. By the proposition, there exists a filtration: $\{0\} \subset \mathcal{V}(\omega) \subset \mathbb{R}^4$, such that:

$$\begin{aligned} &\forall v \in \mathbb{R}^4 \setminus \mathcal{V}(\omega), \lim_{n\to+\infty} \tfrac{1}{n} \log \|\tau(T(1,S^n(\omega))) \dots \tau(T(1,\omega)v\| = \gamma_1, \\ &\forall v \in \mathcal{V}(\omega) \setminus \{0\}, \lim_{n\to+\infty} \tfrac{1}{n} \log \|\tau(T(1,S^n(\omega))) \dots \tau(T(1,\omega))v\| = -\gamma_1. \end{aligned} \tag{5.29}$$

The properties (5.1), (5.13) and (5.14) imply that $\forall v \in \mathbb{R}^4$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log \|\tau(T(1,S^n(\omega))) \dots \tau(T(1,\omega))v\| \\ &\quad = \lim_{n\to\infty} \frac{1}{n} \log \|T(1,S^n(\omega)) \dots T(1,\omega)\rho^{-1}(v)\|. \end{aligned} \tag{5.30}$$

Let $v_0 \in \mathcal{V}(\omega)$, $u_0 = \rho^{-1}(v_0)$ and $\mathcal{V}_0(\omega) = \mathbb{C}u_0$. The equation above proves the first assertion. Consider $u \in \mathbb{C}^2 \setminus \mathcal{V}_0(\omega)$. Then

$$u = \alpha u_0 + u_0^{\perp}, \quad u_0^{\perp} \neq 0, \ \ \alpha \in \mathbb{C}, \tag{5.31}$$

$$\rho(u) = \Re(\alpha)\rho(u_0) + \Im(\alpha)\rho(iu_0) + \rho(u_0^{\perp}). \tag{5.32}$$

The three components are non-zero and mutually orthogonal. Therefore $\rho(u) \in \mathbb{R}^4 \setminus \mathcal{V}(\omega)$, from which the second assertion follows.

We can proceed along the same lines in order to prove the statements concerning the existence and properties of a splitting of $\mathbb{C}^2 = E_2^0(\omega) \oplus E_1^0(\omega)$ with $E_j^0(\omega) = \mathbb{C}\rho^{-1}(v_j)$, $v_j \in E_j(\omega)$. Indeed, let $v_1 \in E_1(\omega)$ and $u_1 = \rho^{-1}(v_1)$. We define $v_1' = \rho(iu_1)$, so that $\langle v_1 | v_1' \rangle = 0$ and $\lim_{n\to\pm\infty} \frac{1}{n} \ln \|\Phi(n,\omega)v_1'\| = \lim_{n\to\pm\infty} \ln \frac{1}{n} \|\Psi(n,\omega)iu_1\| = \gamma_1$. Hence $v_1' \in E_1(\omega)$. Let $v_2 \in E_2(\omega)$ such that $u_2 := \rho^{-1}(v_2)$ is not colinear to u_1. There exists such a v_2, otherwise, $u_2 = \alpha u_1$ implies $\rho(u_2) = v_2 = \Re(\alpha)v_1 + \Im(\alpha)v_1' \in E_2(\omega)$, which is a contradiction. Hence, $v_2' = \rho(iu_2) \in E_2(\omega)$. Now $u = \alpha u_1 + \beta_1 u_2$ and $\rho(u) = \Re(\alpha)v_1 + \Im(\alpha)v_1' + \Re(\beta)v_2 + \Im(\beta)v_2'$. So that

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln \|\Phi(n,\omega)u\| = \gamma_j = \lim_{n\to\pm\infty} \frac{1}{n} \ln \|\Psi(n,\omega)\rho(u)\| \tag{5.33}$$

is equivalent to $\beta = 0$ if $j = 1$ and $\alpha = 0$ if $j = 2$. □

Positivity of the Lyapunov exponent. In order to assess the positivity of the first Lyapunov exponent, we use Furstenberg's Theorem. Let us introduce, according to [BL] III.2.1., the following notions.

Let S be a subset of $GL_d(\mathbb{R})$, $d > 0$. Such a set S is said *irreducible* if there is no strict subspace V of $\mathbb{R}^d$ such that $\forall M \in S$, $M(V) = V$. A set will be called *strongly irreducible* if there is no finite family $V_1, \dots, V_N$ of strict subspaces of $\mathbb{R}^d$, such that: $\forall M \in S$, $M(V_1 \cup \dots \cup V_N) = V_1 \cup \dots \cup V_N$.

The basic theorem is then

Theorem 5.1 (Furstenberg). *If μ is a probability measure on $\mathcal{M} = \{M \in GL_d(\mathbb{R});\ |\det M| = 1\}$ such that: $\int \log \|M\| d\mu(M) < +\infty$ and the group $\mathcal{G}_\mu$ generated by the support of μ is strongly irreducible and non-compact, then the first Lyapunov exponent associated with any sequence of i.i.d. matrices in $\mathcal{M}$ satisfies $\gamma_1 > 0$.*

See [BL] Theorem III.6.1 for a proof.

We note the following property (Exercise IV.2.9 of [BL]) reducing strong irreducibility to irreducibility in some cases.

Lemma 5.3. *Let $1 < d \in \mathbb{N}$ and $\mathcal{S}$ be a connected subset of $GL_d(\mathbb{R})$. Then $\mathcal{S}$ is strongly irreducible if and only if $\mathcal{S}$ is irreducible.*

In our case, the measure μ is the image by the map (4.7) of the uniform measure $\mathbb{P}_0 \otimes \mathbb{P}_0$ on $\mathbb{T}^2$. In order to study the properties of the corresponding set $\mathcal{G}_\mu$, we introduce the connected set of matrices given by the range of the smooth map from $\mathbb{T}^2 \to \mathbb{C}^2$ which to (θ, η) assigns the matrix

$$T_{(\theta,\eta)} = \begin{pmatrix} -e^{-i\theta} & \frac{ir}{t}\left(e^{-i\theta} - 1\right) \\ -\frac{ir}{t}\left(e^{-i\theta} - e^{i(\eta-\theta)}\right) & -\frac{r^2}{t^2}\left(e^{-i\theta} - 1 - e^{i(\eta-\theta)}\right) - \frac{1}{t^2}e^{i\eta} \end{pmatrix}. \tag{5.34}$$

Let $\mathcal{G} \subset \mathcal{G}_\mu$ denote the smallest group generated by the support of the measure image by $\mathbb{P}_0 \otimes \mathbb{P}_0$ on $SL_4(\mathbb{R})$ by $(\theta, \eta) \to \tau(T_{(\theta,\eta)})$.

Proposition 5.2. *The group $\mathcal{G}$ is not compact.*

Proof. The matrix $\tau(T_{(\pi,\pi)})$ belongs to the support of the image of $\mathbb{P}_0 \otimes \mathbb{P}_0$ by (5.34) and it has eigenvalues

$$\frac{(r-1)^2}{t^2} \text{ and } \frac{(r+1)^2}{t^2}. \tag{5.35}$$

The second one is strictly larger than 1, if $t < 1$ so that, since for any $n \in \mathbb{N}$, $\tau(T_{(\pi,\pi)})^n \in \mathcal{G}$, $\mathcal{G}$ cannot be bounded. $\square$

Proposition 5.3. *The group $\mathcal{G}$ is strongly irreducible.*

Proof. It is enough to exhibit an irreducible, connected, subset of $\mathcal{G}$. The map $\tau(T_{\cdot,\cdot})$ is smooth, hence the set $\{\tau(T_{(\theta,\eta)}), (\theta, \eta) \in [0, 2\pi[^2\}$, included in $\mathcal{G}$, is connected. We now show that there exists no strict subspace of $\mathbb{R}^4$ invariant under this set of matrices. We first note that choosing $\eta = \theta \in [0, 2\pi[$ we get

$$\tau(T_{(\theta,\theta)}) = M_0 + \sin(\theta) M_1 + \cos(\theta) M_2, \tag{5.36}$$

where

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{r}{t} \\ 0 & 0 & \frac{r}{t} & 0 \\ 0 & \frac{r}{t} & 2\frac{r^2}{t^2} & 0 \\ -\frac{r}{t} & 0 & 0 & 2\frac{r^2}{t^2} \end{pmatrix}, \tag{5.37}$$

$$M_1 = \begin{pmatrix} 0 & 1 & \frac{r}{t} & 0 \\ -1 & 0 & 0 & \frac{r}{t} \\ -\frac{r}{t} & 0 & 0 & -1 \\ 0 & -\frac{r}{t} & 1 & 0 \end{pmatrix}, \tag{5.38}$$

$$M_2 = -(M_0 + \mathbb{I}), \tag{5.39}$$

where $\mathbb{I}$ denotes the identity matrix. If there exists a strict invariant subspace $\mathcal{E}$ for the set $\tau(T_{(\theta,\theta)})_{\theta \in [0,2\pi[}$, this subspace $\mathcal{E}$ is also invariant for the matrices M_j, $j = 0, 1, 2$. Similarly, choosing $-\eta = \theta \in [0, 2\pi[$, we have

$$\tau(T_{(\theta,-\theta)}) = N_0 + \sin(\theta) N_1 + \cos(\theta) N_2 + \sin(2\theta) N_3 + \cos(2\theta) N_4, \tag{5.40}$$

where, in particular,

$$N_1 = \begin{pmatrix} 0 & 1 & \frac{r}{t} & 0 \\ -1 & 0 & 0 & \frac{r}{t} \\ -\frac{r}{t} & 0 & 0 & \frac{r^2+1}{t^2} \\ 0 & -\frac{r}{t} & -\frac{r^2+1}{t^2} & 0 \end{pmatrix}. \tag{5.41}$$

Again $\mathcal{E}$ must be invariant under N_1.

