

Adiabatic transitions in a two-level system coupled to a free Boson reservoir

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Abstract

We consider a time-dependent two-level quantum system interacting with a free Boson reservoir. The coupling is energy conserving and depends slowly on time, as does the system Hamiltonian, with a common adiabatic parameter ε . Assuming that the system and reservoir are initially decoupled, with the reservoir in equilibrium at temperature $T \geq 0$, we compute the transition probability from one eigenstate of the two-level system to the other eigenstate as a function of time, in the regime of small ε and small coupling constant λ . We analyse the deviation from the adiabatic transition probability obtained in absence of the reservoir.

1 Introduction

In this paper we study the transition probability between the energy eigenstates of a driven two-level system in contact with an environment, a bosonic reservoir at zero or at positive temperatures. The Hamiltonian of the two-level system and the coupling with the reservoir both depend on time, varying on a slow time scale $1/\varepsilon$; that is, they are functions of the rescaled time $t = \varepsilon t_p$, where t_p is the physical time. We consider interaction Hamiltonians which are linear in the bosonic field operators and for which the system and reservoir do not exchange energy instantaneously, meaning that the system Hamiltonian commutes with the interaction at any given time.

The initial system-reservoir state is taken to be disentangled, with the two-level system in an eigenstate of its Hamiltonian and the reservoir in equilibrium at temperature $T \geq 0$. Such an initial state is very natural from an open quantum system perspective; it corresponds to the situation in which the system is put in contact with the reservoir at $t = 0$. Our main goal is to determine the probability, denoted $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$, to find the system in the other eigenstate at some fixed rescaled time $t > 0$. We do this in the adiabatic and weak coupling regime, meaning that ε and the system-reservoir coupling constant λ are both small.

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The adiabatic regime yields rather detailed and precise approximations of the true quantum dynamics in a variety of physically relevant situations and its study has a long history. The adiabatic theorem of quantum mechanics was first stated for self-adjoint time-dependent Hamiltonians with isolated eigenvalues in [BF, K1], and then extended to accommodate isolated parts of the spectrum, see [N1, ASY]. This version applies to the two-level system we consider, in absence of coupling. Adiabatic approximations for gapless Hamiltonians, where the eigenvalues are not isolated from the rest of the spectrum, were later established in [AE, T]. This is in particular the situation for the total Hamiltonian of the two-level system coupled to a free boson reservoir. Then, adiabatic theorems were formulated in [A-SF, J2, AFGG1] for dynamics generated by non-self-adjoint operators, leading to extensions of the gapless, non self-adjoint case in [S]. Such results apply to the dynamics of open quantum system within the markovian approximation, by means of time dependent Lindblad generators. Finally, the adiabatic approximation was also shown to be exponentially accurate for analytic time dependence [JKP, JoPf, N2, J1], in line with the famous Landau-Zener formula; see [HJ] for more details.

Applied to our two-level system without coupling to the reservoir ($\lambda = 0$, isolated eigenvalues), the adiabatic theorem says that the transition probability $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t)$ is of order ε^2 . By contrast, the gapless adiabatic theorem applied to the total Hamiltonian of the system and reservoir in general tells us merely that the transition probability is $o(\varepsilon)$ [AE, T].

We show that in our model, $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ differs from the transition probability $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t)$ (no coupling) by a term of order $\varepsilon \lambda^2$, which we determine explicitly. At zero temperature, this term turns out to be nonzero when the transition is from the upper to the lower energy level, while it vanishes for the reverse transition, up to corrections of higher orders in ε and λ . At positive temperatures, it is nonzero for both transitions. We also identify parameter regimes in which this correction term is the leading one of $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$.

To our knowledge, the problem we address here has been studied in the mathematical physics literature only by means of an effective description of the open system, namely, employing a time-dependent Lindblad operator [AFGG1, AFGG2, FH]. For a dephasing Lindblad operator, commuting with the generator of the system Hamiltonian, the authors there determine the transition probability between distinct energy levels of the system in the adiabatic limit. Like in our microscopic model, they find that this probability is of order ε , but in their Lindbladian approach, the dependence of the probability on whether it is up- or downwards does not show.

With the goal to extend the weak limit procedure to time-dependent Hamiltonian systems, the paper [DS] addresses a similar model for $\varepsilon = \lambda^2$ and time-independent interaction Hamiltonian. In this regime, the authors derive an effective master equation on the system, the solutions of which coincide with those of an adiabatic problem driven to leading order by the systems's Hamiltonian. See [DF, DD] for more on the weak coupling limit. Note also that modelisations of the environment by quantum noises, giving rise to quantum stochastic differential equations, provide by construction an exact Markovian effective equation on the system. The PhD thesis [H] is partly devoted to the analysis of such quantum stochastic models in an adiabatic regime.

Let us finally mention that the general theme addressed here is relevant for the discussion of the validity of the Born-Oppenheimer approximation in presence of a scalar photon field. See for example [TW] which provides a detailed analysis of this type of questions in a regime where the effect of the field is a lower order correction. In spirit, it corresponds in our setting to the regime $\lambda \ll \sqrt{\varepsilon}$ (see the discussion in Remark 2.3).

2 Model and main result

2.1 The model

Let us start by describing the model at zero temperature, see Section 2.4 for the positive temperature case. To account for its slowly-varying nature, the self-adjoint system Hamiltonian $H_S(\varepsilon t_p) \in M_2(\mathbb{C})$ at physical time t_p is assumed to be a function of the rescaled time $t = \varepsilon t_p \in [0, 1]$, with ε a small, positive parameter; $\varepsilon \rightarrow 0$ is the adiabatic limit. The Hilbert space of the total system is

$$\mathcal{H}_{\text{tot}} = \mathbb{C}^2 \otimes \mathcal{F}_+(L^2(\mathbb{R}^3)), \quad (2.1)$$

where $\mathcal{F}_+(L^2(\mathbb{R}^3))$ denotes the bosonic Fock space on $L^2(\mathbb{R}^3)$, the Hilbert space in three dimensional momentum space. The coupling to the reservoir is linear in the bosonic field operator

$$\phi(g) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} d^3k (\overline{g(k)} a(k) + g(k) a^*(k)), \quad (2.2)$$

where $g \in L^2(\mathbb{R}^3)$ is the form factor and $a^*(k)$, $a(k)$ are the creation and annihilation operators of a boson with momentum k . The system-reservoir interaction Hamiltonian is

$$H_{\text{int}}(\varepsilon t_p) = \lambda B(\varepsilon t_p) \otimes \phi(g), \quad (2.3)$$

where $\lambda > 0$ is the coupling constant and $B(\varepsilon t_p)$ is a slowly-varying self-adjoint operator on \mathbb{C}^2 , varying on the same timescale as the system Hamiltonian. We assume that $[H_S(t), B(t)] = 0$ for all $t \in [0, 1]$. This means that there are no instantaneous energy exchanges between the system and reservoir. The two self-adjoint operators $H_S(t)$ and $B(t)$ can thus be diagonalized simultaneously,

$$H_S(t) = \sum_{j=1}^2 e_j(t) P_j(t), \quad B(t) = \sum_{j=1}^2 b_j(t) P_j(t), \quad (2.4)$$

where $\{P_j(t)\}_{j=1}^2$ is a complete set of orthogonal projections on \mathbb{C}^2 and $e_j(t)$, $b_j(t)$ are real eigenvalues depending on the rescaled time t .

In what follows, we set

$$e_{21}(t) = e_2(t) - e_1(t) = -e_{12}(t), \quad b_{21}(t) = b_2(t) - b_1(t) = -b_{12}(t). \quad (2.5)$$

We shall rely on standard assumptions in the context of adiabatic theorems on both self-adjoint operators H_S and B :

- (A.1) Gap hypothesis: $\delta = \inf_{t \in [0, 1]} |e_{21}(t)| > 0$.
- (A.2) The eigenvalues $e_j(t)$ and $b_j(t)$ and spectral projectors $P_j(t)$ are of class $C^4([0, 1])$, with all derivatives having well defined limits at $\{0, 1\}$.
- (A.3) The eigenprojectors satisfy $\lim_{t \rightarrow 0^+} \partial_t^n P_j(t) = 0$ for all $n \in \{1, \dots, 4\}$.

Thanks to assumption (A.1), the spectral projectors $P_j(t)$ are rank one at all times, so that $P_j(t) = |\psi_j(t)\rangle\langle\psi_j(t)|$ with $\{\psi_j(t)\}_{j=1,2}$ an orthonormal common eigenbasis of $H_S(t)$ and $B(t)$ that can be chosen to be $C^4([0, 1])$. When clear from the context, we often write $Z(t)$ instead of $Z(t) \otimes \mathbb{1}$ for operators $Z(t)$ on \mathbb{C}^2 .

The time-independent Hamiltonian of the bosonic reservoir reads

$$H_R = \int d^3k \omega(k) a^*(k) a(k). \quad (2.6)$$

We will assume ω to depend only on the modulus $|k|$ of the wave vector k .

The system and bosons are coupled together at time $t = 0$, the system being initially in the eigenstate $\psi_1(0)$ of $H_S(0)$ and the reservoir in the vacuum state $\chi \in \mathcal{F}_+(L^2(\mathbb{R}^3))$. Hence, the initial state of the system and of the zero-temperature reservoir is the product state

$$\rho(0) = |\psi_1(0)\rangle\langle\psi_1(0)| \otimes |\chi\rangle\langle\chi|, \quad (2.7)$$

where, for any vectors μ, ν , $|\mu\rangle\langle\nu|$ denotes the rank one operator $\eta \mapsto \langle\nu|\eta\rangle\mu$. The system-reservoir evolution operator $U_{\lambda,\varepsilon}(t)$ is given by the time-rescaled Schrödinger equation

$$i\varepsilon\partial_t U_{\lambda,\varepsilon}(t) = \left(H_S(t) \otimes \mathbb{1} + \lambda B(t) \otimes \phi(g) + \mathbb{1} \otimes H_R \right) U_{\lambda,\varepsilon}(t), \quad U_{\lambda,\varepsilon}(0) = \mathbb{1}, \quad t \in (0, 1), \quad (2.8)$$

the operator inside the brackets being the total Hamiltonian. Here and below, all derivatives are understood in the strong sense, on $\mathcal{D} = \mathbb{C}^2 \otimes \mathcal{D}_R$, where \mathcal{D}_R is the domain of H_R . The reduced state on the two-level system is given by taking the partial trace over the reservoir,

$$\rho_S(t) = \text{tr}_R[U_{\lambda,\varepsilon}(t)\rho(0)U_{\lambda,\varepsilon}^*(t)]. \quad (2.9)$$

We note here that the coupling with the reservoir in our model leads to a pure dephasing type evolution for the reduced state $\rho_S(t)$ when H_S and B are time-independent [PSE, JZKGKS, MSB, MBSWM]. In this case, for any initial system-reservoir product state $\rho(0) = \rho_S \otimes \rho_R$, if $(\rho_S(t))_{kj}$ denotes the matrix elements of the reduced state in a common eigenbasis of H_S and B , the level populations $(\rho_S(t))_{jj}$ are time independent while the off-diagonal elements decay with time. Although there is no energy exchange and no relaxation towards an equilibrium, the coupling with the reservoir induces decoherence in the system, so one says that the system undergoes a pure dephasing dynamics. The situation is different for driven systems where $H_S(t)$ and $B(t)$ are time dependent and commute at all times, and we seek to quantify the transition probability between instantaneous energy levels.

2.2 Adiabatic transition probability

The transition probability of the system from level 1 with the reservoir initially in the vacuum, to level 2, irrespectively of the reservoir's final state, is given at the rescaled time $t = t_p\varepsilon$ by

$$p_{1 \rightarrow 2}^{(\lambda,\varepsilon)}(t) = \text{tr} \left((P_2(t) \otimes \mathbb{1}) U_{\lambda,\varepsilon}(t) (P_1(0) \otimes |\chi\rangle\langle\chi|) U_{\lambda,\varepsilon}^*(t)^* \right). \quad (2.10)$$

We define the Kato unitary intertwining operator $W_K(t)$ by

$$\partial_t W_K(t) = K(t) W_K(t), \quad W_K(0) = \mathbb{1}, \quad (2.11)$$

where

$$K(t) = \sum_{j=1}^2 (\partial_t P_j(t)) P_j(t) = -K^*(t) \quad (2.12)$$

satisfies

$$P_j(t) K(t) P_j(t) = 0. \quad (2.13)$$

The operator $W_K(t)$ possesses the well known intertwining property [K1]

$$W_K(t) P_j(0) = P_j(t) W_K(t). \quad (2.14)$$

In absence of the system-reservoir coupling ($\lambda = 0$) the transition probability (2.10) reduces to the adiabatic transition between the levels of a driven system isolated from its environment. As we shall recover along the way, the latter is known to be equal to

$$p_{1 \rightarrow 2}^{(0, \varepsilon)}(t) = \varepsilon^2 q_{1 \rightarrow 2}(t) + \mathcal{O}(\varepsilon^3), \quad q_{1 \rightarrow 2}(t) \equiv \frac{|\langle \psi_2(0) | W_K(t)^* \partial_t W_K(t) | \psi_1(0) \rangle|^2}{e_{21}(t)^2}. \quad (2.15)$$

One can motivate our choice of an instantaneous pure-dephasing model as follows. If H_S and B are time-independent, then the system prepared initially in the state $\psi_1(0)$ remains in that state for all times. This mimics what happens when studying adiabatic transitions in closed systems (i.e. systems uncoupled to their environment), prepared initially in an eigenstate of their Hamiltonian $H_S(0)$. If we considered a model including energy exchanges such as absorption or emission processes of a boson from the reservoir, then transitions from one eigenstate to another induced by these processes would come into play, thus adding contributions to $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ that might not vanish in the adiabatic limit and blur the adiabatic transition we are interested in.

2.3 Reservoir autocorrelation function

The reservoir autocorrelation function for the zero temperature reservoir is defined by

$$\gamma(t) = 2 \langle \chi | e^{itH_R} \phi(g) e^{-itH_R} \phi(g) \chi \rangle = \langle \chi | a(e^{it\omega} g) a^*(g) \chi \rangle = \langle e^{it\omega} g, g \rangle, \quad (2.16)$$

where g is the form factor and $\langle f, g \rangle = \int_{\mathbb{R}^3} d^3k \overline{f(k)} g(k)$ stands for the scalar product in $L^2(\mathbb{R}^3)$. Assuming for concreteness a photonic dispersion relation $\omega(k) = |k|$, we get

$$\gamma(t) = \int_0^\infty d\omega e^{-it\omega} \omega^2 \int_{S^2} d^2\sigma |g(\omega, \sigma)|^2, \quad (2.17)$$

where $g(\omega, \sigma)$ is the expression of $g(k)$ in spherical coordinates and $d^2\sigma$ stands for the uniform measure on the sphere S^2 . Hence, $\gamma(t)$ is the Fourier transform of the non-negative function

$$\hat{\gamma}(\omega) = 2\pi\omega^2 \int_{S^2} d^2\sigma |g(\omega, \sigma)|^2 1_{\{\omega \geq 0\}}, \quad (2.18)$$

where $1_{\{\omega \geq 0\}} = 1$ if $\omega \geq 0$ and 0 otherwise. As a consequence of the non-negativity of $\hat{\gamma}$, γ is a positive definite function and thus satisfies $\gamma(t) = \overline{\gamma(-t)}$ and $|\gamma(t)| \leq \gamma(0)$. (A function f is positive definite by definition if the $n \times n$ matrix with elements $a_{ij} = f(t_i - t_j)$ is positive definite, for any $t_1, \dots, t_n \in \mathbb{R}$, and $n \in \mathbb{N}$.) In the physics literature $\hat{\gamma}(\omega)$ is also called the power spectrum or reservoir spectral density, sometimes denoted $J(\omega)$ [W]. If $\gamma \in L^1(\mathbb{R})$, then $\hat{\gamma}(\omega) = \int_{\mathbb{R}} dt e^{i\omega t} \gamma(t)$.

Remark 2.1 We may as well consider non relativistic massive bosons with mass $M > 0$, for which $\omega(k) = |k|^2/(2M)$. Then

$$\begin{aligned}\gamma(t) &= \sqrt{2}M^{\frac{3}{2}} \int_0^\infty d\omega e^{-i\omega t} \sqrt{\omega} \int_{S^2} d^2\sigma |g(\sqrt{2M\omega}, \sigma)|^2 \\ \widehat{\gamma}(\omega) &= \pi(2M)^{\frac{3}{2}} \sqrt{\omega} \int_{S^2} d^2\sigma |g(\sqrt{2M\omega}, \sigma)|^2 1_{\{\omega \geq 0\}}\end{aligned}$$

and the aforementioned properties of γ still hold.

We shall make the following hypothesis, which implies in particular that $\gamma \in L^1(\mathbb{R})$.

(A.4) $\sup_{t \in \mathbb{R}} (1 + t^2)^{\frac{m+1}{2}} |\gamma(t)| < \infty$ and $\lim_{\omega \rightarrow 0^+} \frac{\widehat{\gamma}(\omega)}{\omega^m} \equiv \gamma_0 \geq 0$ exists and is finite, with $m > 0$ a positive real number.

These assumptions are fulfilled for instance for the photon dispersion relation $\omega(k) = |k|$ and for a rotation-invariant form factor g of the form

$$g(k) = g_0 |k|^{\frac{m}{2}-1} \exp\left(-\frac{|k|}{2\omega_D}\right) \quad (2.19)$$

with $m > 0$, $g_0 \in \mathbb{R}$, and $\omega_D > 0$ a Debye cut-off frequency. Then $\gamma(t)$ and $\widehat{\gamma}(\omega)$ can be calculated explicitly,

$$\gamma(t) = 4\pi g_0^2 \omega_D^{m+1} \frac{\Gamma(m+1)}{(1 + i\omega_D t)^{m+1}}, \quad \widehat{\gamma}(\omega) = 8\pi^2 g_0^2 \omega^m e^{-\frac{\omega}{\omega_D}} 1_{\{\omega \geq 0\}} \quad (2.20)$$

with Γ the Gamma function.

Let us point out that the low-frequency behaviour $\widehat{\gamma}(\omega) \sim \gamma_0 \omega^m$ of the spectral density determines the time decay of the system coherences in the energy eigenbasis $\{\psi_1, \psi_2\}$ (that is, of the off-diagonal elements of the reduced density matrix $\rho_S(t)$) when H_S and B are time-independent: for zero temperature reservoirs, the decoherence is incomplete when $m > 1$ (that is, the off-diagonal elements do not converge to 0 as $t \rightarrow \infty$), whereas it is complete when $0 < m \leq 1$ (the off-diagonal elements tend to zero). The case $m > 1$ is called the super-Ohmic regime, while $m = 1$ and $0 < m < 1$ are termed the Ohmic and sub-Ohmic regimes, respectively (see e.g. [W]).

While the derivation of an approximate Markovian evolution equation for the reduced density matrix ρ_S is not addressed in this paper, note that the assumed time decay of the reservoir autocorrelation function is compatible with the conditions required in [DS] to derive an effective master equation for certain time-dependent systems.

