Dynamical Localization for Unitary Anderson Models

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Abstract

This paper establishes dynamical localization properties of certain families of unitary random operators on the d-dimensional lattice in various regimes. These operators are generalizations of one-dimensional physical models of quantum transport and draw their name from the analogy with the discrete Anderson model of solid state physics. They consist in a product of a deterministic unitary operator and a random unitary operator. The deterministic operator has a band structure, is absolutely continuous and plays the role of the discrete Laplacian. The random operator is diagonal with elements given by i.i.d. random phases distributed according to some absolutely continuous measure and plays the role of the random potential. In dimension one, these operators belong to the family of CMV-matrices in the theory of orthogonal polynomials on the unit circle.

We implement the method of Aizenman-Molchanov to prove exponential decay of the fractional moments of the Green function for the unitary Anderson model in the following three regimes: In any dimension, throughout the spectrum at large disorder and near the band edges at arbitrary disorder and, in dimension one, throughout the spectrum at arbitrary disorder. We also prove that exponential decay of fractional moments of the Green function implies dynamical localization, which in turn implies spectral localization.

These results complete the analogy with the self-adjoint case where dynamical localization is known to be true in the same three regimes.

1 Introduction

The spectral theory of Schrödinger operators and other selfadjoint operators H used to model hamiltonians of quantum mechanical systems has a long history. It can be argued that on physical grounds the main motivation for studying spectral properties is their close connection (e.g. via the RAGE-Theorem) with dynamical properties of the corresponding time evolution e^{-iHt}, i.e. the propagation of wave packets under the time-dependent Schrödinger equation $i\psi'(t) = H\psi(t)$.

However, the dynamical information following from spectral properties is not very accurate and examples have been found where spectral properties are misleading about the dynamics. In particular, this is the case for operators with singular continuous spectrum or dense point spectrum, spectral types quite common in quantum mechanical models of disordered media such as quasiperiodic or random Schrödinger operators, see for instance [20], [53], [29].

As a consequence, much of the recent work on hamiltonians governing disordered systems has focused on directly establishing dynamical properties. For example, it has been shown that Anderson-type random hamiltonians exhibit *dynamical localization* in various energy regimes, the property that wave packets $\psi(t)$ stay localized in space for all times, see [43], [1], [2], [31], [23], [32], for example.

The central object of interest in understanding dynamics is the unitary group $U(t) = e^{-iHt}$, rather than the hamiltonian H itself. As short time fluctuations will generally not have a major impact on long time dynamics, one may discretize time by choosing a time unit $T > 0$ and study $U(nT) = U^n = e^{-inTH}$ as $n \to \infty$, with the fixed unitary propagator $U = U(T)$.

To further stress the role of the propagator as the central object in studies of dynamics, consider hamiltonians $H(t)$ which depend periodically on time, $H(t+T) = H(t)$ for some $T > 0$ and all t. In this case the large time behavior of solutions of $i\psi'(t) = H(t)\psi(t)$ is governed by the unitary propagator $U(nT, 0) = U^n$, where $U = U(T, 0)$ is commonly referred to as the monodromy operator of the time-periodic system. Note that in this case U does not have a meaningful representation of the form e^{iA} any more. While such representations with selfadjoint operators A exist for abstract reasons, the operator A may have little to do with the time-dependent hamiltonian $H(t)$. Actually, in the periodic or quasi-periodic cases, the operator A is linked to the so called quasienergy or Floquet operator [37], [57], [10], [39].

One consequence of this last fact is that it becomes legitimate to study time periodic systems by directly modeling the monodromy operator U based on physical properties. For example, time dependent analyses of electronic transport in disordered metallic rings have been considered within such a framework, [44], [6], [8], [7], [11]. Kicked systems, often used in the study of quantum chaos, provide another example of such models, see e.g. [18], [21], [22], [24], [26], [12], [45], [48]. Similar studies were performed on the dynamical properties of pulsed systems, given by smooth Floquet operators, in [9], [38], [39], [28].

Our central goal here is to investigate the dynamics of one such model U_ω, which we call the *unitary Anderson model*, indicating the presence of disorder by the random parameter ω. The name is chosen by analogy to the selfadjoint Anderson model, which, in its discrete version on $\ell^2(\mathbb{Z}^d)$, takes the form

$$h_\omega = h_0 + V_\omega. \tag{1.1}$$

The potential in (1.1) is given by real-valued i.i.d. random variables $\{V_\omega(k)\}_{k\in\mathbb{Z}^d}$ and h_0 is a deterministic selfadjoint operator, most commonly the discrete Laplacian on $\mathbb{Z}^d$. By comparison, following our previous works [40, 41, 35], for the unitary Anderson model we choose a unitary operator on $\ell^2(\mathbb{Z}^d)$ of the form

$$U_\omega = D_\omega S. \tag{1.2}$$

Here S is a deterministic unitary operator and D_ω a multiplication operator by random phases, i.e. for every $\phi \in \ell^2(\mathbb{Z}^d)$ and $k \in \mathbb{Z}^d$,

$$(D_\omega \phi)(k) = e^{-i\theta_k^\omega}\phi(k), \tag{1.3}$$

with i.i.d. random phases θ_k^ω taking values in $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Note that, despite the formal analogy, there is no simple relation between selfadjoint and unitary Anderson models. In particular, due to non-commutativity, $e^{-i(h_0+V_\omega)}$ is not a unitary Anderson model and, vice versa, $D_\omega S$ can generally not be written in the form $e^{-i(h_0+V_\omega)}$.

For S we choose what we consider to be a unitary analog of the discrete Laplacian. For $d = 1$ it has a five-diagonal structure which finds its roots in some physics models, see [13]. It turns out that S has the same five-diagonal structure as the so-called CMV-matrices, which have recently found much interest in the theory of orthogonal polynomials on the unit circle, where they arise as unitary analogs of Jacobi matrices, see [49, 50, 52] and references therein. For $d > 1$ we define S as a d-fold tensor product of its one-dimensional version. For details see Section 2. One of the reasons for this choice of S is that one can view the CMV-matrix structure as the simplest non-trivial band structure which a unitary operator on $\ell^2(\mathbb{Z})$ can have, see e.g. [13], similar to the role of the discrete Laplacian among selfadjoint band matrices. Moreover, we choose S such that it will be invariant under translations by multiples of 2, yielding ergodicity of the unitary random operator U_ω.

Monodromy operators of the form (1.2), though not necessarily incorporating Anderson-type randomness, have also been proposed and studied in the physics literature [44], [6], [11] (see also [48]). A mathematical investigation of these models was initiated in [13] and continued in [40, 41, 35, 36, 27]. In particular, spectral localization for the unitary Anderson model was established in [35] for the one-dimensional model and in [41] for arbitrary dimension in the presence of large disorder. Here *spectral localization* refers to the property that U_ω has pure point spectrum for almost every ω.

We will provide proofs of dynamical localization (as formally defined in Section 3 below) for the unitary Anderson model in three different regimes: At arbitrary disorder and throughout the spectrum for the one-dimensional model as well as in the large disorder and band-edge regimes in arbitrary dimension (see Section 3 for a detailed description of these regimes). This coincides with the regimes where localization has been found to hold for selfadjoint Anderson models.

Our approach to localization proofs will be via a unitary version of the fractional moment method, which was initiated as a tool in the theory of selfadjoint Anderson models by Aizenman and Molchanov in [4]. Dynamical localization will follow as a general consequence of exponential decay of spatial correlations in the fractional moments of Green's function (Section 5). To complete the proof of dynamical localization in the three regimes described above, the latter property of Green's function will then be established in those regimes.

In fact, in the large disorder regime this has already been done in [41]. Its proof for the one-dimensional model is one of the main results of the thesis [34], from where we borrow the proofs presented here (Section 7). Some of our general results, in particular the proof that exponential decay of fractional moments of Green's function implies dynamical localization (Section 5) and the proof that fractional moments of Green's function are bounded (Section 4), are also essentially taken from [34].

The hardest, but possibly also most rewarding part of our work, is the proof of exponential decay of fractional moments of Green's function in the band edge regime, which is carried out in Sections 8 to 14. Several preparatory sections are devoted to building up various mathematical tools which do not seem to be known in the context of unitary operators, such as the Feynman-Hellmann theorem from perturbation theory (Section 9) and Combes-Thomas type bounds on eigenfunctions (Section 11). Along the way to localization we establish the spectral theoretic precursor of Lifshits tails of the integrated density of states for the unitary Anderson model (Section 12) as well as a decoupling procedure required in the iterative proof of exponential decay of fractional moments near the edges of the spectrum (Sections 10 and 13).

In Section 6 we also include a proof of the general fact that, in the context of the unitary Anderson model, dynamical localization implies spectral localization, as previously known for selfadjoint Anderson models. In the unitary context, this follows from a version of the RAGE-Theorem provided in [30], whose proof we reproduce here.

As already mentioned, when $d = 1$ and when restricted to $l^2(\mathbb{N})$ our unitary Anderson matrices U_ω bear close resemblance with the CMV matrices in the theory of orthogonal polynomials on the unit circle, see [40]. These polynomials are determined by an infinite set of complex numbers on the unit disc that are called Verblunsky coefficients. Actually, U_ω corresponds to a choice of Verblunsky coefficients characterized by constant moduli r and correlated random phases, see [35] for details. Other choices of random Verblunsky coefficients have been studied in the literature, see e.g. [51], [52], [55] and references therein. We note that for i.i.d. Verblunsky coefficients in the unit disc with rotation invariant distribution, Simon proves dynamical localization in [51]. While not spelled out explicitly, our results for the one dimensional case show that dynamical localization also holds for the CMV matrices considered in [35] with constant moduli Verblunsky coefficients and correlated phases.

Finally, there is an underlying pedagogical goal to our paper: We use the unitary models considered here to give a self-contained presentation of the mathematical theory of Anderson localization via the fractional moment approach. Making use of state-of-the-art techniques from localization theory, we revisit the peculiarities of the one-dimensional case and techniques covering various regimes in the multi-dimensional case within a unitary framework. This requires developing and adapting all necessary background, which we do in a widely self-contained fashion.

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2 The Unitary Anderson Model

As the *unitary Anderson model* we denote a unitary random operator of the form

$$U_\omega = D_\omega S \tag{2.1}$$

in $\ell^2(\mathbb{Z}^d)$. Motivated by earlier investigations in the physics literature, e.g. [11], this model was studied mathematically in [13], [35], [40], [41] and [36], from where we take the following definitions and basic results.

A deterministic unitary operator S on $l^2(\mathbb{Z}^d)$, sometimes referred to as the "free" unitary operator or "unitary Laplacian", is constructed as follows:

Starting with $d = 1$, let B_1 and B_2 be unitary 2×2 matrices given by

$$B_1 = \begin{pmatrix} r & t \\ -t & r \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} r & -t \\ t & r \end{pmatrix}, \tag{2.2}$$

with the real parameters t and r linked by $r^2 + t^2 = 1$ to ensure unitarity. Now let U_e be the unitary matrix operator in $l^2(\mathbb{Z})$ found as the direct sum of identical B_1-blocks with blocks starting at even indices. Similarly, construct U_o with identical B_2-blocks, where blocks start at odd indices. Define $S_0 = U_e U_o$, which will serve as the operator S in (2.1) for dimension $d = 1$. The operator S_0 is unitary, with band structure

$$S_0 = \begin{pmatrix} \ddots & rt & -t^2 & & & \\ & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -tr & r^2 & -rt & \\ & & & rt & r^2 & \\ & & & -t^2 & -tr & \ddots \end{pmatrix}, \tag{2.3}$$

where the position of the origin in $\mathbb{Z}$ is fixed by $\langle e_{2k-2} | S e_{2k} \rangle = -t^2$, with e_k ($k \in \mathbb{Z}$) denoting the canonical basis vectors in $l^2(\mathbb{Z})$. Note also that S_0 is invariant under translations by multiples of 2. Due to elementary unitary equivalences it will suffice to consider $0 \leq t, r \leq 1$. Thus S_0 is determined by t. We shall sometimes write $S_0(t)$ to emphasize this parameter dependence. The spectrum of $S_0(t)$ is given by the arc

$$\sigma(S_0(t)) = \Sigma(t) = \{e^{i\vartheta} : \vartheta \in [-\lambda_0, \lambda_0]\}, \tag{2.4}$$

with $\lambda_0 := \arccos(r^2 - t^2)$. The spectrum is symmetric about the real axis and grows from the single point $\{1\}$ for $t = 0$ to the entire unit circle for $t = 1$. The spectrum is purely absolutely continuous for $t > 0$, e.g. Proposition 6.1 in [13].

To define the multidimensional unitary Laplacian, we follow [41] in viewing $l^2(\mathbb{Z}^d)$ as $\otimes_{j=1}^d l^2(\mathbb{Z})$ so that for all $k \in \mathbb{Z}^d$, $e_k \simeq e_{k_1} \otimes ... \otimes e_{k_d}$. Using $S_0 = S_0(t)$ from above we define $S = S(t)$ by

$$S(t) = \otimes_{j=1}^d S_0(t). \tag{2.5}$$

The spectrum of $S(t)$ is

$$\sigma(S(t)) = \{e^{i\vartheta} : \vartheta \in [-d\lambda_0, d\lambda_0]\}. \tag{2.6}$$

Throughout this paper $|\cdot|$ will denote the maximum norm on $\mathbb{Z}^d$. Using this norm it is easy to see that $S(t)$ inherits the band structure of S_0 such that

$$\langle e_k | S(t) e_l \rangle = 0 \qquad \text{if } |k - l| > 2. \tag{2.7}$$

For the definition of the random phase matrix D_ω in (2.1), introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathbb{T}^{\mathbb{Z}^d}$ ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), $\mathcal{F}$ is the σ-algebra generated by cylinders of Borel sets, and $\mathbb{P} = \bigotimes_{k \in \mathbb{Z}^d} \mu$, where μ is a non trivial probability measure on $\mathbb{T}$. The expectation with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$. We will assume throughout that μ is absolutely continuous with bounded density,

$$d\mu(\theta) = \tau(\theta)\, d\theta, \quad \tau \in L^\infty(\mathbb{T}). \tag{2.8}$$

The random variables θ_k on $(\Omega, \mathcal{F}, \mathbb{P})$ are defined by

$$\theta_k : \Omega \to \mathbb{T}, \quad \theta_k^\omega = \omega_k, \quad k \in \mathbb{Z}^d. \tag{2.9}$$

In other words, the $\{\theta_k^\omega\}_{k \in \mathbb{Z}^d}$ are $\mathbb{T}$-valued i.i.d. random variables with common distribution μ.