As M_0, M_1, N_1 are real (anti) symmetric, they all leave $\mathcal{E}^\perp$ invariant as well so that these matrices are reduced by the orthogonal spaces $\mathcal{E} \oplus \mathcal{E}^\perp = \mathbb{R}^4$. In particular, these invariant subspaces must be generated by the eigenvectors $\{u_1, u_2, u_3, u_4\}$ of M_0 which form a basis of $\mathbb{R}^4$. Explicitly,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{r+1}{t} \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ \frac{1-r}{t} \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{1-r}{t} \end{pmatrix}, u_4 = \begin{pmatrix} 0 \\ -\frac{r+1}{t} \\ 1 \\ 0 \end{pmatrix}, \tag{5.42}$$

the first two vectors being associated with the eigenvalue $r(r+1)/t^2$ while the last two are associated with $r(r-1)/t^2$. We further compute, repeatedly using $r^2 + t^2 = 1$, that

$$M_1 u_1 = \frac{1}{t} u_4, \;\; M_1 u_2 = \frac{1}{t} u_3, \;\; M_1 u_3 = -\frac{1}{t} u_2, \;\; M_1 u_4 = -\frac{1}{t} u_1 \tag{5.43}$$

and

$$N_1 u_1 = -\frac{1+r}{t(1-r)} u_2, \;\; N_1 u_2 = \frac{1}{t} u_1, \;\; N_1 u_3 = \frac{1-r}{t(1+r)} u_2, \;\; N_1 u_4 = -\frac{1}{t} u_3. \tag{5.44}$$

Clearly no one dimensional subspace $\mathcal{E} = \langle u_j \rangle$ (or $\mathcal{E}^{\perp} = \langle u_j \rangle$) can be invariant under M_0, M_1 and N_1. And by inspection, one checks that no two dimensional subspace $\mathcal{E} = \langle u_j, u_k \rangle$ can be invariant under M_0, M_1 and N_1. The irreducible set $\{\tau(T_{(\theta,\eta)}), (\theta, \eta) \in [0, 2\pi[^2\}$ being contained in the group $\mathcal{G}$, the latter and $\mathcal{G}_\mu$ are *a fortiori* irreducible. □

Therefore,

Proposition 5.4. *The Lyapunov exponent $\gamma_1(\lambda)$ associated to the ergodic linear dynamical system (5.18) is strictly positive for any $\lambda \in \mathbb{T}$.*

Ishii-Pastur. The link between Lyapunov exponents and a.c. spectrum is provided in the self adjoint random case by the Ishii-Pastur-Kotani Theorem. We provide a unitary version of the Ishii-Pastur part of the result, which is enough for our purpose. In order to adapt the proof of [CFKS], it is only necessary to show that it is spectrally true that the generalized eigenvectors of U are polynomially bounded.

We first show that generalized eigenvectors corresponding to spectral parameters outside the spectrum cannot be polynomially bounded for bounded normal operators with a band structure. We'll say that a matrix $\{M_{j,k}\}_{j,k\in\mathbb{Z}}$ has a *band structure of order* $2p+1$, $p \in \mathbb{N}$ if $|j-k| > p$ implies $M_{j,k} = 0$. Note that if this is so, then

$$(Mv)_k = \sum_{j\in\mathbb{Z}} M_{k,j} v_j = \sum_{j=k-p}^{k+p} M_{k,j} v_j \tag{5.45}$$

makes sense for an *arbitrary* vector $v = \{v_j\}_{j\in\mathbb{Z}}$, since the sum is finite. Define the projections

$$P_{[a,b]} = \sum_{a\le j\le b} |\varphi_j\rangle\langle\varphi_j| \tag{5.46}$$

and note that

$$\begin{aligned} P_{[a,b]} U &= P_{[a,b]} U P_{[a-p,b+p]}, \\ U P_{[a,b]} &= P_{[a-p,b+p]} U P_{[a,b]}. \end{aligned} \tag{5.47}$$

That is, in fact, just another way of saying that U has band structure.

Lemma 5.4. *Let $(\varphi_n)_{n\in\mathbb{Z}}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$ on which a normal operator U acts. Assume U has a band structure of order $2p+1$ and consider an arbitrary nontrivial sequence ϕ such that $U\phi = z\phi$, where $z \in \mathbb{C}$ is in the resolvent set of U. Then the sequence $(\langle\varphi_k|\phi\rangle)_{k\in\mathbb{Z}}$ is not polynomially bounded.*

Proof. The operator U being normal, for any z in the resolvent set, $(U-z)^{-1}$ is normal too. Therefore $\|(U-z)^{-1}\| = r_\sigma((U-z)^{-1})$, where $r_\sigma(A)$ is the spectral radius of the operator A. As $r_\sigma((U-z)^{-1}) = 1/\text{dist}\,(z,\sigma(U))$ ([K] (III 6.16 p.177)), we deduce

$$\forall\psi \in \mathcal{H},\quad \|\psi\| \le \frac{1}{\text{dist}\,(z,\sigma(U))}\|(U-z)\psi\| \ . \tag{5.48}$$

Consider the generalized eigenvector ϕ. Since $z \notin \sigma(U)$, ϕ cannot be in l^2, so it must fail to be in l^2 either at $+\infty$ or at $-\infty$. We will assume that it fails at $+\infty$, and focus on the coefficients $\langle\varphi_k|\phi\rangle$, with $k \ge 0$ and large enough.

Let $n > 3p$ and let

$$P_n = P_{[p,n]} = \mathbb{I} - Q_n. \tag{5.49}$$

Since $P_n\phi \in l^2(\mathbb{Z})$, we have by (5.48),

$$\|P_n\phi\| \le c_z\|(U-z)P_n\phi\|, \tag{5.50}$$

where $c_z^{-1} = \text{dist}\,(z,\sigma(U))$. Since we have assumed that ϕ is not in l^2 as $k \ge 0$, necessarily

$$\|P_{n-p}\phi\|^2 = \|P_{[p,n-p]}\phi\|^2 \to \infty \text{ as } n \to \infty. \tag{5.51}$$

So there exists an n_0 such that, given $\epsilon > 0$, $n \ge n_0$ implies

$$\|P_{n-p}\phi\|^2 \ge \epsilon^{-1}\|P_{[0,2p]}\phi\|^2. \tag{5.52}$$

Since $(U-z)\phi = 0$, it follows for any finite projection P that

$$(U-z)P\phi = -(U-z)Q\phi, \tag{5.53}$$

where $Q = \mathbb{I} - P$, and hence that

$$\begin{aligned}(U-z)P\phi &= P(U-z)P\phi + Q(U-z)P\phi\\ &= -P(U-z)Q\phi + Q(U-z)P\phi\\ &= -PUQ\phi + QUP\phi.\end{aligned} \tag{5.54}$$

Take in (5.50), $P = P_n = P_{[p,n]} = \mathbb{I} - Q_n$. By (5.47), we get

$$\begin{aligned}P_nUQ_n &= P_{[p,n]}UP_{[0,p+n]}Q_n\\ &= P_{[p,n]}U(P_{[0,p-1]} + P_{[n+1,p+n]}).\end{aligned} \tag{5.55}$$

Also,

$$Q_nUP_n = Q_nU(P_{[p,2p]} + P_{[2p+1,n-p]} + P_{[n-p+1,n]}). \tag{5.56}$$

But

$$Q_nUP_{[2p+1,n-p]} = Q_nP_{[p+1,n]}UP_{[2p+1,n-p]} = 0 \tag{5.57}$$

so that

$$Q_n U P_n = Q_n U (P_{[p,2p]} + P_{[n-p+1,n]}). \tag{5.58}$$

Since the ranges of the appropriate projectors are orthogonal, we have with $A = \|U\|^2$,

$$\begin{aligned}
\|(U-z)P_n\phi\|^2 &= \|P_n U Q_n \phi\|^2 + \|Q_n U P_n \phi\|^2 \\
&= \|P_n U (P_{[0,p-1]} + P_{[n+1,p+n]})\phi\|^2 + \|Q_n U (P_{[p,2p]} + P_{[n-p+1,n]})\phi\|^2 \\
&\le A\left(\|P_{[0,p-1]}\phi\|^2 + \|P_{[n+1,p+n]}\phi\|^2 + \|P_{[p,2p]}\phi\|^2 + \|P_{[n-p+1,n]}\phi\|^2\right) \\
&= A\left(\|P_{[0,2p]}\phi\|^2 + \|P_{[p,n+p]}\phi\|^2 - \|P_{[p,n-p]}\phi\|^2\right) \\
&= A\left(\|P_{[0,2p]}\phi\|^2 + \|P_{n+p}\phi\|^2 - \|P_{n-p}\phi\|^2\right).
\end{aligned} \tag{5.59}$$

Thus, by (5.52) and (5.50), for $n > \max(n_0, 3p)$, we have

$$\begin{aligned}
\|P_{n-p}\phi\|^2 \le \|P_n\phi\|^2 &\le c_z^2 \|(U-z)\phi\|^2 \\
&\le c_z^2 A\left(\epsilon\|P_{n-p}\phi\|^2 + \|P_{n+p}\phi\|^2 - \|P_{n-p}\phi\|^2\right),
\end{aligned} \tag{5.60}$$

which implies that

$$\|P_{n+p}\phi\|^2 \ge \|P_{n-p}\phi\|^2 \left(\frac{1}{Ac_z^2} + 1 - \epsilon\right) \equiv B\|P_{n-p}\phi\|^2, \tag{5.61}$$

where $B > 1$, if $\epsilon < 1/(Ac_z^2)$. Iterating the argument, we get $\forall k \in \mathbb{N}$,

$$\|P_{n+p2k}\phi\| \ge B^{\frac{k}{2}} \|P_n\phi\| \ , \tag{5.62}$$

which ensures the existence of an exponentially growing subsequence of coefficients.
□

The second element is the construction of generalized solutions corresponding to spectral parameters in the spectrum of U which are polynomially bounded, *à la* Berezanskii. This is done in our unitary setting following, *mutatis mutandis*, the arguments given in [S] for the self-adjoint case. We only quote the end result here, including a proof in the Appendix for completeness.

Recall that a measure ρ is in the measure class of a unitary operator U with spectral projection $E(\cdot)$ if for any Borel set $\Delta \subset \mathbb{T}$: $\rho(\Delta) = 0 \Leftrightarrow E(\Delta) = 0$.

Theorem 5.2. *Let U be a unitary operator with a band structure defined on $l^2(\mathbb{Z})$ and $\delta > 1$. Then there exists a measure ρ in the spectral measure class of U and a family of disjoint measurable sets $(\Delta_n)_{n\in\mathbb{N}^*}$ whose union supports ρ such that for $\lambda \in \Delta_n$, there exist n vectors $\phi_j(\lambda)$ satisfying*

- $(U - e^{i\lambda})\phi_j(\lambda) = 0$.
- $\forall n \in \mathbb{Z}$, $|\langle \varphi_n | \phi_j(\lambda)\rangle| \le C < n >^{\delta}$.
- *For any λ fixed, the family $\{\phi_j(\lambda)\}_j$ is linearly independent.*

Remark. The result is also true if the operator U is defined on $l^2(\mathbb{N})$ or $l^2(\mathbb{N})^*$.

Corollary 5.2. *$\sigma(U)$ is the essential closure of the set*

$$S = \{\lambda \in \mathbb{T}^1; U\phi = e^{i\lambda}\phi \text{ admits a polynomially bounded solution}\} \tag{5.63}$$

and $E_{[0,2\pi[\setminus S}(U) = 0$.