2.4 Positive temperatures

At positive temperatures $T = 1/\beta > 0$, the reservoir equilibrium momentum distribution is given by Planck's law as $1/(e^{\beta|k|} - 1)$, where we assume that $\omega(k) = |k|$. Let $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ again denote the probability of transition from the system state on level 1, where now the reservoir is

initially in the temperature state, to the system state on level 2, irrespective of the reservoir's final state. Formally, it is given by (compare with (2.10))

$$p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \text{tr} \left((P_2(t) \otimes \mathbb{1}) U_{\lambda, \varepsilon}(t) (P_1(0) \otimes \rho_{R, \beta}) U_{\lambda, \varepsilon}(t)^* \right), \quad (2.21)$$

where $\rho_{R, \beta}$ is the reservoir Gibbs density matrix. The expression (2.21) is formal in the sense that we consider the reservoir to be infinitely extended (having continuous momentum modes), so that $\rho_{R, \beta}$ has to be interpreted as an operator in a modified Hilbert space, see Section 6. Another way of saying this is that in (2.21), we understand that the thermodynamic limit is taken, that is, we replace $\rho_{R, \beta}$ by $\rho_{R, \beta}^\Lambda$, where Λ is a compact box in position space \mathbb{R}^3 (then $\rho_{R, \beta}^\Lambda$ is a well defined operator on $\mathcal{F}_+(L^2(\mathbb{R}^3))$) and we take the limit $\Lambda \nearrow \mathbb{R}^3$.

Now the Fourier transform of the reservoir autocorrelation function is

$$\widehat{\gamma}^\beta(\omega) = \frac{1}{2} \widehat{\gamma}(|\omega|) (\coth(\beta|\omega|/2) + \text{sgn}(\omega)) \geq 0, \quad (2.22)$$

where $\widehat{\gamma}(\omega)$ is the spectral density (2.18) and sgn is the sign function (see Section 6.3 for a derivation of (2.22)). As above (see (2.17), (2.18)), we set

$$\gamma^\beta(t) = \frac{1}{2\pi} \int_{\mathbb{R}} d\omega e^{-i\omega t} \widehat{\gamma}^\beta(\omega). \quad (2.23)$$

We show in Section 6.3 that, in the positive temperature case, condition (A.4) with γ^β , $\widehat{\gamma}^\beta$ in place of γ , $\widehat{\gamma}$, is satisfied for

$$g(k) = g_0 |k|^{\frac{\mu}{2}-1} \exp\left(-\frac{|k|}{2\omega_D}\right) \quad \text{with } \mu > m + 1 > 1.$$

Physically, the form factors we can deal with at positive temperature correspond to the super-Ohmic regime, i.e. $\mu > 1$.

2.5 Main result

Here is our main result valid for both the zero and positive temperature reservoirs.

Theorem 2.2 *Suppose the reservoir is initially in equilibrium at zero temperature or at temperature $1/\beta > 0$. Assume that conditions (A.1)-(A.4) are satisfied for γ and $\widehat{\gamma}$ in the former case or for γ^β and $\widehat{\gamma}^\beta$ in the latter case and let $m_1 \equiv \min\{m, 1\} > 0$ and $\alpha_0 = (2m - m_1 + 2)^{-1}$. Then*

(i) *At positive temperature $1/\beta > 0$, we have*

$$\begin{aligned} p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= p_{1 \rightarrow 2}^{(0, \varepsilon)}(t) + \frac{\lambda^2}{2\varepsilon} \int_0^t ds p_{1 \rightarrow 2}^{(0, \varepsilon)}(s) b_{12}^2(s) \widehat{\gamma}^\beta(e_{12}(s)) \\ &\quad + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda \varepsilon^{(3+m_1)/2} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}}) + \mathcal{O}(\lambda^2 \varepsilon^{1+m\alpha_0}) + \mathcal{O}(\lambda^3 \varepsilon^{(1+m_1)/2} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}}) \\ &\quad + \mathcal{O}(\lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}}) + \mathcal{O}(\lambda^5 \varepsilon^{\frac{1}{2}m_1} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}}) + \mathcal{O}(\lambda^6), \end{aligned} \quad (2.24)$$

with $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t) = \varepsilon^2 q_{1 \rightarrow 2}(t) + \mathcal{O}(\varepsilon^3)$ the transition probability (2.15) in the absence of coupling to the reservoir. (Here, $\delta_{a,b}$ is the Kronecker delta.)

(ii) At zero temperature, the same expression (2.24) holds with $\hat{\gamma}$ in place of $\hat{\gamma}^\beta$.

(iii) In both cases, $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t)$ can be replaced by $\varepsilon^2 q_{1 \rightarrow 2}(t)$ in (2.24), without altering the error terms.

Remark 2.3 The theorem shows the following.

- (0) Recall that no assumption on the sign of $e_{12}(t)$ is made, so that our analysis holds for transitions from the lower to the upper level, when $e_{12}(t) < 0$, and also from the upper to the lower one, when $e_{12}(t) > 0$.
- (i) The second term in the right-hand side of (2.24) describes, to leading order in (ε, λ) , the modification of the transition probability due to the coupling with the reservoir with respect to the case without coupling. This term is always nonnegative, as $\hat{\gamma} \geq 0$ and $\hat{\gamma}^\beta \geq 0$. At zero temperature, it vanishes for transitions from the ground to the excited state, since $\hat{\gamma}(e_{12}) = 0$ when $e_{12} < 0$, see (2.18). By contrast, at positive temperature, we have $\hat{\gamma}^\beta(e_{12}) > 0$ even if $e_{12} < 0$, (assuming the interior of $|e_{12}|([0, 1]) \cap \text{supp } \hat{\gamma}$ is not empty), see (2.22), so that the coupling with the reservoir always enhances the transition probability, be it from ground to excited state or vice versa. The asymmetry in the transitions to the upper and to the lower energy levels decreases with the temperature $1/\beta$, since we have

$$\hat{\gamma}^\beta(-\omega) = e^{-\beta\omega} \hat{\gamma}^\beta(\omega) \quad \text{for } \omega > 0$$

and so for large β , $\hat{\gamma}^\beta(-\omega)$ decreases exponentially quickly in β .

- (ii) To insure that the error terms be much smaller than both the first and second terms in the right-hand side of (2.24), the coupling constant and adiabatic parameter must satisfy $\varepsilon \ll \lambda \ll \varepsilon^{1/3}$ when $m > 1$ and $\varepsilon^{(1+m)/2} |\ln(\varepsilon)|^{\delta_{1,m}/2} \ll \lambda \ll \varepsilon^{(3-m)/6} |\ln \varepsilon|^{-\delta_{1,m}/6}$ when $0 < m \leq 1$. One can further distinguish the following regimes:

1. If λ scales like $\sqrt{\varepsilon}$, the transition probability is larger than its value $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t)$ in absence of coupling to the reservoir by an amount of the same order, ε^2 , with overall error term $o(\varepsilon^2)$, save in the zero temperature case for the transition to the excited state.
2. By contrast, when $\lambda \ll \sqrt{\varepsilon}$, the system-reservoir interactions have a negligible effect on the transition probabilities: $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = p_{1 \rightarrow 2}^{(0, \varepsilon)}(t) + o(\varepsilon^2)$.
3. For stronger coupling constants λ such that $\sqrt{\varepsilon} \ll \lambda \ll \varepsilon^{\max\{1/4, (1-m_1)/2\}}$, the transition probability is asymptotically larger than in the absence of reservoir:

$$p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \frac{\lambda^2}{2\varepsilon} \int_0^t ds p_{1 \rightarrow 2}^{(0, \varepsilon)}(s) b_{12}^2(s) \hat{\gamma}^\beta(e_{12}(s)) + o(\lambda^2 \varepsilon), \quad (2.25)$$

save in the zero temperature case for the transition to the upper level. This means that the reservoir significantly helps the system to tunnel.

- (iii) If $H_S(s)$ is constant in a neighborhood of t (but not on the whole time interval $[0, t]$), then $q_{1 \rightarrow 2}(t) = 0$ and the transition probability is given by (2.25) in the wider range of coupling constants $\varepsilon^{(1+m_1)/2} |\ln \varepsilon|^{\delta_{m,1}/2} \ll \lambda \ll \varepsilon^{\max\{1/4, (1-m_1)/2\}}$. Note that, save for the transition to the upper level in the zero temperature case, the integral in the right-hand side of (2.25) is nonzero as $q_{1 \rightarrow 2}(s) > 0$ on $[0, t]$ except for times s close to 0 and t .

Remark 2.4 The second term in (2.24) – describing the effect of the reservoir on the transition probability, depends linearly on the adiabatic parameter ε and quadratically on the coupling constant λ . A similar linear dependence on ε of adiabatic transition probabilities in open quantum system dynamics governed by so-called time-dependent dephasing Lindbladians has been found in [AFGG2], see also [FH]. Such Lindbladians share with our model the property that they instantaneously generate pure dephasing dynamics with no energy exchange. For static Hamiltonians, they describe the evolution of the system under the Born-Markov and rotating wave approximations (van Hove weak coupling limit). These approximations are not obvious to justify from a microscopic approach even for time-independent open systems, see e.g. [D, M2], let alone when the system Hamiltonian and the coupling depend on time.

As pointed out in [AFGG2, FH], the dephasing Lindbladians should be considered as phenomenological models. Although the same dependence on ε and λ^2 (the latter corresponding to the amplitude of the dephasing dissipator of the Lindbladian) is found for both our microscopic and the Lindbladian models, we stress that the Lindbladian approach does not feature any asymmetry in the transition probabilities to the upper and lower energy levels.

The papers [AFGG2] and [FH] actually consider as the system generator the emblematic Landau-Zener Hamiltonian

$$H_{\text{LZ}}(t) = \frac{1}{2} \begin{pmatrix} t & \Delta \\ \Delta & -t \end{pmatrix},$$

which gives rise, in a scattering regime, to the exact Landau-Zener formula. This formula tells us that the transition probability is exponentially small, $p_{1 \rightarrow 2}^{(0, \varepsilon)} = e^{-\pi \Delta^2 / (2\varepsilon)}$, see [L, Z] and [J1] for generalisations. When dephasing is included within the Lindbladian approach, the scattering limit of the transition probability is shown in [FH] to be given by $e^{-\pi \Delta^2 / (2\varepsilon)}$ plus an explicit term of order $\gamma_{\text{deph}} \varepsilon$, up to error terms $\mathcal{O}(\gamma_{\text{deph}} \varepsilon^2)$, where γ_{deph} is the dephasing rate (amplitude of the dephasing dissipator). Hence, unless the dephasing rate is exponentially small, the Landau-Zener term is buried in the error terms. In our approach, we consider general two-level Hamiltonians $H_S(t)$ and finite rescaled time intervals over which the explicit leading order of the transition probability $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t)$ is of order ε^2 . This enables us compare this contribution with that induced by the coupling to the reservoir to the full probability $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ in (2.24).

Remark 2.5 *Prima facie* the proofs of the results for zero and positive temperatures might be expected to look quite different. However, by using the so-called Gelfand-Naimark-Segal (GNS) representation of the reservoir equilibrium state at positive temperature, its density matrix is simply a rank-one projection on a vacuum state, but in a different Fock space. We explain this in Section 6 and we show that, once the replacement of the Hilbert space is made, the proof of Theorem 2.2 for zero temperature is straightforwardly altered to accommodate for positive temperatures.

2.6 Organization of the paper

The remaining part of the paper is devoted to the proof of Theorems 2.2 and we start with the zero temperature case. In Section 3 we introduce the adiabatic evolution and the corresponding wave operator and we give a Dyson series expansion of the latter. The first term in this series produces the main term in the expression for the transition probability (see (3.12)). We analyze its adiabatic and weak coupling limit in Section 4, where the main result is Proposition 4.7. In Section 5 we control the remaining terms in the Dyson series. The main result is Proposition 5.1. We explain in Section 6 the positive temperature formalism and the necessary changes in the previous arguments.

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3 Exact calculations and adiabatic Dyson expansion

3.1 Expansion of the wave operator

Let us consider the adiabatic evolution operator $V_{\lambda,\varepsilon}(t)$ solution of

$$i\varepsilon\partial_t V_{\lambda,\varepsilon}(t) = (H(t) + i\varepsilon K(t) \otimes \mathbb{1})V_{\lambda,\varepsilon}(t), \quad V_{\lambda,\varepsilon}(0) = \mathbb{1}, \quad t \in (0, 1), \quad (3.1)$$

where $K(t)$ is given by (2.12) and $H(t) = H_S(t) \otimes \mathbb{1} + H_{\text{int}}(t) + \mathbb{1} \otimes H_R$ is the total Hamiltonian of the system and reservoir. Since $i\varepsilon K(t)$ is self-adjoint, the operators $V_{\lambda,\varepsilon}(t)$ are unitary. For later purposes, we define the unitary dynamical phase operator $\Psi_{\lambda,\varepsilon}(t)$ given by

$$V_{\lambda,\varepsilon}(t) = (W_K(t) \otimes \mathbb{1}) \Psi_{\lambda,\varepsilon}(t). \quad (3.2)$$

The justification of this designation will be provided by Lemma 3.1, which shows that $\Psi_{\lambda,\varepsilon}(t)$ can be computed exactly, commutes with $P_j(0) \otimes \mathbb{1}$ and contains fast-oscillating terms as $\varepsilon \rightarrow 0$.

Before showing that, let us introduce the unitary wave operator

$$\Omega_{\lambda,\varepsilon}(t) = V_{\lambda,\varepsilon}^*(t)U_{\lambda,\varepsilon}(t), \quad t \in [0, 1]. \quad (3.3)$$

In view of (2.8), (3.1), and $K(t)^* = -K(t)$, this operator satisfies $\partial_t \Omega_{\lambda,\varepsilon}(t) = -V_{\lambda,\varepsilon}^*(t)(K(t) \otimes \mathbb{1})V_{\lambda,\varepsilon}(t)\Omega_{\lambda,\varepsilon}(t)$ and $\Omega_{\lambda,\varepsilon}(0) = \mathbb{1}$ or, equivalently,

$$\partial_t \Omega_{\lambda,\varepsilon}(t) = -\Psi_{\lambda,\varepsilon}(t)^* \tilde{K}(t) \Psi_{\lambda,\varepsilon}(t) \Omega_{\lambda,\varepsilon}(t), \quad \Omega_{\lambda,\varepsilon}(0) = \mathbb{1}, \quad (3.4)$$

where $\tilde{K}(t) = W_K^*(t)K(t)W_K(t) \otimes \mathbb{1}$ is independent of ε and acts trivially on the reservoir. Note that the dependence of $\Omega_{\lambda,\varepsilon}(t)$ on ε comes from the fast-oscillating factors in the dynamical phase operator only.

Upon substituting the right-hand side of (3.4) into $\Omega_{\lambda,\varepsilon}(t) = \mathbb{1} + \int_0^t ds \partial_s \Omega_{\lambda,\varepsilon}(s)$ and iterating, one obtains the norm-convergent Dyson expansion

$$\begin{aligned} \Omega_{\lambda,\varepsilon}(t) &= \sum_{k \geq 0} \Omega_{\lambda,\varepsilon}^{(k)}(t) \\ &= \sum_{k \geq 0} (-1)^k \int \dots \int_{0 \leq s_k \leq \dots \leq s_1 \leq t} ds_1 \dots ds_k \Psi_{\lambda,\varepsilon}^*(s_1) \tilde{K}(s_1) \Psi_{\lambda,\varepsilon}(s_1) \dots \Psi_{\lambda,\varepsilon}^*(s_k) \tilde{K}(s_k) \Psi_{\lambda,\varepsilon}(s_k) \end{aligned} \quad (3.5)$$

such that for $k \geq 1$

$$\Omega_{\lambda,\varepsilon}^{(k)}(t) = - \int_0^t ds \Psi_{\lambda,\varepsilon}^*(s) \tilde{K}(s) \Psi_{\lambda,\varepsilon}(s) \Omega_{\lambda,\varepsilon}^{(k-1)}(s), \quad \Omega_{\lambda,\varepsilon}^{(0)}(t) = \mathbb{1}. \quad (3.6)$$

We may now rewrite the transition probability (2.10) in terms of $\Omega_{\lambda,\varepsilon}(t)$ by proceeding as follows

$$\begin{aligned} p_{1 \rightarrow 2}^{(\lambda,\varepsilon)}(t) &= \text{tr} \left[P_2(t) V_{\lambda,\varepsilon}(t) \Omega_{\lambda,\varepsilon}(t) (P_1(0) \otimes |\chi\rangle\langle\chi|) \Omega_{\lambda,\varepsilon}(t)^* V_{\lambda,\varepsilon}(t)^* \right] \\ &= \text{tr} \left[W_K(t)^* P_2(t) W_K(t) \Psi_{\lambda,\varepsilon}(t) \Omega_{\lambda,\varepsilon}(t) (P_1(0) \otimes |\chi\rangle\langle\chi|) \Omega_{\lambda,\varepsilon}(t)^* \Psi_{\lambda,\varepsilon}(t)^* \right] \\ &= \left\| P_2(0) \Omega_{\lambda,\varepsilon}(t) \psi_1(0) \otimes \chi \right\|^2, \end{aligned} \quad (3.7)$$

where the intertwining property (2.14) and the commutation of the dynamical phase operator with the projectors $P_j(0)$ have been used to get the last expression.

Introduce now for $i, j \in \{1, 2\}$

$$\tilde{K}_{ij}(t) \equiv P_i(0) \tilde{K}(t) P_j(0) = W_K^*(t) P_i(t) K(t) P_j(t) W_K(t) \quad (3.8)$$

and observe that $\tilde{K}_{jj}(t) = 0$ (due to (2.13)). Using once again the commutation of the dynamical phase operator with the projectors $P_j(0)$, this implies, (dropping $\otimes \mathbb{1}$ from the notation),

$$\Omega_{\lambda,\varepsilon}^{(k)}(t) P_1(0) = \begin{cases} P_2(0) \Omega_{\lambda,\varepsilon}^{(k)}(t) P_1(0) & \text{for } k \text{ odd,} \\ P_1(0) \Omega_{\lambda,\varepsilon}^{(k)}(t) P_1(0) & \text{for } k \text{ even.} \end{cases} \quad (3.9)$$

Hence the substitution of the series (3.5) into (3.7) yields

$$p_{1 \rightarrow 2}^{(\lambda,\varepsilon)}(t) = \left\| P_2(0) \sum_{k \text{ odd}} \omega_{\lambda,\varepsilon}^{(k)}(t) \right\|^2 = \sum_{\substack{j \text{ odd} \\ k \text{ odd}}} \langle \omega_{\lambda,\varepsilon}^{(j)}(t) | \omega_{\lambda,\varepsilon}^{(k)}(t) \rangle \quad (3.10)$$

with

$$\omega_{\lambda,\varepsilon}^{(k)}(t) \equiv \Omega_{\lambda,\varepsilon}^{(k)}(t) \psi_1(0) \otimes \chi. \quad (3.11)$$

Thus

$$\left| p_{1 \rightarrow 2}^{(\lambda,\varepsilon)}(t) - \|\omega_{\lambda,\varepsilon}^{(1)}(t)\|^2 \right| \leq 2 \|\omega_{\lambda,\varepsilon}^{(1)}(t)\| \sum_{k \geq 3, k \text{ odd}} \|\omega_{\lambda,\varepsilon}^{(k)}(t)\| + \left(\sum_{k \geq 3, k \text{ odd}} \|\omega_{\lambda,\varepsilon}^{(k)}(t)\| \right)^2. \quad (3.12)$$

Therefore, if one is able to show that

$$\sum_{k \geq 3, k \text{ odd}} \|\omega_{\lambda, \varepsilon}^{(k)}(t)\| \ll \|\omega_{\lambda, \varepsilon}^{(1)}(t)\|, \quad (3.13)$$

it will follow that $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2 + o(\|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2)$. This is what we set out to prove (see Sec. 5 below).