The diagonal operator D_ω in $\ell^2(\mathbb{Z}^d)$ is given by

$$D_\omega e_k = e^{-i\theta_k^\omega} e_k. \tag{2.10}$$

With this choice for D_ω and $S = S(t)$ we define the unitary Anderson model U_ω via (2.1).

This definition and the periodicity of S ensures that the operator U_ω is ergodic with respect to the 2-shift in Ω. U_ω also inherits the band structure of the original operator S. The general theory of ergodic operators, as for example presented in Chapter V of [17] for the self-adjoint case, carries over to the unitary setting. In particular, it follows that the spectrum of U_ω is almost surely deterministic, i.e. there is a subset Σ of the unit circle such that $\sigma(U_\omega) = \Sigma$ for almost every ω. The same holds for the absolutely continuous, singular continuous and pure point parts of the spectrum: There are Σ_{ac}, Σ_{sc} and Σ_{pp} such that almost surely $\sigma_{ac}(U_\omega) = \Sigma_{ac}$, $\sigma_{sc}(U_\omega) = \Sigma_{sc}$ and $\sigma_{pp}(U_\omega) = \Sigma_{pp}$. Moreover, we can characterize Σ in terms of the support of μ and of the spectrum of S;

$$\Sigma = \exp(-i \operatorname{supp} \mu)\, \sigma(S) = \{e^{i\alpha} : \alpha \in [-d\lambda_0, d\lambda_0] - \operatorname{supp} \mu\}. \tag{2.11}$$

Here $\operatorname{supp} \mu$ denotes the support of the probability measure μ, defined as

$$\operatorname{supp} \mu := \{a \mid \mu((a - \epsilon, a + \epsilon)) > 0 \text{ for all } \epsilon > 0\}. \tag{2.12}$$

The identity (2.11) is shown in [40] for the one-dimensional model, but the argument carries over to arbitrary dimension.

As $t \to 0$, $S_0(t)$ tends to the identity operator, whereas as $t \to 1$, $S_0(t)$ tends to a direct sum of shift operators. Accordingly, if t is zero then the operators U_ω are diagonal and thus trivially have pure point spectrum. On the other hand, if $t = 1$ then it is not hard to see that all U_ω are purely absolutely continuous (in fact, they are unitarily equivalent to a direct sum of shift operators). Thus, excluding the trivial special cases, we shall from now on assume that $0 < t < 1$. For the unitary Anderson model the parameter t takes the role of a disorder parameter with small t corresponding to large disorder, as this means that U_ω is dominated by its diagonal part.

3 The Results

As discussed in the introduction, our main goal is to study regimes in which a quantum mechanical system governed by the unitary propagator U_ω is dynamically localized. This can be expressed in terms of the transition amplitudes $\langle e_k | U_\omega^n e_l \rangle$, whose squares measure the probability that a system initially in state e_l evolves into state e_k after time n. By *dynamical localization* we will refer to the property that the expectation of these amplitudes stays exponentially small in the distance of k and l, uniformly for all times, i.e. the existence of constants $C < \infty$ and $\alpha > 0$ such that

$$\mathbb{E}(\sup_{n \in \mathbb{Z}} |\langle e_k | U_\omega^n e_l \rangle|) \le C e^{-\alpha |k-l|}. \tag{3.1}$$

In fact, what we will prove in several corresponding regimes is the stronger result that

$$\mathbb{E}(\sup_{f} |\langle e_k | f(U_\omega) e_l \rangle|) \le C e^{-\alpha |k-l|}, \tag{3.2}$$

where the supremum is taken over all functions $f \in C(\mathbb{S})$ with $\|f\|_\infty := \sup_{z \in \mathbb{S}} |f(z)| \le 1$. Here $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

Also, dynamical localization may only hold in an arc $\{e^{i\theta} : \theta \in [a, b]\}$ of the spectrum of U_ω. In this case, the spectral projection $P_{[a,b]}^\omega$ of U_ω onto this arc will be applied to the state e_l in (3.2), restricting the initial state to the localized part of the spectrum.

Our detailed results, stated in the following two subsections, fall into two categories: We start with results which show that dynamical localization can be established by a unitary version of the fractional moments method, using as a criterion the exponential decay of fractional moments of spatial correlations of the Green function. We also show that dynamical localization generally implies spectral localization, i.e. that U_ω almost surely has pure point spectrum in the corresponding part of the spectrum. Finally, we state that dynamical localization implies almost sure finiteness of all quantum moments of the position operator. All these results hold for arbitrary dimension d, for general values of the disorder parameter $t \in (0,1)$, and without restriction on the spectral parameter of the unitary operators U_ω.

Our second set of results concerns the proof of dynamical localization in three concrete regimes: for the one-dimensional unitary Anderson model, as well as for large disorder and at band edges in arbitrary dimensions. In each case this will be done by verifying exponential decay of the fractional moments of the Green function.

3.1 Fractional Moment Criteria for Localization

For $z \in \mathbb{C}$ with $|z| \neq 1$ let

$$G(z) = G_\omega(z) = (U_\omega - z)^{-1}, \tag{3.3}$$

and

$$G(k,l;z) = \langle e_k | G(z) e_l \rangle, \quad k,l \in \mathbb{Z}^d \tag{3.4}$$

be the Green function of U_ω (to use a term from the theory of selfadjoint hamiltonians in the unitary setting).

The Green function becomes singular as z approaches the spectrum of U_ω. The first insight which makes the fractional moment method a useful tool in localization theory is that these singularities are fractionally integrable with respect to the random parameters. This means that for $s \in (0,1)$ the fractional moments $\mathbb{E}(|G(k,l;z)|^s)$ have bounds which are uniform for z arbitrarily close to the spectrum. This is the content of our first result. In fact, we will need a somewhat stronger result later, namely that it suffices to average over the random variables θ_k and θ_l to get uniform bounds on $|G(k,l;z)|^s$. The role of bounds of this type in localization proofs via the fractional moment method roughly corresponds to the use of *Wegner estimates* in the approach to localization by the method of multi scale analysis.

Theorem 3.1. *Assume that the random variables $\{\theta_k\}_{k\in\mathbb{Z}^d}$ are i.i.d. with distribution μ satisfying (2.8). Then for every $s \in (0,1)$ there exists $C(s) < \infty$ such that*

$$\iint |G(k,l;z)|^s d\mu(\theta_k) d\mu(\theta_l) \le C(s) \tag{3.5}$$

for all $z \in \mathbb{C}$, $|z| \neq 1$, all $k,l \in \mathbb{Z}^d$, and arbitrary values of θ_j, $j \notin \{k,l\}$. Consequently,

$$\mathbb{E}(|G(k,l;z)|^s) \le C(s), \tag{3.6}$$

for all $z \in \mathbb{C}$, $|z| \neq 1$.

The second statement simply derives from the bound (3.5), which uniform in the random variables θ_j, $j \notin \{k, l\}$, and the independence of the θ_j. The proof of (3.5) is given in Section 4.

In the next subsection we will identify several situations where the fractional moments $\mathbb{E}(|G(k, l; z)|^s)$ are not just uniformly bounded, but decay exponentially in the distance of k and l. The following general result shows that this can be used as a criterion for dynamical localization of U_ω.

Theorem 3.2. *Assume that the random variables $\{\theta_k\}_{k\in\mathbb{Z}^d}$ satisfy (2.8) and that for some $s \in (0, 1)$, $C < \infty$, $\alpha > 0$, $\varepsilon > 0$ and an interval $[a, b] \in \mathbb{T}$,*

$$\mathbb{E}(|G(k, l; z)|^s) \leq Ce^{-\alpha|k-l|} \tag{3.7}$$

for all $k, l \in \mathbb{Z}^d$ and all $z \in \mathbb{C}$ such that $1 - \varepsilon < |z| < 1$ and $\arg z \in [a, b]$.

Then there exists $\tilde{C}$ such that

$$\mathbb{E}[\sup_{\substack{f\in C(\mathbb{S}) \\ \|f\|_\infty\leq 1}} |\langle e_k|f(U_\omega)P^\omega_{[a,b]}e_l\rangle|] \leq \tilde{C}e^{-\alpha|k-l|/4} \tag{3.8}$$

for all $k, l \in \mathbb{Z}^d$.

Here, as usual $\arg z \in \mathbb{T}$ refers to the polar representation $z = |z| \exp(i \arg z)$ of a complex number. We prove Theorem 3.2 in Section 5.

It has been shown in [41] that fractional moment bounds of the form (3.7) for a unitary Anderson model imply spectral localization via a spectral averaging technique, following an approach to localization which is due to Simon and Wolff [54] for the selfadjoint Anderson model. For completeness, we present a direct proof of the fact that dynamical localization expressed by (3.8) implies spectral localization in the unitary setup. This follows from a simple adaptation of arguments borrowed from Enss-Veselic [30], see also [15], on the geometric characterization of bound states, i.e. a RAGE-type theorem for unitary operators. We prove

Proposition 3.1. *Assume that for an interval $[a, b]$ there exist constants $C < \infty$ and $\alpha > 0$ such that*

$$\mathbb{E}[\sup_{\substack{f\in C(\mathbb{S}) \\ \|f\|_\infty\leq 1}} |\langle e_k|f(U_\omega)P^\omega_{[a,b]}e_l\rangle|] \leq Ce^{-\alpha|k-l|} \tag{3.9}$$

for all $k, l \in \mathbb{Z}^d$. Then

$$(a, b) \cap \Sigma_{cont} = \emptyset, \tag{3.10}$$

where $\Sigma_{cont} = \Sigma_{sc} \cup \Sigma_{ac}$. In other words, almost surely $P^\omega_{[a,b]}U_\omega$ has pure point spectrum.

Another direct consequence of the dynamical localization estimate (3.8), is that it prevents the spreading of wave packets from $P^{\omega}_{[a,b]} l^2(\mathbb{Z}^d)$ under the discrete dynamics generated by U_ω. This dynamical localization property is measured in terms of the boundedness in time of all quantum moments of the position operator on the lattice. More precisely, for $p > 0$ let $|X|_e^p$ be the maximal multiplication operator such that

$$|X|_e^p e_j = |j|_e^p e_j, \quad \text{for } j \in \mathbb{Z}^d, \tag{3.11}$$

where $|j|_e$ denotes the Euclidean norm on $\mathbb{Z}^d$. We have

Proposition 3.2. *Assume that there exist $C < \infty$ and $\alpha > 0$ such that*

$$\mathbb{E}[\sup_{\substack{f \in C(\mathbb{S}) \\ \|f\|_\infty \leq 1}} |\langle e_k | f(U_\omega) P^{\omega}_{[a,b]} e_l \rangle|] \leq C e^{-\alpha |k-l|} \tag{3.12}$$

for all $k, l \in \mathbb{Z}^d$. Then, for any $p \geq 0$ and for any ψ in $l^2(\mathbb{Z}^d)$ of compact support,

$$\sup_{n \in \mathbb{Z}} \| |X|_e^p U_\omega^n P^{\omega}_{[a,b]} \psi \| < \infty \quad \textit{almost surely.} \tag{3.13}$$

Similar results hold under weaker support conditions on ψ. Our choice is made to keep things simple. The proofs of Propositions 3.1 and 3.2 are given in Section 6.

3.2 Localization Regimes

The other main goal of our work is to identify three different regimes, where the fractional moment condition (3.7) can be verified, and thus dynamical localization follows by Theorem 3.2.

3.2.1 Large Disorder

As explained at the end of Section 2, the parameter $t \in (0, 1)$ used in the definition of $S(t)$ and thus, implicitly, also in the definition of U_ω, can be thought of as a measure of the degree of disorder in the unitary Anderson model. Thus one can expect a tendency toward localization for small values of t. This was confirmed in [41] where the following was proven:

Theorem 3.3. *Suppose that the i.i.d. random variables $\{\theta_k\}_{k \in \mathbb{Z}^d}$ have distribution μ satisfying (2.8) and let $s \in (0, 1)$. Then there exists $t_0 > 0$ and $C < \infty$ such that if $t < t_0$, there exists $\alpha > 0$ so that*

$$\mathbb{E}(|\langle e_j | U_\omega (U_\omega - z)^{-1} e_k \rangle|^s) \leq C e^{-\alpha |j-k|} \tag{3.14}$$

for all $j, k \in \mathbb{Z}^d$ and all $z \in \mathbb{C}$ with $|z| \neq 1$.

In fact, [41] considers a more general model in which different parameters t_i are chosen in each factor of (2.5) and shows (3.14) under the condition that $\sum_i t_i$ is sufficiently small.

Using (4.1) below this implies that (3.7) holds for all $|z| \neq 1$. Thus in the large disorder regime $t < t_0$ dynamical localization holds on the entire spectrum of U_ω by Theorem 3.2 (the spectral projection $P^\omega_{[a,b]}$ in (3.8) can be dropped).