Proof. If $e^{i\lambda} \in \sigma(U)$ then for any $\epsilon > 0$, $E(]\lambda-\epsilon, \lambda+\epsilon[) > 0$ and $\rho(]\lambda-\epsilon, \lambda+\epsilon[) > 0$. Hence, by Theorem 5.2, for λ' arbitrarily close to λ there exists a polynomially bounded solution $\phi_j(\lambda')$. Thus $\sigma(U) \subset \bar{S}$. The reverse inclusion follows from Lemma 5.4 and the fact that $\sigma(U)$ is closed. The last statement follows immediately. □

Putting these arguments together, we get the unitary version of the Ishii-Pastur theorem suited to our monodromy operator:

Theorem 5.3. *Let U_ω be unitary with a band structure. Assume that the corresponding transfer matrix at spectral parameter $e^{i\lambda}$ induces two Lyapunov exponents $\gamma_1(\lambda) \geq \gamma_2(\lambda) = -\gamma_1(\lambda)$ which are constant almost surely. Then*

$$\sigma_{ac}(U_\omega) \subseteq \overline{\{e^{i\lambda} \in S^1; \gamma_1(\lambda) = 0\}}\ . \tag{5.64}$$

Proof. Identical to that given in [CFKS] Thm 9.13. □

Therefore, Theorem 4.1 follows from the above theorem and Proposition 5.4.

6. Coherent Setting

In this section we consider situations where the behavior of the matrix coefficients of U in (2.5) are periodic functions of k as the result of a coherent behavior of the phases. We first show that this implies purely absolutely continuous spectrum. Then we prove that when restricted to $l^2(\mathbb{N}^*)$, these operators have no singular continuous spectrum and may possess finitely many simple eigenvalues only.

Coherence on $l^2(\mathbb{Z})$. As a first particular case of coherent dependence of the scattering phases, we consider the simple situation where the θ_k's and α_k's take alternatively two values, up to a linear term. This corresponds to a monodromy operator $U = U_o U_e$, where U_e, U_o are direct sums of constant blocks $S_{2k} = S_e,\ S_{2k+1} = S_o$.

Proposition 6.1. *Let $t \in]0, 1[$, let the sequence $\{\gamma_k\}$ be arbitrary and*

$$\theta_k = \begin{cases} \theta_e \text{ if } k \text{ is even} \\ \theta_o \text{ if } k \text{ is odd} \end{cases},\ \alpha_k = ak + \begin{cases} \alpha_e \text{ if } k \text{ is even} \\ \alpha_o \text{ if } k \text{ is odd} \end{cases} \quad \forall k \in \mathbb{Z},$$

where $\theta_e, \theta_o, \alpha_e, \alpha_o, a \in \mathbb{R}$. Define $\Delta = \alpha_e - \alpha_o$, $\Theta = \theta_e + \theta_o$. Then, with the identification $U \equiv M(\{\theta_k\}, \{\alpha_k\}, \{\gamma_k\})$, U is purely absolutely continuous and

$$\sigma_{ac}(U) = \{e^{-i(a+\Theta)} e^{\pm i(\arccos(r^2\cos\Delta - t^2\cos(2x+\Delta)))}, x \in \mathbb{T}\}.$$

Proof. By Lemma 3.2, we can replace γ_k by $(-1)^{k+1}\alpha_k$ so that, with our choice of phases, $U \simeq \mathrm{e}^{-i(a+\Theta)}V$, where

$$\begin{aligned} V\varphi_{2k} &= irt\mathrm{e}^{-i\Delta}\varphi_{2k-1} + r^2\mathrm{e}^{-i\Delta}\varphi_{2k} + irt\mathrm{e}^{i\Delta}\varphi_{2k+1} - t^2\mathrm{e}^{i\Delta}\varphi_{2k+2}, \\ V\varphi_{2k+1} &= -t^2\mathrm{e}^{-i\Delta}\varphi_{2k-1} + itr\mathrm{e}^{-i\Delta}\varphi_{2k} + r^2\mathrm{e}^{i\Delta}\varphi_{2k+1} + irt\mathrm{e}^{i\Delta}\varphi_{2k+2}. \end{aligned} \tag{6.1}$$

Let us map $l^2(\mathbb{Z})$ unitarily to $L^2(\mathbb{T})$ via

$$W : \varphi_k \mapsto \mathrm{e}^{ikx}, \tag{6.2}$$

such that for any $\psi = \sum_k c_k\varphi_k$, $c_k \in \mathbb{C}$,

$$(W\psi)(x) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{ikx} \in L^2(\mathbb{T}). \tag{6.3}$$

We further introduce $L^2_\pm(\mathbb{T}) = WL_\pm$, where L_+, L_- are the subspaces of $l^2(\mathbb{Z})$ generated by the basis vectors with even, respectively odd, indices. It is easily checked that V is unitarily equivalent on $L^2(\mathbb{T}) = L^2_+(\mathbb{T}) \oplus L^2_-(\mathbb{T})$ to the matrix valued multiplication operator

$$V \simeq \begin{pmatrix} r^2\mathrm{e}^{-i\Delta} - t^2\mathrm{e}^{i\Delta}\mathrm{e}^{2ix} & 2itr\cos(x+\Delta) \\ 2itr\cos(x+\Delta) & r^2\mathrm{e}^{i\Delta} - t^2\mathrm{e}^{-i\Delta}\mathrm{e}^{-2ix} \end{pmatrix}. \tag{6.4}$$

This matrix is analytic in x and has non-constant eigenvalues given by

$$\lambda_\pm(x) = r^2\cos\Delta - t^2\cos(\Delta+2x) \pm i\sqrt{1-(r^2\cos\Delta - t^2\cos(\Delta+2x))^2}, \tag{6.5}$$

from which the result follows. □

Using basically the same strategy, we can consider the general case where the elements of U display an arbitrary periodicity.

Theorem 6.1. *Let $t \in]0,1[$, let the sequence $\{\gamma_k\}$ be arbitrary and $\{\theta_k\}$, $\{\alpha_k\}$ be such that for some $2 \le N \in \mathbb{N}$, and all $k \in \mathbb{Z}$,*

$$\theta_{k+N} = \theta_k, \quad \alpha_k = ak + \pi_k, \quad \text{where } \pi_{k+N} = \pi_k \text{ and } a \in \mathbb{R}.$$

Then, with the identification $U \equiv M(\{\theta_k\},\{\alpha_k\},\{\gamma_k\})$, U is purely absolutely continuous.

Proof. As above, we first replace γ_k by $(-1)^{k+1}\alpha_k$ and we introduce

$$L^2(\mathbb{T}) = \left[\oplus_{q=0}^{N-1} L^2_{2q}(\mathbb{T})\right] \bigoplus \left[\oplus_{q=0}^{N-1} L^2_{2q+1}(\mathbb{T})\right], \tag{6.6}$$

$$L^2_{2q}(\mathbb{T}) = \overline{\mathrm{span}\{(\mathrm{e}^{(2Nk+2q)ix})_{k\in\mathbb{Z}}, x\in\mathbb{T}\}}, \tag{6.7}$$

$$L^2_{2q+1}(\mathbb{T}) = \overline{\mathrm{span}\{(\mathrm{e}^{(2Nk+2q+1)ix})_{k\in\mathbb{Z}}, x\in\mathbb{T}\}}\ . \tag{6.8}$$

If P_{2q} and P_{2q+1} denote the orthogonal projections on these subspaces, we get for $\psi = \sum_{k\in\mathbb{Z}} c_k\varphi_k$, with the same notations as above,

$$(P_{2q}W\psi)(x) = \sum_{k\in\mathbb{Z}} c_{2(Nk+q)}\mathrm{e}^{i2(Nk+q)x} \in L^2_{2q}(\mathbb{T}), \tag{6.9}$$

and similarly for $(P_{2q+1}W\Psi)(x)$. To determine the image of U under the unitary mapping W, we introduce

$$\nu_k^{\pm} = \theta_{2k} + \theta_{2k\pm 1} \mp (\pi_{2k} - \pi_{2k\pm 1}), \tag{6.10}$$

such that $\nu_k^{\pm} = \nu_{k+N}^{\pm}$ for any $k \in \mathbb{Z}$. Hence $U \simeq \mathrm{e}^{-ia}V$ where, this time, V acts on $l^2(\mathbb{Z})$ according to

$$\begin{aligned} V\varphi_{2k} &= irt\mathrm{e}^{-i\nu_k^-}\varphi_{2k-1} + r^2\mathrm{e}^{-i\nu_k^-}\varphi_{2k} + irt\mathrm{e}^{-i\nu_k^+}\varphi_{2k+1} - t^2\mathrm{e}^{-i\nu_k^+}\varphi_{2k+2}, \\ V\varphi_{2k+1} &= -t^2\mathrm{e}^{-i\nu_k^-}\varphi_{2k-1} + itr\mathrm{e}^{-i\nu_k^-}\varphi_{2k} + r^2\mathrm{e}^{-i\nu_k^+}\varphi_{2k+1} + irt\mathrm{e}^{-i\nu_k^+}\varphi_{2k+2}. \end{aligned} \tag{6.11}$$

The phases $\nu_k^{\pm}$ being N-periodic, by manipulations similar to those performed above, one gets that $V \simeq T$, where T is a matrix valued multiplication operator on the decomposition of the Hilbert space (6.6) by the $2N \times 2N$ matrix

$$T(\mathrm{e}^{ix}) = \sum_{k=-2}^{2} \mathrm{e}^{ikx} T_k, \tag{6.12}$$

where the T_k's have a $N \times N$ block structure of the form

$$\begin{aligned} &T_2 = -t^2\begin{pmatrix} 0 & 0 \\ 0 & W_u \end{pmatrix}, T_1 = irt\begin{pmatrix} 0 & D_- \\ W_u & 0 \end{pmatrix}, \\ &T_0 = r^2\begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}, T_{-1} = irt\begin{pmatrix} 0 & W_l \\ D_+ & 0 \end{pmatrix}, T_{-2} = -t^2\begin{pmatrix} W_l & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{6.13}$$

with

$$\begin{aligned} D_{\pm} &= \mathrm{diag}(\mathrm{e}^{-i\nu_0^{\pm}}, \mathrm{e}^{-i\nu_1^{\pm}}, \dots, \mathrm{e}^{-i\nu_{N-1}^{\pm}}) \\ W_u &= \begin{pmatrix} 0 & \mathrm{e}^{-i\nu_1^-} & & & \\ & 0 & \mathrm{e}^{-i\nu_2^-} & & \\ & & \ddots & & \\ & & & 0 & \mathrm{e}^{-i\nu_{N-1}^-} \\ \mathrm{e}^{-i\nu_0^-} & & & & 0 \end{pmatrix}, \\ W_l &= \begin{pmatrix} 0 & & & & \mathrm{e}^{-i\nu_{N-1}^+} \\ \mathrm{e}^{-i\nu_0^+} & 0 & & & \\ & \mathrm{e}^{-i\nu_1^+} & \ddots & & \\ & & & 0 & \\ & & & \mathrm{e}^{-i\nu_{N-2}^+} & 0 \end{pmatrix}. \end{aligned} \tag{6.14}$$