3.2 Exact expression of the dynamical phase operator

For our free boson reservoir model, an exact expression of $\Psi_{\lambda, \varepsilon}(t)$ can be obtained in terms of the Weyl operators. Recall that the latter are unitary operators on $\mathcal{F}_+(L^2(\mathbb{R}^3))$, defined by $W(F) = e^{i\phi(F)}$ for any $F \in L^2(\mathbb{R}^3)$ (see [BR] for more details).

Lemma 3.1 *Let $\varphi_j(t) = \varepsilon^{-1} \int_0^t ds e_j(s)$, $j = 1, 2$ denote the dynamical phases of the system Hamiltonian H_S . Then, the dynamical phase operator defined by (3.2) is given by*

$$\Psi_{\lambda, \varepsilon}(t) = \sum_{j=1}^2 e^{-i\varphi_j(t)} P_j(0) \otimes e^{-\frac{it}{\varepsilon} H_R} X_j(t), \quad (3.14)$$

where the unitary operators $X_j(t)$ on $\mathcal{F}_+(L^2(\mathbb{R}^3))$ are given by

$$X_j(t) = e^{i\zeta_j(t)} W(F_j(t)), \quad j = 1, 2 \quad (3.15)$$

with

$$\begin{aligned} F_j(t) &= \int_0^t ds f_j(s) \\ \zeta_j(t) &= \frac{1}{2} \operatorname{Im} \int_0^t ds \langle F_j(s), f_j(s) \rangle = \frac{1}{2} \operatorname{Im} \int_0^t ds \int_0^s d\tau \langle f_j(\tau), f_j(s) \rangle \end{aligned} \quad (3.16)$$

and

$$f_j(t) = -\frac{\lambda}{\varepsilon} b_j(t) e^{i\omega(\cdot)t/\varepsilon} g(\cdot) \in L^2(\mathbb{R}^3). \quad (3.17)$$

To simplify notation we do not indicate the dependence of $\varphi_j(t)$, $\zeta_j(t)$, $F_j(t)$, $f_j(t)$, and $X_j(t)$ on ε and λ .

Remark 3.2 The operator $\Psi_{\lambda, \varepsilon}(t)$ is diagonal in the eigenbasis of $H_S(0)$ and in absence of coupling to the reservoir $P_j(0) \Psi_{0, \varepsilon}(t) = e^{-\frac{i}{\varepsilon} \int_0^t ds e_j(s)} P_j(0)$ coincides with the dynamical phase, see [K1]. For $\lambda > 0$, $P_j(0) \Psi_{\lambda, \varepsilon}(t)$ has a non-trivial action on the reservoir degrees of freedom and contains other fast-oscillating factors depending on the interaction and reservoir Hamiltonians $H_{\text{int}}(t)$ and H_R .

Proof. Plugging (3.2) into (3.1) and taking advantage of (2.11), (2.14) and (2.4), one finds that $\Psi_{\lambda,\varepsilon}(t)$ satisfies

$$\begin{aligned} i\varepsilon\partial_t\Psi_{\lambda,\varepsilon}(t) &= W_K(t)^*H(t)W_K(t)\Psi_{\lambda,\varepsilon}(t) \\ &= \sum_j P_j(0) \otimes \left(e_j(t)\mathbb{1} + \lambda b_j(t)\phi(g) + H_R \right) \Psi_{\lambda,\varepsilon}(t), \quad \Psi_{\lambda,\varepsilon}(0) = \mathbb{1}. \end{aligned} \quad (3.18)$$

The solution of this equation is given by (3.14) with $X_j(t)$ the unitary operators on $\mathcal{F}_+(L^2(\mathbb{R}^3))$ given by

$$i\varepsilon\partial_t X_j(t) = \lambda b_j(t) e^{\frac{it}{\varepsilon}H_R} \phi(g) e^{-\frac{it}{\varepsilon}H_R} X_j(t), \quad X_j(0) = \mathbb{1}. \quad (3.19)$$

The expression of $X_j(t)$ in terms of the Weyl operators in the lemma is obtained from this equation, using also $e^{\frac{it}{\varepsilon}H_R} \phi(g) e^{-\frac{it}{\varepsilon}H_R} = \phi(e^{i\omega t/\varepsilon}g)$ and the commutation relation [BR]

$$[W(F), \phi(G)] = -\text{Im} \{ \langle F, G \rangle \} W(F), \quad \forall F, G \in L^2(\mathbb{R}^3), \quad (3.20)$$

which yields

$$\begin{aligned} \partial_t W(F(t)) &= i \int_0^1 du e^{iu\phi(F(t))} \phi(\partial_t F(t)) e^{i(1-u)\phi(F(t))} \\ &= \left(i\phi(\partial_t F) - \frac{i}{2} \text{Im} \langle F(t), \partial_t F(t) \rangle \right) W(F(t)). \end{aligned} \quad (3.21)$$

□

3.3 Iterative formula for the vectors $\omega_{\lambda,\varepsilon}^{(k)}$

Using the property of the Weyl operators

$$W(f)W(g) = e^{-\frac{i}{2}\text{Im} \langle f, g \rangle} W(f+g), \quad f, g \in L^2(\mathbb{R}^3), \quad (3.22)$$

one infers from (3.6), (3.8), (3.9), (3.11), and Lemma 3.1 that

$$\omega_{\lambda,\varepsilon}^{(k)}(t) = \begin{cases} -\int_0^t ds e^{-i(\varphi_{12}(s) - \zeta_{12}(s) + \frac{1}{2}\text{Im} \langle F_1(s), F_2(s) \rangle)} \tilde{K}_{21}(s) \otimes W(F_{12}(s)) \omega_{\lambda,\varepsilon}^{(k-1)}(s) & \text{if } k \text{ is odd} \\ -\int_0^t ds e^{-i(\varphi_{21}(s) - \zeta_{21}(s) + \frac{1}{2}\text{Im} \langle F_2(s), F_1(s) \rangle)} \tilde{K}_{12}(s) \otimes W(F_{21}(s)) \omega_{\lambda,\varepsilon}^{(k-1)}(s) & \text{if } k \text{ is even,} \end{cases} \quad (3.23)$$

where we have introduced the dynamical Bohr frequencies

$$\varphi_{ij}(t) \equiv \varphi_i(t) - \varphi_j(t) = \frac{1}{\varepsilon} \int_0^t du (e_i(u) - e_j(u)), \quad i \neq j \in \{1, 2\} \quad (3.24)$$

and we have set similarly $\zeta_{ij}(t) \equiv \zeta_i(t) - \zeta_j(t)$ and $F_{ij}(t) \equiv F_i(t) - F_j(t)$.

An iteration formula for $\omega_{\lambda,\varepsilon}^{(k)}(t)$ for odd k 's, $k \geq 3$, is obtained by plugging the second equation into the first one in (3.23). Using (3.22) again, this yields ($k \geq 3$, k odd)

$$\omega_{\lambda,\varepsilon}^{(k)}(t) = \int_0^t ds \int_0^s d\tau e^{-i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s,\tau)} \tilde{K}_{21}(s) \tilde{K}_{12}(\tau) \otimes W(F_{12}(s, \tau)) \omega_{\lambda,\varepsilon}^{(k-2)}(\tau), \quad (3.25)$$

where

$$\theta_{12}^{\pm}(s, \tau) \equiv \frac{1}{2} \text{Im} \left\{ \langle F_1(s), F_2(s) \rangle - \langle F_1(\tau), F_2(\tau) \rangle \pm \langle F_{12}(s), F_{12}(\tau) \rangle \right\} = -\theta_{12}^{\pm}(\tau, s) = -\theta_{21}^{\mp}(s, \tau) \quad (3.26)$$

and we have introduced the notation

$$h_{ij}(s, \tau) \equiv h_{ij}(s) - h_{ij}(\tau) \quad \text{for} \quad h = \varphi, \zeta, F, \text{ etc.} \quad (3.27)$$

4 Contribution of the first term in the Dyson expansion

4.1 Integration by parts

We now use the identity

$$\langle \chi | W(f) \chi \rangle = e^{-\|f\|^2/4}, \quad f \in L^2(\mathbb{R}^3) \quad (4.1)$$

to obtain an exact expression of the main term $\|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2$ in (3.12). The latter is given by (3.23) with $\omega_{\lambda, \varepsilon}^{(0)}(t) = \psi_1(0) \otimes \chi$. Taking advantage of the antisymmetry of φ_{12} , ζ_{12} , and θ_{12}^{\pm} under the exchange of s and τ , a simple calculation yields

$$\|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2 = 2\text{Re} \left\{ \int_0^t ds \int_0^s d\tau e^{-i\varphi_{12}(s, \tau) + i\zeta_{12}(s, \tau) - \eta_{12}(s, \tau)} e_{21}(\tau)^2 q_{1 \rightarrow 2}(s, \tau) \right\} \quad (4.2)$$

with $\varphi_{12}(s, \tau)$, $\zeta_{12}(s, \tau)$ defined in (3.24), (3.27) and

$$\begin{aligned} q_{1 \rightarrow 2}(s, \tau) &= \frac{\langle \psi_1(0) | \tilde{K}_{21}(\tau)^* \tilde{K}_{21}(s) \psi_1(0) \rangle}{e_{21}(\tau)^2} \\ \eta_{12}(s, \tau) &= \frac{1}{4} \|F_{12}(s, \tau)\|^2 + i\theta_{12}^+(s, \tau). \end{aligned} \quad (4.3)$$

Observe that by the gap and smoothness assumptions (A.1) and (A.2), $q_{1 \rightarrow 2}(s, \tau)$ and its first three derivatives are bounded uniformly by

$$q_{\infty}^{(n)} \equiv \sup_{0 \leq \tau \leq s \leq 1} |\partial_{\tau}^n q_{1 \rightarrow 2}(s, \tau)| \leq c_n \max_{\nu=0, \dots, n+1} \sup_{0 \leq \tau \leq 1} \|\partial_{\tau}^{\nu} P_1(\tau)\|, \quad 0 \leq n \leq 3, \quad (4.4)$$

where the positive constant c_n depends on $\max_{\nu=0, \dots, n} \sup_{0 \leq \tau \leq 1} |\partial_{\tau}^{\nu} e_{21}|$ and δ . In what follows, we write $q_{\infty} \equiv q_{\infty}^{(0)}$.

Our main tool to estimate the right-hand side of (4.2) is the following integration by part formula.

Proposition 4.1 *Under assumptions (A.1)-(A.3), one has*

$$\begin{aligned} \|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2 &= \varepsilon^2 q_{1 \rightarrow 2}(t, t) - 2\varepsilon^2 \text{Re} \int_0^t ds \int_0^s d\tau e^{-i\varphi_{12}(s, \tau)} \\ &\quad \times \partial_{\tau} \left(\frac{1}{e_{21}(\tau)} \partial_{\tau} \left(e^{(i\zeta_{12} - \eta_{12})(s, \tau)} e_{21}(\tau) q_{1 \rightarrow 2}(s, \tau) \right) \right). \end{aligned} \quad (4.5)$$

Proof. The statement follows by integrating by parts twice the τ -integral in (4.2), using $e_{21}(\tau)e^{-i\varphi_{12}(s,\tau)} = i\varepsilon\partial_\tau(e^{-i\varphi_{12}(s,\tau)})$. Noting that

$$(i) \lim_{t \rightarrow 0^+} \partial_t^n K(t) = \lim_{t \rightarrow 0^+} \partial_t^n \tilde{K}_{ij}(t) = 0 \text{ for any } n = 0, \dots, 3 \text{ and } i, j = 1, 2, \text{ by assumption (A.3), (2.11) and (2.12), and (3.8);}$$

$$(ii) \varphi_{12}(s, s) = \zeta_{12}(s, s) = \eta_{12}(s, s) = 0,$$

we get $(\partial_\tau^n q_{1 \rightarrow 2})(s, 0) = 0$ for $n \in \{0, \dots, 3\}$ and the boundary term in the first integration by parts is equal to $i\varepsilon e_{21}(s)q_{1 \rightarrow 2}(s, s) \in i\mathbb{R}$ which disappears after taking the real part. We arrive at

$$\|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2 = -2\varepsilon \operatorname{Re} \left\{ i \int_0^t ds \int_0^s d\tau e^{-i\varphi_{12}(s,\tau)} \partial_\tau \left(e^{i\zeta_{12}(s,\tau) - \eta_{12}(s,\tau)} e_{21}(\tau) q_{1 \rightarrow 2}(s, \tau) \right) \right\}. \quad (4.6)$$

The contribution of the boundary term in the second integration by parts is estimated as follows:

$$\begin{aligned} & 2\varepsilon^2 \operatorname{Re} \left\{ \int_0^t ds \left[\frac{e^{-i\varphi_{12}(s,\tau)}}{e_{21}(\tau)} \partial_\tau \left(e^{i\zeta_{12} - \eta_{12}}(s,\tau) e_{21}(\tau) q_{1 \rightarrow 2}(s, \tau) \right) \right]_{\tau=0}^{\tau=s} \right\} \\ &= 2\varepsilon^2 \int_0^t ds \left(-\operatorname{Re} \{ \partial_\tau \eta_{12} \}(s, s) q_{1 \rightarrow 2}(s, s) + \frac{\partial_\tau e_{21}(s)}{e_{21}(s)} q_{1 \rightarrow 2}(s, s) + \operatorname{Re} \{ \partial_\tau q_{1 \rightarrow 2} \}(s, s) \right) \\ &= 2\varepsilon^2 \int_0^t ds \frac{d}{ds} \left(\frac{q_{1 \rightarrow 2}(s, s)}{2} \right) = \varepsilon^2 q_{1 \rightarrow 2}(t, t). \end{aligned}$$

In the third equality, we have used that $\operatorname{Re} \{ \partial_\tau \eta_{12} \}(s, s) = 0$, which follows from differentiating

$$\|F_{12}(s, \tau)\|^2 = \left\| \int_\tau^s du f_{12}(u) \right\|^2 = \frac{\lambda^2}{\varepsilon^2} \int_\tau^s du \int_\tau^s dv b_{12}(u) b_{12}(v) \gamma \left(\frac{u-v}{\varepsilon} \right), \quad (4.7)$$

see (2.16) and (3.17). The integral term of the second integration by parts gives rise to the double integral in (4.5). \square

In the absence of coupling with the reservoir, $e^{(i\zeta_{12} - \eta_{12})(s,\tau)} \equiv 1$ and another integration by parts shows that the double integral in (4.5) is of order ε^3 or smaller. When $\lambda > 0$ however, the integral term after such a third integration by parts is not small, due to the presence of the third derivatives of ζ_{12} and η_{12} that make factors $1/\varepsilon$ appear upon differentiating $f_j(t)$ in (3.17). It is then necessary to analyze more carefully the different contributions coming from the first and second derivatives of the ε -dependent exponential $e^{(i\zeta_{12} - \eta_{12})(s,\tau)}$. This will be done in Subsection 4.3. We will show that the second derivative $\partial_\tau^2 \eta_{12}(s, \tau)$ yields a contribution $\mathcal{O}(\lambda^2 \varepsilon)$ in (4.5), while the other derivatives yield much smaller contributions in the limit $\varepsilon \ll 1$, $\lambda \ll \varepsilon^\alpha$ with $\alpha \in (0, 1/2)$ some fixed exponent. Our analysis is based on preliminary estimations of integrals involving the derivatives of η_{12} and ζ_{12} , which are spelled out in the next subsection.

4.2 Estimations on the derivatives of η_{12} and ζ_{12}

4.2.1 Preliminaries

It is convenient to rewrite the expression on the right-hand side of (4.3) as

$$\eta_{12}(s, \tau) = \frac{1}{4} \langle F_{12}(s, \tau), F_{12}(s, \tau) \rangle + \frac{i}{2} \text{Im} \langle F_{12}(s, \tau), F_1(s) \rangle - \frac{i}{2} \text{Im} \langle F_2(s, \tau), F_{12}(\tau) \rangle .$$

Let us set $\gamma_R(x) = \text{Re} \gamma(x)$ and $\gamma_I(x) = \text{Im} \gamma(x)$. By (2.16), (3.16), and (3.17), we get

$$\begin{aligned} \eta_{12}(s, \tau) = & \frac{\lambda^2}{2\varepsilon^2} \int_{\tau}^s du \left[\frac{1}{2} \int_{\tau}^s dv b_{12}(u) b_{12}(v) \gamma_R\left(\frac{u-v}{\varepsilon}\right) + i \int_0^s dv b_{12}(u) b_1(v) \gamma_I\left(\frac{u-v}{\varepsilon}\right) \right. \\ & \left. - i \int_0^{\tau} dv b_2(u) b_{12}(v) \gamma_I\left(\frac{u-v}{\varepsilon}\right) \right] . \end{aligned} \quad (4.8)$$

Similarly,

$$\zeta_{12}(s, \tau) = -\frac{\lambda^2}{2\varepsilon^2} \int_{\tau}^s du \int_0^u dv (b_1(u) b_1(v) - b_2(u) b_2(v)) \gamma_I\left(\frac{u-v}{\varepsilon}\right) . \quad (4.9)$$

At first sight, $\eta_{12}(s, \tau)$, $\zeta_{12}(s, \tau)$, and their first derivatives with respect to τ seem to be of order λ^2/ε^2 , while their second derivatives seem to be of order λ^2/ε^3 . This would imply that the factor ε^2 gained upon integrating by parts in lemma 4.1 is lost because of the fast oscillations and damping in the integral induced by the reservoir. Actually, this is not true for regular enough form factors: as we shall prove below, $\partial_{\tau} \eta_{12}$ and $\partial_{\tau} \zeta_{12}$ turn out to be, after suitable integrations over τ , of order $\lambda^2 \varepsilon^{\min\{m-1, 0\}}$ and λ^2/ε , respectively, while $\partial_{\tau}^2 \eta_{12}$ and $\partial_{\tau}^2 \zeta_{12}$ are of order λ^2/ε .

It is clear from the formulas above that η_{12}, ζ_{12} and their derivatives depend essentially on integrals of the real and imaginary parts of the reservoir autocorrelation function γ . The crucial property that we will use below, which follows from assumption (A.4), is that $\gamma \in L^1(\mathbb{R})$ and

$$\int_0^{\infty} \gamma_R(x) dx = 0 . \quad (4.10)$$

Indeed, (A.4) implies that

$$0 = \widehat{\gamma}(0) = \int_{\mathbb{R}} \gamma_R(x) dx + i \int_{\mathbb{R}} \gamma_I(x) dx = 2 \int_0^{\infty} \gamma_R(x) dx,$$

since γ_R and γ_I are even and odd integrable functions, respectively.