3.2.2 The One-Dimensional Model

For the one-dimensional self-adjoint Anderson model, localization holds throughout the spectrum, independent of the amount of disorder. The same is true for the unitary Anderson model, as implied by the following result:

Theorem 3.4. *Let $d = 1$ and suppose that the i.i.d. random variables $\{\theta_k\}_{k\in\mathbb{Z}}$ have distribution μ satisfying (2.8). Then for every $t < 1$ there exist $s > 0$, $C < \infty$ and $\alpha > 0$ such that*

$$\mathbb{E}(|G(k,l;z)|^s) \leq Ce^{-\alpha|k-l|} \tag{3.15}$$

for all $z \in \mathbb{C}$ such that $0 < ||z| - 1| < 1/2$ and all $k, l \in \mathbb{Z}^d$.

By Theorem 3.2 this implies dynamical localization for the one-dimensional unitary Anderson model throughout the spectrum.

Many of the special tools which have been heavily exploited in studies of the one-dimensional self-adjoint Anderson model, are also available for the one-dimensional Unitary model. First of all, there is a transfer-matrix formalism which allows the definition of Lyapunov exponents. In particular, it has been shown in [36] that under assumption (2.8) (in fact, for much more general distributions) the Lyapunov exponent is positive on the entire spectrum and continuous in the spectral parameter. This is the central ingredient into Lemma 7.1 stated in Section 7 below. For a proof of Lemma 7.1, which is a close analog of a result proven for the selfadjoint one-dimensional Anderson model in [16], we will refer to [34]. In Section 7 we will then explain in detail how this leads to (3.15).

3.2.3 Band Edge Localization

For notational simplicity and without restriction we assume for the following result that $\operatorname{supp}\mu \subset [-a, a]$ with $a \in (0, \pi)$ and $-a, a \in \operatorname{supp}\mu$. Furthermore, we assume that

$$a + d\lambda_0 < \pi, \tag{3.16}$$

which by (2.11) guarantees the existence of a gap in the almost sure spectrum Σ of U_ω,

$$\{e^{i\vartheta} : \vartheta \in (d\lambda_0 + a, 2\pi - d\lambda_0 - a)\} \cap \Sigma = \emptyset, \tag{3.17}$$

and that $e^{i(d\lambda_0+a)}$ and $e^{i(2\pi-d\lambda_0-a)}$ are band edges of Σ. Our main result is

Theorem 3.5. *Assume (2.8) and let* $0 < s < 1/3$*. There exist* $\delta > 0$*,* $\alpha > 0$ *and* $C < \infty$ *such that*

$$\mathbb{E}(|G(k,l;z)|^s) \leq Ce^{-\alpha|k-l|} \tag{3.18}$$

for all $k, l \in \mathbb{Z}^d$ *and all* $z \in \mathbb{C}$ *with* $|z| \neq 1$ *and* $\arg z \in [d\lambda_0 + a - \delta, d\lambda_0 + a] \cup [2\pi - d\lambda_0 - a, 2\pi - d\lambda_0 - a + \delta]$*.*

Note that by Theorem 3.2 this implies dynamical localization for U_ω near the edges $d\lambda_0 + a$ and $2\pi - d\lambda_0 - a$ of its almost sure spectrum.

The strategy of the proof of Theorem 3.5 is the following. We control the expectation value of fractional moments of the infinite volume Green function in terms of the expectation value of fractional moments of the Green function of a *finite* volume restriction of the operator. This requires addressing several distinct issues.

The first one is the definition of an appropriate finite volume restriction. We restrict the problem to a finite but large box, by introducing appropriate boundary conditions in Section 8. Our choice of boundary conditions is governed by the fact that we need monotoncity properties which are similar to those of Neumann conditions in the selfadjoint case. The link between the infinite and finite volume resolvents is provided by a geometric resolvent estimate and a decoupling argument, similar to the self-adjoint case. Provided one has good estimates on the expectation value of the fractional moments of the finite volume Green function, this allows to lift such estimates to the fractional moments of the infinite volume resolvent by means of an iteration, for large but fixed size of the box, in a neighborhood of the band edges. This second step is addressed in Section 13.

To get the sought for estimates on the resolvent of the finite volume restriction, in the band edge regime, we need to control the probability that the finite volume restriction of U_ω has eigenvalues close to the band edge. Quantitatively, this amounts to showing that the probability of a small distance, algebraic in the inverse size of the box, between the eigenvalues of this finite volume restriction and the band edges is exponentially small, as the size of the box increases, see Proposition 12.1. This is an expression of the fact that the spectrum close to the band edges is very fluctuating which gives rise to Lifshits tails in the density of states. To prove this, we follow the self-adjoint route, see [56]. We first study the effect of Neumann boundary conditions on the spectrum of S in Section 8, and we make use of a unitary version of the Feynman-Hellmann formula, Proposition 9.1, to control this effect on the spectrum of the random operator U_ω. Then, the Lifshits tail estimate together with a unitary version of the Combes-Thomas estimate, Proposition 11.1, allow to show that the expectation of the moments of the finite volume Green function is exponentially small in a power of the size of the box, Proposition 14.1.

4 Boundedness of Fractional Moments

In this section we prove Theorem 3.1.

As the bound (3.5) is trivial for $|z| < 1/2$, it suffices to consider $|z| \geq 1/2$ and $|z| \neq 1$. Below we prefer to work with the modified resolvent $(U_\omega + z)(U_\omega - z)^{-1}$. Since

$$(U_\omega - z)^{-1} = \frac{1}{2z}[(U_\omega + z)(U_\omega - z)^{-1} - I], \tag{4.1}$$

it easy to see that the existence of $0 < C < \infty$ for which

$$\iint |\langle e_k|(U_\omega + z)(U_\omega - z)^{-1} e_l\rangle|^s \, d\mu(\theta_k) \, d\mu(\theta_l) \leq C, \tag{4.2}$$

for all z with $|z| \neq 1$, all $k, l \in \mathbb{Z}^d$, and uniformly in θ_j, $j \notin \{k, l\}$, gives the required bound.

Key to the proof of (4.2) is knowledge of the exact algebraic dependence of the Green function on the two parameters θ_k and θ_l. Similar formulas for rank one and rank two perturbations of the resolvents of unitary operators have been derived in [49], Section 4.5, from where we took guidance.

We mostly focus on the proof of (4.2) for the case $k \neq l$. At the end of the proof we comment on the simpler case $k = l$, where (4.2) only requires averaging over one parameter.

For $k \neq l$ weborrow an idea from [3] and introduce the change of variables $\alpha = \frac{1}{2}(\theta_k + \theta_l)$, $\beta = \frac{1}{2}(\theta_k - \theta_l)$. This will have the effect of essentially reducing (4.2) to averaging over the single parameter α (although this still corresponds to a rank two perturbation).

Let $A = \{k, l\} \subset \mathbb{Z}^d$ and define

$$\eta_j = \begin{cases} \alpha, & j \in A \\ 0, & j \notin A \end{cases} \tag{4.3}$$

$$\xi_j = \begin{cases} \beta, & j = k \\ -\beta, & j = l \\ 0, & j \notin A \end{cases} \tag{4.4}$$

$$\widehat{\theta}_j^\omega = \begin{cases} 0, & j \in A \\ \theta_j^\omega, & j \notin A \end{cases} \tag{4.5}$$

Next, we define the diagonal operators D_α, D_β and $\widehat{D}$ by

$$D_\alpha e_j = e^{-i\eta_j} e_j, \quad D_\beta e_j = e^{-i\xi_j} e_j, \quad \widehat{D} e_j = e^{-i\hat{\theta}_j} e_j. \tag{4.6}$$

Using these definitions we can write

$$U_\omega = D_\alpha V_\omega, \tag{4.7}$$

with the unitary operator $V_\omega = D_\beta \widehat{D} S$. In what follows we explore the relation between the modified resolvents of U_ω and V_ω.

Let P_A be the orthogonal projection onto the span of $\{V_\omega^{-1} e_j : j \in A\}$. Using that $\{V_\omega^{-1} e_j : j \in \mathbb{Z}^d\}$ is an orthonormal basis of $l^2(\mathbb{Z}^d)$, simple calculations show that $(U_\omega - V_\omega)(I - P_A) = 0$ and $V_\omega^{-1} U_\omega = e^{-i\alpha} I$ on range P_A. In particular, $U_\omega - V_\omega = (U_\omega - V_\omega) P_A$ is a finite rank operator. Therefore,

$$U_\omega = V_\omega (I - P_A) + e^{-i\alpha} V_\omega P_A. \tag{4.8}$$

For $z \in \mathbb{C} \setminus \{0\}$ with $|z| \neq 1$, let $F_z = P_A(U_\omega + z)(U_\omega - z)^{-1} P_A$ while $\widehat{F}_z = P_A(V_\omega + z)(V_\omega - z)^{-1} P_A$, both viewed as operators on the range of P_A (i.e. 2×2-matrices). We see that

$$\widehat{F}_z + \widehat{F}_z^* = P_A(2I - 2|z|^2)(V_\omega - z)^{-1}[(V_\omega - z)^{-1}]^* P_A. \tag{4.9}$$

This shows that $\widehat{F}_z + \widehat{F}_z^*$ is invertible and $\widehat{F}_z + \widehat{F}_z^* < 0$ for $|z| > 1$. Therefore, $-i\widehat{F}_z$ is a dissipative operator, *i.e.* an operator A such that $(A - A^*)/2i > 0$. Similarly, $-i\widehat{F}_z^{-1}$ is also a dissipative operator. In the case $|z| < 1$, we have that $i\widehat{F}_z$, $i\widehat{F}_z^{-1}$ are dissipative.

Next we explore the relation between $\widehat{F}_z$ and F_z. Following Section 4.5 of [49] we use the fact that $(x+z)(x-z)^{-1} = 1 + 2z(x-z)^{-1}$ along with (4.8) to obtain

$$F_z - \widehat{F}_z = -2z P_A (V_\omega - z)^{-1} V_\omega P_A (e^{-i\alpha} - 1) P_A (U_\omega - z)^{-1} P_A. \tag{4.10}$$

Using the definitions of F_z and $\widehat{F}_z$, this can be rewritten in the form

$$F_z - \widehat{F}_z = \frac{1}{2}(1 + \widehat{F}_z)(e^{-i\alpha} - 1)(1 - F_z). \tag{4.11}$$

For $\alpha \notin \{0, \pi\}$, let $m(\alpha) = i\frac{1+e^{-i\alpha}}{1-e^{-i\alpha}} \in \mathbb{R}$. A straightforward calculation shows that

$$F_z = -i(-i\widehat{F}_z + m(\alpha))^{-1} - i(-i\widehat{F}_z^{-1} - m^{-1}(\alpha))^{-1}. \tag{4.12}$$

Note that $\widehat{F}_z$ depends on β and the $\widehat{\theta}_j$, but not on α.

From the definitions of F_z and P_A and the fact that $V_\omega^{-1} U_\omega = e^{-i\alpha} I$ on $\mathrm{span}\{V_\omega^{-1} e_j : j \in \{k, l\}\}$, a simple calculation shows that

$$\langle e_k | (U_\omega + z)(U_\omega - z)^{-1} e_l \rangle \;=\; \langle V_\omega^{-1} e_k | F_z V_\omega^{-1} e_l \rangle. \tag{4.13}$$

Therefore,

$$\begin{aligned} &\iint |\langle e_k | (U_\omega + z)(U_\omega - z)^{-1} e_l \rangle|^s \, d\mu(\theta_k)\, d\mu(\theta_l) \\ &\leq \; ||\tau||_\infty^2 \int_0^{2\pi} \int_0^{2\pi} |\langle V_\omega^{-1} e_k | F_z V_\omega^{-1} e_l \rangle|^s \, d\theta_k \, d\theta_l \\ &\leq \; \|\tau\|_\infty^2 \int_0^{2\pi} \int_0^{2\pi} \|F_z\|^s \, d\theta_k \, d\theta_l \\ &\leq \; 2||\tau||_\infty^2 \int_{-\pi}^{\pi} \left(\int_0^{2\pi} \|F_z\|^s \, d\alpha \right) d\beta, \end{aligned} \tag{4.14}$$

where we have changed to the variables α and β and slightly enlarged the integration domain into the rectangle $0 \leq \alpha \leq 2\pi$, $-\pi \leq \beta \leq \pi$.

We split the α-integral according to (4.12),

$$\int_0^{2\pi} \|F_z\|^s \, d\alpha \leq \int_0^{2\pi} \|(-i\widehat{F}_z + m(\alpha))^{-1}\|^s \, d\alpha + \int_0^{2\pi} \|(-i\widehat{F}_z^{-1} - m^{-1}(\alpha))^{-1}\|^s \, d\alpha. \tag{4.15}$$

Recalling that $m(\alpha)$ has singularities at $\alpha \in \{0, 2\pi\}$, we make the change of variables $x = m(\alpha)$,

$$
\begin{aligned}
&\int_0^{2\pi} ||(-i\widehat{F}_z + m(\alpha))^{-1}||^s d\alpha \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{2}{x^2+1} ||(-i\widehat{F}_z + x)^{-1}||^s \, dx \\
&= 2\sum_{n\in\mathbb{Z}} \int_n^{n+1} \frac{1}{x^2+1} ||(-i\widehat{F}_z + x)^{-1}||^s \, dx \\
&\leq 2\sum_{n\in\mathbb{Z}} \frac{1}{(|n|-1)^2+1} \int_n^{n+1} ||(-i\widehat{F}_z + x)^{-1}||^s \, dx.
\end{aligned}
\tag{4.16}
$$

We can now complete the proof of (4.2) by treating the cases $|z| > 1$ and $|z| < 1$ separately. If $|z| > 1$ then $-i\widehat{F}_z$ is dissipative and Lemma 4.1 shows boundedness of the integral on the right of (4.16), uniform in n, $|z| > 1$, β and $\widehat{\theta}_j$. Thus, after summation, $\int_0^{2\pi} ||(-i\widehat{F}_z + m(\alpha))^{-1}||^s \leq C(s)$. The second term on the right of (4.15) can be bounded in a similar way, using that $-i\widehat{F}_z^{-1}$ is dissipative as well. Inserting these bounds into (4.14) makes the β-integration trivial and completes the proof of (4.2) for the case $|z| > 1$.