Now, the operator T being unitary, the matrix $T(\mathrm{e}^{ix})$ is unitary as well for almost every $x \in \mathbb{R}$. But this matrix being analytic in a neighborhood of the real axis, it must be unitary everywhere on the real axis. By classical results in analytic perturbation theory, see [K], it is therefore diagonalizable with analytic eigenprojectors in a neighborhood of the real axis, and identically zero eigennilpotents. In order to prove the absolutely continuous nature of the spectrum of U, it is then enough to show that the analytic eigenvalues of the matrix $T(\mathrm{e}^{ix})$ are non-constant in $x \in \mathbb{R}$. But this is immediate, because otherwise, an infinitely degenerate eigenvalue would exist, which is forbidden by (3.25). □

Remarks. i) The formulae (6.13) above are the starting point of a detailed analysis of the band spectrum of U as a function of $t \in]0, 1[$, which we shall not perform here. We only note that for $t = 0$ (and with no loss of generality $a = 0$),

$$\sigma(U) = \left\{e^{i\nu_0^+}, \nu_1^+ + \ldots + \nu_{N-1}^+ + \pi e^{i\nu_0^-} + \nu_1^- + \ldots + \nu_1^- + \ldots + \nu_{N-1}^- + \pi)\right\}, \tag{6.15}$$

where each eigenvalue is infinitely degenerate, whereas, for $t = 1$,

$$\sigma(U) = \bigcup_{k=0,\ldots,N-1} \mathrm{Ran}\left\{e^{-2ix} e^{i(\nu_0^+ + \nu_1^+ + \cdots + \nu_{N-1}^+ + \pi)/N} e^{ik2\pi/N}, x \in \mathbb{T}\right\}$$
$$\cup\ \mathrm{Ran}\left\{e^{2ix} e^{i(\nu_0^- + \nu_1^- + \cdots + \nu_{N-1}^- + \pi)/N} e^{ik2\pi/N}, x \in \mathbb{T}\right\}. \tag{6.16}$$

Perturbation theory as $t \to 0$ and $t \to 1$ can now be applied to get information on the corresponding band functions in these regimes.
ii) It is not difficult to check that a unitary band matrix of order $2p + 1$ with periodic coefficients, in the sense that there exists $N > 0$ such that $\langle \varphi_j | U \varphi_k \rangle = \langle \varphi_{j+N} | U \varphi_{k+N} \rangle$, is always unitarily equivalent to a multiplication by an $pN \times pN$ unitary matrix $T(e^{ix})$ on $\oplus_{q=0,\ldots,pN-1} L^2_q(\mathbb{T})$, where $T(e^{ix})$ is a polynomial of degree p in $e^{\pm ix}$. However, in general, one cannot rule out the existence of finitely many infinitely degenerate eigenvalues.

Coherence on $l^2(\mathbb{N}^*)$. Let us now turn to the study of U^+ defined on $l^2(\mathbb{N}^*)$ by (3.1) in case the phases $\{\gamma_k\}$ are arbitrary whereas $\{\theta_k\}$ and $\{\alpha_k\}$ are eventually coherent: *i.e.*, there exists $k_0 \in \mathbb{N}$ and $2 \le N \in \mathbb{N}$ such that for all $k \ge k_0 \in \mathbb{N}^*$,

$$\theta_{k+N} = \theta_k, \quad \alpha_k = ak + \pi_k, \quad \text{where} \quad \pi_{k+N} = \pi_k \quad \text{and} \quad a \in \mathbb{R}. \tag{6.17}$$

We can replace without loss γ_k by $(-1)^{k+1}\alpha_k$ and assume $a = 0$, since we are working up to unitary equivalence. Our coherent comparison operator U_0 on $l^2(\mathbb{Z})$ is defined by (2.5) with phases $\{\theta_k\}$ and $\{\alpha_k\}$ obtained by extending (6.17) (with $a = 0$) to $\mathbb{Z}$. Therefore we can write on $l^2(-\mathbb{N}^*) \oplus \mathbb{C} \oplus l^2(\mathbb{N}^*)$

$$U_0 = \begin{pmatrix} W^- & & \\ & 1 & \\ & & U^+ \end{pmatrix} - F, \tag{6.18}$$

where absent elements denote zeros, W^- is an operator defined on $l^2(-\mathbb{N}^*)$ which is eventually periodic and F is a finite rank operator. It is always possible to construct U_0 this way with $\dim\ \mathrm{Ran}\ F = M$ depending on N and k_0.

Theorem 6.2. *Let U^+ and U_0 be as above. Then*

$$\sigma_{s.c.}(U^+) = \emptyset \quad \textit{and} \quad \sigma_{a.c.}(U^+) = \sigma_{a.c.}(U_0).$$

The point spectrum of U^+ consists of finitely many simple eigenvalues in the resolvent set of U_0.

Remark. As the proof below shows, the same statement holds if U^+ denotes a doubly infinite coherent matrix perturbed by a finite rank operator.

Proof. Let us first show that the finite rank perturbation F of the unitary U_0 does not produce any singular continuous spectrum. By Weyl's Theorem, this cannot happen in the gaps (on S^1) of the absolutely continuous spectrum of U_0. Therefore we focus on $\sigma(U_0)$. Depending on k_0 and N, we have for some finite $M > 0$,

$$F = \sum_{|j|,|k|\leq M} c_{j,k}|\varphi_j\rangle\langle\varphi_k|. \tag{6.19}$$

We know from (6.12) that U_0 is unitarily equivalent to the multiplication by a $2N \times 2N$ unitary matrix $V(x)$ on the decomposition (6.6), where $V(x)$ is a polynomial in $e^{\pm ix}$ whose eigenvalues are not constant in x. Therefore, $V(x)$ is analytic in a neighborhood of the real axis and we can write

$$V(x) = \sum_{j=1}^{2N} P_j(x)\lambda_j(x), \tag{6.20}$$

where the eigenprojections P_j and eigenvalues λ_j are analytic in a neighborhood of the real axis as well (see [K]). We know that

$$\sigma(U_0) = \cup_{j=1}^{2N} \text{ Ran } \{\lambda_j(x),\ \ x \in \mathbb{T}\}. \tag{6.21}$$

Note that

$$F \simeq \sum_{|j|,|k|\leq M} c_{j,k}|(\mathrm{e}^{ijx})\rangle\langle(\mathrm{e}^{ikx})|, \tag{6.22}$$

where the r.h.s. is to be understood as a multiplication operator on the decomposition (6.6) and (e^{ijx}) is a vector in $\mathbb{C}^{2N}$ with zero elements except at some line, depending on j, where the entry is e^{ijx}. We follow the perturbation theory of unitary operators presented in [KK] to study the unitary operator $U_1 \equiv U_0 + F$. Let $\zeta = \rho e^{i\beta}$ with $\rho \neq 1$ and $\beta \in \mathbb{T}$. We set for $j = 0, 1$,

$$R_j(\zeta) = U_j(U_j - \zeta)^{-1} = (\mathbb{I} - \zeta U_j{}^*)^{-1}, \tag{6.23}$$

$$\begin{aligned} G(\zeta) &= \mathbb{I} + \zeta(U_1^* - U_0^*)R_1(\zeta) \\ &= (\mathbb{I} + \zeta(U_0^* - U_1^*)R_0(\zeta))^{-1}. \end{aligned} \tag{6.24}$$

These quantities are analytic in $\mathbb{C} \setminus S^1$. We know from [KK] that for any vectors f, g,

$$\lim_{\rho\to 1^-} \langle g|\delta_\rho(E_j, \beta)f\rangle = \frac{d}{d\beta}\langle g|E_{a.c.,j}(\beta)f\rangle \quad \text{a.e. } \beta \in \mathbb{T}, \quad j = 0, 1, \tag{6.25}$$

where

$$2\pi\delta_\rho(E_j, \beta) = R_j(\zeta) - R_j(\zeta') \ \text{ with } \zeta' = 1/\bar{\zeta}, \tag{6.26}$$

and $E_{a.c.,j}(\beta)$ is the absolutely continuous part of the the spectral projector of U_j at $\mathrm{e}^{i\beta}$. Also,

$$\begin{aligned} \delta_\rho(E_1, \beta) &= G(\zeta)^*\delta_\rho(E_0, \beta)G(\zeta) \\ &= (\mathbb{I} - \zeta F^* R_0(\zeta))^{-1*}\delta_\rho(E_0, \beta)(\mathbb{I} - \zeta F^* R_0(\zeta))^{-1}. \end{aligned} \tag{6.27}$$

In order to get information on the nature of the spectral measure of U_1, it is sufficient to consider (6.25) on the cyclic subspace for U_0 generated by the range of F^*. Indeed, the spectral measures of U_0 and U_1 associated with vectors in the orthogonal complement of this subspace coincide and it is cyclic also for U_1. Let P denote the projector on Ran F^*. We first note that $\mathbb{I} - \zeta F^* R_0(\zeta)$ is invertible on Ran P if and only if

$$\det(P(\mathbb{I} - \zeta F^* R_0(\zeta))P) \neq 0 \tag{6.28}$$

and

$$(\mathbb{I} - \zeta F^* R_0(\zeta)|_{\text{Ran } P})^{-1} = (P(\mathbb{I} - \zeta F^* R_0(\zeta))P)^{-1} P. \tag{6.29}$$

So we need to consider the finite matrix whose elements are given for $|n|, |m| \leq M$ by

$$\begin{aligned} \langle \varphi_n | F^* R_0(\zeta) \varphi_m \rangle &= \sum_{|j| \leq M} \bar{c}_{j,n} \langle \varphi_j | U_0 (U_0 - \zeta)^{-1} \varphi_m \rangle \\ &= \sum_{|j| \leq M} \sum_{l=1}^{2N} \bar{c}_{j,n} \int_0^{2\pi} dx \left\langle (\mathrm{e}^{ijx}), \frac{P_l(x) \lambda_l(x)}{\lambda_l(x) - \zeta} (\mathrm{e}^{imx}) \right\rangle, \end{aligned} \tag{6.30}$$

where $< \cdot, \cdot >$ denotes here the scalar product in $\mathbb{C}^{2N}$. Therefore, (6.30) is a finite sum of the form

$$\sum_{l=1}^{2N} \int_0^{2\pi} dx \frac{f_{n,m}^{(l)}(x)}{\lambda_l(x) - \zeta}, \tag{6.31}$$

where $f_{n,m}^{(l)}$ is analytic in an open strip of finite width, independent of l, n, m containing the real axis.