4.2.2 Estimations on the derivatives of η_{12}

Proposition 4.2 *Suppose that $\gamma \in L^1(\mathbb{R})$ and that (4.10) and (A.2) hold true and set*

$$r(z) \equiv \int_0^z dy \int_y^{\infty} dx |\gamma(x)| + \int_0^z dx x |\gamma(x)| , \quad z \geq 0. \quad (4.11)$$

Let $h(s, \tau)$ be a continuous function on $[0, 1]^2$, which may depend on ε and λ , bounded uniformly by an (ε, λ) -independent constant N , $\sup_{0 \leq \tau \leq s \leq 1} |h(s, \tau)| \leq N < \infty$. Then there exists a constant $c < \infty$ independent of λ and ε such that for any $0 < s \leq t \leq 1$, the following bounds hold:

$$\sup_{\tau, s \in [0, t]} |\eta_{12}(s, \tau)| \leq c\lambda^2 r\left(\frac{t}{\varepsilon}\right), \quad (4.12)$$

$$\int_0^t d\tau |\partial_\tau \eta_{12}(s, \tau)| \leq c\lambda^2 r\left(\frac{t}{\varepsilon}\right), \quad \int_0^t d\tau |\partial_\tau \eta_{12}(s, \tau)|^2 \leq c\frac{\lambda^4}{\varepsilon} r\left(\frac{t}{\varepsilon}\right), \quad (4.13)$$

and

$$\left| \int_0^s d\tau h(s, \tau) \partial_\tau^2 \eta_{12}(s, \tau) - \frac{\lambda^2}{2\varepsilon} \int_0^{\frac{s}{\varepsilon}} dx \left(h(s, s - \varepsilon x) b_{12}(s)^2 \gamma(-x) - i h(s, \varepsilon x) b_{12}(0)^2 \gamma_I(x) \right) \right| \leq c\lambda^2 r\left(\frac{s}{\varepsilon}\right). \quad (4.14)$$

Corollary 4.3 Suppose that assumptions (A.2) and (A.4) hold and let us set $m_1 \equiv \min\{m, 1\}$. Let $0 \leq t \leq 1$. If $m \neq 1$, then

$$\int_0^t ds \int_0^t d\tau |\partial_\tau \eta_{12}(s, \tau)| \leq c\lambda^2 \varepsilon^{m_1-1}, \quad \int_0^t ds \int_0^t d\tau |\partial_\tau \eta_{12}(s, \tau)|^2 \leq c\lambda^4 \varepsilon^{m_1-2},$$

and if $m = 1$, then

$$\int_0^t ds \int_0^{t'} d\tau |\partial_\tau \eta_{12}(s, \tau)| \leq c\lambda^2 |\ln \varepsilon|, \quad \int_0^t ds \int_0^{t'} d\tau |\partial_\tau \eta_{12}(s, \tau)|^2 \leq c\lambda^4 \frac{|\ln \varepsilon|}{\varepsilon},$$

where c is a constant independent of λ , ε , t' and t .

Proof of Corollary 4.3. By Assumption (A.4) one has $|\gamma(x)| \leq \kappa x^{-m-1}$ for $x \geq 1$, with κ a positive constant. An explicit calculation then shows that for any $0 \leq t \leq 1$,

$$r\left(\frac{t}{\varepsilon}\right) \leq r\left(\frac{1}{\varepsilon}\right) \leq \begin{cases} r_m & \text{if } m > 1 \\ r_1 |\ln \varepsilon| & \text{if } m = 1 \\ r_m \varepsilon^{m-1} & \text{if } 0 < m < 1 \end{cases} \quad (4.15)$$

with r_m a positive finite constant independent of λ , ε and t . \square

Before proving the Proposition, let us discuss a Corollary which gives rise to the second term in our formula (2.24) for the transition probability. Consider the term obtained by spelling out the τ -derivatives in formula (4.5) and keeping only the term involving $\partial_\tau^2 \eta_{12}(s, \tau)$

$$J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) \equiv 2\varepsilon^2 \text{Re} \int_0^t ds \int_0^s d\tau e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, \tau)} \partial_\tau^2 \eta_{12}(s, \tau) q_{1 \rightarrow 2}(s, \tau). \quad (4.16)$$

Corollary 4.4 *Suppose that assumptions (A.1)-(A.4) hold. Then*

$$J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \frac{\lambda^2 \varepsilon}{2} \left(\int_0^t ds b_{12}(s)^2 q_{1 \rightarrow 2}(s, s) \widehat{\gamma}(e_{12}(s)) + \mathcal{O}(\varepsilon^{m\alpha}) + \mathcal{O}(\varepsilon^{1-2\alpha}) + \mathcal{O}(\lambda^2 \varepsilon^{-\alpha}) \right), \quad (4.17)$$

where the exponent α can be chosen arbitrarily in $(0, 1/2)$.

Proof of Corollary 4.4. In view of (4.4) and $\operatorname{Re} \eta_{12}(s, \tau) \geq 0$, the function $h = e^{-i\varphi_{12} + i\zeta_{12} - \eta_{12}} q_{1 \rightarrow 2}$ satisfies the hypotheses of Proposition 4.2 with $N = q_\infty$. Thus if $m \neq 1$,

$$\begin{aligned} J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= \lambda^2 \varepsilon \operatorname{Re} \left\{ \int_0^t ds \left(b_{12}(s)^2 \int_0^{\frac{s}{\varepsilon}} dx e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, s - \varepsilon x)} q_{1 \rightarrow 2}(s, s - \varepsilon x) \gamma(-x) \right. \right. \\ &\quad \left. \left. - i b_{12}(0)^2 \int_0^{\frac{s}{\varepsilon}} dx e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, \varepsilon x)} \gamma_I(x) q_{1 \rightarrow 2}(s, \varepsilon x) \right) \right\} + O(\lambda^2 \varepsilon^{1+m_1}). \end{aligned} \quad (4.18)$$

For $m = 1$, the same result holds with an error term of order $\lambda^2 \varepsilon^2 |\ln \varepsilon|$. Now, recalling that $q_{1 \rightarrow 2}(s, 0) = 0$ by assumptions (A.3), one has $|q_{1 \rightarrow 2}(s, \varepsilon x)| \leq q_\infty^{(1)} \varepsilon x$, with $q_\infty^{(1)}$ given by (4.4). Hence the last integral in (4.18) is bounded in modulus by

$$q_\infty^{(1)} \varepsilon \int_0^{\frac{s}{\varepsilon}} dx x |\gamma(x)| \leq q_\infty^{(1)} \varepsilon r \left(\frac{s}{\varepsilon} \right).$$

Similarly, $q_{1 \rightarrow 2}(s, s - \varepsilon x)$ can be substituted by $q_{1 \rightarrow 2}(s, s)$ in the first integral over x in (4.18), making an error bounded by the same expression. Thus if $m \neq 1$,

$$J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \lambda^2 \varepsilon \operatorname{Re} \int_0^t ds b_{12}(s)^2 q_{1 \rightarrow 2}(s, s) \int_0^{\frac{s}{\varepsilon}} dx e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, s - \varepsilon x)} \gamma(-x) + O(\lambda^2 \varepsilon^{1+m_1}). \quad (4.19)$$

As before, when $m = 1$ the error term must be replaced by $O(\lambda^2 \varepsilon^2 |\ln \varepsilon|)$.

Let us introduce an exponent $\alpha \in (0, 1/2)$. Dividing the integration range of the s -integral in the right-hand side of (4.19) into $[0, \varepsilon^{1-\alpha}]$ and $[\varepsilon^{1-\alpha}, t]$ and noting that the integral over $[0, \varepsilon^{1-\alpha}]$ can be bounded by $C \varepsilon^{1-\alpha}$ with $C = \sup_{0 \leq u \leq 1} |b_{12}(u)|^2 q_\infty^{(0)} \int_0^\infty |\gamma| < \infty$, one has

$$\begin{aligned} J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= \lambda^2 \varepsilon \operatorname{Re} \int_{\varepsilon^{1-\alpha}}^t ds b_{12}(s)^2 q_{1 \rightarrow 2}(s, s) \int_0^{\frac{s}{\varepsilon}} dx e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, s - \varepsilon x)} \gamma(-x) \\ &\quad + O(\lambda^2 \varepsilon^{2-\alpha}) + O(\lambda^2 \varepsilon^{1+m_1}), \end{aligned} \quad (4.20)$$

with the aforementioned substitution of the last error term when $m = 1$.

We now divide the integration range of the x -integral in (4.20) into $[0, \varepsilon^{-\alpha}]$ and $[\varepsilon^{-\alpha}, s/\varepsilon]$. The integral over $[\varepsilon^{-\alpha}, s/\varepsilon]$ is bounded by a constant times $\varepsilon^{\alpha m}/m$ by using the inequality $|\gamma(t)| \leq \kappa t^{-m-1}$, which follows from Assumption (A.4). Now, for any $s \geq \varepsilon x \geq 0$ one has

$$\varphi_{12}(s, s - \varepsilon x) = x e_{12}(s) + O(\varepsilon x^2) \quad (4.21)$$

(see (3.24)). Furthermore, it follows from (4.9) and (4.8) that

$$\begin{aligned}
|\zeta_{12}(s, s - \varepsilon x)| &= \frac{\lambda^2}{2\varepsilon} \left| \int_{s-\varepsilon x}^s du \int_0^{\frac{u}{\varepsilon}} dy (b_1(u)b_1(u - \varepsilon y) - b_2(u)b_2(u - \varepsilon y)) \gamma_I(y) \right| \\
&\leq x\lambda^2 \sup_{0 \leq u \leq 1, i=1,2} \{|b_i(u)|^2\} \int_0^\infty |\gamma| \\
|\eta_{12}(s, s - \varepsilon x)| &\leq 5x\lambda^2 \sup_{0 \leq u \leq 1, i=1,2} \{|b_i(u)|^2\} \int_0^\infty |\gamma|. \tag{4.22}
\end{aligned}$$

Taking advantage of (4.21) and (4.22), one obtains for any $s > \varepsilon^{1-\alpha}$

$$\int_0^{\frac{s}{\varepsilon}} dx e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, s - \varepsilon x)} \gamma(-x) = \int_0^\infty dx e^{-ixe_{12}(s)} \gamma(-x) + O(\varepsilon^{1-2\alpha}) + O(\lambda^2 \varepsilon^{-\alpha}) + O(\varepsilon^{m\alpha}).$$

The real part of the integral in the right-hand side is easily found using the symmetry properties of γ to be equal to half of the Fourier transform of γ evaluated at $\omega = e_{12}(s)$,

$$\widehat{\gamma}(e_{12}(s)) \equiv \int_{-\infty}^\infty dx e^{ixe_{12}(s)} \gamma(x).$$

Noting that both error terms in (4.19) can be dropped (actually, $\varepsilon^{m_1} \ll \max\{\varepsilon^{m\alpha}, \varepsilon^{1-2\alpha}\}$ for $m \neq 1$ and $\varepsilon \ln |\varepsilon| \ll \max\{\varepsilon^{m\alpha}, \varepsilon^{1-2\alpha}\}$ for $m = 1$), the statement of the corollary follows. \square

Proof of Proposition 4.2. Let us denote by $\dot{b}_i(t)$ and $\ddot{b}_i(t)$ the first and second derivatives of $b_i(t)$, with $i = 1, 2$ or 12 , and let us set

$$M = \sup_{0 \leq u \leq 1} \max \{ |b_i(u)|, |\dot{b}_i(u)|, |\ddot{b}_i(u)| ; i = 1, 2 \}, \tag{4.23}$$

which is finite by assumption (A.2). By means of a change of variables $v \rightarrow x = (u - v)/\varepsilon$ and the Taylor expansion $b_i(u - \varepsilon x) = b_i(u) - \varepsilon x \dot{b}_i(w_{u,\varepsilon x})$ with $w_{u,\varepsilon x}$ lying between u and $u - \varepsilon x$, one deduces from (4.8) that

$$\begin{aligned}
\eta_{12}(s, \tau) &= \frac{\lambda^2}{2\varepsilon} \int_\tau^s du \left[\frac{1}{2} b_{12}(u) \int_{\frac{u-s}{\varepsilon}}^{\frac{u-\tau}{\varepsilon}} dx (b_{12}(u) - \varepsilon x \dot{b}_{12}(w_{u,\varepsilon x})) \gamma_R(x) \right. \\
&\quad \left. + i b_{12}(u) \int_{\frac{u-s}{\varepsilon}}^{\frac{u}{\varepsilon}} dx (b_1(u) - \varepsilon x \dot{b}_1(w_{u,\varepsilon x})) \gamma_I(x) - i b_2(u) \int_{\frac{u-\tau}{\varepsilon}}^{\frac{u}{\varepsilon}} dx (b_{12}(u) - \varepsilon x \dot{b}_{12}(w_{u,\varepsilon x})) \gamma_I(x) \right].
\end{aligned}$$

It follows from (4.10) and the symmetry properties $\gamma_R(-t) = \gamma_R(t)$, $\gamma_I(-t) = -\gamma_I(t)$ of the auto-correlation function that for $\tau \leq u \leq s$,

$$\int_{\frac{u-s}{\varepsilon}}^{\frac{u-\tau}{\varepsilon}} \gamma_R(x) dx = - \int_{\frac{s-u}{\varepsilon}}^\infty \gamma_R(x) dx - \int_{\frac{u-\tau}{\varepsilon}}^\infty \gamma_R(x) dx, \quad \int_{\frac{u-s}{\varepsilon}}^{\frac{u}{\varepsilon}} \gamma_I(x) dx = \int_{\frac{s-u}{\varepsilon}}^{\frac{u}{\varepsilon}} \gamma_I(x) dx$$

which implies

$$\begin{aligned}
\left| \int_{\frac{u-s}{\varepsilon}}^{\frac{u-\tau}{\varepsilon}} \gamma_R(x) dx \right| &\leq \int_{\frac{s-u}{\varepsilon}}^\infty |\gamma_R(x)| dx + \int_{\frac{u-\tau}{\varepsilon}}^\infty |\gamma_R(x)| dx \\
\left| \int_{\frac{u-s}{\varepsilon}}^{\frac{u}{\varepsilon}} \gamma_I(x) dx \right| &\leq \max \left\{ \int_{\frac{s-u}{\varepsilon}}^\infty |\gamma_I(x)| dx, \int_{\frac{u}{\varepsilon}}^\infty |\gamma_I(x)| dx \right\}.
\end{aligned}$$

Let $0 \leq \tau \leq s \leq t$. Then $|\eta_{12}(s, \tau)|$ can be bounded by

$$\begin{aligned}
|\eta_{12}(s, \tau)| &\leq \frac{\lambda^2 M^2}{\varepsilon} \int_{\tau}^s du \left(\int_{\frac{s-u}{\varepsilon}}^{\infty} (|\gamma_R(x)| + |\gamma_I(x)|) dx + \int_{\frac{u-\tau}{\varepsilon}}^{\infty} (|\gamma_R(x)| + |\gamma_I(x)|) dx \right. \\
&\quad \left. + \int_{\frac{u}{\varepsilon}}^{\infty} |\gamma_I(x)| dx \right) \\
&\quad + \lambda^2 M^2 \int_{\tau}^s du \left(\int_{\frac{u-s}{\varepsilon}}^{\frac{u-\tau}{\varepsilon}} dx |x\gamma_R(x)| + \int_{\frac{u-s}{\varepsilon}}^{\frac{u}{\varepsilon}} dx |x\gamma_I(x)| + \int_{\frac{u-\tau}{\varepsilon}}^{\frac{u}{\varepsilon}} dx |x\gamma_I(x)| \right) \\
&\leq \lambda^2 M^2 \left(4 \int_0^{\frac{s-\tau}{\varepsilon}} dy \int_y^{\infty} |\gamma(x)| dx + \int_{\frac{\tau}{\varepsilon}}^{\frac{s}{\varepsilon}} dy \int_y^{\infty} |\gamma(x)| dx + 5(s-\tau) \int_0^{\frac{s}{\varepsilon}} dx x |\gamma(x)| \right),
\end{aligned}$$

where the last inequality is obtained by making the changes of variables $u \rightarrow y = (s-u)/\varepsilon$, $u \rightarrow y = (u-\tau)/\varepsilon$, and $u \rightarrow y = u/\varepsilon$ and by using the parity of $|x\gamma(x)|$. Noting that $0 \leq s-\tau \leq s \leq t \leq 1$, this gives $|\eta_{12}(s, \tau)| \leq 5\lambda^2 M^2 r(t/\varepsilon)$.

If $0 \leq s \leq \tau \leq t$, all the estimates above remain valid provided s and τ are exchanged. This yields the first bound in the Proposition.

Similarly, the derivative of $\eta_{12}(s, \tau)$ is found from (4.8) to be given by

$$\begin{aligned}
\partial_{\tau} \eta_{12}(s, \tau) &= -\frac{\lambda^2}{2\varepsilon} \left[\int_{\frac{\tau-s}{\varepsilon}}^0 dx b_{12}(\tau) b_{12}(\tau - \varepsilon x) \gamma_R(x) + i \int_0^{\frac{\tau}{\varepsilon}} dx (b_{12}(\tau) b_1(\tau - \varepsilon x) \right. \\
&\quad \left. - b_2(\tau) b_{12}(\tau - \varepsilon x)) \gamma_I(x) + i \int_0^{\frac{s-\tau}{\varepsilon}} dx (-b_{12}(\tau) b_1(\tau + \varepsilon x) + b_2(\tau + \varepsilon x) b_{12}(\tau)) \gamma_I(x) \right],
\end{aligned} \tag{4.24}$$

which can be bounded by proceeding as above,

$$|\partial_{\tau} \eta_{12}(s, \tau)| \leq \frac{2\lambda^2 M^2}{\varepsilon} \left[2 \int_{\frac{|s-\tau|}{\varepsilon}}^{\infty} |\gamma| + \int_{\frac{\tau}{\varepsilon}}^{\infty} |\gamma| + 2\varepsilon \int_0^{\frac{|s-\tau|}{\varepsilon}} dx x |\gamma(x)| + \varepsilon \int_0^{\frac{\tau}{\varepsilon}} dx x |\gamma(x)| \right].$$

Integrating this expression with respect to τ from 0 to t and making the changes of variables $\tau \rightarrow y = |s-\tau|/\varepsilon$ and $\tau \rightarrow y = \tau/\varepsilon$, the second estimate in the Proposition is obtained. Using (4.24) again, one easily proves that

$$\sup_{\tau, s \in [0, t]} |\partial_{\tau} \eta_{12}(s, \tau)| \leq \frac{6\lambda^2 M^2}{\varepsilon} \int_0^{\infty} |\gamma|.$$

This inequality together with the first bound yields the third bound in the Proposition.

The second derivative of $\eta_{12}(s, \tau)$ is found to be equal to

$$\begin{aligned}
\partial_{\tau}^2 \eta_{12}(s, \tau) &= \frac{\lambda^2}{2\varepsilon^2} \left(b_{12}(\tau) b_{12}(s) \gamma\left(\frac{\tau-s}{\varepsilon}\right) - i(b_{12}(\tau) b_1(0) - b_2(\tau) b_{12}(0)) \gamma_I\left(\frac{\tau}{\varepsilon}\right) \right) \\
&\quad - \frac{\lambda^2}{2\varepsilon} \partial_{\tau} (b_{12}^2)(\tau) \left(\int_{\frac{\tau-s}{\varepsilon}}^0 \gamma_R - i \int_{-\frac{\tau}{\varepsilon}}^{\frac{s-\tau}{\varepsilon}} \gamma_I \right) \\
&\quad + O\left(\lambda^2 \int_0^{\frac{s}{\varepsilon}} dx x |\gamma(x)| \right).
\end{aligned} \tag{4.25}$$

In this expression, we have approximated $b_i(\tau - \varepsilon x)$ and $\dot{b}_i(\tau - \varepsilon x)$ by $b_i(\tau)$ and $\dot{b}_i(\tau)$, for $i = 1, 2$, and 12. The error incurred on $\partial_\tau^2 \eta_{12}(s, \tau)$ is estimated, using $|b_i(\tau - \varepsilon x) - b_i(\tau)| \leq M\varepsilon|x|$ and $|\dot{b}_i(\tau - \varepsilon x) - \dot{b}_i(\tau)| \leq M\varepsilon|x|$ with M given by (4.23), to be less than $12M^2\lambda^2 \int_0^{\frac{s}{\varepsilon}} dx x |\gamma(x)|$.