If $|z| < 1$, then $-i\widehat{F}_z$ and $-i\widehat{F}_z^{-1}$ and anti-dissipative. As Lemma 4.1 obviously also holds for anti-dissipative matrices A, the proof of (4.2) goes through with the same argument.

The proof of (4.2) for the case $k = l$ is similar but simpler. We don't need a change of variables, but directly work with one of the parameters θ_l instead of α, leading to rank one perturbations. The objects corresponding to F_z and $\widehat{F}_z$ become scalars and we only have to use the trivial scalar version of Lemma 4.1 to conclude.

We finally provide an elementary proof of the following Lemma which was used above. A much more general result of this form is given as Lemma 3.1 in [3].

Lemma 4.1. *For every $s \in (0, 1)$ there exists $C(s) < \infty$ such that*

$$\int_E \|(A + xI)^{-1}\|^s \, dx \leq C(s) \tag{4.17}$$

for every dissipative 2×2-matrix A and every unit interval E.

Proof: First observe that a general dissipative 2×2-matrix A is unitarily equivalent to an upper triangular dissipative matrix (choose as unitary transformation any matrix whose first column is given by a normalized eigenvector of A). Thus we may assume that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \tag{4.18}$$

which implies

$$(A + xI)^{-1} = \begin{pmatrix} \frac{1}{a_{11}+x} & -\frac{a_{12}}{(a_{11}+x)(a_{22}+x)} \\ 0 & \frac{1}{a_{22}+x} \end{pmatrix}. \tag{4.19}$$

The bound (4.17) follows if we can establish a corresponding fractional integral bound for the absolute value of each entry of (4.19) separately. For the diagonal entries this is obvious.

We bound the upper right entry of (4.19) by

$$\begin{aligned} \left|\frac{a_{12}}{(a_{11}+x)(a_{22}+x)}\right| &\leq \frac{|a_{12}|}{|\mathrm{Im}\,((a_{11}+x)(a_{22}+x))|} \\ &= \frac{1}{\left|x\frac{\mathrm{Im}\,a_{11}+\mathrm{Im}\,a_{22}}{|a_{12}|}+\frac{\mathrm{Im}(a_{11}a_{22})}{|a_{12}|}\right|}. \end{aligned} \tag{4.20}$$

The positive matrix

$$\mathrm{Im}\,A = \frac{1}{2i}(A - A^*) = \begin{pmatrix} \mathrm{Im}\,a_{11} & \frac{1}{2i}a_{12} \\ -\frac{1}{2i}\bar{a}_{12} & \mathrm{Im}\,a_{22} \end{pmatrix} \tag{4.21}$$

has positive determinant, i.e. det $\mathrm{Im}\,A = \mathrm{Im}\,a_{11}\mathrm{Im}\,a_{22} - |a_{12}|^2/4$. We thus get

$$\left|\frac{\mathrm{Im}\,a_{11}+\mathrm{Im}\,a_{22}}{a_{12}}\right|^2 \geq \frac{2\mathrm{Im}\,a_{11}\mathrm{Im}\,a_{22}}{|a_{12}|^2} \geq \frac{1}{2}. \tag{4.22}$$

The latter allows to conclude the required integral bound for (4.20). ■

5 Dynamical Localization via Green's Function

Here we will prove Theorem 3.2, i.e. that exponential decay of fractional moments of Green's function implies dynamical localization.

Our proof uses an idea which in the context of selfadjoint Anderson models is due to Graf [33], namely that second moments of an Anderson model's Green function can be bounded in terms of its fractional moments (including, however, a scalar factor which becomes singular as the spectral parameter approaches the spectrum). While the details of the proof are more involved than in the selfadjoint case, we find a bound of this form for unitary Anderson models in Section 5.1.

Another tool we use is the integral formula (5.28), which expresses operator functions $f(U)$ in terms of the resolvent of U. In Section 5.2 we provide a proof of this formula, which combines the spectral theorem for unitary operators with the representation of Borel measures on $\mathbb{T}$ by Poisson integrals.

Equipped with these tools we complete the proof of dynamical localization in Section 5.3.

5.1 A Second Moment Estimate

We start by a bound of second moments of Green's function in terms of its fractional moments, which holds pointwise in the spectral parameter.

Proposition 5.1. *Assume that the* $\{\theta_k^\omega\}_{k\in\mathbb{Z}^d}$ *satisfy (2.8). Then for every* $s \in (0,1)$ *there exists* $C(s) < \infty$ *such that*

$$\mathbb{E}((1-|z|^2)|G(k,l;z)|^2) \leq C(s) \sum_{|m-k|\leq 4} \mathbb{E}(|G(m,l;z)|^s) \tag{5.1}$$

for all $|z| < 1$ *and* $k, l \in \mathbb{Z}^d$.

Proof: Throughout the proof we will assume $z \neq 0$. The bound (5.1) carries over to $z = 0$ by continuity.

For $\delta \in \mathbb{T}$, let $\eta_k = e^{-i(\theta_k^\omega+\delta)} - e^{-i\theta_k^\omega}$. Then define $D_\omega^\delta = D_\omega + \eta_k P_k$, where P_k is the orthogonal projection into the span of e_k. Let $U_\omega^\delta = D_\omega^\delta S$. Using the resolvent identity we have

$$(U_\omega^\delta - z)^{-1} - (U_\omega - z)^{-1} = -(U_\omega^\delta - z)^{-1}\eta_k P_k S(U_\omega - z)^{-1}, \tag{5.2}$$

for all $z \in \mathbb{C}$ such that $0 < |z| < 1$. Letting $F(z) = S(U_\omega - z)^{-1}$ and $F_\delta(z) = S(U_\omega^\delta - z)^{-1}$, the last equation takes the form

$$F_\delta(z) - F(z) = -F_\delta(z)\eta_k P_k F(z). \tag{5.3}$$

Denoting $F(i,j,z) = \langle e_i | F(z) e_j\rangle$ it is easy to see that

$$F_\delta(z) - F(z) = -\eta_k \frac{F(z)P_kF(z)}{1+\eta_k F(k,k,z)}. \tag{5.4}$$

Therefore, for all $l \in \mathbb{Z}^d$

$$F_\delta(k,l,z) = \frac{F(k,l,z)}{1+\eta_k F(k,k,z)}. \tag{5.5}$$

On the other hand, we also have that

$$\begin{aligned} |F_\delta(k,l,z)|^2 &\leq \sum_{y\in\mathbb{Z}^d} |F_\delta(k,y,z)|^2 \\ &= \langle e_k | S(U_\omega^\delta - z)^{-1}[(U_\omega^\delta - z)^{-1}]^* S^* e_k\rangle. \end{aligned} \tag{5.6}$$

Since U_ω^δ is a unitary operator, the following identity holds

$$[(U_\omega^\delta - z)^{-1}]^* = \frac{-1}{\bar{z}}(U_\omega^\delta - \frac{1}{\bar{z}})^{-1} U_\omega^\delta. \tag{5.7}$$

Thus, it follows that

$$\begin{aligned} |F_\delta(k,l,z)|^2 &\leq \frac{-1}{\bar{z}}\langle e_k | S(U_\omega^\delta - z)^{-1}(U_\omega^\delta - \frac{1}{\bar{z}})^{-1} D_\omega^\delta e_k\rangle \\ &= \frac{-e^{-i(\theta_k^\omega+\delta)}}{\bar{z}}\langle e_k | S(U_\omega^\delta - z)^{-1}(U_\omega^\delta - \frac{1}{\bar{z}})^{-1} e_k\rangle. \end{aligned} \tag{5.8}$$

Again using the resolvent identity, we see that

$$(U_\omega^\delta - z)^{-1}(U_\omega^\delta - \frac{1}{\bar{z}})^{-1} = \frac{\bar{z}}{|z|^2-1}\{(U_\omega^\delta - z)^{-1} - (U_\omega^\delta - \frac{1}{\bar{z}})^{-1}\}. \tag{5.9}$$

Hence,

$$|F_\delta(k,l,z)|^2 \le \frac{e^{-i(\theta_k^\omega+\delta)}}{1-|z|^2}\{\langle e_k|S(U_\omega^\delta - z)^{-1}e_k\rangle - \langle e_k|S(U_\omega^\delta - \frac{1}{\bar{z}})^{-1}e_k\rangle\}. \tag{5.10}$$

From (5.7), the definition of U_ω^δ and the fact that $(U_\omega^\delta - z)^{-1} = -\frac{1}{z}[I - U_\omega^\delta(U_\omega^\delta - z)^{-1}]$, it follows that

$$\begin{aligned}\langle e_k|S(U_\omega^\delta - \frac{1}{\bar{z}})^{-1}e_k\rangle &= -\bar{z}e^{i(\theta_k^\omega+\delta)}\langle e_k|[(U_\omega^\delta - z)^{-1}]^*e_k\rangle \\ &= -\bar{z}e^{i(\theta_k^\omega+\delta)}\{1 - \langle U_\omega^\delta(U_\omega^\delta - z)^{-1}e_k|e_k\rangle\} \\ &= e^{i(\theta_k^\omega+\delta)}\{1 - e^{i(\theta_k^\omega+\delta)}\overline{F_\delta(k,k,z)}\}.\end{aligned} \tag{5.11}$$

Therefore, we now obtain that

$$\begin{aligned}|F_\delta(k,l,z)|^2 &\le \frac{1}{1-|z|^2}\{2\Re[e^{-i(\theta_k^\omega+\delta)}F_\delta(k,k,z)] - 1\} \\ &= \frac{1}{1-|z|^2}\{|F_\delta(k,k,z)|^2 - |e^{i(\theta_k^\omega+\delta)} - F_\delta(k,k,z)|^2\},\end{aligned} \tag{5.12}$$

since $|x-y|^2 = |x|^2 + |y|^2 - 2\Re[\bar{x}y]$. Using (5.5), to rewrite $F_\delta(k,k,z)$ in terms of elements of F, along with the definition of η_k we get

$$|F_\delta(k,l,z)|^2 \le \frac{1}{1-|z|^2}\{\frac{|F(k,k,z)|^2 - |e^{i\theta_k^\omega} - F(k,k,z)|^2}{|1+\eta_k F(k,k,z)|^2}\}. \tag{5.13}$$

This inequality gives, in particular, that $F(k,k,z) \neq 0$. Therefore

$$(1-|z|^2)|F_\delta(k,l,z)|^2 \le \frac{1 - |1 - e^{i\theta_k}F(k,k,z)^{-1}|^2}{|\eta_k + F(k,k,z)^{-1}|^2}. \tag{5.14}$$

Finally note that the last inequality allows us to conclude that $|1 - e^{i\theta_k}F(k,k,z)^{-1}| \le 1$.

One can also use the fact that

$$|F_\delta(k,k,z)| \le ||(U_\omega^\delta - z)^{-1}| \le \frac{1}{1-|z|}, \tag{5.15}$$

for all $\delta \in \mathbb{T}$, to get a different upper bound on $(1-|z|^2)|F_\delta(k,l,z)|^2$. Since (5.5) can be rewritten as

$$F_\delta(k,l,z) = \frac{1}{\eta_k + F(k,k,z)^{-1}}\frac{F(k,l,z)}{F(k,k,z)}, \tag{5.16}$$

it follows that $1 - |\eta_k + F(k,k,z)^{-1}| \le |z|$. Then by choosing δ such that $e^{-i\delta} = \frac{1-e^{i\theta_k}F(k,k,z)^{-1}}{|1-e^{i\theta_k}F(k,k,z)^{-1}|}$, we see that

$$|1 - e^{i\theta_k}F(k,k,z)^{-1}| \le |z|. \tag{5.17}$$

Using this along with (5.16) we obtain the following upper bound

$$(1-|z|^2)|F_\delta(k,l,z)|^2 \leq \frac{1-|1-e^{i\theta_k}F(k,k,z)^{-1}|^2}{|\eta_k+F(k,k,z)^{-1}|^2}\frac{|F(k,l,z)|^2}{|F(k,k,z)|^2}. \tag{5.18}$$

Combining the two estimates (5.14) and (5.18) and using that for $0<s<1$ we have $\min(1,|x|^2)\leq |x|^s$, it follows that

$$(1-|z|^2)|F_\delta(k,l,z)|^2 \leq \frac{1-|1-e^{i\theta_k}F(k,k,z)^{-1}|^2}{|e^{-i\delta}-(1-e^{i\theta_k}F(k,k,z)^{-1})|^2}\frac{|F(k,l,z)|^s}{|F(k,k,z)|^s}. \tag{5.19}$$

Letting $y=1-e^{i\theta_k}F(k,k,z)^{-1}$, this can be rewritten as

$$(1-|z|^2)|F_\delta(k,l,z)|^2 \leq \frac{(1-|y|^2)|1-y|^s}{|e^{-i\delta}-y|^2}|F(k,l,z)|^s. \tag{5.20}$$

Since the expectations of F and F_δ are related by

$$\mathbb{E}[|F(k,l,z)|^2] = \mathbb{E}[\int d\mu(\theta_k+\delta)|F_\delta(k,l,z)|^2], \tag{5.21}$$

it follows that

$$\begin{aligned}
&\mathbb{E}[(1-|z|^2)|F(k,l,z)|^2] \\
&\quad\leq\ ||\tau||_\infty \mathbb{E}\Big[|F(k,l,z)|^s \sup_{\{y\in\mathbb{C}:|y|<1\}}\int_0^{2\pi} d\delta \frac{(1-|y|^2)|1-y|^s}{|e^{-i\delta}-y|^2}\Big] \\
&\quad\leq\ 2^s||\tau||_\infty \mathbb{E}\Big[|F(k,l,z)|^s \sup_{\{y\in\mathbb{C}:|y|<1\}}\int_0^{2\pi} d\delta \frac{(1-|y|^2)}{|e^{-i\delta}-y|^2}\Big].
\end{aligned} \tag{5.22}$$