Fix an $l \in \{1, \ldots, 2N\}$ and let $x_\beta \in \mathbb{T}$ be such that $\mathrm{e}^{i\beta} = \lambda_l(x_\beta)$. There is only a finite number of such points. Assume $\lambda_l'(x_\beta) \neq 0$. Then we can deform the contour of integration in x to control the integrals (6.31) when $\rho \to 1$ as follows. There exists a neighborhood $\mathbb{C} \supset N_\beta$ of x_β which is mapped by λ_l bijectively on its image M_β which contains $\mathrm{e}^{i\beta}$ in its interior. Let $D_\beta \subset M_\beta$ be a smooth deformation of the unit circle towards the exterior which avoids $\mathrm{e}^{i\beta}$. Taking the inverse image $\lambda_l^{-1}(D_\beta) \subset N_\beta$ and connecting it at both ends with the real axis, we get a smooth path C_β along which

$$\int_0^{2\pi} dx \frac{f_{n,m}^{(l)}(x)}{\lambda_l(x) - \zeta} = \int_{C_\beta} dz \frac{f_{n,m}^{(l)}(z)}{\lambda_l(z) - \zeta}. \tag{6.32}$$

By construction, the last integral is now analytic in ζ in a neighborhood $\tilde{M}_\beta \subset M_\beta$ containing $\mathrm{e}^{i\beta}$. Therefore, the matrix (6.30) has an analytic continuation in ζ in a neighborhood of S^1 except at a finite set of points. Hence there is only a countable set of points of S^1, call it Z, where the determinant (6.28) is zero.

Then, for any $\psi = P\psi$ and any $\mathrm{e}^{i\beta} \in S^1 \setminus Z$, we can write

$$(\mathbb{I} - \zeta F^* R_0(\zeta))^{-1} \psi = \sum_{|k| \leq M} d_k(\zeta) \varphi_k, \tag{6.33}$$

where the $d_k(\zeta)$'s are analytic in a neighborhood of $e^{i\beta}$ ‘ and the φ_k's span the range of F^*. Thus, we deduce from the relation

$$\langle\psi|\delta_\rho(E_1,\beta)\psi\rangle = \sum_{|k|,|j|\leq M} \bar{d}_j(\zeta)d_k(\zeta)\langle\varphi_j|\delta_\rho(E_0,\beta)\varphi_k\rangle, \tag{6.34}$$

that the limit $\rho \to 1^-$ yields the derivative of an absolutely continuous measure on $S^1\backslash Z$, as the limits $\lim_{\rho\to 1^-}\langle\varphi_j|\delta_\rho(E_0,\beta)\varphi_k\rangle \in L^1(\mathbb{T})$. As a countable set of point cannot support a continuous measure, we get that $\sigma_{s.c.}(U_1) = \emptyset$.

Let us consider the point spectrum of U^+. From the relation (3.25), we get that the eigenvalues have multiplicity two at most. Except for a finite number of them, the transfer matrices $T(k)$ are periodic in k, of period N. Therefore we define

$$R = T(k_0+1+N)T(k_0+N)\cdots T(k_0+1) \tag{6.35}$$

and set

$$d(k) = \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} \tag{6.36}$$

so that

$$d(jN+k_0) = R^j d(k_0) = R^j T(k_0)T(k_0-1)\cdots T(2)d(1). \tag{6.37}$$

We will use the notation $D(j) = d(jN+k_0)$. Note also that $\det R = e^{i\kappa}$, where $\kappa \in \mathbb{T}$ is independent of λ, due to (3.24), and that the matrix R is analytic in λ since it is a polynomial in $e^{+i\lambda}$ and $e^{-i\lambda}$.

Assume that an eigenvector of U^+ exists in $l^2(\mathbb{N}^*)$ for the eigenvalue $e^{i\lambda}$. This implies that the sequence $\{\|D(j)\|\}_{j\in\mathbb{Z}}$ belongs to l^2 at $+\infty$. We are thus lead to the study of (6.37). This is done by means of the following elementary lemma whose proof we omit.

Lemma 6.1. *Let R be a 2×2 matrix with $|\det R| = 1$, and let E_1 be an eigenvalue of R. Consider $D(j) = R^j D(0)$, where $D(0) \in \mathbb{C}^2$. Then,*
1) there exists $K > 0$, such that for all vectors $D(0)$ of norm 1, for all $j \in \mathbb{Z}$, $K \leq \|D(j)\| \leq |j|/K$ if and only if $|E_1| = 1$.
2) When $|E_1| \neq 1$, there exists another eigenvalue $E_2 \neq E_1$. We can assume $|E_1| > 1 > |E_2| = |E_1|^{-1}$ and we get

$$D(j) = AE_1^j v_1 + BE_2^j v_2, \quad j \in \mathbb{Z},$$

where $v_1, v_2 \in \mathbb{C}^2$ are the corresponding eigenvectors of R and $A, B \in \mathbb{C}$ are the coefficient of $D(0)$ in the basis they form.

A direct consequence is that $\{\|D(j)\|\} \in l^2(\mathbb{N})$ implies exponential decay at $+\infty$, thus $D(0) = v_2$ and any eigenvalue is simple. Now we use $D(0)$ as an initial vector to construct a generalized vector for the coherent operator U_0 on $l^2(\mathbb{Z})$. Note that considered as a functions of λ, $R(\lambda)$ is analytic in a neighborhood of the real axis, therefore, $E_1(\lambda)$ is analytic on $\mathbb{T}$, except at the finite set X of exceptional points in $\mathbb{T}$ where the eigenvalues of $R(\lambda)$ cross. At such exceptional points, $|E_1| = 1$. Then observe that if the second statement of Lemma 6.1 is true for some $\lambda \in \mathbb{T}$, it still true in a neighborhood of λ by continuity. This implies that all generalized eigenvectors corresponding to eigenvalues in the corresponding neighborhood grow exponentially at one end or the other. Due to Corollary 5.2, this can take place only in the resolvent set of U_0. Also, as the spectrum of U_0 contains no isolated point, the argument above shows that X must belong to the closure of the set of points in $\mathbb{T} \setminus X$, where $|E_1(\lambda)| = 1$. Therefore $|E_1|$ is continuous on the whole of $\mathbb{T}$ and $\sigma(U_0) = |E_1|^{-1}(\{1\})$. The band edges are also excluded from the point spectrum of U^+, since they correspond to points where $|E_1| = 1$.

We now study the number of eigenvalues of U^+. The boundary condition that $d(1)$ has to meet reads, according to (3.1),

$$\tilde{T}^{-1}d(1) = c_1 b(\lambda), \tag{6.38}$$

where c_1 is the non zero first coefficient of the generalized eigenvector and

$$\tilde{T}^{-1} = \mathrm{e}^{-i(\theta_1+\theta_2+\alpha_2-\alpha_1)} \begin{pmatrix} irt & -t^2 \\ r^2 & itr \end{pmatrix} - \mathrm{e}^{i\lambda} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6.39}$$

$$b(\lambda) = \mathrm{e}^{i\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathrm{e}^{-i(\theta_0+\theta_1+\alpha_1)} \begin{pmatrix} r \\ it \end{pmatrix}\ , \tag{6.40}$$

with $|\det \tilde{T}^{-1}| = 1$. Therefore, the condition to have an eigenvalue $\mathrm{e}^{i\lambda}$ for U^+ is equivalent to

$$b(\lambda) \ \parallel \ \tilde{T}^{-1}T^{-1}(2)T^{-1}(3)\cdots T^{-1}(k_0)v_2(\lambda), \tag{6.41}$$

where $v_2(\lambda)$ is an eigenvector of $R(\lambda)$ and all matrices involved are analytic in $\lambda \in \mathbb{T}$. In other words, $\mathrm{e}^{i\lambda}$ is an eigenvalue if and only if

$$\det(v_2(\lambda); T(k_0)T(k_0-1)\cdots T(2)\tilde{T}b(\lambda)) \equiv \det(v_2(\lambda); a(\lambda)) = 0, \tag{6.42}$$

where a is analytic on $\mathbb{T}$ and v_2 can be chosen analytic on $\mathbb{T} \setminus X$, see [K]. Therefore, to show that the number of eigenvalues of U^+ is finite, it is enough to prove that, as a function of λ on $\mathbb{T}$, the determinant above has finitely many zeros. This is a consequence of the next lemma we prove in the Appendix.

Lemma 6.2. *If $\lambda_0 \in X$, the eigenvectors $v_j(\lambda)$, $j = 1, 2$, have at worst a square root branch point at λ_0.*

It follows that the function $\lambda \mapsto \det(v_2(\lambda); a(\lambda))$ is analytic on $\mathbb{T} \setminus X$ and possesses square root branch point singularities at X as well. Therefore it only possesses finitely many zeros on $\mathbb{T}$.

Finally, we show that $\sigma(U_0) \subset \sigma(U^+)$. Let $\mathrm{e}^{i\lambda}$ be in the interior of the set $\sigma_{a.c}(U_0)$ and consider the relation (3.25) yielding the coefficients $d(k)$ of the corresponding generalized eigenvector. Up to a finite number of transfer matrices $T(k_0)T(k_0-1)\cdots T(1)$, this relation is identical to that yielding the coefficients with positive indices of a corresponding generalized eigenvector for U^+. The discussion above shows that $d(k)$ is polynomially bounded at both ends, so that by Corollary 5.2, $\mathrm{e}^{i\lambda}$ belongs the spectrum of U^+ as well. This finishes the proof of the theorem. □

Remark. In keeping with the last remark following the proof of Theorem 6.1, let U_0 denote a periodic band matrix of order $2p + 1$. Then, it is also true that a finite rank perturbation of the form (6.19) produces no singular continuous spectrum, since the first part of the above proof goes through without changes.

7. An Almost Periodic Example

In order to complete the picture of the spectral properties such matrices can possess, we briefly describe below an example of deterministic unitary band matrices which is almost periodic and purely singular continuous. This example is constructed in analogy with the random discrete Schrödinger case according to the approaches of Herman and Gordon, see e.g. [CFKS].

We consider again the matrix $M(\{\theta_k\}, \{\alpha_k\}, \{\gamma_k\})$, where the phases α_k are taken as constants, while the γ_k's are arbitrary and can be replaced by $(-1)^{k+1}\alpha_k$, as above. The almost periodicity lies with the phases θ_k defined according to

$$\theta_k = 2\pi\beta k + \theta, \quad \forall k \in \mathbb{N}, \tag{7.1}$$

where β is irrational, and $\theta \in [0, 2\pi[$.