Let us set

$$I(s) \equiv \int_0^s d\tau h(s, \tau) \partial_\tau^2 \eta_{12}(s, \tau).$$

The contribution of the first line in (4.25) to this integral is given after making the changes of variables $x = (s - \tau)/\varepsilon$ and $x = \tau/\varepsilon$ by

$$\frac{\lambda^2}{2\varepsilon} \int_0^{\frac{s}{\varepsilon}} dx \left(h(s, s - \varepsilon x) b_{12}(s - \varepsilon x) b_{12}(s) \gamma(-x) - i h(s, \varepsilon x) (b_{12}(\varepsilon x) b_1(0) - b_2(\varepsilon x) b_{12}(0)) \gamma_I(x) \right).$$

In this expression, $b_i(s - \varepsilon x)$ and $b_i(\varepsilon x)$ can be replaced by $b_i(s)$ and $b_i(0)$, respectively, making an error $\mathcal{O}(\lambda^2 \int_0^{s/\varepsilon} dx x |\gamma(x)|)$. The contribution to the integral $I(s)$ of the second line in (4.25) is bounded by $8NM^2\lambda^2 \int_0^{s/\varepsilon} dy \int_y^\infty |\gamma|$, making use once again of (4.10) and the symmetry properties of γ_R and γ_I . The error term of order $\lambda^2 \int_0^{\frac{s}{\varepsilon}} dx x |\gamma(x)|$ produces an error of the same order in $I(s)$. This proves the last bound in the Proposition. \square

4.2.3 Estimations on the derivatives of ζ_{12}

We show in this subsection that $\zeta_{12}(s, \tau)$ is of order λ^2/ε . More precisely, it behaves as

$$\zeta_{12}(s, \tau) = -\frac{\lambda^2 \beta_0}{2\varepsilon} \int_\tau^s du (b_1^2 - b_2^2)(u) + \mathcal{O}(\lambda^2 \varepsilon^{m-1}) \quad (4.26)$$

as $(\varepsilon, \lambda) \rightarrow (0, 0)$, with

$$\beta_0 \equiv \int_0^\infty dx \gamma_I(x). \quad (4.27)$$

Furthermore, we prove that $\partial_\tau^2 \zeta_{12}(s, \tau)$ can be approximated by the second derivative of the first term in the right-hand side of (4.26) in certain integrals $\int_0^s d\tau h(s, \tau) \partial_\tau^2 \zeta_{12}(s, \tau)$, if $h(s, \tau)$ is a smooth enough function vanishing for $\tau = 0$. Note that $\partial_\tau^n \zeta_{12}(s, \tau) = -\partial_\tau^n \zeta_{12}(\tau)$ for $n = 1, 2, \dots$, since by definition $\zeta_{12}(s, \tau) = \zeta_{12}(s) - \zeta_{12}(\tau)$.

Proposition 4.5 *Suppose that $\gamma \in L^1(\mathbb{R})$ and that assumption (A.2) holds. Let $h_{\varepsilon, \lambda}(s, \tau) \equiv q(s, \tau) g_{\varepsilon, \lambda}(s, \tau)$ be such $q, g_{\varepsilon, \lambda} \in C^1([0, 1]^2)$, q is independent of (ε, λ) , and*

$$q(s, 0) = 0, \quad \sup_{0 \leq \tau \leq s \leq 1} |q(s, \tau)| < \infty, \quad \sup_{0 \leq \tau \leq s \leq 1} |\partial_\tau q(s, \tau)| < \infty \quad \text{and} \quad \sup_{0 \leq \tau \leq s \leq 1} |g_{\varepsilon, \lambda}(s, \tau)| \leq 1.$$

Then there is a constant c' independent on ε, λ such that for any $0 \leq t \leq 1$,

$$\sup_{\tau \in [0, t]} |\partial_\tau \zeta_{12}(\tau)| \leq c' \frac{\lambda^2}{\varepsilon}, \quad \int_0^t d\tau |\partial_\tau^2 \zeta_{12}(\tau)| \leq c' \frac{\lambda^2}{\varepsilon} \quad (4.28)$$

and for any $0 \leq \tau \leq s \leq 1$,

$$\left| \zeta_{12}(s, \tau) + \frac{\lambda^2 \beta_0}{2\varepsilon} \int_\tau^s du (b_1^2 - b_2^2)(u) \right| \leq c' \lambda^2 r \left(\frac{s}{\varepsilon} \right) \quad (4.29)$$

$$\left| \int_0^s d\tau h_{\varepsilon, \lambda}(s, \tau) \partial_\tau^2 \zeta_{12}(s, \tau) - \frac{\lambda^2 \beta_0}{2\varepsilon} \int_0^s d\tau h_{\varepsilon, \lambda}(s, \tau) \partial_\tau (b_1^2 - b_2^2)(\tau) \right| \leq c' \lambda^2 r \left(\frac{s}{\varepsilon} \right)$$

with $r(z)$ defined in (4.11).

In analogy with the developments above, let us consider the term involving $\partial_\tau^2 \zeta_{12}(s, \tau)$ in the formula (4.5) for the transition probability,

$$L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) \equiv -2\varepsilon^2 \operatorname{Re} \left\{ i \int_0^t ds \int_0^s d\tau e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, \tau)} \partial_\tau^2 \zeta_{12}(s, \tau) q_{1 \rightarrow 2}(s, \tau) \right\}. \quad (4.30)$$

Corollary 4.6 *Assume (A.1)-(A.4). Then*

$$\begin{aligned} L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= -\lambda^2 \varepsilon \beta_0 \operatorname{Re} \left\{ i \int_0^t ds \int_0^s d\tau e^{(-i\varphi_{12} + i\zeta_{12} - \eta_{12})(s, \tau)} \partial_\tau (b_1^2 - b_2^2)(\tau) q_{1 \rightarrow 2}(s, \tau) \right\} \\ &\quad + \mathcal{O}(\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}}) \end{aligned} \quad (4.31)$$

with $\delta_{m,1} = 1$ if $m = 1$ and 0 otherwise.

Proof of Corollary 4.6. One applies Proposition 4.5 with $g_{\varepsilon, \lambda} = e^{-i\varphi_{12} + i\zeta_{12} - \eta_{12}}$ and $q = q_{1 \rightarrow 2}$, which satisfy the required assumptions since $q_{1 \rightarrow 2}(s, 0) = 0$ and $\operatorname{Re} \eta_{12} \geq 0$, see also (4.4). The statement then follows from (4.15). \square

Proof of Proposition 4.5. The derivatives of $\zeta_{12}(s, \tau)$ are equal to (see (4.9))

$$\begin{aligned} \partial_\tau \zeta_{12}(s, \tau) &= -\partial_\tau \zeta_{12}(\tau) = \frac{\lambda^2}{2\varepsilon} \int_0^{\frac{\tau}{\varepsilon}} dx \left(b_1(\tau) b_1(\tau - \varepsilon x) - b_2(\tau) b_2(\tau - \varepsilon x) \right) \gamma_I(x) \\ \partial_\tau^2 \zeta_{12}(s, \tau) &= -\partial_\tau^2 \zeta_{12}(\tau) = \frac{\lambda^2}{2\varepsilon} \int_0^{\frac{\tau}{\varepsilon}} dx \partial_\tau \left(b_1(\tau) b_1(\tau - \varepsilon x) - b_2(\tau) b_2(\tau - \varepsilon x) \right) \gamma_I(x) \\ &\quad + \frac{\lambda^2}{2\varepsilon^2} \left(b_1(\tau) b_1(0) - b_2(\tau) b_2(0) \right) \gamma_I\left(\frac{\tau}{\varepsilon}\right). \end{aligned} \quad (4.32)$$

Clearly, for any $0 \leq \tau \leq t \leq 1$, $|\partial_\tau \zeta_{12}(\tau)|$ and $\int_0^t d\tau |\partial_\tau^2 \zeta_{12}(\tau)|$ are bounded from above by $c'(\lambda^2/\varepsilon)$ and by $3c'(\lambda^2/\varepsilon)$, respectively, with $c' = M^2 \int_0^\infty |\gamma_I| < \infty$, M being given by (4.23). The two last bounds in the Proposition follow from (4.9) and (4.32), by relying on similar arguments as those of the proof of Proposition 4.2 and making use of the Taylor expansion $q(s, \varepsilon x) = \varepsilon x \partial_\tau q(s, w_{s, \varepsilon x})$ with $w_{s, \varepsilon x} \in [0, \varepsilon x]$. \square

4.3 Adiabatic limit of the first term of the Dyson series

We are now ready to estimate the right-hand side of the integration by part formula (4.5). The contributions of the different terms generated by the derivatives with respect to τ can be estimated by relying on Propositions 4.2 and 4.5. The strategy is as follows:

- (I) show with the help of Corollary 4.3 that the terms involving the first derivative of $\eta_{12}(s, \tau)$ contribute to order $\mathcal{O}(\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}}) + \mathcal{O}(\lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}})$;
- (II) show by means of another integration by parts that the terms involving derivatives of e_{12} and $q_{1 \rightarrow 2}$ and the first derivative of ζ_{12} only contribute to order $\mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda^2 \varepsilon^2) + \mathcal{O}(\lambda^4 \varepsilon) + \mathcal{O}(\lambda^6)$;

(III) show with the help of Corollary 4.4 that the terms involving the second derivative of $\eta_{12}(s, \tau)$ can be approximated by the second term in the right-hand side of (2.24), which is of order $\lambda^2 \varepsilon$;

(IV) show with the help of Corollary 4.6 and an integration by parts that the terms involving the second derivative of $\zeta_{12}(s, \tau)$ contribute to order $\mathcal{O}(\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}}) + \mathcal{O}(\lambda^4 \varepsilon)$.

Note that the error terms obtained in (I) are small with respect to both contributions to the transition probability in (2.24) (that is, to the transition probability in the absence of reservoir and to our estimated correction coming from the coupling to the reservoir) provided that $\lambda^2 \ll \varepsilon^{1-\frac{m_1}{2}} \ll 1$ if $m \neq 1$ and $\lambda^2 \ll (\varepsilon/|\ln \varepsilon|)^{1/2}$ if $m = 1$. Similarly, the error terms obtained in (II) are negligible with respect to both contributions provided that $\varepsilon^2 \ll \lambda^2 \ll \varepsilon^{2/3} \ll 1$.

We implement our strategy by first expressing $\|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2$ as a sum of six terms. From (4.5),

$$\|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2 = \varepsilon^2 q_{1 \rightarrow 2}(t, t) + I_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) + J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) + L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) + R_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) + S_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) \quad (4.33)$$

with

$$\begin{aligned} I_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= -2\varepsilon^2 \operatorname{Re} \int_0^t ds \int_0^s d\tau \left(e^{-i\varphi_{12} + i\zeta_{12} - \eta_{12}} (\partial_\tau v_{12} + i(\partial_\tau \zeta_{12})w_{12} - (\partial_\tau \zeta_{12})^2 q_{1 \rightarrow 2}) \right)(s, \tau) \\ R_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= 2\varepsilon^2 \operatorname{Re} \int_0^t ds \int_0^s d\tau \left(e^{-i\varphi_{12} + i\zeta_{12} - \eta_{12}} (\partial_\tau \eta_{12})w_{12} \right)(s, \tau) \\ S_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= -2\varepsilon^2 \operatorname{Re} \int_0^t ds \int_0^s d\tau \left(e^{-i\varphi_{12} + i\zeta_{12} - \eta_{12}} \left(-2i(\partial_\tau \zeta_{12})(\partial_\tau \eta_{12}) + (\partial_\tau \eta_{12})^2 \right) q_{1 \rightarrow 2} \right)(s, \tau) \end{aligned}$$

and $J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ and $L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ are defined in (4.16) and (4.30). Here, we have set

$$v_{12} = \partial_\tau (\ln |e_{21}|) q_{1 \rightarrow 2} + \partial_\tau q_{1 \rightarrow 2} \quad , \quad w_{12} = \partial_\tau (\ln |e_{21}|) q_{1 \rightarrow 2} + 2\partial_\tau q_{1 \rightarrow 2} .$$

We assume in the sequel that Assumptions (A.1)-(A.4) are satisfied. Then

$$\sup_{0 \leq \tau \leq 1} \max_{n=1,2,3} |\partial_\tau^n (\ln |e_{21}|)(\tau)| < \infty \quad , \quad a \equiv \sup_{0 \leq \tau \leq s \leq 1} \max_{n=0,1} \{ |\partial_\tau^{n+1} v_{12}(s, \tau)|, |\partial_\tau^n w_{12}(s, \tau)| \} < \infty . \quad (4.34)$$

Actually, one can bound e.g. $|\partial_\tau (\ln |e_{12}|)(\tau)|$ by $2 \max_{i=1,2} \sup_{0 \leq \tau \leq 1} |\partial_\tau e_i(\tau)|/\delta$; the boundedness of the second supremum is a consequence of that of the first one and of (4.4).

We now proceed to prove the statements (I)-(IV).

(I) Using (4.34), applying Corollary 4.3 and recalling that $\operatorname{Re} \eta_{12}(s, \tau) \geq 0$, one has

$$|R_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)| \leq 2ac\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}} .$$

Similarly, using the first bound in Proposition 4.5 and the same Corollary,

$$|S_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)| \leq 2(cc' + c)q_\infty \lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}} .$$

Hence $R_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ and $S_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ are of order $\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}}$ and $\lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}}$, respectively, as announced above.

(II) We now estimate $I_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ by performing an integration by parts in the τ -integral, using $e_{21}(\tau)e^{-i\varphi_{12}(s, \tau)} = i\varepsilon \partial_\tau(e^{-i\varphi_{12}(s, \tau)})$. By relying on observation (i) in the proof of Proposition 4.1, one has $\partial_\tau^n q_{1 \rightarrow 2}(s, 0) = 0$ for $n = 0, 1, 2$, showing that $\partial_\tau v_{1 \rightarrow 2}(s, 0) = w_{1 \rightarrow 2}(s, 0) = 0$. Using also the fact that $q_{1 \rightarrow 2}(s, s)$ and $\partial_\tau \zeta_{12}(s)$ are real and the observation (ii) of the same proof, one finds that the boundary term in the integration by parts reads

$$2\varepsilon^3 \int_0^t ds \frac{1}{e_{21}(s)} \left(-i\partial_\tau v_{12}(s, s) + (\partial_\tau \zeta_{12})(s)w_{12}(s, s) \right).$$

One then deduces from (4.34), the gap hypothesis (A.1), and Proposition 4.5 that the boundary term is of order $\mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda^2\varepsilon^2)$. Thus the integration by parts yields

$$\begin{aligned} I_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= 2\varepsilon^3 \operatorname{Re} \int_0^t ds \int_0^s d\tau e^{-i\varphi_{12}(s, \tau)} \partial_\tau \left[\frac{e^{i\zeta_{12} - \eta_{12}}}{e_{21}(\tau)} (i\partial_\tau v_{12} - (\partial_\tau \zeta_{12})w_{12} \right. \\ &\quad \left. - i(\partial_\tau \zeta_{12})^2 q_{1 \rightarrow 2}) \right](s, \tau) + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda^2\varepsilon^2). \end{aligned} \quad (4.35)$$

Computing the derivative of the expression inside the square brackets and using (4.4), (4.34) and the gap hypothesis (A.1), one finds that for any $0 \leq \tau \leq 1$, the integrand in (4.35) is bounded in modulus by a constant factor times the sum of terms

$$\sum_{n=0}^3 |\partial_\tau \zeta_{12}(\tau)|^n + |\partial_\tau^2 \zeta_{12}(\tau)| + |\partial_\tau^2 \zeta_{12}(\tau)| |\partial_\tau \zeta_{12}(\tau)| + |\partial_\tau \eta_{12}(s, \tau)| \sum_{n=0}^2 |\partial_\tau \zeta_{12}(\tau)|^n.$$

Thanks to Corollary 4.3 and Proposition 4.5, the double integrals of this sum in the triangle $\{0 \leq \tau \leq s \leq t\}$ is bounded from above by

$$\frac{t^2}{2} \sum_{n=0}^3 (c')^n \frac{\lambda^{2n}}{\varepsilon^n} + tc' \frac{\lambda^2}{\varepsilon} + t(c')^2 \frac{\lambda^4}{\varepsilon^2} + \lambda^2 \varepsilon^{m_1-1} |\ln \varepsilon|^{\delta_{m,1}} \sum_{n=0}^2 (c')^n \frac{\lambda^{2n}}{\varepsilon^n}.$$

This proves that

$$I_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda^2\varepsilon^2) + \mathcal{O}(\lambda^4\varepsilon) + \mathcal{O}(\lambda^6), \quad (4.36)$$

as announced above.

(III) By virtue of Corollary 4.4,

$$J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \frac{\lambda^2 \varepsilon}{2} \int_0^t ds b_{12}^2(s) q_{1 \rightarrow 2}(s, s) \widehat{\gamma}(e_{12}(s)) + \mathcal{O}(\lambda^2 \varepsilon^{\min\{1+m\alpha, 2-2\alpha\}}) + \mathcal{O}(\lambda^4 \varepsilon^{1-\alpha}) \quad (4.37)$$

with $\alpha \in (0, 1/2)$ an arbitrary exponent. We choose α as follows. First, since we want that the aforementioned errors to be much smaller than both contributions to the transition probability in (2.24), we require that in the limit $\varepsilon \ll 1$,

$$\begin{cases} \lambda^2 \varepsilon^{1+m\alpha} & \ll \min\{\varepsilon^2, \lambda^2 \varepsilon\} \\ \lambda^2 \varepsilon^{2-2\alpha} & \ll \min\{\varepsilon^2, \lambda^2 \varepsilon\} \\ \lambda^4 \varepsilon^{1-\alpha} & \ll \min\{\varepsilon^2, \lambda^2 \varepsilon\} \end{cases} \Leftrightarrow \begin{cases} \lambda^2 & \ll \varepsilon^{1-m\alpha} \\ \lambda^2 & \ll \varepsilon^{2\alpha} \\ \lambda^2 & \ll \min\{\varepsilon^\alpha, \varepsilon^{(1+\alpha)/2}\}. \end{cases}$$

The optimal value α_0 minimizes the maximal exponent $\max\{1 - m\alpha, 2\alpha, (1 + \alpha)/2\}$ of ε in the bounds on λ^2 . One easily finds

$$\alpha_0 = \frac{1}{2 + 2m - m_1} = \begin{cases} \frac{1}{1 + 2m} & \text{if } m \geq 1 \\ \frac{1}{2 + m} & \text{if } 0 < m \leq 1. \end{cases} \quad (4.38)$$

For the choice $\alpha = \alpha_0$, the error terms in the estimation (4.37) are $\mathcal{O}(\lambda^2 \varepsilon^{1+m\alpha_0})$.