Next we evaluate the integral

$$\begin{aligned}
\int_0^{2\pi} d\delta \frac{(1-|y|^2)}{|e^{-i\delta}-y|^2} &= \int_0^{2\pi} d\delta \Re[\frac{e^{-i\delta}+y}{e^{-i\delta}-y}] \\
&= \Re\int_0^{2\pi} d\delta[\frac{2e^{-i\delta}}{e^{-i\delta}-y}-1].
\end{aligned} \tag{5.23}$$

The latter integral can be easily evaluated using Cauchy integral formula, by simply substituting $z=e^{-i\delta}$ and gives

$$\int_0^{2\pi} d\delta \frac{(1-|y|^2)}{|e^{-i\delta}-y|^2} = 2\pi. \tag{5.24}$$

Therefore,

$$\mathbb{E}[(1-|z|^2)|F(k,l,z)|^2] \leq 2^{s+1}\pi||\tau||_\infty \mathbb{E}[|F(k,l,z)|^s]. \tag{5.25}$$

Since S is a unitary operator with band structure, we see that for $s\in(0,1)$

$$\begin{aligned}
\mathbb{E}[|F(k,l,z)|^s] &= \mathbb{E}[|\langle S^* e_k|(U_\omega-z)^{-1}e_l\rangle|^s] \\
&\leq C_1(s)\sum_{|m-k|\leq 2}\mathbb{E}[|\langle e_m|(U_\omega-z)^{-1}e_l\rangle|^s].
\end{aligned} \tag{5.26}$$

Finally, in order to get the required bound of the second moment of elements of $(U_\omega - z)^{-1}$ we use again that S is a unitary operator with band structure. Therefore, for $k, l \in \mathbb{Z}^d$

$$\begin{aligned} \mathbb{E}[(1-|z|^2)|\langle e_k|(U_\omega - z)^{-1} e_l\rangle|^2] &= \mathbb{E}[(1-|z|^2)|\langle Se_k|S(U_\omega - z)^{-1} e_l\rangle|^2] \\ &\leq C_2(s) \sum_{|m-k|\leq 2} \mathbb{E}[(1-|z|^2)|F(m,l,z)|^2]. \end{aligned} \quad (5.27)$$

Combining (5.27), (5.25) and (5.26) gives (5.1). ∎

5.2 An Integral Representation for $f(U)$

We will reduce bounds for $f(U)$ to bounds on resolvents by the formula

$$f(U) = w - \lim_{r\to 1^-} \frac{1-r^2}{2\pi} \int_0^{2\pi} (U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} f(e^{i\theta}) d\theta \quad (5.28)$$

for $f \in C(\mathbb{S})$ and U a unitary operator. This is a simple consequence of the representation of Borel measures on $\mathbb{T}$ by Poisson integrals:

Let $\varphi \in \mathcal{H}$ be normalized and consider the non negative spectral measure $d\mu_\varphi$ on $\mathbb{T}$ such that

$$\langle \varphi | U\varphi\rangle = \int_{\mathbb{T}} e^{i\alpha} d\langle \varphi | E(\alpha)\varphi\rangle = \int_{\mathbb{T}} e^{i\alpha} d\mu_\varphi(\alpha), \quad (5.29)$$

where $E(\cdot)$ denotes the spectral family of U. We can thus rewrite for $0 \leq r < 1$

$$\begin{aligned} (1-r^2)\langle \varphi|(U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1}\varphi\rangle &= \int_{\mathbb{T}} \frac{1-r^2}{|e^{i\alpha} - re^{i\theta}|^2} d\mu_\varphi(\alpha) \\ = \int_{\mathbb{T}} \frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)} d\mu_\varphi(\alpha) &= u(r,\theta), \end{aligned} \quad (5.30)$$

which coincides with the Poisson integral of the measure $d\mu_\varphi$. As a function of $z = x+iy = re^{i\theta}$, with the standard abuses of notations, the RHS of (5.30) is non negative and harmonic in the open unit disc ([47] Sect. 11.17), but it is not bounded as $r \to 1^-$ at the atoms of $d\mu_\varphi$. Now, Theorem 3.9.8 in [46] on the representation of non negative Borel measures on $\mathbb{T}$ says that for any $f \in C(\mathbb{S})$ we have

$$\lim_{r\to 1^-} \int_0^{2\pi} u(r,\theta) f(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{\mathbb{T}} f(e^{i\alpha}) d\mu_\varphi(\alpha) \equiv \langle \varphi | f(U)\varphi\rangle. \quad (5.31)$$

Considering the matrix element $\langle \varphi | \cdot \psi\rangle$ for arbitrary φ and ψ in $\mathcal{H}$, we get the equivalent result by polarization, which proves (5.28).

If $P_{[a,b]}$ denotes the spectral projector of U on the arc $[a,b] \subset \mathbb{T}$ and $\varphi \in \mathcal{H}$ is normalized, then

$$d\langle \varphi | P_{[a,b]} E(\alpha)\varphi\rangle \equiv d\mu_\varphi^{[a,b]}(\alpha) = d\mu_\varphi(\alpha)|_{[a,b]}. \quad (5.32)$$

By the same argument with $P_{[a,b]}\varphi$ in place of φ, i.e. with $d\mu_\varphi^{[a,b]}$ in place of $d\mu_\varphi$, we have for any $f \in C(\mathbb{S})$

$$
\begin{aligned}
P_{[a,b]}f(U) &= w - \lim_{r\to 1^-} \frac{1-r^2}{2\pi}\int_a^b (U-re^{i\theta})^{-1}(U^{-1}-re^{-i\theta})^{-1}f(e^{i\theta})d\theta \\
&\equiv w - \lim_{r\to 1^-} f_r(U).
\end{aligned} \tag{5.33}
$$

5.3 Conclusion of the Proof of Theorem 3.2

Let $f \in C(\mathbb{S})$ such that $\|f\|_\infty \le 1$ and consider

$$
f_r(U) = \frac{1-r^2}{2\pi}\int_a^b (U-re^{i\theta})^{-1}(U^{-1}-re^{-i\theta})^{-1}f(e^{i\theta})d\theta. \tag{5.34}
$$

We compute

$$
\begin{aligned}
&|\langle e_k|f_r(U)e_j\rangle| \\
&\le \frac{1-r^2}{2\pi}\int_a^b |\langle e_k|(U-re^{i\theta})^{-1}(U^{-1}-re^{-i\theta})^{-1}e_j\rangle||f(e^{i\theta})|d\theta \\
&\le \frac{1-r^2}{2\pi}\int_a^b \sum_{l\in\mathbb{Z}^d} |\langle e_k|(U-re^{i\theta})^{-1}e_l\rangle||\langle e_j|(U-re^{i\theta})^{-1}e_l\rangle|d\theta,
\end{aligned} \tag{5.35}
$$

which is independent of f. Hence, using Fatou's Lemma and Fubini's Theorem and taking the supremum over all $f \in C(\mathbb{S})$ such that $\|f\|_\infty \le 1$,

$$
\begin{aligned}
&\mathbb{E}[\sup_f |\langle e_k|P_{[a,b]}f(U_\omega)e_j\rangle|] \\
&= \mathbb{E}[\sup_f \lim_{r\to 1^-} |\langle e_k|f_r(U)e_j\rangle|] \\
&\le \mathbb{E}[\liminf_{r\to 1^-} \sup_{\|f\|_\infty\le 1} |\langle e_k|f_r(U)e_j\rangle|] \\
&\le \liminf_{r\to 1^-} \frac{1-r^2}{2\pi}\int_a^b \sum_{l\in\mathbb{Z}^d} \mathbb{E}[|\langle e_k|(U-re^{i\theta})^{-1}e_l\rangle||\langle e_j|(U-re^{i\theta})^{-1}e_l\rangle|]|d\theta.
\end{aligned} \tag{5.36}
$$

By the Cauchy-Schwarz inequality and the second moment bound (5.1), we have

$$
\begin{aligned}
&\mathbb{E}[|(1-r^2)\langle e_k|(U-re^{i\theta})^{-1}e_l\rangle||\langle e_j|(U-re^{i\theta})^{-1}e_l\rangle|] \\
&\le (\mathbb{E}[|(1-r^2)\langle e_k|(U-re^{i\theta})^{-1}e_l\rangle|^2])^{1/2}(\mathbb{E}[|(1-r^2)\langle e_j|(U-re^{i\theta})^{-1}e_l\rangle|^2])^{1/2} \\
&\le C_1\left(\sum_{m:|m-k|\le 4}\mathbb{E}(G(m,l;re^{i\theta})|^s)\right)^{1/2}\left(\sum_{m:|m-j|\le 4}\mathbb{E}(G(m,l;re^{i\theta})|^s)\right)^{1/2} \\
&\le C_2 e^{-\alpha|k-l|/2}e^{-\alpha|j-l|/2},
\end{aligned} \tag{5.37}
$$

where ultimately the assumption (3.7) on exponential decay of fractional moments of Green's function was used. Since there exists a finite c such that

$$
\sum_{l\in\mathbb{Z}^d} e^{-\alpha(|k-l|+|j-l|)/2} \le ce^{-\alpha|k-j|/4} \tag{5.38}
$$

we finally get

$$\mathbb{E}[\sup_f |\langle e_k | P_{[a,b]} f(U_\omega) e_j \rangle|] \leq \tilde{C} e^{-\alpha|k-j|/4}, \tag{5.39}$$

with $\tilde{C} = c\, C_2$. This completes the proof of Theorem 3.2.

6 Dynamical Localization implies Spectral Localization

We start by recalling the geometrical concepts and results introduced in [30] in a general framework.

Let U be a unitary operator in a separable Hilbert space $\mathcal{H}$ and let $\mathcal{H}_{pp}(U)$, respectively $\mathcal{H}_c(U)$ denote the pure point, respectively continuous, spectral subspace of U. We will denote the spectral resolution of U by E_U.

A practical characterization of the vectors of $\mathcal{H}_{pp}(U)$ and $\mathcal{H}_c(U)$ in terms of their behaviour under the discrete dynamics with respect to families of finite dimensional subspaces of $\mathcal{H}$ is provided in [30] and reads as follows:

Let $P = \{P_r\}_{r\geq 0}$ be a family of projections on $\mathcal{H}$ such that

$$P_r = P_r^2 = P_r^*, \;\; \text{Rank } P_r < \infty, \;\; \text{and } s - \lim_{r\to\infty} P_r = I. \tag{6.1}$$

We define the set of bounded trajectories with respect to the family P by

$$\mathcal{M}_U^b(P) = \{\psi \in \mathcal{H} \mid \lim_{r\to\infty} \sup_{n\in\mathbb{Z}} \|(I - P_r)U^n\psi\| = 0\}. \tag{6.2}$$

Similarly, the set of propagating trajectories with respect to P is defined by

$$\mathcal{M}_U^p(P) = \{\psi \in \mathcal{H} \mid \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \|P_r U^n \psi\| = 0, \ \forall r \geq 0\}, \tag{6.3}$$

Both $\mathcal{M}_U^b(P)$ and $\mathcal{M}_U^p(P)$ are easily seen to be closed subspaces of $\mathcal{H}$.

Then the following holds.

Lemma 6.1. *For any unitary operator U on a separable Hilbert space $\mathcal{H}$, and any family P as in (6.1),*

$$\mathcal{M}_U^b(P) = \mathcal{H}_{pp}(U) \quad \textit{and} \quad \mathcal{M}_U^p(P) = \mathcal{H}_c(U) \tag{6.4}$$

While the definition above suffices for our purpose, more general families of operators than P can be considered, see [30]. For completeness, we provide a detailed proof of this Lemma, which is a slight adaptation of the argument in [30].

Proof of Lemma 6.1 in three steps:

First, $\mathcal{H}_{pp}(U) \subset \mathcal{M}_U^b(P)$:

If $U\varphi = e^{i\alpha}\varphi$, then

$$\|(I-P_r)U^n\varphi\| = \|(I-P_r)\varphi\| \to 0, \quad \text{uniformly in } n \text{ as } r \to \infty. \tag{6.5}$$

We conclude using that $\mathcal{H}_{pp}(U) = \overline{\{\varphi \mid U\varphi = e^{i\alpha}\varphi\}}$ and the fact that $\mathcal{M}_U^b(P)$ is closed.

Second, $\mathcal{M}_U^b(P) \perp \mathcal{M}_U^p(P)$:

Let $\varphi \in \mathcal{M}_U^p(P)$ and $\psi \in \mathcal{M}_U^b(P)$. Then

$$\begin{aligned}\langle\varphi|\psi\rangle &= \frac{1}{2N+1}\sum_{n=-N}^{N}\langle\varphi|\psi\rangle = \frac{1}{2N+1}\sum_{n=-N}^{N}\langle U^n\varphi|U^n\psi\rangle \\ &= \frac{1}{2N+1}\sum_{n=-N}^{N}\langle U^n\varphi|(I-P_r)U^n\psi\rangle + \langle U^n\varphi|P_rU^n\psi\rangle. \end{aligned}\tag{6.6}$$

By our choice of ψ, for any $\epsilon > 0$ there exists $r(\epsilon)$, uniform in n, such that $|\langle U^n\varphi|(I - P_{r(\epsilon)})U^n\psi\rangle| < \epsilon$. And our choice of φ and the selfadjointness of $P_{r(\epsilon)}$ imply

$$\left|\frac{1}{2N+1}\sum_{n=-N}^{N}\langle U^n\varphi|P_{r(\epsilon)}U^n\psi\rangle\right| \leq \frac{1}{2N+1}\sum_{n=-N}^{N}\|P_{r(\epsilon)}U^n\varphi\|\|\psi\| \to 0 \text{ as } N \to \infty, \tag{6.7}$$

which shows that $\langle\varphi|\psi\rangle = 0$.