Consider the uniform measure $\mathbb{P}_0$ on the $\mathbb{T}$, and the translation $\tau : \mathbb{T} \to \mathbb{T}$ defined by

$$\tau(\theta) = 2i\pi\beta + \theta. \tag{7.2}$$

Then the set of iterates τ^k, $k \in \mathbb{Z}$ is ergodic. The corresponding transfer matrices $T(k)^\theta$ generated by this set of translations are then given by (see (4.8))

$$\begin{aligned} T(k)^\theta_{11} &= -\mathrm{e}^{-i(\lambda+2\theta+8k\pi\beta-6\beta\pi)}, \\ T(k)^\theta_{12} &= \frac{ir}{t}\left(\mathrm{e}^{-i(\lambda+2\theta+8k\pi\beta-6\beta\pi)} - 1\right), \\ T(k)^\theta_{21} &= \frac{ir}{t}\left(\mathrm{e}^{2i\theta} - \mathrm{e}^{-i(\lambda+2\theta+8k\pi\beta-6\beta\pi)}\right), \\ T(k)^\theta_{22} &= \frac{r^2}{t^2}\left(\mathrm{e}^{4i\beta} + 1 - \mathrm{e}^{-i(\lambda+2\theta+8k\pi\beta-6\beta\pi)}\right) \\ &\quad -\frac{1}{t^2}\mathrm{e}^{i(\lambda+2\theta+8k\pi\beta-2\beta\pi)}. \end{aligned} \tag{7.3}$$

Following Herman [He], we first get the positivity of the Lyapunov exponent.

Proposition 7.1. *Let $T(k)^\theta$ be the transfer matrices (7.3) at spectral parameter $\lambda \in \mathbb{T}$ corresponding to $U \equiv M(\{\theta_k\}, \{\alpha\}, \{\gamma_k\})$, where the θ_k's are given by (7.1). For β irrational, the Lyapunov exponent $\gamma(\lambda)$ satisfies for almost all θ:*

$$\gamma(\lambda) \geq \ln\frac{1}{t^2} > 0, \quad \textit{therefore} \quad \sigma_{ac}(U) = \emptyset.$$

Proof. We first note that the sub-additive ergodic theorem applies to $F_N(\theta) = \ln \|\Pi_{k=1}^N T(k)^\theta\|$ and since τ is ergodic,

$$\lim_{N\to\infty} \frac{F_N(\theta)}{N} = \gamma(\lambda) \tag{7.4}$$

almost surely with respect to $\mathbb{P}_0$. Setting $z = \mathrm{e}^{-i\theta}$, we write our transfer matrices $T(k, z)$, expliciting the dependence in $z \in \mathbb{C}^*$, and we define three matrices $(R_j(k))$, $j = -2, 0, 2$, by

$$T(k, z) \equiv z^2 R_2(k) + R_0(k) + z^{-2} R_{-2}(k), \tag{7.5}$$

where

$$R_2(k) = \mathrm{e}^{-i(\lambda+8k\pi\beta-6\beta\pi)} \begin{pmatrix} -1 & \frac{ir}{t} \\ -\frac{ir}{t} & -\frac{r^2}{t^2} \end{pmatrix}, \tag{7.6}$$

$$R_0(k) = \begin{pmatrix} 0 & -\frac{ir}{t} \\ \frac{ir}{t}\mathrm{e}^{4i\pi\beta} & \frac{r^2}{t^2}(\mathrm{e}^{i4\pi\beta}+1) \end{pmatrix}, \tag{7.7}$$

$$R_{-2}(k) = -\frac{1}{t^2}\mathrm{e}^{i(\lambda+8k\pi\beta-2\beta\pi)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.8}$$

Then we consider $S_k(z) = z^2 T(k, z)$ which is analytic in $\mathbb{C}$ and such that if $z \in S^1$, $\|S_k(z)\| = \|T_k(z)\|, \forall k \in \mathbb{Z}$. Hence, the function $\| \ln \prod_{k=1}^{N} S_k(z)\|$ is sub-harmonic and as $S_k(0) = R_{-2}$, we get the estimate

$$\frac{1}{2\pi}\int_0^{2\pi} \ln \| \prod_{k=1}^{N} S_k(e^{i\theta})\| d\theta \geq \ln \| \prod_{k=1}^{N} S_k(0)\| = N \ln \frac{1}{t^2}. \tag{7.9}$$

We finally note that (7.4) implies

$$\gamma(\lambda) = \lim_{N\to+\infty} \frac{1}{N}\int_0^{2\pi} \ln \left\| \prod_{k=1}^{N} T_k(e^{i\theta}) \right\| \frac{d\theta}{2\pi}. \tag{7.10}$$

The second statement then follows from Theorem 5.3. □

Next, we adapt the argument of Gordon to our setting in order to exclude eigenvalues in $\sigma(U)$, for β a Liouville number. That is if for any $k \in \mathbb{N}$, there exist $p_k, q_k \in \mathbb{N}$ such that

$$|\beta - p_k/q_k| \leq k^{-q_k}. \tag{7.11}$$

Proposition 7.2. *Assume the phases (θ_k) are given by (7.1) and (α_k) are zero. Moreover, suppose β is a Liouville number and $((\theta^m)_k)$ a family of periodic sequence of period q_m. For each sequence, the corresponding family of transfer matrices (7.3) is denoted by $(T(k)^\theta)_{k\in\mathbb{Z}}$ and $(T_m^\theta(k))_{k\in\mathbb{Z}}$ respectively. Assume the period of the sequence (θ^m) obeys $\lim_{m\to+\infty} q_m = +\infty$ and the following estimates hold:*

$$\sup_{k,m} \|T_m^\theta(k)\| < \infty, \qquad \sup_{|k|\leq 2q_m} \|T^\theta(k) - T_m^\theta(k)\| \leq C m^{-q_m}.$$

Then, any non-zero solution $\phi = \sum c_k\varphi_k$ of $U\phi = \mathrm{e}^{i\lambda}\phi$ satisfies

$$\limsup_{|k|\to+\infty} \frac{c_{k+1}^2 + c_k^2}{c_1^2 + c_0^2} \geq \frac{1}{4}. \tag{7.12}$$

Its proof is identical to that given in [CFKS], Theorem 10.3, noting that the norm of any transfer matrix (7.3) is bounded by a constant depending on t, r only.

Theorem 7.1. *Let U be as in Proposition 7.1. If β is a Liouville number, then for a.e. θ, U is purely singular continuous.*

Proof. Let β be a Liouville number. It can be approximated by a sequence of irreducible fractions $\left(\frac{p_m}{q_m}\right)$ obeying (7.11). Define the sequence $((\theta^m)_k)$ by: $\forall k \in \mathbb{Z}$, $(\theta^m)_k = 2\pi \frac{p_m}{q_m} k + \theta$. A simple computation shows $\forall k \in \mathbb{Z}$,

$$\begin{aligned} &T^\theta(k) - T^\theta_m(k) \\ &\quad = 2i \sin\left((4k-3)\pi\left(\beta - \frac{p_m}{q_m}\right)\right) e^{-i\left(\lambda+2\theta+(4k-3)\pi\left(\beta+\frac{p_m}{q_m}\right)\right)} \begin{pmatrix} 1 & -irt^{-1} \\ irt^{-1} & r^2t^{-2} \end{pmatrix} \\ &\qquad -2i \sin\left((4k-1)\pi\left(\beta - \frac{p_m}{q_m}\right)\right) e^{i\left(\lambda+2\theta+(4k-1)\pi\left(\beta+\frac{p_m}{q_m}\right)\right)} \begin{pmatrix} 0 & 0 \\ 0 & t^{-2} \end{pmatrix} \end{aligned} \tag{7.13}$$

Then, β being a Liouville number allows us to check the hypotheses of Proposition 7.2. Therefore, the generalized eigenvalue equation cannot have l^2 solution and the point spectrum of U is empty. Combining this result with Theorem 7.1 yields the conclusion.

8. Appendix

Proof of Lemma 3.1. Assume U is unitary and, for all $k \in \mathbb{Z}$,

$$U\varphi_k = \alpha_k\varphi_{k-1} + \beta_k\varphi_k + \gamma_k\varphi_{k+1}, \tag{8.1}$$

so that

$$U = \begin{pmatrix} \ddots & \alpha_{k-1} & & & \\ & \beta_{k-1} & \alpha_k & & \\ & \gamma_{k-1} & \beta_k & \alpha_{k+1} & \\ & & \gamma_k & \beta_{k+1} & \\ & & & \gamma_{k+1} & \ddots \end{pmatrix}. \tag{8.2}$$

Then, for all $k \in \mathbb{Z}$,

$$\begin{aligned} &|\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2 = 1, \\ &|\alpha_{k+1}|^2 + |\beta_k|^2 + |\gamma_{k-1}|^2 = 1, \\ &\alpha_k\beta^*_{k-1} + \beta_k\gamma^*_{k-1} = 0, \\ &\gamma_{k-1}\beta^*_{k-1} + \beta_k\alpha^*_k = 0, \\ &\alpha_k\gamma^*_k = 0, \\ &\alpha_k\gamma^*_{k-2} = 0. \end{aligned} \tag{8.3}$$

Let us start by noting that $|\beta_{k_0}| = 1$ is equivalent to $\alpha_{k_0} = \gamma_{k_0} = \alpha_{k_0+1} = \gamma_{k_0-1} = 0$, which creates an isolated 1×1 block in the matrix structure of U.

We assume now that one off-diagonal element is non-zero. By considering the transpose of U instead of U if necessary, we assume without loss $\alpha_{k_0} \neq 0$. The last two relations impose $\gamma_{k_0} = \gamma_{k_0-2} = 0$ and the two middle ones yield

$$|\beta_{k_0}\alpha_{k_0+1}| = |\beta_{k_0-1}\alpha_{k_0-1}| = |\beta_{k_0+1}\alpha_{k_0+1}| = |\beta_{k_0-2}\alpha_{k_0-1}| = 0. \tag{8.4}$$

On the one hand, if $\beta_{k_0} \neq 0$, then $\beta_{k_0-1} \neq 0$. Otherwise we would get from the first two relations in (8.3) $|\alpha_{k_0}| = 1$ and $\beta_{k_0} = 0$. Hence, from (8.4), $\alpha_{k_0-1} = \alpha_{k_0+1} = 0$, showing that an isolated block of the form

$$\begin{pmatrix} \beta_{k_0-1} & \alpha_{k_0} \\ \gamma_{k_0-1} & \beta_{k_0} \end{pmatrix} \tag{8.5}$$

exists in the matrix U.