In fact, if $m > 1$ then $\lambda^2 \varepsilon^{2-2\alpha_0} \ll \lambda^2 \varepsilon^{1+m\alpha_0}$ (since $1 + m\alpha_0 < 2 - 2\alpha_0$) and $\lambda^4 \varepsilon^{1-\alpha_0} \ll \lambda^2 \varepsilon^{1+m\alpha_0}$ (since by construction $\lambda^2 \ll \varepsilon^{1-m\alpha_0}$ and $1 - m\alpha_0 = \alpha_0 + m\alpha_0$). Similarly, if $0 < m \leq 1$ then $\lambda^2 \varepsilon^{2-2\alpha_0} = \lambda^2 \varepsilon^{1+m\alpha_0}$ (since $1 + m\alpha_0 = 2 - 2\alpha_0$) and $\lambda^4 \varepsilon^{1-\alpha_0} \ll \lambda^2 \varepsilon^{1+m\alpha_0}$ (since by construction $\lambda^2 \ll \varepsilon^{(1+\alpha_0)/2}$ and $(1 + \alpha_0)/2 > \alpha_0 + m\alpha_0$). Thus in all cases

$$J_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \frac{\lambda^2 \varepsilon}{2} \int_0^t ds b_{12}^2(s) q_{1 \rightarrow 2}(s, s) \widehat{\gamma}(e_{12}(s)) + \mathcal{O}(\lambda^2 \varepsilon^{1+m\alpha_0}). \quad (4.39)$$

(IV) It remains to estimate $L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$. For this, we use Corollary 4.6 and make an integration by parts to get

$$\begin{aligned} L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) &= \beta_0 \lambda^2 \varepsilon^2 \left\{ \int_0^t ds \partial_s (b_1^2 - b_2^2)(s) \frac{q_{1 \rightarrow 2}(s, s)}{e_{21}(s)} \right. \\ &\quad \left. - \operatorname{Re} \int_0^t ds \int_0^s d\tau e^{-i\varphi_{12}(s, \tau)} \partial_\tau \left(e^{i(\zeta_{12} - \eta_{12})(s, \tau)} \partial_\tau (b_1^2 - b_2^2)(\tau) \frac{q_{1 \rightarrow 2}(s, \tau)}{e_{21}(\tau)} \right) \right\} \\ &\quad + \mathcal{O}(\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}}), \end{aligned} \quad (4.40)$$

where we have used again $q_{1 \rightarrow 2}(s, 0) = \varphi_{12}(s, s) = \zeta_{12}(s, s) = \eta_{12}(s, s) = 0$ as well as $q_{1 \rightarrow 2}(s, s) \in \mathbb{R}$. The boundary term in the first line of (4.40) is obviously of order $\lambda^2 \varepsilon^2$. The integral in the second line, in turn, is bounded by

$$\frac{4M^2}{\delta} \int_0^t ds \int_0^s dt \left(\left(\sup_{0 \leq \tau \leq 1} |\partial_\tau (\ln |e_{21}|)(\tau)| + \sup_{0 \leq \tau \leq 1} |\partial_\tau \zeta_{12}(\tau)| + |\partial_\tau \eta_{12}(s, \tau)| + 2 \right) q_\infty + q_\infty^{(1)} \right).$$

In view of Corollary 4.3 and Proposition 4.5, the contribution to $L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ of the integral in the second line of (4.40) is thus of order $\mathcal{O}(\lambda^2 \varepsilon^2) + \mathcal{O}(\lambda^4 \varepsilon)$. This gives

$$L_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t) = \mathcal{O}(\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}}) + \mathcal{O}(\lambda^4 \varepsilon). \quad (4.41)$$

Altogether, collecting (4.33), (4.36), (4.39) and (4.41) and using $\lambda^2 \varepsilon^{1+m_1} |\ln \varepsilon|^{\delta_{m,1}} \ll \lambda^2 \varepsilon^{1+m\alpha_0}$ (since $m\alpha_0 < m_1$), we deduce that

$$\begin{aligned} \|\omega_{\lambda, \varepsilon}^{(1)}(t)\|^2 &= \varepsilon^2 q_{1 \rightarrow 2}(t, t) + \frac{\lambda^2 \varepsilon}{2} \int_0^t ds q_{1 \rightarrow 2}(s, s) b_{12}^2(s) \widehat{\gamma}(e_{12}(s)) \\ &\quad + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda^2 \varepsilon^{1+m\alpha_0}) + \mathcal{O}(\lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}}) + \mathcal{O}(\lambda^6). \end{aligned} \quad (4.42)$$

Since $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t) = \varepsilon^2 q_{1 \rightarrow 2}(t, t) + \mathcal{O}(\varepsilon^3)$ (see (2.15)), one may substitute $p_{1 \rightarrow 2}^{(0, \varepsilon)}(t)$ for $\varepsilon^2 q_{1 \rightarrow 2}(t, t)$ in (4.42), making an error of order $\lambda^2 \varepsilon^2 \ll \lambda^2 \varepsilon^{1+m\alpha_0}$. We conclude that

Proposition 4.7 *Under assumptions (A.1)-(A.4),*

$$\begin{aligned} \|\omega_{\lambda,\varepsilon}^{(1)}(t)\|^2 &= p_{1\rightarrow 2}^{(0,\varepsilon)}(t) + \frac{\lambda^2}{2\varepsilon} \int_0^t ds p_{1\rightarrow 2}^{(0,\varepsilon)}(s) b_{12}^2(s) \widehat{\gamma}(e_{12}(s)) \\ &\quad + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\lambda^2 \varepsilon^{1+m\alpha_0}) + \mathcal{O}(\lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}}) + \mathcal{O}(\lambda^6), \end{aligned}$$

where α_0 is given by (4.38).

Observe that the exponent of ε in the second error term, $1 + m\alpha_0$, belongs to $(1, 4/3]$ when $0 < m \leq 1$ and to $(4/3, 3/2)$ when $m > 1$.

5 Contribution of higher-order terms in the Dyson expansion

Recall that the transition probability between distinct energy levels of the system is given by

$$p_{1\rightarrow 2}^{(\lambda,\varepsilon)}(t) = \left\| \sum_{k \geq 1} P_2(0) \omega_{\lambda,\varepsilon}^{(k)}(t) \right\|^2 = \left\| \sum_{k \text{ odd}} \omega_{\lambda,\varepsilon}^{(k)}(t) \right\|^2,$$

see (3.10). In this section, we show that the terms of this series of order $k > 1$ do not contribute to the transition probability to lowest order in ε and λ . The main result is summarized in the following Proposition.

Proposition 5.1 *Under assumptions (A.1)-(A.4), we have*

$$\sum_{j=1}^{\infty} \sup_{0 \leq t \leq 1} \|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\| = \mathcal{O}(\varepsilon^2 + \lambda \varepsilon^{\frac{1+m_1}{2}} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}} + \lambda^2 \varepsilon^{\frac{m_1}{2}} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}} + \lambda^5), \quad (5.1)$$

where as above $m_1 = \min\{m, 1\}$.

In view of (3.12) and since

$$\|\omega_{\lambda,\varepsilon}^{(1)}(t)\| = \mathcal{O}(\varepsilon + \lambda \varepsilon^{\frac{1}{2}} + \lambda^2 \varepsilon^{\frac{1}{2}m_1} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}} + \lambda^3)$$

as shown in the previous section, one deduces from Proposition 5.1 that

$$\begin{aligned} p_{1\rightarrow 2}^{(\lambda,\varepsilon)}(t) &= \|\omega_{\lambda,\varepsilon}^{(1)}(t)\|^2 + \mathcal{O}(\varepsilon^3 + \lambda \varepsilon^{\frac{3+m_1}{2}} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}} + \lambda^2 \varepsilon^{1+\frac{m_1}{2}} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}} + \lambda^3 \varepsilon^{\frac{1+m_1}{2}} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}}) \\ &\quad + \mathcal{O}(\lambda^4 \varepsilon^{m_1} |\ln \varepsilon|^{\delta_{m,1}} + \lambda^5 \varepsilon^{\frac{1}{2}m_1} |\ln \varepsilon|^{\frac{1}{2}\delta_{m,1}} + \lambda^8). \end{aligned} \quad (5.2)$$

Together with Proposition 4.7, this yields the result stated in Theorem 2.2. Actually, given that $m\alpha_0 < m_1/2$, the error term proportional to λ^2 in (5.2) is much smaller than $\lambda^2 \varepsilon^{1+m\alpha_0}$.

To prove Proposition 5.1, we proceed analogously as in the previous section. We first integrate by parts the recursion relation (3.25) and then rely on the estimations of Subsection 4.2 to bound $\sup_{0 \leq t \leq 1} \|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\|^2$ in terms of its value for $j \rightarrow j-1$ up to some remainder terms (Subsection 5.1). The remainder terms involve multiple integrals of first and second derivatives of quantum expectations in the vacuum state of products of $2j+2$ Weyl operators. The latter are controled in Subsections 5.2 and 5.3 by using similar arguments as in Subsection 4.2. With the help of these results, we conclude the proof of Proposition 5.1 in Subsection 5.4.

5.1 Integration by parts

One easily deduces from the recursion relation (3.25) that for any integer $j \geq 1$,

$$\begin{aligned} \|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\|^2 &= \\ \int_0^t ds \int_0^t ds' \int_0^s d\tau \int_0^{s'} d\tau' e_{21}(s)e_{21}(s') e^{i(\varphi_{12}-\zeta_{12}+\theta_{12}^-)(s,\tau)} e^{-i(\varphi_{12}-\zeta_{12}+\theta_{12}^-)(s',\tau')} Q_j(s,\tau;s',\tau') \end{aligned} \quad (5.3)$$

with

$$\begin{aligned} Q_j(s,\tau;s',\tau') &\equiv \frac{1}{e_{21}(s)e_{21}(s')} \langle \omega_{\lambda,\varepsilon}^{(2j-1)}(\tau) | \tilde{K}_{12}(\tau)^* \tilde{K}_{21}(s)^* \tilde{K}_{21}(s') \tilde{K}_{12}(\tau') \otimes \\ &W(-F_{12}(s,\tau)) W(F_{12}(s',\tau')) | \omega_{\lambda,\varepsilon}^{(2j-1)}(\tau') \rangle . \end{aligned} \quad (5.4)$$

We start by deriving an exact formula for $\|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\|^2$, obtained through two integrations by parts in the integrals in (5.3).

Proposition 5.2 *Under assumptions (A.1)-(A.2), one has for any integer $j \geq 1$ and any rescaled time $t \in (0, 1]$,*

$$\begin{aligned} \|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\|^2 &= \varepsilon^2 \text{Re} \left\{ \int_0^t d\tau \int_0^t d\tau' \left(Q_j(\tau,\tau;\tau',\tau') - 2e^{-i(\varphi_{12}-\zeta_{12}+\theta_{12}^-)(t,\tau')} Q_j(\tau,\tau;t,\tau') \right. \right. \\ &+ e^{i(\varphi_{12}-\zeta_{12}+\theta_{12}^-)(t,\tau)} e^{-i(\varphi_{12}-\zeta_{12}+\theta_{12}^-)(t,\tau')} Q_j(t,\tau;t,\tau') \left. \right) \\ &+ 2 \int_0^t ds \int_0^s d\tau \int_0^t d\tau' e^{i\varphi_{12}(s,\tau)} \partial_s \left(e^{i(-\zeta_{12}+\theta_{12}^-)(s,\tau)} (Q_j(s,\tau;\tau',\tau') \right. \\ &- e^{-i(\varphi_{12}-\zeta_{12}+\theta_{12}^-)(t,\tau')} Q_j(s,\tau;t,\tau')) \left. \right) \\ &+ \int_0^t ds \int_0^s d\tau \int_0^t ds' \int_0^{s'} d\tau' e^{i(\varphi_{12}(s,\tau)-\varphi_{12}(s',\tau'))} \times \\ &\left. \partial_s \partial_{s'} \left(e^{i(-\zeta_{12}+\theta_{12}^-)(s,\tau)} e^{-i(-\zeta_{12}+\theta_{12}^-)(s',\tau')} Q_j(s,\tau;s',\tau') \right) \right\} . \end{aligned} \quad (5.5)$$

Proof. This follows by integrating the two integrals over s and s' in (5.3) by parts, using $e_{21}(s)e^{i\varphi_{12}(s,\tau)} = i\varepsilon \partial_s (e^{i\varphi_{12}(s,\tau)})$. The calculation is simplified by relying on

$$Q_j(s,\tau;s',\tau') = \overline{Q_j(s',\tau';s,\tau)} \quad (5.6)$$

to recognize complex conjugate terms. \square

Combining Proposition 5.2 with the results of Section 4.2, one can derive the following bound on $\|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\|^2$.

Proposition 5.3 *Let assumptions (A.1)-(A.4) hold. Then for any integer $j \geq 1$ and $t \in (0, 1]$,*

$$\begin{aligned} \|\omega_{\lambda,\varepsilon}^{(2j+1)}(t)\|^2 &\leq c_1^2(\varepsilon^2 + \lambda^2\varepsilon + \lambda^4) \sup_{0 \leq \tau \leq t} \|\omega_{\lambda,\varepsilon}^{(2j-1)}(\tau)\|^2 + c_2^2(\varepsilon^2 + \lambda^2\varepsilon) D_{\lambda,\varepsilon}^{(j)}(t) \\ &+ \varepsilon^2 |E_{\lambda,\varepsilon}^{(j)}(t)| , \end{aligned} \quad (5.7)$$

where the positive constants c_1 and c_2 are independent of $(\lambda, \varepsilon, j, t)$ and we have set

(5.8)

$$D_{\lambda, \varepsilon}^{(j)}(t) \equiv \sup_{0 \leq \tau' \leq s' \leq t} \int_0^t ds \int_0^s d\tau \left(\left| \partial_\sigma Q_j(s, \sigma, \tau; s', s', \tau') \Big|_{\sigma=s} \right| + \left| \partial_\sigma \partial_{\nu'} Q_j(s, \sigma, \tau; \nu', s', \tau') \Big|_{\sigma=s, \nu'=s'} \right| \right) \quad (5.9)$$

$$E_{\lambda, \varepsilon}^{(j)}(t) \equiv \int_0^t ds \int_0^s d\tau \int_0^t ds' \int_0^{s'} d\tau' e^{i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s, \tau)} e^{-i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s', \tau')} \times \partial_\sigma \partial_{\sigma'} Q_j(s, \sigma, \tau; s', \sigma', \tau') \Big|_{\sigma=s, \sigma'=s'} \quad (5.10)$$

with

$$Q_j(s, \sigma, \tau; s', \sigma', \tau') \equiv \frac{1}{e_{21}(s)e_{21}(s')} \langle \omega_{\lambda, \varepsilon}^{(2j-1)}(\tau) | \tilde{K}_{12}(\tau) * \tilde{K}_{21}(s) * \tilde{K}_{21}(s') \tilde{K}_{12}(\tau') \otimes W(-F_{12}(\sigma, \tau)) W(F_{12}(\sigma', \tau')) | \omega_{\lambda, \varepsilon}^{(2j-1)}(\tau') \rangle . \quad (5.11)$$

Proof. By Assumption (A.2) and the definition (2.12) of $K(s)$, the three suprema

$$k_\infty \equiv \sup_{0 \leq s \leq 1} \|\tilde{K}_{21}(s)\| \quad , \quad k'_\infty \equiv \sup_{0 \leq s \leq 1} \|\partial_s \tilde{K}_{21}(s)\| \quad \text{and} \quad \ell_\infty \equiv \sup_{0 \leq s \leq 1} \left| \partial_s \ln |e_{21}(s)| \right| \quad (5.12)$$

are finite (in fact, $\|\tilde{K}_{21}(s)\| \leq \|K(s)\|$ and $\|\partial_s \tilde{K}_{21}(s)\| \leq 2k_\infty^2 + \|\partial_s K(s)\|$ for any $s \in [0, 1]$). Let us fix $t \in (0, 1]$. Thanks to (5.4), one has

$$\sup_{0 \leq \tau \leq s \leq t} \sup_{0 \leq \tau' \leq s' \leq t} |Q_j(s, \tau; s', \tau')| \leq \tilde{c}_1 \sup_{0 \leq \tau \leq t} \|\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)\|^2 \quad (5.13)$$

with $\tilde{c}_1 = k_\infty^4 / \delta^2$. One then deduces from (5.5) and (5.6) that

$$\begin{aligned} \|\omega_{\lambda, \varepsilon}^{(2j+1)}(t)\|^2 &\leq \tilde{c}_1 \varepsilon^2 (4t^2 + 2tZ_t + Z_t^2) \sup_{0 \leq \tau \leq t} \|\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)\|^2 \\ &+ \varepsilon^2 (4t + 2Z_t) \sup_{0 \leq \tau' \leq s' \leq t} \int_0^t ds \int_0^s d\tau |\partial_s Q_j(s, \tau; s', \tau')| \\ &+ \varepsilon^2 \left| \int_0^t ds \int_0^s d\tau \int_0^t ds' \int_0^{s'} d\tau' e^{i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s, \tau)} e^{-i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s', \tau')} \partial_s \partial_{s'} Q_j(s, \tau; s', \tau') \right| \end{aligned} \quad (5.14)$$

with

$$Z_t \equiv \int_0^t ds \int_0^s d\tau |\partial_s (\zeta_{12} - \theta_{12}^-)(s, \tau)| .$$

In what follows, $\tilde{c}_1, \tilde{c}_2, \dots$, denote constants independent of $(\lambda, \varepsilon, j, t)$. Decomposing the derivative of Q_j as

$$\partial_s Q_j(s, \tau; s', \tau') = \partial_\nu Q_j(\nu, s, \tau; s', s', \tau') \Big|_{\nu=s} + \partial_\sigma Q_j(s, \sigma, \tau; s', s', \tau') \Big|_{\sigma=s}$$

and using

$$\left| \partial_\nu Q_j(\nu, s, \tau; s', s', \tau') \Big|_{\nu=s} \right| \leq \frac{k_\infty^3}{\delta^2} \left(\ell_\infty k_\infty + k'_\infty \right) \sup_{0 \leq \tau \leq t} \|\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)\|^2 ,$$

the supremum in the second line of (5.14) can be bounded from above by

$$\tilde{c}_2 \sup_{0 \leq \tau \leq t} \|\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)\|^2 + \sup_{0 \leq \tau' \leq s' \leq t} \int_0^t ds \int_0^s d\tau \left| \partial_\sigma Q_j(s, \sigma, \tau; s', s', \tau') \Big|_{\sigma=s} \right|.$$

Similarly, the integral in the last line of (5.14) is bounded by

$$\begin{aligned} & \tilde{c}_3 \sup_{0 \leq \tau \leq t} \|\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)\|^2 + \sup_{0 \leq \tau' \leq s' \leq t} \int_0^t ds \int_0^s d\tau \left| \partial_\sigma \partial_{\nu'} Q_j(s, \sigma, \tau; \nu', s', \tau') \Big|_{\sigma=s, \nu'=s'} \right| \\ & + \left| \int_0^t ds \int_0^s d\tau \int_0^t ds' \int_0^{s'} d\tau' e^{i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s, \tau)} e^{-i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s', \tau')} \times \right. \\ & \left. \partial_\sigma \partial_{\sigma'} Q_j(s, \sigma, \tau; s', \sigma', \tau') \Big|_{\sigma=s, \sigma'=s'} \right|, \end{aligned}$$

where we have taken advantage of $Q_j(\nu, \sigma, \tau; \nu', \sigma', \tau') = \overline{Q_j(\nu', \sigma', \tau'; \nu, \sigma, \tau)}$. But by (3.26) and (4.3), one has $|\partial_s \theta_{12}^-(s, \tau)| = |\partial_s \theta_{21}^+(\tau, s)| \leq |\partial_s \eta_{21}(\tau, s)|$ ¹ and thus

$$\int_0^t ds \int_0^s d\tau |\partial_s \theta_{12}^-(s, \tau)| \leq \int_0^t ds \int_s^t d\tau |\partial_\tau \eta_{21}(s, \tau)|.$$

Applying Corollary 4.3 and Proposition 4.5, this yields

$$Z_t \leq c' \frac{t^2 \lambda^2}{\varepsilon} + c \lambda^2 \varepsilon^{m_1-1} |\ln \varepsilon|^{\delta_{m,1}} = \mathcal{O}(\lambda^2 \varepsilon^{-1})$$

(note that the labels 1, 2 of the energy levels can be exchanged without altering the results of Corollary 4.3). Collecting the results above, the desired bound follows from (5.14). \square

5.2 Estimation of $D_{\lambda, \varepsilon}^{(j)}$

Proposition 5.4 *Let assumptions (A.2)-(A.4) hold. Then for any integer $j \geq 1$ and $t \in (0, 1]$, one has*

$$\sum_{j=1}^{\infty} (D_{\lambda, \varepsilon}^{(j)}(t))^{\frac{1}{2}} \leq c \lambda \sqrt{r\left(\frac{1}{\varepsilon}\right)} = \mathcal{O}\left(\lambda \varepsilon^{\frac{m_1-1}{2}} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}}\right). \quad (5.15)$$

with $c > 0$ independent on $(\lambda, \varepsilon, j, t)$ and $r(1/\varepsilon)$ defined by (4.11).