Third, $\mathcal{H}_c(U) \subset \mathcal{M}_U^p(P)$:

Let $\psi \in \mathcal{H}_c(U) = P_c(U)\mathcal{H}$, where $P_c(U)$ is the spectral projector of U on the continuous spectral subspace of U. Since the orthogonal projector P_r is finite dimensional for any r, there exists vectors $\varphi_i \in \text{Ran } P_r$ such that we can write for some $m < \infty$

$$P_r = \sum_{i=1}^{m}|\varphi_i\rangle\langle\varphi_i|, \quad \text{where} \quad \langle\varphi_i|\varphi_j\rangle = \delta_{ij}. \tag{6.8}$$

Hence, $\|P_rU^n\psi\|^2 = \sum_{i=1}^m |\langle\varphi_i|U^n\psi\rangle|^2$. Now, by the Spectral Theorem we have

$$f_{\varphi_i,\psi}(n) := \langle\varphi_i|U^n\psi\rangle = \int_{\mathbb{T}} e^{ixn}d\mu_{\varphi_i,\psi}(x), \tag{6.9}$$

where $d\mu_{\varphi,\psi}(x) = d\langle P_c(U)\varphi_i|E_U(x)P_c(U)\psi\rangle$ is a continuous complex valued measure. Thus, by Wiener's Theorem,

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|f_{\varphi_i,\psi}(n)|^2 = \sum_{x\in\mathbb{T}}|\mu_{\varphi_i,\psi}(\{x\})|^2 = 0. \tag{6.10}$$

Consequently, making use of Cauchy-Schwarz,

$$\begin{aligned}\frac{1}{2N+1}\sum_{n=-N}^{N}\|P_rU^n\psi\| &\leq \left\{\frac{1}{2N+1}\sum_{n=-N}^{N}\|P_rU^n\psi\|^2\right\}^{1/2} \\ &= \left\{\sum_{i=1}^{m}\frac{1}{2N+1}\sum_{n=-N}^{N}|f_{\varphi_i,\psi}(n)|^2\right\}^{1/2}\end{aligned}\tag{6.11}$$

which tends to zero as $N \to 0$.

Finally the decomposition $\mathcal{H} = \mathcal{H}_{pp}(U) \oplus \mathcal{H}_c(U)$ leads to the conclusion. ∎

Proof of Proposition 3.1

Coming back to the random situation at hand, we construct a suitable family of projectors $P = \{P_r\}_{r\geq 0}$ by means of

$$P_r = \sum_{\substack{k\in\mathbb{Z}^d \\ |k|\leq r}} |e_k\rangle\langle e_k|, \quad r \geq 0, \tag{6.12}$$

and consider the vectors of the form $P^{\omega}_{[a,b]} e_j$, $j \in \mathbb{Z}^d$. We have for all $n \in \mathbb{Z}$

$$\|(I - P_r)U^n_\omega P^\omega_{[a,b]} e_j\| = \{\sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|^2\}^{1/2}. \tag{6.13}$$

Therefore, since $|\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle| \leq 1$,

$$\begin{aligned}\sup_n \|(I - P_r)U^n_\omega P^\omega_{[a,b]} e_j\| &= \{\sup_n \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|^2\}^{1/2} \\ &\leq \{\sup_n \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|\}^{1/2}. \end{aligned} \tag{6.14}$$

Thus, by Fatou's Lemma and Cauchy-Schwarz inequality,

$$\begin{aligned}&\mathbb{E}(\lim_{r\to\infty} \sup_n \|(I - P_r)U^n_\omega P^\omega_{[a,b]} e_j\|) \\ &\quad\leq \liminf_{r\to\infty} \mathbb{E}(\{\sup_n \sum_{k\in\mathbb{Z}^d |k|>r} |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|\}^{1/2}) \\ &\quad\leq \liminf_{r\to\infty} \{\mathbb{E}(\sup_n \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|)\}^{1/2}. \end{aligned} \tag{6.15}$$

Now, by Fubini's Theorem,

$$\begin{aligned}\mathbb{E}(\sup_n \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|) &\leq \mathbb{E}(\sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} \sup_n |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|) \\ &= \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} \mathbb{E}(\sup_n |\langle e_k| P^\omega_{[a,b]} U^n_\omega e_j\rangle|) \\ &\leq \tilde{C} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} e^{-\alpha|k-j|/4} \leq \tilde{C} e^{\alpha|j|/4} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>r}} e^{-\alpha|k|/4}, \end{aligned} \tag{6.16}$$

where the right hand side decays exponentially fast to zero as $r \to \infty$. As a consequence,

$$\mathbb{E}(\lim_{r\to\infty} \sup_n \|(I - P_r)U_\omega^n P^\omega_{[a,b]} e_j\|) = 0, \tag{6.17}$$

so that there exists a set $\Omega_j \subset \Omega$ such that $\mathbb{P}(\Omega_j) = 1$ and for all $\omega \in \Omega_j$

$$\lim_{r\to\infty} \sup_n \|(I - P_r)U_\omega^n P^\omega_{[a,b]} e_j\| = 0. \tag{6.18}$$

In other words, $P^\omega_{[a,b]} e_j \in \mathcal{H}_{pp}(U_\omega), \quad \forall \omega \in \Omega_j$. Then, $\tilde{\Omega} = \cap_{j\in\mathbb{Z}^d}\Omega_j$ is a set of probability one such that for all $\omega \in \tilde{\Omega}$, $P^\omega_{[a,b]} e_j \in \mathcal{H}_{pp}(U_\omega), \quad \forall j \in \mathbb{Z}^d$. Hence,

$$P^\omega_{[a,b]} l^2(\mathbb{Z}^d) \subset \mathcal{H}_{pp}(U_\omega), \quad \text{a.s.} \tag{6.19}$$

■

Proof of Proposition 3.2
By assumption, $\psi = \sum_k \psi_k e_k$ satisfies $\psi_k = 0$ if $|k| > R$, for some $R > 0$. Hence, by Cauchy-Schwarz

$$\begin{aligned}
\||X|_e^p U_\omega^n P^\omega_{[a,b]}\psi\|^2 &= \sum_j |\langle e_j||X|_e^p U_\omega^n P^\omega_{[a,b]}\psi\rangle|^2 \\
&= \sum_j |j|_e^{2p} \left|\sum_k \langle e_j|U_\omega^n P^\omega_{[a,b]} e_k\rangle\psi_k\right|^2 \\
&\leq \sum_j \sum_{|k|\leq R} |j|_e^{2p} |\langle e_j|U_\omega^n P^\omega_{[a,b]} e_k\rangle|^2 \|\psi\|^2 \\
&\leq \sum_j \sum_{|k|\leq R} |j|_e^{2p} |\langle e_j|U_\omega^n P^\omega_{[a,b]} e_k\rangle| \|\psi\|^2,
\end{aligned} \tag{6.20}$$

since $|\langle e_j|U_\omega^n P^\omega_{[a,b]} e_k\rangle| \leq 1$. By the same steps as those performed in the previous proof, one gets that

$$\mathbb{E}(\sup_{n\in\mathbb{Z}^d} \||X|_e^p U_\omega^n P^\omega_{[a,b]}\psi\|) < \infty \quad \text{if} \quad \sum_j \sum_{|k|\leq R} |j|_e^{2p} \mathbb{E}(\sup_n |\langle e_j|U_\omega^n P^\omega_{[a,b]} e_k\rangle|) < \infty. \tag{6.21}$$

That the latter sum is finite follows from (3.8), which ends the proof. ■

7 One-Dimensional Localization

In this section we prove Theorem 3.4.

7.1 Basic Properties of the One-Dimensional Model

The proof of Theorem 3.4 uses a number of tools, which are specific to the one-dimensional model, where the operators studied here can be considered as unitary analogs of Jacobi matrices. We start this section by briefly presenting the properties which we will need. While we have a somewhat different point of view, much of this is related or equivalent to facts on CMV-matrices, which can be found throughout [49, 50].

Due to the specific band structure of U_ω, the generalized eigenvectors can be studied using complex 2×2-transfer matrices. Specifically, generalized eigenvectors are solutions (not necessarily in l^2) of the eigenvalue equation

$$U_\omega \psi = z\psi, \quad \psi = \sum_{k\in\mathbb{Z}} \psi_k e_k, \tag{7.1}$$

for $z \in \mathbb{C}\backslash\{0\}$ and characterized by the relations

$$\begin{pmatrix} \psi_{2k+1} \\ \psi_{2k+2} \end{pmatrix} = T_z(\theta^\omega_{2k}, \theta^\omega_{2k+1}) \begin{pmatrix} \psi_{2k-1} \\ \psi_{2k} \end{pmatrix}, \tag{7.2}$$

for all $k \in \mathbb{Z}$. Here the transfer matrices $T_z : \mathbb{T}^2 \to GL(2, \mathbb{C})$ are defined by

$$T_z(\theta, \eta) = \begin{pmatrix} -\dfrac{e^{-i\eta}}{z} & \dfrac{r}{t}\left(e^{i(\theta-\eta)} - \dfrac{e^{-i\eta}}{z}\right) \\ \dfrac{r}{t}\left(1 - \dfrac{e^{-i\eta}}{z}\right) & -\dfrac{ze^{i\theta}}{t^2} + \dfrac{r^2}{t^2}\left(1 + e^{i(\theta-\eta)} - \dfrac{e^{-i\eta}}{z}\right) \end{pmatrix}. \tag{7.3}$$

Note that $\det T_z(\theta^\omega_{2k}, \theta^\omega_{2k+1}) = e^{i(\theta^\omega_{2k} - \theta^\omega_{2k+1})}$ has modulus one and is independent of z. We have for any $n \in \mathbb{N}$

$$\begin{pmatrix} \psi_{2n-1} \\ \psi_{2n} \end{pmatrix} = T_z(\theta^\omega_{2(n-1)}, \theta^\omega_{2(n-1)+1}) \cdots T_z(\theta^\omega_0, \theta^\omega_1) \begin{pmatrix} \psi_{-1} \\ \psi_0 \end{pmatrix} \equiv T_z(\omega, n) \begin{pmatrix} \psi_{-1} \\ \psi_0 \end{pmatrix}, \tag{7.4}$$

$$\begin{pmatrix} \psi_{-2n-1} \\ \psi_{-2n} \end{pmatrix} = T_z(\theta^\omega_{-2n}, \theta^\omega_{-2n+1})^{-1} \cdots T_z(\theta^\omega_{-2}, \theta^\omega_{-1})^{-1} \begin{pmatrix} \psi_{-1} \\ \psi_0 \end{pmatrix} \equiv T_z(\omega, -n) \begin{pmatrix} \psi_{-1} \\ \psi_0 \end{pmatrix}. \tag{7.5}$$

We also set $T_z(\omega, 0) = I$.

The transfer matrix formalism allows to introduce the Lyapunov exponent $\gamma(z)$;

$$\gamma(z) = \lim_{n\to\infty} \frac{\mathbb{E}(\ln ||T_z(\omega, n)||)}{n}. \tag{7.6}$$

Positivity and continuity of the Lyapunov exponent for all values of z under the current assumptions was proven in [36]. A consequence of these properties of $\gamma(z)$ is the following unitary version of Lemma 5.1 of [16].

Lemma 7.1. *Assume that the random variables $\{\theta_k\}_{k\in\mathbb{Z}^d}$ satisfy (2.8), then for each compact subset Λ of $\mathbb{C}$ there exist $\alpha = \alpha(\Lambda) > 0$ and $0 < \delta = \delta(\Lambda) < 1$ and $C = C(\Lambda) < \infty$ such that*

$$\mathbb{E}[||T_z(\omega, n)v||^{-\delta}] \leq Ce^{-\alpha n} \tag{7.7}$$

for all $z \in \Lambda$, $n \geq 0$ and unit vector $v \in \mathbb{C}^2$.

We omit the proof of this lemma which is very similiar to the one given for the self-adjoint Anderson model in [16], see Appendix A in [34] for details.

We will frequently work with restrictions of the infinite matrices U_ω to discrete finite intervals $[a, b]$ or half-lines $[a, \infty)$ and $(-\infty, b]$, respectively. Here we slightly abuse notation and write, for example, $[a, b]$ for $[a, b] \cap \mathbb{Z}$. To guarantee that the restrictions remain unitary we have to choose suitable boundary conditions. At each finite endpoint these boundary conditions can be labeled by a parameter in $\mathbb{T}$.

For $a \in \mathbb{Z}$ and $\eta \in \mathbb{T}$, the unitary operator $S_\eta^{[a,\infty)}$ on $l^2([a, \infty))$ is constructed as follows: If $a = 2n$ is even, let the 2×2 matrices B_1 and B_2 be defined as in (2.2), then let $U_e^{[2n,\infty)}$ be the unitary operator in $l^2([2n, \infty))$ found as the direct sum of identical B_1-blocks with blocks starting at $2n$. On the other hand construct $U_o^{[2n,\infty)}$ starting with a single 1×1 block $e^{i\eta}$, then identical B_2-blocks starting at $2n + 1$. Now let $S_\eta^{[2n,\infty)} = U_e^{[2n,\infty)} U_o^{[2n,\infty)}$. The operator $S_\eta^{[2n,\infty)}$ on $l^2([2n, \infty))$ will have a band structure

$$S_\eta^{[2n,\infty)} = \begin{pmatrix} re^{i\eta} & rt & -t^2 & & & \\ -te^{i\eta} & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -tr & r^2 & -rt & \\ & & & rt & r^2 & \\ & & & -t^2 & -tr & \ddots \end{pmatrix}. \tag{7.8}$$

The parameter η can be thought of as a boundary condition at $2n$.