If, on the other hand, $\beta_{k_0} = 0$, together with $\alpha_{k_0} \neq 0$ this implies $|\alpha_{k_0}| = 1$ and, in turn $\beta_{k_0-1} = 0$.

We first assume $\gamma_{k_0-1} \neq 0$. Hence the last two equations in (8.3) yield $\alpha_{k_0-1} = \alpha_{k_0+1} = 0$, which again yields an isolated block of the form

$$\begin{pmatrix} 0 & \alpha_{k_0} \\ \gamma_{k_0-1} & 0 \end{pmatrix}. \tag{8.6}$$

in U with $|\gamma_{k_0-1}| = |\alpha_{k_0}| = 1$.

If $\gamma_{k_0-1} = 0$ and $|\alpha_{k_0}| = 1$, we get $|\alpha_{k_0-1}| = |\alpha_{k_0+1}| = 1$. In turn, this imposes $\beta_{k_0-1} = \gamma_{k_0-1} = 0$ and $\beta_{k_0+1} = \gamma_{k_0+1} = 0$. Thus, U is of the form

$$U = \begin{pmatrix} \ddots & \alpha_{k_0-1} & & & \\ & 0 & \alpha_{k_0} & & \\ & & 0 & \alpha_{k_0+1} & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \tag{8.7}$$

and is therefore unitarily equivalent to the shift operator, using a unitary defined similarly to (3.18).

Hence, except in the last case, iteration of the above arguments, shows that U has the block structure announced. □

Proof of Lemma 4.1. We can set the value λ at zero without loss. Let

$$\Phi_u(n) = \mathbb{E}(e^{-in\theta}) = \delta_{n,0} \tag{8.8}$$

be the characteristic function of the common uniform distribution of the phases θ_k and α_k. Consider the characteristic function of the set of random vectors $\{\delta_{k_1}, \delta_{k_2}, \cdots \delta_{k_j}\}$ given by

$$\begin{aligned} &\Phi_{\delta_{k_1},\delta_{k_2},\cdots\delta_{k_j}}(n_1, n_2, \cdots, n_j) = \mathbb{E}(\exp(-i(n_1 \cdot \delta_{k_1} + n_2 \cdot \delta_{k_2} + \cdots + n_j \cdot \delta_{k_j}))) \\ &\quad = \mathbb{E}\big(\exp(-i(n_1^1\theta_{2k_1} + (n_1^1 + n_1^2)\theta_{2k_1-1} + n_1^2\theta_{2k_1-2} + \cdots + n_j^1\theta_{2k_j} \\ &\qquad + (n_j^1 + n_j^2)\theta_{2k_j-1} + n_j^2\theta_{2k_j-2}))\big) \\ &\quad \times \mathbb{E}\big(\exp(-i(n_1^1\alpha_{2k_1} + (n_1^2 - n_1^1)\alpha_{2k_1-1} - n_1^2\alpha_{2k_1-2} + \cdots + n_j^1\alpha_{2k_j} \\ &\qquad + (n_j^2 - n_j^1)\alpha_{2k_j-1} - n_j^2\alpha_{2k_j-2}))\big), \end{aligned} \tag{8.9}$$

where $n_k = (n_k^1, n_k^2) \in \mathbb{Z}^2$. We used independence of the θ's and α's to factorize the expectations over these random variables. We can assume the k_l's are ordered and we deal with the θ's only. The argument is similar for the α's. From the expression above, one sees that one can factorize the expectations over θ_l with $l \leq 2k_r$ from those with $l \geq 2k_{r+1} - 2$ as soon as $k_r < k_{r+1} + 1$. Therefore, it is enough to consider consecutive indices $k_1 = m, k_2 = m + 1, \ldots, k_j = m + j$. As (8.8) shows, in such a case, the expectation over the θ's equals zero unless

$$n_1^1 = 0, \quad n_1^1 + n_1^2 = 0, \ldots, n_j^1 + n_j^2 = 0, \quad n_j^2 = 0, \tag{8.10}$$

when it equals one. But this is equivalent to $n_k^l = 0$ for all $k = 1, \dots, j$, $l = 1, 2$. Hence, we have proven that

$$\Phi_\delta(n) = \Phi_u(n^1)\Phi_u(n^2) \tag{8.11}$$

for $j = 1$ and

$$\Phi_{\delta_{k_1},\delta_{k_2},\dots,\delta_{k_j}}(n_1, n_2, \dots, n_j) = \Phi_{\delta_{k_1}}(n_1)\Phi_{\delta_{k_2}}(n_2)\cdots\Phi_{\delta_{k_j}}(n_j), \tag{8.12}$$

which is equivalent to independence of the random vectors $\delta_{k_1}, \delta_{k_2}, \cdots \delta_{k_j}$. □

Proof of Lemma 4.2. Let us consider $Y_k = Y_k^+ = X_k + X_{k-1}$ only, the other case being similar. Let the measure μ_X denote the distribution of the X_k's. Then the Y_k's are identically distributed according to the measure $\mu_Y = \mu_X * \mu_X$. Let Φ_X be the characteristic function of the random variable X. Then $\Phi_Y(n) = \Phi_X^2(n)$. Given Lemma 4.1, we need only prove that independence of the Y_k's imposes μ_X is uniform on the torus. Then, the characteristic function of the variables $\{Y_k, Y_{k+1}\}$ must satisfy for all $(n_1, n_2) \in \mathbb{Z}^2$,

$$\begin{aligned}\Phi_{Y_k,Y_{k+1}}(n_1, n_2) &= \mathbb{E}(\exp(-in_1(X_k + X_{k-1}) - in_2(X_{k+1} + X_k)))\\ &= \Phi_X(n_1)\Phi_X(n_1 + n_2)\Phi_X(n_2) \equiv \Phi_X^2(n_1)\Phi_X^2(n_2). \end{aligned}\tag{8.13}$$

In case $\Phi_X(n_1)\Phi_X(n_2) = 0$, this relation is fulfilled. Otherwise, we have for all other cases

$$\Phi_X(n_1)\Phi_X(n_2) = \Phi_X(n_1 + n_2). \tag{8.14}$$

If N is the smallest positive integer such that $\Phi_X(N) \neq 0$, we get that

$$1 = \Phi_X(0) = \Phi_X(N)\Phi_X(-N) = |\Phi_X(N)|^2 \iff \Phi_X(N) = \mathrm{e}^{-i\nu}, \tag{8.15}$$

for some $\nu \in \mathbb{T}$. Iteration of (8.14) implies that for any $m \in \mathbb{Z}$,

$$\Phi_X(mN) = \mathrm{e}^{-im\nu}. \tag{8.16}$$

One checks with (8.14) that there can be no integer $M > N$, $M \neq kN$, $k \in \mathbb{N}$, such that $\Phi_X(M) \neq 0$. That implies that $\mu_X = \delta(x - \nu/N)$, which is a contradiction to our hypothesis. Hence we must have $\Phi_X(n) = 0$ for all $n \neq 0$, which corresponds to a uniform distribution. □

Proof of Theorem 5.2. We develop here the arguments yielding polynomially bounded generalized eigenfunctions associated with spectral parameters in the spectrum of U. We state the starting point result, Theorem C.5.1 in [S], specialized to our setting.

Theorem 8.1. *Let $\mathcal{H}$ be a separable Hilbert space. Assume that to any Borel set $\Delta \subset [0, 2\pi[$ we have a positive trace class operator $A(\Delta)$ on $\mathcal{H}$ satisfying: the condition if $\Delta = \cup_{n=1}^{+\infty}\Delta_n$ with $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$, then $A(\Delta) = s - \lim \sum A(\Delta_n)$.*

Then there exists a Borel measure $d\rho$ and a positive, trace class, operator valued measurable function $a(\lambda)$ such that:

- $\forall \phi \in \mathcal{H}$, $\langle\phi|A(\Delta)\phi\rangle = \int_\Delta \langle\phi|a(\lambda)\phi\rangle d\rho(\lambda)$.
- $Tr(a(\lambda)) = 1$, *$d\rho$-ae.*

These two conditions characterize the operator valued function a.

Let us introduce weighted $l^2(\mathbb{Z})$ spaces

$$l^2_\delta(\mathbb{Z}) = \{\phi = (\phi_n)_n \in l^2(\mathbb{Z}^*); \sum_{n\in\mathbb{Z}} < n >^\delta |\phi_n|^2 < +\infty\}, \tag{8.17}$$

where $< n >=< 1 + n^2 >^{1/2}$. We prove the equivalent of Theorem C.5.2. in [S].

Proposition 8.1. *Let U be a unitary operator defined on $l^2(\mathbb{Z})$ and $\delta > 1$. Then there exists a spectral measure $d\rho$ and, for $d\rho$ almost all λ, there exists a function $F_{.,.}(\lambda)$ defined on $\mathbb{Z} \times \mathbb{Z}$ such that:*

- *$F_{n,m}$ is measurable in λ.*
- *$\sum_{n,m} < n >^{-\delta} |F_{n,m}(\lambda)|^2 < m >^{-\delta} \leq 1$, $d\rho$-ae.*
- *$|F_{n,m}(\lambda)| \leq C < n >^{-\frac{\delta}{2}} < m >^{-\frac{\delta}{2}}$.*
- *For any bounded Borel function g on S^1, and for any vectors ϕ, ψ in $l^2_\delta(\mathbb{Z})$,*

$$\langle \phi | g(U) \psi \rangle = \int g(\lambda) \left(\sum_{n,m} F_{n,m}(\lambda) {\phi_n}^* \psi_m \right) d\rho(\lambda). \tag{8.18}$$

- *For any fixed m, $(U - e^{i\lambda}) F_{.,m}(\lambda) = 0$, where $F_{.,m}(\lambda) = \sum_{n\in\mathbb{Z}} F_{n,m}(\lambda)\varphi_n$.*

Proof. We denote the spectral projectors of U by $(E(\Delta))_{\Delta\in\mathcal{B}([0,2\pi[)}$, where $\mathcal{B}([0, 2\pi[)$ denotes the Borel sets on the interval $[0, 2\pi[$. Let x be the self adjoint operator, diagonal on the orthonormal basis $(\varphi_n)_{n\in\mathbb{Z}}$, defined $\forall n \in \mathbb{Z}$, by $x\varphi_n =< n > \varphi_n$. The operators $(A(\Delta))_{\Delta\in\mathcal{B}([0,2\pi[)}$ defined $\forall \Delta \in \mathcal{B}([0, 2\pi[)$ by

$$A(\Delta) = x^{\frac{-\delta}{2}} E(\Delta) x^{\frac{-\delta}{2}}, \tag{8.19}$$

are positive and trace class:

$$\sum_{n\in\mathbb{Z}} \left\langle \varphi_n | x^{\frac{-\delta}{2}} E(\Delta) x^{\frac{-\delta}{2}} \varphi_n \right\rangle \leq \sum_{n\in\mathbb{Z}} \langle n \rangle^{-\delta} \langle \varphi_n | E(\Delta) \varphi_n \rangle < +\infty \ . \tag{8.20}$$