Proof. We divide the proof into three steps.

STEP 1. Let us show that for any $t \in (0, 1]$,

$$\left| D_{\lambda, \varepsilon}^{(j)}(t) \right| \leq c \sup_{0 \leq \tau' \leq s' \leq t} \int_{0 \leq v_{2j-1} \leq \dots \leq v_1 \leq \tau \leq s \leq t} \dots \int ds d\tau d^{2j-1} \underline{v} \int_{0 \leq v'_{2j-1} \leq \dots \leq v'_1 \leq \tau'} \dots \int d^{2j-1} \underline{v}' \left| \partial_s \mathcal{R}_{\lambda, \varepsilon}^{(j)}(s, \tau, \underline{v}; s', \tau', \underline{v}') \right| \quad (5.16)$$

¹The same bound holds in the positive temperature case, still with the zero temperature expression for the upper bound $|\partial_s \eta_{21}(\tau, s)|$, because the left side of the inequality (namely θ_{12}^\pm) is independent of the temperature, see Section 6.3.

with c as in the Proposition and, for any $\underline{v} = (v_1, \dots, v_{2j-1})$, $\underline{v}' = (v'_1, \dots, v'_{2j-1}) \in \mathbb{R}_+^{2j-1}$,

$$\begin{aligned} \mathcal{R}_{\lambda, \varepsilon}^{(j)}(s, \tau, \underline{v}; s', \tau', \underline{v}') &\equiv \langle \chi | W(-F_{12}(v_{2j-1}, v_{2j})) \cdots W(-F_{12}(v_{-1}, v_0)) \\ &W(F_{12}(v'_{-1}, v'_0)) \cdots W(F_{12}(v'_{2j-1}, v'_{2j})) \chi \rangle, \end{aligned} \quad (5.17)$$

where we have set $v_{2j} = 0$, $v_0 = \tau$, $v_{-1} = s$ and, similarly, $v'_{2j} = 0$, $v'_0 = \tau'$, $v'_{-1} = s'$. We shall freely pass from (s, τ) to (v_{-1}, v_0) and so on, wherever convenient in the sequel.

Actually, thanks to (3.25) and (3.23), the vector $\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)$ is given by the multiple integral

$$\begin{aligned} \omega_{\lambda, \varepsilon}^{(2j-1)}(\tau) &= - \int \cdots \int_{0 \leq v_{2j-1} \leq \cdots \leq v_1 \leq \tau} d^{2j-1} \underline{v} \exp \left\{ -i \sum_{k=1}^j (\varphi_{12} - \zeta_{12} + \theta_{12}^-)(v_{2k-1}, v_{2k}) \right\} \times \\ &\tilde{K}_{21}(v_1) \tilde{K}_{12}(v_2) \cdots \tilde{K}_{21}(v_{2j-1}) \otimes W(F_{12}(v_1, v_2)) \cdots W(F_{12}(v_{2j-1}, v_{2j})) \psi_1(0) \otimes \chi. \end{aligned} \quad (5.18)$$

Plugging this formula into (5.11) gives

$$\begin{aligned} Q_j(s, \sigma, \tau; s', \sigma', \tau') &= \frac{1}{e_{21}(s)e_{21}(s')} \int \cdots \int_{0 \leq v_{2j-1} \leq \cdots \leq v_1 \leq \tau} d^{2j-1} \underline{v} \int \cdots \int_{0 \leq v'_{2j-1} \leq \cdots \leq v'_1 \leq \tau'} d^{2j-1} \underline{v}' \\ &\exp \left\{ i \sum_{k=1}^j \left((\varphi_{12} - \zeta_{12} + \theta_{12}^-)(v_{2k-1}, v_{2k}) - (\varphi_{12} - \zeta_{12} + \theta_{12}^-)(v'_{2k-1}, v'_{2k}) \right) \right\} \times \\ &\langle \psi_1(0) | \tilde{K}_{21}(v_{2j-1})^* \cdots \tilde{K}_{12}(v_0)^* \tilde{K}_{21}(v_{-1})^* \tilde{K}_{21}(v'_{-1}) \tilde{K}_{12}(v'_0) \cdots \tilde{K}_{21}(v'_{2j-1}) \psi_1(0) \rangle \times \\ &\mathcal{R}_{\lambda, \varepsilon}^{(j)}(\sigma, \tau, \underline{v}; \sigma', \tau', \underline{v}'). \end{aligned} \quad (5.19)$$

The inequality (5.16) then follows from Assumption (A.2) and the boundedness of k_∞ , k'_∞ , and ℓ_∞ in (5.12).

STEP 2: Exact formula for $\mathcal{R}_{\lambda, \varepsilon}^{(j)}(s, \tau, \underline{v}; s', \tau', \underline{v}')$.

Lemma 5.5 *One has*

$$\begin{aligned} \mathcal{R}_{\lambda, \varepsilon}^{(j)}(s, \tau, \underline{v}; s', \tau', \underline{v}') &= e^{i\theta(\underline{v}, \underline{v}')} \exp \left\{ -\frac{1}{2} \sum_{k=1}^j \langle F_{12}(v_{2k-1}, v_{2k}), F_{12}(s, \tau) - F_{12}(s', \tau') \rangle \right\} \\ &\times \exp \left\{ \frac{1}{2} \sum_{k=1}^j \langle F_{12}(s, \tau) - F_{12}(s', \tau'), F_{12}(v'_{2k-1}, v'_{2k}) \rangle + \frac{1}{2} \langle F_{12}(s, \tau), F_{12}(s', \tau') \rangle \right\} \\ &\times \exp \left\{ -\frac{1}{4} \left\| \sum_{k=1}^j \left(F_{12}(v_{2k-1}, v_{2k}) - F_{12}(v'_{2k-1}, v'_{2k}) \right) \right\|^2 - \frac{1}{4} \|F_{12}(s, \tau)\|^2 - \frac{1}{4} \|F_{12}(s', \tau')\|^2 \right\}, \end{aligned} \quad (5.20)$$

where the function $\theta(\underline{v}, \underline{v}') : \mathbb{R}_+^{4j-2} \rightarrow \mathbb{R}$ is independent of s, τ, s' , and τ' .

Proof of Lemma 5.5. This follows from repeated applications of the properties (3.22) and (4.1) of the Weyl operators. To get formula (5.20), one may apply the following identity, which is a consequence of these two properties: for any $F, G, H \in L^2(\mathbb{R}^3)$, it holds

$$\langle \chi | W(-G)W(F)W(H) \chi \rangle = \exp \left\{ \frac{1}{2} (-\langle F, H \rangle + \langle G, F \rangle + i \operatorname{Im} \langle G, H \rangle) - \frac{1}{4} (\|G-H\|^2 + \|F\|^2) \right\}.$$

Use this formula with $G = \sum_{k=1}^j F_{12}(v_{2k-1}, v_{2k})$, $H = \sum_{k=1}^j F_{12}(v'_{2k-1}, v'_{2k})$ and $F = -F_{12}(s, \tau) + F_{12}(s', \tau')$. The real phase $\theta(\underline{v}, \underline{v}')$ comes from the phases generated by (3.22) when grouping the Weyl operators into the terms with G and H . \square

STEP 3: We conclude the proof by using similar arguments as in the proof of Proposition 4.2.

Let $0 = v_{2j} \leq v_{2j-1} \leq \dots \leq v_{-1} \leq t \leq 1$ and $0 = v'_{2j} \leq v'_{2j-1} \leq \dots \leq v'_{-1} \leq t$. We denote by $\underline{\mathbf{v}} = (v_{-1}, v_0, \dots, v_{2j-1}) \in \mathbb{R}_+^{2j+1}$, where we recall that $v_{-1} = s$ and $v_0 = \tau$, and use a similar notation with the primes. Recalling that $F_{12}(s, \tau) = \int_\tau^s dx f_{12}(x)$, one finds thanks to Lemma 5.5 that

$$\begin{aligned} \partial_s \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}') &= \frac{1}{2} \left(- \sum_{k=1}^j \langle F_{12}(v_{2k-1}, v_{2k}), f_{12}(s) \rangle - \operatorname{Re} \langle F_{12}(s, \tau), f_{12}(s) \rangle \right. \\ &\quad \left. + \sum_{k=0}^j \langle f_{12}(s), F_{12}(v'_{2k-1}, v'_{2k}) \rangle \right) \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}'). \end{aligned}$$

(Note that the second sum starts with $k = 0$.) Using $|\mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}')| \leq 1$ (see (5.17)),

$$\begin{aligned} 2|\partial_s \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}')| &\leq \sum_{k=1}^j |\langle F_{12}(v_{2k-1}, v_{2k}), f_{12}(s) \rangle| + |\operatorname{Re} \langle F_{12}(s, \tau), f_{12}(s) \rangle| \\ &\quad + \sum_{k=0}^j |\langle f_{12}(s), F_{12}(v'_{2k-1}, v'_{2k}) \rangle|. \end{aligned}$$

Arguing as in the proof of Proposition 4.2, one finds

$$\begin{aligned} |\partial_s \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}')| &\leq \frac{2M^2 \lambda^2}{\varepsilon} \left\{ \sum_{k=1}^j \left| \int_{\frac{v_{2k-s}}{\varepsilon}}^{\frac{v_{2k-1-s}}{\varepsilon}} dx \gamma(x) \right| + \left| \int_{\frac{\tau-s}{\varepsilon}}^0 dx \gamma_R(x) \right| \right. \\ &\quad \left. + \sum_{k=0}^j \left| \int_{\frac{v'_{2k-s}}{\varepsilon}}^{\frac{v'_{2k-1-s}}{\varepsilon}} dx \gamma(x) \right| \right\} + 4M^2 \lambda^2 \left\{ \int_{-\frac{s}{\varepsilon}}^0 dx |x \gamma(x)| + \int_{-\frac{s'}{\varepsilon}}^{\frac{s'-s}{\varepsilon}} dx |x \gamma(x)| \right\}. \end{aligned} \quad (5.21)$$

One has

$$\sum_{k=1}^j \left| \int_{\frac{v_{2k-s}}{\varepsilon}}^{\frac{v_{2k-1-s}}{\varepsilon}} dx \gamma(x) \right| \leq \Gamma\left(\frac{s-\tau}{\varepsilon}\right)$$

and, for any $k = 0, \dots, j$,

$$\left| \int_{\frac{v'_{2k-s}}{\varepsilon}}^{\frac{v'_{2k-1-s}}{\varepsilon}} dx \gamma(x) \right| \leq \Gamma\left(\frac{|v'_{2k-s} - s|}{\varepsilon}\right) + \Gamma\left(\frac{|v'_{2k-1-s} - s|}{\varepsilon}\right),$$

where we have set

$$\Gamma(y) \equiv \int_y^\infty dx |\gamma(x)| = \int_{-\infty}^{-y} dx |\gamma(x)| \quad (5.22)$$

and the second bound follows from $\int_{-\infty}^{\infty} dx \gamma(x) = 0$ see (4.10). One shows with the help of the change of variables $y = |v' - s|/\varepsilon$ that for any $0 \leq v' \leq t$,

$$\int_0^t ds \Gamma\left(\frac{|v' - s|}{\varepsilon}\right) \leq 2\varepsilon \int_0^{\frac{t}{\varepsilon}} dy \Gamma(y).$$

Thanks to (5.21), the three last bounds, and $\int_0^\infty dx \gamma_R(x) = 0$, one is led to

$$\begin{aligned} & \sup_{0 \leq \tau' \leq s' \leq t} \int_{0 \leq v_{2j-1} \leq \dots \leq v_{-1} \leq t} \dots \int_{0 \leq v'_{2j-1} \leq \dots \leq v'_1 \leq t} d^{2j+1} \underline{\mathbf{v}} \int \dots \int d^{2j-1} \underline{\mathbf{v}'} |\partial_s \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}')| \\ & \leq \frac{2M^2 \lambda^2}{((2j-1)!)^2} \left((6+4j) \int_0^{\frac{1}{\varepsilon}} dy \Gamma(y) + 6 \int_0^{\frac{1}{\varepsilon}} dx x |\gamma(x)| \right) \leq c_j \lambda^2 r \left(\frac{1}{\varepsilon} \right), \end{aligned}$$

where $c_j > 0$ depends on j only and satisfies $\sum_{j \geq 1} \sqrt{c_j} < \infty$. Substituting this bound into (5.16) and relying on (4.15), one gets the result of Proposition 5.4. \square

5.3 Estimation of $E_{\lambda, \varepsilon}^{(j)}$

Replacing (5.18), (5.11), and (5.17) into (5.10), it follows

$$\begin{aligned} E_{\lambda, \varepsilon}^{(j)}(t) &= \int_{0 \leq v_{2j-1} \leq \dots \leq v_{-1} \leq t} \dots \int_{0 \leq v'_{2j-1} \leq \dots \leq v'_{-1} \leq t} d^{2j+1} \underline{\mathbf{v}} \int \dots \int d^{2j+1} \underline{\mathbf{v}'} \frac{1}{e_{21}(v_{-1}) e_{21}(v'_{-1})} \times \\ & \exp \left\{ i \sum_{k=0}^j \left((\varphi_{12} - \zeta_{12} + \theta_{12}^-)(v_{2k-1}, v_{2k}) - (\varphi_{12} - \zeta_{12} + \theta_{12}^-)(v'_{2k-1}, v'_{2k}) \right) \right\} \times \quad (5.23) \\ & \langle \psi_1(0) | \tilde{K}_{21}(v_{2j-1})^* \dots \tilde{K}_{21}(v_{-1})^* \tilde{K}_{21}(v'_{-1}) \dots \tilde{K}_{21}(v'_{2j-1}) \psi_1(0) \rangle \partial_{v_{-1}} \partial_{v'_{-1}} \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}'). \end{aligned}$$

Now, according to Lemma 5.5 (recall that $s = v_{-1}$, $\tau = v_0$, $s' = v'_{-1}$, and $\tau' = v'_0$), one has

$$\begin{aligned} \partial_s \partial_{s'} \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}') &= \frac{1}{4} \left\{ \left[- \sum_{k=1}^j \langle F_{12}(v_{2k-1}, v_{2k}), f_{12}(s) \rangle - \text{Re} \langle F_{12}(s, \tau), f_{12}(s) \rangle \right. \right. \\ & + \sum_{k=0}^j \langle f_{12}(s), F_{12}(v'_{2k-1}, v'_{2k}) \rangle \left. \right] \left[\sum_{k=0}^j \langle F_{12}(v_{2k-1}, v_{2k}), f_{12}(s') \rangle - \text{Re} \langle F_{12}(s', \tau'), f_{12}(s') \rangle \right. \\ & \left. \left. - \sum_{k=1}^j \langle f_{12}(s'), F_{12}(v'_{2k-1}, v'_{2k}) \rangle \right] + 2 \langle f_{12}(s'), f_{12}(s) \rangle \right\} \mathcal{R}_{\lambda, \varepsilon}^{(j)}(\underline{\mathbf{v}}, \underline{\mathbf{v}}'). \quad (5.24) \end{aligned}$$

The term involving the scalar product $\langle f_{12}(s'), f_{12}(s) \rangle$ requires some special care. Its contribution to $E_{\lambda, \varepsilon}^{(j)}(t)$ is given by

$$\begin{aligned} G_{\lambda, \varepsilon}^{(j)}(t) &\equiv \frac{1}{2} \int_0^t ds \int_0^s d\tau \int_0^t ds' \int_0^{s'} d\tau' e^{i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s, \tau)} e^{-i(\varphi_{12} - \zeta_{12} + \theta_{12}^-)(s', \tau')} \langle f_{12}(s'), f_{12}(s) \rangle \times \\ & Q_j(s, \tau; s', \tau') \end{aligned}$$

and in view of (5.13) can be bounded for any $t \in (0, 1]$ as follows

$$|G_{\lambda,\varepsilon}^{(j)}(t)| \leq \frac{1}{2} \tilde{c}_1 \sup_{0 \leq \tau \leq t} \|\omega_{\lambda,\varepsilon}^{(2j-1)}(\tau)\|^2 \int_0^1 ds \int_0^1 ds' |\langle f_{12}(s'), f_{12}(s) \rangle| \leq c_3^2 \frac{\lambda^2}{\varepsilon} \sup_{0 \leq \tau \leq t} \|\omega_{\lambda,\varepsilon}^{(2j-1)}(\tau)\|^2 \quad (5.25)$$

with $c_3^2 = 4M^2 \tilde{c}_1 \int_0^\infty dx |\gamma|(x) < \infty$. The contribution to $E_{\lambda,\varepsilon}^{(j)}(t)$ of the other terms in the derivative (5.24) is controlled in the following Proposition.