Similarly, $U_e^{(-\infty,2n+1]}$ is found as the direct sum of identical B_1-blocks with blocks starting at even indices, while $U_o^{(-\infty,2n+1]}$ has identical B_2-blocks starting at odd indices, with $(U_o^{(-\infty,2n+1]})(2n + 1, 2n + 1) = e^{i\eta}$. Thus,

$$S_\eta^{(-\infty,2n+1]} = \begin{pmatrix} \ddots & rt & -t^2 & & & \\ & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -rt & r^2 & -rt & \\ & & & rt & r^2 & te^{i\eta} \\ & & & -t^2 & -rt & re^{i\eta} \end{pmatrix}. \tag{7.9}$$

To define $S_\eta^{(-\infty,2n]}$ and $S_\eta^{[2n+1,\infty)}$ we slightly modify this construction, this time filling up $U_e^{[2n+1,\infty)}$ and $U_e^{(-\infty,2n]}$ with a 1×1-block $e^{i\eta}$, respectively, yielding

$$S_\eta^{(-\infty,2n]} = \begin{pmatrix} \ddots & rt & -t^2 & & \\ & r^2 & -rt & & \\ & rt & r^2 & rt & -t^2 \\ & -t^2 & -rt & r^2 & -rt \\ & & & te^{i\eta} & re^{i\eta} \end{pmatrix}, \tag{7.10}$$

while

$$S_\eta^{[2n+1,\infty)} = \begin{pmatrix} re^{i\eta} & -te^{i\eta} & & & \\ rt & r^2 & rt & -t^2 & \\ -t^2 & -rt & r^2 & -rt & \\ & & rt & r^2 & \\ & & -t^2 & -rt & \ddots \end{pmatrix}. \tag{7.11}$$

In similar fashion, for integers $-\infty < a < b < \infty$ we can construct the unitary operator $S_{\eta_a,\eta_b}^{[a,b]}$ with η_a boundary condition at a and η_b boundary condition at b, for example we have

$$S_{\eta_{2n},\eta_{2m}}^{[2n,2m]} = \begin{pmatrix} re^{i\eta_{2n}} & rt & -t^2 & & & \\ -te^{i\eta_{2n}} & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -rt & r^2 & -rt & \\ & & & & \ddots & \\ & & & & te^{i\eta_{2m}} & re^{i\eta_{2m}} \end{pmatrix}. \tag{7.12}$$

Finally, we define

$$U_{\omega,\eta_a,\eta_b}^{[a,b]} = D_\omega^{[a,b]} S_{\eta_a,\eta_b}^{[a,b]}, \tag{7.13}$$

where the diagonal operator $D_\omega^{[a,b]}$ on $l^2([a,b])$ is defined as in (2.10). Similarly, we define $U_{\omega,\eta}^{[a,\infty)}$ and $U_{\omega,\eta}^{(-\infty,b]}$.

As before, the generalized eigenvectors of $U_{\omega,\eta_a,\eta_b}^{[a,b]}$, $U_{\omega,\eta}^{[a,\infty)}$ and $U_{\omega,\eta}^{(-\infty,b]}$ are characterized by the relations (7.2) are supplemented with appropriate relations to reflect the boundary conditions, see [34] for details. In the proof of Theorem 3.4 we only use $\eta = 0$, so for the rest of this section we write $U_{\omega,0,0}^{[a,b]} = U^{[a,b]}$, $U_{\omega,0}^{[a,\infty)} = U^{[a,\infty)}$ and $U_{\omega,0}^{(-\infty,b]} = U^{(-\infty,b]}$ for simplicity, also frequently leaving the ω-dependence implicit. Other boundary conditions will be used later, see Section 8.

In the following discussion we use the notation $U^{[a,b]}$ for general $-\infty \le a < b \le \infty$, i.e. we write $U^{[-\infty,\infty]}$, $U^{[a,\infty]}$ and $U^{[-\infty,b]}$ for U, $U^{[a,\infty)}$ and $U^{(-\infty,b]}$.

Another feature of the one-dimensional model is that Green's function $G(k,l;z)$ can be expressed in terms of two generalized eigenfunctions to z which, separately at each endpoint a, b, are square-summable or satisfy the boundary condition at the endpoint.

For a solution φ of $(U-z)\varphi = 0$, we define $\widetilde{\varphi}$ by

$$\begin{pmatrix} \widetilde{\varphi}_{2n} \\ \widetilde{\varphi}_{2n+1} \end{pmatrix} = \begin{pmatrix} t^2 & rt \\ rt & r^2 - ze^{i\theta_{2n}} \end{pmatrix} \begin{pmatrix} \varphi_{2n-1} \\ \varphi_{2n} \end{pmatrix}. \tag{7.14}$$

A straightforward calculation shows that $\widetilde{\varphi}$ is characterized by the relations

$$\begin{pmatrix} \widetilde{\varphi}_{2k} \\ \widetilde{\varphi}_{2k+1} \end{pmatrix} = \widetilde{T_z}(\theta_{2k-1},\theta_{2k}) \begin{pmatrix} \widetilde{\varphi}_{2k-2} \\ \widetilde{\varphi}_{2k-1} \end{pmatrix}, \tag{7.15}$$

for all $k \in \mathbb{Z}$. Here the transfer matrices $\widetilde{T_z} : \mathbb{T}^2 \to GL(2, \mathbb{C})$ are defined by

$$\widetilde{T_z}(\theta, \eta) = T_z^t(\eta, \theta), \tag{7.16}$$

where $T_z(\eta, \theta)$ is given by (7.3) and T^t denotes the transpose of T.

For $-\infty \leq a < b \leq \infty$ let

$$G^{[a,b]}(z) = (U^{[a,b]} - z)^{-1}. \tag{7.17}$$

To any z not in the spectrum of $U^{[a,b]}$ choose generalized eigenvectors φ^a and φ^b as follows:

If a is finite, then φ^a is the unique solution of $(U^{[a,\infty)} - z)\varphi^a = 0$ with $\varphi^a(a) = 1$, i.e. a generalized eigenvector to z which satisfies the boundary condition at a. If $a = -\infty$, then for φ^a we choose a non-trivial solution of $(U - z)\varphi^a = 0$, which is square-summable at $-\infty$. In the latter case φ^a is determined up to a constant (one can construct if from the tail of $(U - z)^{-1}e_0$ at $-\infty$ and the fact that the transfer matrices (7.3) have determinant of modulus one shows that there can't be two linearly independent solutions which are square-summable at $-\infty$). Similar we choose φ^b with prescribed boundary behavior at b.

The following proposition gives an expression of the elements of $G^{[a,b]}(z)$ in terms of φ^a and φ^b and the corresponding $\widetilde{\varphi}^a$, $\widetilde{\varphi}^b$ defined as in (7.14).

Proposition 7.1. *For all finite k, l with $a \leq k, l \leq b$, if $l = 2n$ or $l = 2n + 1$,*

$$G^{[a,b]}(k, l; z) = \begin{cases} c_l \widetilde{\varphi}_l^b \varphi_k^a & \text{if } k < l \text{ or } k = l \text{ are even}, \\ c_l \widetilde{\varphi}_l^a \varphi_k^b & \text{if } k > l \text{ or } k = l \text{ are odd}, \end{cases} \tag{7.18}$$

where $c_l = \dfrac{e^{i\theta_l}}{\widetilde{\varphi}_{2n+1}^a \widetilde{\varphi}_{2n}^b - \widetilde{\varphi}_{2n}^a \widetilde{\varphi}_{2n+1}^b}$.

Proof: A straightforward, if rather tedious, calculation shows that the matrix whose entries are given by the right hand side of (7.18) is indeed the inverse of $U_\omega^{[a,b]} - z$, see [34] for details. ■

We conclude this section by proving the following lemma that is used later in the proof of Theorem 3.4.

Lemma 7.2. *For $z \in \mathbb{C}$ with $0 < ||z| - 1| < 1/2$, $a \in \mathbb{Z}$ and $s \in (0, 1)$ there exists $0 < C_\mu(t, s) < \infty$ such that*

$$\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^a + (r - ze^{i\theta_{2m}})\varphi_{2m}^a|^s} \leq C_\mu(t, s) \left\| \begin{pmatrix} \varphi_{2m-1}^a \\ \varphi_{2m}^a \end{pmatrix} \right\|^{-s}, \tag{7.19}$$

for all $m \geq a + 2$.

Proof: First note that both φ_{2m-1}^a and φ_{2m}^a are independent of θ_{2m} and can not vanish simultaneously. This follows from the fact that the transfer matrices needed to construct them via (7.2) from $\varphi^a(a) = 1$ only contain $\theta_a, \ldots, \theta_{2m-1}$.

Therefore we have the following cases:

Case 1: $\varphi^a_{2m} = 0$, using that $\left\| \begin{pmatrix} \varphi^a_{2m-1} \\ \varphi^a_{2m} \end{pmatrix} \right\| \leq 2|\varphi^a_{2m-1}| + 2|\varphi^a_{2m}|$, the bound follows directly.

Case 2: $\varphi^a_{2m} \neq 0$. In this case

$$\begin{aligned}
&\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi^a_{2m-1} + (r - ze^{i\theta_{2m}})\varphi^a_{2m}|^s} \\
&= \quad \frac{1}{|\varphi^a_{2m}|^s} \int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\frac{\varphi^a_{2m-1}}{\varphi^a_{2m}} + (r - ze^{i\theta_{2m}})|^s}. \qquad (7.20)
\end{aligned}$$

Let $M = \sup\{|r - ze^{i\theta_{2m}}| : \theta \in [0, 2\pi], 0 < \big||z| - 1\big| < 1/2\} < \infty$ and distinguish between two subcases;

Case 2a: If $t\left|\frac{\varphi^a_{2m-1}}{\varphi^a_{2m}}\right| > 2M$, it follows that

$$\left|t\frac{\varphi^a_{2m-1}}{\varphi^a_{2m}} + (r - ze^{i\theta_{2m}})\right| > \frac{t}{2}\left|\frac{\varphi^a_{2m-1}}{\varphi^a_{2m}}\right|. \qquad (7.21)$$

On the other hand

$$\left\| \begin{pmatrix} \varphi^a_{2m-1} \\ \varphi^a_{2m} \end{pmatrix} \right\|^s \leq (2 + \frac{1}{M})^s |\varphi^a_{2m-1}|^s. \qquad (7.22)$$

Using these estimates we conclude that

$$\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi^a_{2m-1} + (r - ze^{i\theta_{2m}})\varphi^a_{2m}|^s} \leq \frac{2^{s+1}\pi||\tau||_\infty}{t^s}(2+\frac{1}{M})^s \left\| \begin{pmatrix} \varphi^a_{2m-1} \\ \varphi^a_{2m} \end{pmatrix} \right\|^{-s}. \qquad (7.23)$$

Case 2b: Assume that $t\left|\frac{\varphi^a_{2m-1}}{\varphi^a_{2m}}\right| \leq 2M$. For all $0 < s < 1$ there exists $0 < C_\mu(s) < \infty$ such that for all $\beta \in \mathbb{C}$

$$\int_{\mathbb{T}} d\mu(\theta) \frac{1}{|(e^{\pm i\theta} - \beta)|^s} \leq C_\mu(s), \qquad (7.24)$$

see e.g. [41]. Using this we get the bound

$$\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi^a_{2m-1} + (r - ze^{i\theta_{2m}})\varphi^a_{2m}|^s} \leq \frac{2^s C^{(1)}_\mu(s)}{|\varphi^a_{2m}|^s}. \qquad (7.25)$$

However, under the current assumption we have

$$\left\| \begin{pmatrix} \varphi^a_{2m-1} \\ \varphi^a_{2m} \end{pmatrix} \right\|^s \leq (4\frac{M}{t} + 2)^s |\varphi^a_{2m}|^s, \qquad (7.26)$$

which gives the required bound. ∎

7.2 Proof of Theorem 3.4

We now turn to the proof of Theorem 3.4. The proof is done in three steps: first we prove that it suffices to deal with even matrix elements of Green's function, i.e. that we only need to show (3.15) for the case that $k = 2n$, $l = 2m$. This mainly serves the purpose of avoiding to cover four separate sub-cases. Then we show that proving the bound (3.15) for an element $(2n, 2m)$ of $(U_\omega - z)^{-1}$ can be reduced to proving the same bound for the element $(2n, 2m)$ of the resolvent of the finite volume operator $U_\omega^{[2n,2m]}$ at z. Finally, we prove that expectation of fractional moments of $G^{[2n,2m]}(2n, 2m; z)$ decays exponentially. Based on the Green's function formula (7.18) this will be found by combining Lemma 7.2 with Lemma 7.1.

Step One: The following Lemma shows that the expectation of a fractional moment of any element $G(k, l; z)$ can be reduced to the expectation of fractional moments of even matrix elements of $G(z)$. This comes at the cost of enlarging the fractional exponent s due to the use of Hölder's inequality.

Lemma 7.3. *Let $s \in (0, 1/4)$ and $k, l \in \mathbb{Z}$ such that $|k - l| > 4$. Choose $n, m \in \mathbb{Z}$ such that $k \in \{2n, 2n + 1\}$, $l \in \{2m, 2m + 1\}$. There exists $\kappa(t, s, \mu) < \infty$ such that*

$$\mathbb{E}[|G(k, l; z)|^s] \leq \kappa(t, s, \mu) \sum_{i,j=0}^{1} \left(\mathbb{E}[|G(2n + 2i, 2m - 2j; z)|^{4s}]\right)^{1/4}, \tag{7.27}$$

for all $z \in \mathbb{C}$ with $0 < ||z| - 1| < 1/2$.