By definition, the spectral family $E(.)$ satisfies for any countable disjoint family $(\Delta_i)_{i\in I} \subset \mathcal{B}([0, 2\pi[)$: $E(\cup_{i\in I}\Delta_i) = s - \lim \sum_{i\in I} E(\Delta_i)$. The operators $x^{-\delta}$ being bounded on $l^2(\mathbb{Z})$, we get $A(\cup_{i\in I}\Delta_i) = s - \lim \sum_{i\in I} A(\Delta_i)$. Hence $A(.)$ is a Borel measure with values in positive, trace class operators. and Theorem 8.1 applies. Thus, $\forall (n, m) \in \mathbb{Z}\times\mathbb{Z}$, we get a function defined $d\rho$-ae,

$$\begin{aligned} F_{n,m}(\lambda) &= \left\langle \varphi_n | x^{\frac{\delta}{2}} a(\lambda) x^{\frac{\delta}{2}} \varphi_m \right\rangle = (\langle n \rangle \langle m \rangle)^{\frac{\delta}{2}} \langle \varphi_n | a(\lambda) \varphi_m \rangle \\ &= (\langle n \rangle \langle m \rangle)^{\frac{\delta}{2}} a_{n,m}(\lambda). \end{aligned} \tag{8.21}$$

By construction, the functions $a_{n,m}$ (hence $F_{n,m}$) are measurable. Moreover,

$$\begin{aligned} \sum_{n,m} |F_{n,m}(\lambda)|^2 (\langle n \rangle \langle m \rangle)^{-\delta} &= \sum_{n,m} |a_{n,m}(\lambda)|^2 \\ &= \sum_n \|a(\lambda)\varphi_n\|^2 = \|a(\lambda)\|_2^2 \leq \|a(\lambda)\|_1^2 = Tr(a(\lambda))^2 = 1 \ d\rho - ae. \end{aligned} \tag{8.22}$$

This implies the third statement. Let $A_i \subset [0, 2\pi[$ be the Borel set, $\chi_i : S^1 \to \mathbb{R}$ be its characteristic function and ϕ, ψ be two vectors of $l^2_\delta(\mathbb{Z})$. Then

$$\begin{aligned}
\int_{[0,2\pi[} \chi_i(e^{i\lambda}) \sum_{n,m} \phi_n^* \psi_m F_{n,m}(\lambda) d\rho(\lambda) &= \sum_{n,m} \phi_n^* \psi_m \int_{[0,2\pi[} \chi_i(e^{i\lambda}) F_{n,m}(\lambda) d\rho(\lambda) \\
= \sum_{n,m} \phi_n^* \psi_m \int_{A_i} F_{n,m}(\lambda) d\rho(\lambda) &= \sum_{n,m} \phi_n^* \psi_m \langle \varphi_n | E(A_i) \varphi_m \rangle \\
= \sum_{n,m} \phi_n^* \psi_m \left\langle \varphi_n \middle| \left(\int_{[0,2\pi[} \chi_i(e^{i\lambda}) d\lambda \right) \varphi_m \right\rangle &= \langle \phi | \chi_i(U) \psi \rangle. \qquad (8.23)
\end{aligned}$$

This results holds for step functions by linearity, and for bounded measurable functions on $[0, 2\pi[$. In particular, taking $g = id$ and $\psi = \varphi_m$,

$$\begin{aligned}
\langle \phi | U \varphi_m \rangle &= \int_{[0,2\pi[} e^{i\lambda} \left(\sum_n F_{n,m}(\lambda) \phi_n^* \right) d\rho(\lambda) \ , \\
&= \int_{[0,2\pi[} \langle \phi | e^{i\lambda} F_{\cdot,m}(\lambda) \rangle d\rho(\lambda). \qquad (8.24)
\end{aligned}$$

But,

$$\begin{aligned}
\int_{[0,2\pi[} \langle \phi | U F_{\cdot,m}(\lambda) \rangle d\rho(\lambda) &= \sum_k \int_{[0,2\pi[} \phi_k^* (U F_{\cdot,m}(\lambda))_k d\rho(\lambda) \\
= \sum_k \int_{[0,2\pi[} \phi_k^* \sum_j U_{kj} F_{j,m}(\lambda) d\rho(\lambda) &= \sum_{k,j} U_{kj} \phi_k^* \int_{[0,2\pi[} F_{j,m}(\lambda) d\rho(\lambda) \\
= \sum_{k,j} U_{kj} \phi_k^* (jm)^\delta \int_{[0,2\pi[} a_{j,m}(\lambda) d\rho(\lambda) &= \sum_{k,j} U_{kj} \phi_k^* (jm)^\delta A([0, 2\pi[)_{j,m} \\
= \sum_{k,j} U_{kj} \phi_k^* E([0, 2\pi[)_{j,m} = \sum_{k,j} U_{kj} \phi_k^* \delta_{j,m} &= \langle \phi | U \varphi_m \rangle. \qquad (8.25)
\end{aligned}$$

It follows that $\forall m \in \mathbb{Z}, \forall \phi \in l^2_\delta(\mathbb{Z})$,

$$\int_{[0,2\pi[} \langle \phi | U F_{\cdot,m}(\lambda) \rangle d\rho(\lambda) = \int_{[0,2\pi[} \langle \phi | e^{i\lambda} F_{\cdot,m}(\lambda) \rangle d\rho, \qquad (8.26)$$

and thus

$$\langle \phi | U F_{\cdot,m}(\lambda) \rangle = \langle \phi | e^{i\lambda} F_{\cdot,m}(\lambda) \rangle \ \ d\rho - ae. \qquad (8.27)$$

□

At this point we can prove Theorem 5.2, following closely the arguments of [S]: Let $N(\lambda)$ be the rank of the Hilbert-Schmidt operator $a(\lambda)$, which is a measurable function of λ. For all λ, there exists a set of orthogonal vectors [K], $(f_j(\lambda))_{j\in\{1,\dots,N(\lambda)\}}$, such that d$\rho$-ae:

$$a(\lambda) = \sum_{j=1}^{N(\lambda)} |f_j(\lambda)\rangle\langle f_j(\lambda)| \quad \text{and}$$

$$\sum_{j=1}^{N(\lambda)} \|f_j(\lambda)\|^2 = \sum_{m,j} \frac{1}{\|f_m(\lambda)\|^2}\langle f_m(\lambda)|f_j(\lambda)\rangle\langle f_j(\lambda)|f_m(\lambda)\rangle$$

$$= \sum_{m=1}^{N(\lambda)} \left\langle \frac{f_m(\lambda)}{\|f_m(\lambda)\|}|a(\lambda)\frac{f_m(\lambda)}{\|f_m(\lambda)\|}\right\rangle = Tr(a(\lambda)) = 1.$$

In case of degeneracy of the spectrum, it is always possible [S] to choose the f's so that they are measurable. It is enough to set now

$$\phi_n(\lambda) = x^{\delta/2} f_n(\lambda) \quad, \forall n \in \mathbb{Z}, \forall \lambda \in [0, 2\pi[, \quad \Delta_n = \{\lambda; N(\lambda) = n\}. \tag{8.28}$$

The sets Δ_n are disjoint by construction. For any fixed λ, the vectors $\phi_j(\lambda)$ are linearly independent, as is easily checked. The conditions on the growth of the components of the vectors $\phi_j(\lambda)$ are consequences of their definitions and Proposition 8.1. By construction, $\forall k \in \mathbb{Z}$,

$$\|f_j(\lambda)\|^2(\phi_j(\lambda))_k = \sum_m < m >^{-\delta} F_{k,m}(\lambda)(\phi_j(\lambda))_m. \tag{8.29}$$

Therefore, $\forall n \in \mathbb{Z}, \forall j \in \{1, \dots, N(\lambda)\}$,

$$\langle \varphi_n|U\phi_j(\lambda)\rangle = \sum_k U_{nk}(\phi_j(\lambda))_k$$

$$= \frac{1}{\|f_j(\lambda)\|^2}\sum_{k,m} U_{nk} < m >^{-\delta} F_{k,m}(\lambda)(\phi_j(\lambda))_m$$

$$= \frac{1}{\|f_j(\lambda)\|^2}\sum_{m} < m >^{-\delta} (\phi_j(\lambda))_m\langle\varphi_n|UF_{.,m}(\lambda)\rangle.$$

Using Proposition 8.1, it follows that the previous line equals

$$= \frac{1}{\|f_j(\lambda)\|^2}\sum_{m} < m >^{-\delta} (\phi_j(\lambda))_m e^{i\lambda}\langle\varphi_n|F_{.,m}(\lambda)\rangle = \langle\varphi_n|e^{i\lambda}\phi_j(\lambda)\rangle. \tag{8.30}$$

Thus, $\forall\phi \in l^2_\delta(\mathbb{Z})$, $\langle\phi|U\phi_j(\lambda)\rangle = \langle\phi|e^{i\lambda}\phi_j(\lambda)\rangle$, d$\rho$-ae. □

Proof of Lemma 6.2. Write

$$R(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}, \tag{8.31}$$

where a, b, c, d are analytic on $\mathbb{T}$ and $\det R(\lambda) = \mathrm{e}^{i\kappa}$. The eigenvalues of $R(\lambda)$ are

$$E_j(\lambda) = \frac{\mathrm{Tr}R(\lambda)}{2} + (-1)^j\sqrt{\frac{(\mathrm{Tr}R(\lambda))^2}{4} - e^{i\kappa}} \quad j = 1, 2, \tag{8.32}$$

and the set X consists of the zeros of $(\mathrm{Tr}R)^2 - 4e^{i\kappa}$. Let $\lambda = 0$ belong to X. We can assume that in a punctured neighborhood of 0, $b(\lambda) \neq 0$. Therefore, the eigenvectors can be chosen as

$$v_j(\lambda) = \begin{pmatrix} b(\lambda) \\ E_j(\lambda) - a(\lambda) \end{pmatrix}. \tag{8.33}$$

Since

$$(\mathrm{Tr}R(\lambda))^2/4 - \mathrm{e}^{i\kappa} = \sum_{n\in\mathbb{N}} t_n \lambda^n \tag{8.34}$$

with $t_0 = 0$, E_j and, in turn, v_j admit convergent series expansions in non-negative powers of $\lambda^{1/2}$ in a neighborhood of 0. $\square$

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