Proposition 5.6 *Let assumptions (A.1)-(A.4) hold and let us set $\tilde{E}_{\lambda,\varepsilon}^{(j)}(t) = E_{\lambda,\varepsilon}^{(j)}(t) - G_{\lambda,\varepsilon}^{(j)}(t)$. Then for any integer $j \geq 1$ and $t \in (0, 1]$, one has*

$$\sum_{j=1}^{\infty} |\tilde{E}_{\lambda,\varepsilon}^{(j)}(t)|^{\frac{1}{2}} = \mathcal{O}(\lambda^2 \varepsilon^{\min\{m-1, -\frac{1}{2}\}} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1/2}}). \quad (5.26)$$

Proof. Thanks to (5.23) and (5.24), one has

$$\begin{aligned} |\tilde{E}_{\lambda,\varepsilon}^{(j)}(t)| &\leq \frac{k_\infty^{4j+2}}{4\delta^2} \int_{0 \leq v_{2j-1} \leq \dots \leq v_{-1} \leq t} \dots \int d^{2j+1} \mathbf{v} \int_{0 \leq v'_{2j-1} \leq \dots \leq v'_{-1} \leq t} \dots \int d^{2j+1} \mathbf{v}' \\ &\left[\sum_{k=1}^j |\langle F_{12}(v_{2k-1}, v_{2k}), f_{12}(s) \rangle| + |\operatorname{Re} \langle F_{12}(s, \tau), f_{12}(s) \rangle| + \sum_{k=0}^j |\langle F_{12}(v'_{2k-1}, v'_{2k}), f_{12}(s) \rangle| \right] \times \\ &\left[\sum_{k=0}^j |\langle F_{12}(v_{2k-1}, v_{2k}), f_{12}(s') \rangle| + |\operatorname{Re} \langle F_{12}(s', \tau'), f_{12}(s') \rangle| + \sum_{k=1}^j |\langle F_{12}(v'_{2k-1}, v'_{2k}), f_{12}(s') \rangle| \right]. \end{aligned}$$

Proceeding as in Step 3 of the proof of Proposition 5.4, one is led to

$$\begin{aligned} |\tilde{E}_{\lambda,\varepsilon}^{(j)}(t)| &\leq ck_\infty^{4j+2} \frac{\lambda^4}{\varepsilon^2} \int_{0 \leq v_{2j-1} \leq \dots \leq v_{-1} \leq t} \dots \int d^{2j+1} \mathbf{v} \int_{0 \leq v'_{2j-1} \leq \dots \leq v'_{-1} \leq t} \dots \int d^{2j+1} \mathbf{v}' \left[2\Gamma\left(\frac{v_{-1} - v_0}{\varepsilon}\right) + \right. \\ &+ \sum_{l=-1}^{2j} \Gamma\left(\frac{|v_{-1} - v'_l|}{\varepsilon}\right) + 6\varepsilon \int_0^{\frac{t}{\varepsilon}} dx x |\gamma(x)| \left. \right] \left[2\Gamma\left(\frac{v'_{-1} - v'_0}{\varepsilon}\right) + \right. \\ &+ \sum_{l=-1}^{2j} \Gamma\left(\frac{|v'_{-1} - v_l|}{\varepsilon}\right) + 6\varepsilon \int_0^{\frac{t}{\varepsilon}} dx x |\gamma(x)| \left. \right] \end{aligned}$$

yielding, for any $t \in (0, 1]$,

$$|\tilde{E}_{\lambda,\varepsilon}^{(j)}(t)| \leq \frac{k_\infty^{4j+2} \lambda^4}{((2j-1)!)^2} \left\{ c_j \left[r\left(\frac{1}{\varepsilon}\right) \right]^2 + \frac{2c}{\varepsilon} \int_0^{\frac{1}{\varepsilon}} dy \Gamma(y)^2 \right\},$$

where the constant c_j is quadratic in j . Using Assumption (A.4), one easily shows that

$$\int_0^{\frac{1}{\varepsilon}} dy \Gamma(y)^2 = \begin{cases} \mathcal{O}(1) & \text{if } m > \frac{1}{2} \\ \mathcal{O}(|\ln \varepsilon|) & \text{if } m = \frac{1}{2} \\ \mathcal{O}(\varepsilon^{2m-1}) & \text{if } m < \frac{1}{2}. \end{cases} \quad (5.27)$$

The result follows. \square

5.4 End of the proof of Proposition 5.1

Combining the results of Propositions 5.3, 5.4, and 5.6 and taking advantage of (5.25), one gets

$$\begin{aligned} \sum_{j=1}^{\infty} \sup_{0 \leq t \leq 1} \|\omega_{\lambda, \varepsilon}^{(2j+1)}(t)\| &\leq (c_1 \varepsilon + (c_1 + c_3) \lambda \sqrt{\varepsilon} + c_1 \lambda^2) \sum_{j=1}^{\infty} \sup_{0 \leq \tau \leq 1} \|\omega_{\lambda, \varepsilon}^{(2j-1)}(\tau)\| \\ &\quad + c'_2 \lambda \varepsilon^{\frac{m_1+1}{2}} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}} + c'_2 \lambda^2 \varepsilon^{\frac{1}{2} m_1} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}} + c_4 \lambda^2 \varepsilon^{\min\{m, \frac{1}{2}\}} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1/2}}. \end{aligned} \quad (5.28)$$

Since $\frac{1}{2} m_1 \leq \min\{m, \frac{1}{2}\}$, the last term is much smaller than the previous one. Decomposing the infinite series in the right-hand side of (5.28) as its first term plus the remainder and noting that the latter coincides with the series in the left-hand side this gives

$$\begin{aligned} \sum_{j=1}^{\infty} \sup_{0 \leq t \leq 1} \|\omega_{\lambda, \varepsilon}^{(2j+1)}(t)\| &= \left(1 + \mathcal{O}(\varepsilon + \lambda \sqrt{\varepsilon} + \lambda^2)\right) \left(\sup_{0 \leq t \leq 1} \|\omega_{\lambda, \varepsilon}^{(1)}(t)\| \mathcal{O}(\varepsilon + \lambda \sqrt{\varepsilon} + \lambda^2) + \right. \\ &\quad \left. \mathcal{O}(\lambda \varepsilon^{\frac{m_1+1}{2}} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}} + \lambda^2 \varepsilon^{\frac{1}{2} m_1} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}})\right). \end{aligned}$$

But $\sup_{0 \leq t \leq 1} \|\omega_{\lambda, \varepsilon}^{(1)}(t)\| = \mathcal{O}(\varepsilon + \lambda \varepsilon^{\frac{1}{2}} + \lambda^2 \varepsilon^{\frac{1}{2} m_1} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}} + \lambda^3)$ by Proposition 4.7.

Noting furthermore that $\lambda \varepsilon^{\frac{3}{2}}$ is much smaller than $\lambda \varepsilon^{\frac{m_1+1}{2}} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}}$ and that $\lambda^2 \varepsilon$, $\lambda^3 \sqrt{\varepsilon}$ and $\lambda^4 \varepsilon^{\frac{1}{2} m_1} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}}$ are much smaller than $\lambda^2 \varepsilon^{\frac{1}{2} m_1} |\ln \varepsilon|^{\frac{1}{2} \delta_{m,1}}$, this yields the result of Proposition 5.1. \square

6 Positive temperatures

6.1 Positive temperature representation

To describe the reservoir state $\omega_{R, \beta}$ at positive temperature $T = 1/\beta > 0$ one takes the thermodynamic limit of finite volume Gibbs states of the free bose gas, see [BR] or [M1] for example. In this limit, the expectation of a Weyl operator $W(f)$, $f \in L^2(\mathbb{R}^3)$, is calculated to be

$$\omega_{R, \beta}(W(f)) = \exp \left\{ -\frac{1}{4} \int_{\mathbb{R}^3} |f(k)|^2 \coth(\beta \omega(k)/2) d^3 k \right\}. \quad (6.1)$$

This reduces to the value (4.1) for $\beta \rightarrow \infty$. For simplicity, we restrict attention to $\omega(k) = |k|$ in this positive temperature section. Following [AW, JaPi], a Hilbert space supporting the state $\omega_{R, \beta}$ as a rank-one density matrix $|\chi\rangle\langle\chi|$ is given by $\mathcal{F}_+(L^2(\mathbb{R} \times S^2))$, the Fock space over the (new) single particle Hilbert space $L^2(\mathbb{R} \times S^2)$, with χ denoting its vacuum vector. More precisely, we have

$$\omega_{R, \beta}(W(f)) = \langle \chi | W(f_\beta) \chi \rangle, \quad (6.2)$$

where the function $f_\beta \in L^2(\mathbb{R} \times S^2)$ is constructed from $f \in L^2(\mathbb{R}^3)$ by the rule

$$f_\beta(u, \sigma) = \sqrt{\frac{u}{1 - e^{-\beta u}}} |u|^{1/2} \begin{cases} f(u, \sigma) & u \geq 0 \\ -\bar{f}(-u, \sigma) & u < 0 \end{cases}. \quad (6.3)$$

The function f on the right side in (6.3) is represented in polar coordinates $\mathbb{R}^3 \ni k \mapsto (u, \sigma) \in [0, \infty) \times S^2$. In particular, $u = |k|$ for $u \geq 0$. The radial argument of the function f_β on the left side is $u \in \mathbb{R}$.

The operator $W(f_\beta)$ in (6.2) is the *represented Weyl operator* acting on $\mathcal{F}_+(L^2(\mathbb{R} \times S^2))$, given by

$$W(f_\beta) = e^{i\phi(f_\beta)}, \quad \phi(f_\beta) = \frac{1}{\sqrt{2}}(a^*(f_\beta) + a(f_\beta)). \quad (6.4)$$

Here, $a^*(f_\beta)$ and $a(f_\beta)$ are the creation and annihilation operators acting on $\mathcal{F}_+(L^2(\mathbb{R} \times S^2))$, satisfying the canonical commutation relations $[a(f_\beta), a^*(g_\beta)] = \langle f_\beta, g_\beta \rangle_{L^2(\mathbb{R} \times S^2)}$.

We may write $W(f_\beta) = \pi_\beta(W(f))$, where π_β is a $*$ -representation of the Weyl algebra. In particular, due to (3.22),

$$W(f_\beta)W(g_\beta) = \pi_\beta(W(f)W(g)) = e^{-\frac{i}{2}\text{Im}\langle f, g \rangle} \pi_\beta(W(f+g)) = e^{-\frac{i}{2}\text{Im}\langle f, g \rangle} W(f_\beta + g_\beta).$$

On the other hand, the left hand side equals $e^{-\frac{i}{2}\langle f_\beta, g_\beta \rangle} W(f_\beta + g_\beta)$ and it is indeed easy to see directly from the definition (6.3) that

$$\text{Im}\langle f_\beta, g_\beta \rangle = \text{Im}\langle f, g \rangle. \quad (6.5)$$

(The two inner products are in different spaces but it is clear which ones they are.)

We assume here that the radial function $u \mapsto \omega(u)$, originally defined for $u \geq 0$ (namely, $u = |k|$) extends to $u \in \mathbb{R}$ so that $\omega(-u) = -\omega(u)$, the typical example being $\omega(u) = u$. Then it is readily seen from (6.3) that the dynamics $t \mapsto W(e^{i\omega t} f)$ is implemented as

$$t \mapsto W((e^{i\omega t} f)_\beta) = e^{itL_R} W(f_\beta) e^{-itL_R}, \quad (6.6)$$

where the Liouville operator L_R is the second quantization of the operator of multiplication by $\omega(u)$, which can be written as (compare to (2.6))

$$L_R = \int_{\mathbb{R} \times S^2} \omega(u) a^*(u, \sigma) a(u, \sigma) du d^2\sigma. \quad (6.7)$$

Here, $d^2\sigma$ is the uniform measure on S^2 . The operator L_R is the generator implementing the Bogoliubov transformation $a^*(f_\beta) \mapsto a^*(e^{i\omega t} f_\beta)$.

6.2 Positive temperature setup

According to the previous section, the setup for the positive temperature case is obtained from the zero temperature situation by making the following replacements.

- The Hilbert space (2.1) is replaced by

$$\mathcal{H}_{\text{tot}} = \mathbb{C}^2 \otimes \mathcal{F}_+(L^2(\mathbb{R} \times S^2)). \quad (6.8)$$

- The Hamiltonian H_R , (2.6), is replaced by the Liouvillian L_R , (6.7).

- The interaction (2.3) is replaced by the operator

$$H_{\text{int}}(\varepsilon t_p) = \lambda B(\varepsilon t_p) \otimes \phi(g_\beta), \quad (6.9)$$

where $\phi(g_\beta)$ acts on $\mathcal{F}_+(L^2(\mathbb{R} \times S^2))$.

- The initial state (2.7) is replaced by

$$\rho(0) = |\psi_1(0)\rangle\langle\psi_1(0)| \otimes |\chi\rangle\langle\chi|, \quad (6.10)$$

where χ is the vacuum vector in $\mathcal{F}_+(L^2(\mathbb{R} \times S^2))$.

None of the quantities involving the two-level system only are changed (such as $H_S(t)$, $B(t)$, $K(t)$, $W_K(t)$...). The transition probability $p_{1 \rightarrow 2}^{(\lambda, \varepsilon)}(t)$ is still given by the formula (2.10), where the trace is that of the space (6.8), and in which $U_{\lambda, \varepsilon}(t)$ still obeys equation (2.8), simply with $\phi(g)$ replaced by $\phi(g_\beta)$ and H_R by L_R , see also (2.21).

The reservoir autocorrelation function (2.16) now reads

$$\begin{aligned} \gamma^\beta(t) &= 2\omega_{R, \beta}(\phi(e^{i\omega t} g)\phi(g)) = 2\langle\chi|\phi(e^{i\omega t} g_\beta)\phi(g_\beta)\chi\rangle = \langle\chi|a(e^{i\omega t} g_\beta)a^*(g_\beta)\chi\rangle = \langle e^{i\omega t} g_\beta, g_\beta \rangle \\ &= \int_0^\infty \int_{S^2} \frac{u^2}{e^{\beta u} - 1} |g(u, \sigma)|^2 (e^{-i\omega t} e^{\beta u} + e^{i\omega t}) du d^2\sigma. \end{aligned} \quad (6.11)$$

To obtain the last equality we used (6.3). Taking the real and imaginary parts,

$$\begin{aligned} \gamma^\beta(t) &= \gamma_R^\beta(t) + i\gamma_I^\beta(t) \\ \gamma_R^\beta(t) &= \text{Re} \int_0^\infty e^{-i\omega t} \omega^2 \coth(\beta\omega/2) \int_{S^2} |g(\omega, \sigma)|^2 d\omega d^2\sigma \end{aligned} \quad (6.12)$$

$$\gamma_I^\beta(t) = \text{Im} \int_0^\infty e^{-i\omega t} \omega^2 \int_{S^2} |g(\omega, \sigma)|^2 d\omega d^2\sigma. \quad (6.13)$$

The real part depends on β but the imaginary part does not and is the same as for zero temperature. Compare with (2.16), (2.17).

6.3 Proof of Theorem 2.2 (i)

The analysis of Sections 3, 4 and 5 carries through in the positive temperature case, upon making the changes (6.8)-(6.10). This is so because the contribution of the reservoir is dealt with entirely in a representation independent way. For instance, the crucial result of Lemma 3.1 still holds. Indeed, (3.18) is valid with $\phi(g)$ replaced by $\phi(g_\beta)$ and H_R replaced with L_R . The same holds for (3.19). To solve equation (3.19) we use again the commutation relation (3.20) which holds for $F, G \in L^2(\mathbb{R} \times S^2)$ and the ensuing relation (3.21), where now $F(t)$, $\zeta_j(t)$ and $f_j(t)$ are given by (3.16)-(3.17) but with g replaced by g_β . Explicitly, for example, $f_j(t)$ becomes

$$[f_j(t)]_\beta(u) = -\frac{\lambda}{\varepsilon} b_j(t) e^{\frac{i\omega t}{\varepsilon}} g_\beta(u, \sigma) \in L^2(\mathbb{R} \times S^2)$$

Incidentally, $\zeta_j(t)$, (3.16), is independent of β , as follows from (6.5). In the same vein, $\theta_{12}^\pm(s, \tau)$ defined in (3.26) is independent of β (and takes the same value as the zero temperature case).

The main term, $\|\omega_{\lambda,\varepsilon}^{(1)}(t)\|^2$, is then given in (4.2) and the only difference with the zero temperature case is that the real part of η_{12} now depends on β . The expression (4.8) of η_{12} is still valid but now $\gamma_R(t)$ is replaced by $\gamma_R^\beta(t)$, (6.12), (while $\gamma_I(t)$ is replaced by $\gamma_I^\beta(t)$, (6.13), and is the same as for zero temperature).

In terms of the properties of the reservoir, the analysis in Sections 4 and 5 relies entirely on assumption (A.4) (other than the properties $\gamma_R(-t) = \gamma_R(t)$ and $\gamma_I(-t) = -\gamma_I(t)$ which are satisfied for (6.12), (6.13)). So we should now verify that (A.4) holds for non trivial form factors, *i.e.*, that

$$\sup_{t \in \mathbb{R}} (1+t^2)^{\frac{m+1}{2}} |\gamma^\beta(t)| < \infty \quad \text{and} \quad \lim_{\omega \rightarrow 0^+} \frac{\widehat{\gamma}^\beta(\omega)}{\omega^m} \equiv \gamma_0 \geq 0 \quad (6.14)$$

for some $m > 0$. We consider again a radially symmetric g of the form (see (2.19))

$$g(k) = g_0 |k|^{\frac{\mu}{2}-1} \exp\left(-\frac{|k|}{2\omega_D}\right) \quad (6.15)$$

for some $\mu > 0$.

To show that the first condition in (6.14) is satisfied, we let $\ell \in \mathbb{N}$, use $e^{-i\omega t} = \frac{1}{(-it)^\ell} \partial_\omega^\ell e^{-i\omega t}$ and integrate by parts ℓ times to get that

$$|\gamma_R^\beta(t)| \leq \left| \int_0^\infty e^{-i\omega t} \omega^2 \coth(\beta\omega/2) \int_{S^2} |g(\omega, \sigma)|^2 d\omega d^2\sigma \right| \leq \frac{C}{(1+t^2)^{\ell/2}}, \quad (6.16)$$

provided $\mu > \ell$. More precisely, the boundary terms all vanish,

$$\partial_\omega^r \left(\omega^\mu \coth(\beta\omega/2) e^{-\omega/\omega_D} \right) \Big|_0^\infty = 0, \quad r = 0, 1, \dots, \ell - 1, \quad (6.17)$$

and the final integral left over after the integrations by part is absolutely convergent. Note that $\coth(\beta\omega/2)$ has a $1/\omega$ singularity at the origin and is bounded for large ω . The same argument holds to bound $|\gamma_I^\beta(t)|$, replacing the cotangent by 1 in the integral in (6.16) (in fact then, we only need $\mu > \ell - 1$ since the singularity of the cotangent is absent – we may also use the explicit formula (2.20) in this case). We conclude that by choosing $\mu > m + 1$ in (6.15), the first condition in (6.14) is satisfied.

Next we turn to the second condition in (6.14). We note that $\widehat{\gamma}^\beta(\omega) = \widehat{\gamma}_R^\beta(\omega) + i\widehat{\gamma}_I^\beta(\omega)$ with $\widehat{\gamma}_R^\beta$ and $\widehat{\gamma}_I^\beta$ the Fourier transforms of γ_R^β and γ_I^β , respectively. Now, by (6.12),

$$\widehat{\gamma}_R^\beta(\omega) = \int_{\mathbb{R}} e^{i\omega t} \gamma_R^\beta(t) dt = \frac{1}{4\pi} \int_{\mathbb{R}} e^{-i\omega t} \int_0^\infty (e^{-iut} + e^{iut}) \coth(\beta u/2) \widehat{\gamma}(u) du dt, \quad (6.18)$$

where we recall that $\widehat{\gamma}(\omega) = 2\pi\omega^2 \int_{S^2} |g(\omega, \sigma)|^2 d^2\sigma$ for $\omega \geq 0$. Using the representation $\int_{\mathbb{R}} e^{i\xi t} dt = 2\pi\delta(\xi)$ of the Dirac distribution we obtain from (6.18)

$$\widehat{\gamma}_R^\beta(\omega) = \frac{1}{2} \coth(\beta|\omega|/2) \widehat{\gamma}(|\omega|), \quad \omega \in \mathbb{R}. \quad (6.19)$$

We proceed in the same way to find

$$\widehat{\gamma}_I^\beta(\omega) = -\frac{i}{2} \operatorname{sgn}(\omega) \widehat{\gamma}(|\omega|), \quad \omega \in \mathbb{R}, \quad (6.20)$$

where $\text{sgn}(\omega) = 1$ if $\omega > 0$, $\text{sgn}(\omega) = 0$ if $\omega = 0$, and $\text{sgn}(\omega) = -1$ if $\omega < 0$. To satisfy the second condition in (6.14) we thus require $\widehat{\gamma}(\omega)/\omega^{m+1}$ to have a finite limit as $\omega \rightarrow 0+$. In terms of (6.15), it suffices to take $\mu \geq m + 1$.

We conclude that Assumption (A.4) is satisfied in the positive temperature case for form factors (6.15) with $\mu > m + 1 > 1$.

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