Proof: Using the definition of n, m and that $|k - l| > 4$ one has $|n - m| \geq 2$. Since $\widetilde{\varphi}^{\pm\infty}$, defined by (7.14), satisfies (7.15) and (7.16), a straightforward calculation shows that

$$\begin{aligned} \widetilde{\varphi}^{\pm\infty}_{2m+1} &= \frac{1}{rt(e^{i\theta^\omega_{2m-1}} - 1/z)} \big\{[r^2(e^{i\theta^\omega_{2m}} + e^{i\theta^\omega_{2m-1}} - \frac{1}{z}) \\ &\quad -ze^{i(\theta^\omega_{2m}+\theta^\omega_{2m-1})}]\widetilde{\varphi}^{\pm\infty}_{2m} - t^2 e^{i\theta^\omega_{2m}} \widetilde{\varphi}^{\pm\infty}_{2m-2}\big\}. \end{aligned} \tag{7.28}$$

Using Theorem 7.1 along with (7.15) it follows that for $k \notin \{2m - 1, 2m\}$

$$\begin{aligned} G(k, 2m + 1; z) &= \frac{e^{i\theta^\omega_{2m+1}}}{rt(e^{i\theta^\omega_{2m-1}} - 1/z)} \big\{[r^2(1 + e^{-i(\theta^\omega_{2m}-\theta^\omega_{2m-1})} - \frac{e^{-i\theta^\omega_{2m}}}{z}) - ze^{i\theta^\omega_{2m-1}}] \\ &\quad G(k, 2m; z) - t^2 e^{i(\theta^\omega_{2m-1}-\theta^\omega_{2m-2})} G(k, 2m - 2; z)\big\}. \end{aligned} \tag{7.29}$$

By Hölder's inequality and (7.24) it follows that for $s \in (0, 1/4)$ there exists $0 < C^{(1)}_\mu(s, r) < \infty$ such that

$$\begin{aligned} &\mathbb{E}[|G(k, 2m + 1; z)|^s] \\ &\quad \leq C^{(1)}_\mu(s, r) \left((\mathbb{E}[|G(k, 2m; z)|^{2s}])^{1/2} + (\mathbb{E}[|G(k, 2m - 2; z)|^{2s}])^{1/2}\right). \end{aligned} \tag{7.30}$$

Similarly, using (7.2) and (7.3) we obtain

$$\varphi^{\pm\infty}_{2n+1} = \frac{-t}{r(e^{i\theta^\omega_{2n+1}} - 1/z)} \big\{\frac{1}{z}\varphi^{\pm\infty}_{2n+2} + e^{i\theta^\omega_{2n}} \varphi^{\pm\infty}_{2n}\big\}. \tag{7.31}$$

Thus, for $l \notin \{2n, 2n+1\}$, $s \in (0, 1/2)$ and all $z \in \mathbb{C}$ with $0 < ||z| - 1| < 1/2$,

$$\begin{aligned} &\mathbb{E}[|G(2n+1, l; z)|^s] \\ &\quad \leq \ C_\mu^{(2)}(s, r) \left((\mathbb{E}[|G(2n+2, l; z)|^{2s}])^{1/2} + (\mathbb{E}[|G(2n, l; z)|^{2s}])^{1/2} \right). \end{aligned} \tag{7.32}$$

It readily follows from (7.30) and (7.32) that for $|n - m| \notin \{0, 1\}$ and all $s \in (0, 1/4)$

$$\mathbb{E}[|G(2n+1, 2m+1; z)|^s] \leq \tilde{\kappa}_\mu^{(1)}(s, r) \sum_{i,j=0}^{1} \left(\mathbb{E}[|G(2n+2i, 2m-2j; z)|^{4s}] \right)^{1/4}. \tag{7.33}$$

This proves the Lemma for the case $k = 2n+1$, $l = 2m+1$. The other cases are more direct. ■

Step Two: Let $\Big|[k, l]\Big|$ denote the interval $\big[\min\{k, l\}, \max\{k, l\}\big]$. In what follows we show that the expectation of fractional moments of $G(2n, 2m; z)$ can be reduced to that of $G^{\left|[2n,2m]\right|}(2n, 2m; z)$.

Lemma 7.4. *For $s \in (0, 1/3)$ and $n, m \in \mathbb{Z}$ with $|n - m| \geq 2$, we have*

$$\mathbb{E}[|G(2n, 2m; z)|^s] \leq C_\mu(t, s) (\mathbb{E}[|G^{|[2n,2m]|}(2n, 2m; z)|^{3s}])^{1/3}, \tag{7.34}$$

for all $z \in \mathbb{C}$ with $0 < ||z| - 1| < 1/2$.

Proof: For definiteness, assume that $m \geq n+2$, the case $n \geq m+2$ being similar. Using the definition of $U_\omega^{[x,y]}$ (7.13), we see that

$$U_\omega = U_\omega^{(-\infty, 2n-1]} \oplus U_\omega^{[2n, \infty)} + \Gamma_n^e, \tag{7.35}$$

where Γ_n^e is given by

$$\Gamma_n^e(k, l) = \begin{cases} (rt - t)e^{-i\theta_{2n-2}}, & k = 2n-2, l = 2n-1 \\ -t^2 e^{-i\theta_{2n-2}}, & k = 2n-2, l = 2n \\ (r^2 - r)e^{-i\theta_{2n-1}}, & k = 2n-1, l = 2n-1 \\ -rte^{-i\theta_{2n-1}}, & k = 2n-1, l = 2n \\ rte^{-i\theta_{2n}}, & k = 2n, l = 2n-1 \\ (r^2 - r)e^{-i\theta_{2n}}, & k = 2n, l = 2n \\ -t^2 e^{-i\theta_{2n+1}}, & k = 2n+1, l = 2n-1 \\ (-rt + t)e^{-i\theta_{2n+1}}, & k = 2n+1, l = 2n \\ 0, & \text{otherwise.} \end{cases} \tag{7.36}$$

Denote $G^n(z) = G^{(-\infty, 2n-1]}(z) \oplus G^{[2n, \infty)}(z)$. By the first resolvent identity, we have

$$G(z) - G^n(z) = -G(z)\Gamma_n^e G^n(z). \tag{7.37}$$

Therefore, it follows for all $m \geq n+2$ that

$$\begin{aligned} G(2n,2m;z) &= \{1 + t^2 e^{-i\theta_{2n-2}} G(2n,2n-2;z) + rte^{-i\theta_{2n-1}} G(2n,2n-1;z) \\ &\quad -(r^2-r)e^{-i\theta_{2n}} G(2n,2n;z) - (t-rt)e^{-i\theta_{2n+1}} G(2n,2n+1;z)\} \\ &\quad G^{[2n,\infty)}(2n,2m;z). \end{aligned} \tag{7.38}$$

A similar application of the first resolvent identity, this time to the difference $G^{[2n,\infty)}(z) - G^{[2n,2m]}(z) \oplus G^{[2m+1,\infty)}(z)$, allows to express $G^{[2n,\infty)}(2n,2m;z)$ in terms of $G^{[2n,2m]}(2n,2m;z)$ and ultimately leads to

$$\begin{aligned} G(2n,2m;z) &= \Big\{1 + t^2 e^{-i\theta_{2n-2}} G(2n,2n-2;z) + rte^{-i\theta_{2n-1}} G(2n,2n-1;z) \\ &\quad -(r^2-r)e^{-i\theta_{2n}} G(2n,2n;z) - (t-rt)e^{-i\theta_{2n+1}} G(2n,2n+1;z)\Big\} \\ &\quad \times\Big\{1 - e^{-i\theta_{2m}}[(rt-t)G^{[2n,\infty)}(2m-1,2m;z) + \\ &\quad (r^2-r)G^{[2n,\infty)}(2m,2m;z) + rtG^{[2n,\infty)}(2m+1,2m;z) \\ &\quad -t^2 G^{[2n,\infty)}(2m+2,2m;z)]\Big\} G^{[2n,2m]}(2n,2m;z). \end{aligned} \tag{7.39}$$

If A and B denote the two $\{\cdot\}$-factors on the right hand side of (7.39), then it follows from $s < 1/2$ and Theorem 3.1 that

$$\mathbb{E}(|A|^{3s}) \leq C, \quad \mathbb{E}(|B|^{3s}) \leq C \tag{7.40}$$

uniformly in $|z| \neq 1$. Here we are using that Theorem 3.1 remains true with identical proof for the Green function of $U_\omega^{[2n,\infty)}$. An application of Hölder's inequality to (7.39) yields

$$\mathbb{E}[|G(2n,2m;z)|^s] \leq (\mathbb{E}[|A|^{3s}])^{1/3} (\mathbb{E}[|B|^{3s}])^{1/3} \left(\mathbb{E}[|G^{[2n,2m]}(2n,2m;z)|^{3s}]\right)^{1/3}, \tag{7.41}$$

which gives (7.34) when combined with (7.40). ■

Step Three: Now that we have reduced the problem to dealing with fractional moments of elements of the form $G_z^{|[2n,2m]|}(2n,2m)$, we show that expectations of such objects decay exponentially, in particular we show;

Lemma 7.5. *Assume that $\{\theta_k^\omega\}_{k\in\mathbb{Z}}$ are i.i.d. with probability measure $d\mu(\theta) = \tau(\theta)d\theta$, where $\tau \in L^\infty(\mathbb{T})$. There exist $s_0 \in (0,1)$, $0 < C_1 < \infty$, $\alpha_1 > 0$ such that*

$$\mathbb{E}[|G^{|[2n,2m]|}(2n,2m;z)|^{s_0}] \leq C_1 e^{-\alpha_1 |m-n|}, \tag{7.42}$$

for all $z \in \mathbb{C}$ with $0 < ||z|-1| < 1/2$, and all $m,n \in \mathbb{Z}$ such that $|m-n| \geq 2$.

Proof: For $m \geq n+2$, let φ^{2n} and φ^{2m} be two solutions that satisfy the boundary conditions at $2n$ and $2m$, respectively, such that $\varphi_{2n}^{2n} = 1$ and $\varphi_{2m}^{2m} = 1$. Using (7.18), we have

$$G^{[2n,2m]}(2n,2m;z) = \frac{e^{i\theta_{2m}}}{\widetilde{\varphi}_{2m+1}^{2n}\widetilde{\varphi}_{2m}^{2m} - \widetilde{\varphi}_{2m}^{2n}\widetilde{\varphi}_{2m+1}^{2m}} \widetilde{\varphi}_{2m}^{2m}. \tag{7.43}$$

Since φ^{2m} satisfies the boundary condition at $2m$, it follows that

$$\widetilde{\varphi}^{2m}_{2m} = tze^{i\theta^\omega_{2m}}, \tag{7.44}$$

$$\widetilde{\varphi}^{2m}_{2m+1} = (r-1)ze^{i\theta^\omega_{2m}}. \tag{7.45}$$

Using this along with the definition of $\widetilde{\varphi}^{2n}$, we obtain

$$G^{[2n,2m]}(2n,2m;z) = \frac{e^{i\theta_{2m}}}{t\varphi^{2n}_{2m-1} + (r - ze^{i\theta^\omega_{2m}})\varphi^{2n}_{2m}}. \tag{7.46}$$

Now, for $s \in (0,1)$ the expectation of the s-moment of $G^{[2n,2m]}(2n,2m;z)$ is given by

$$\mathbb{E}[|G^{[2n,2m]}(2n,2m;z)|^s] = \widehat{\mathbb{E}}\left[\int_0^{2\pi} d\mu(\theta_{2m})\frac{1}{|t\varphi^{2n}_{2m-1} + (r - ze^{i\theta^\omega_{2m}})\varphi^{2n}_{2m}|^s}\right], \tag{7.47}$$

where $\widehat{\mathbb{E}}$ is the expectation with respect to the random variables $\{\theta^\omega_k\}_{k\in\mathbb{Z}\setminus\{2m\}}$. By Lemma 7.2, we have

$$\mathbb{E}(|G^{[2n,2m]}_z(2n,2m)|^s) \leq C_\mu(s,t)\mathbb{E}\left[\left\|T_z(\omega,m-n)\begin{pmatrix}\varphi^{2n}_{2n-1}\\ \varphi^{2n}_{2n}\end{pmatrix}\right\|^{-s}\right]. \tag{7.48}$$

Using Lemma 7.1, it follows that there exist $\alpha_1 > 0$ and $s_0 \in (0,1)$ and $\widetilde{C}^{(1)}_\mu(s_0,t) < \infty$ such that

$$\mathbb{E}[|G^{[2n,2m]}_z(2n,2m)|^{s_0}] \leq \widetilde{C}^{(1)}_\mu(s_0,t)e^{-\alpha_1(m-n)}, \tag{7.49}$$

for all $z \in \mathbb{C}$ with $0 < ||z|-1| < 1/2$, $0 < \epsilon < 1/2$ and $m,n \in \mathbb{Z}$ such that $m-n \geq 2$.

In the case $n \geq m+2$ let ψ^{2n} and ψ^{2m} be solutions of $(U-z)\psi = 0$ that satisfy the boundary condition at $2n$ and $2m$, respectively, with $\psi^{2n}_{2n} = \psi^{2m}_{2m} = 1$. Using (7.18), (7.14), we obtain that

$$G^{[2m,2n]}(2n,2m;z) = \frac{1}{z(t\psi^{2n}_{2m-1} + (1-r)\psi^{2n}_{2m})}. \tag{7.50}$$

In order to bound the expectation we first integrate with respect to θ^ω_{2m-1} and use the same procedure as before. ∎

Proof of Theorem 3.4: Without restriction we may assume $|k-l| > 4$ (as (3.15) for $|k-l| \leq 4$ only requires the a-priori bound (3.5)). Pick $m,n \in \mathbb{Z}$ such that $k \in \{2n, 2n+1\}$, $l \in \{2m, 2m+1\}$ and $|m-n| > 1$. Thus using the results of Lemma 7.3 and Lemma 7.4, there exists $0 < \kappa(t,s,\mu) < \infty$ such that

$$\mathbb{E}[|G(k,l;z)|^s] \leq \kappa(t,s,\mu)\sum_{i,j=0}^{1}\left(\mathbb{E}[|G^{|[2n+2i,2m-2j]|}(2n+2i,2m-2j;z)|^{12s}]\right)^{1/12}. \tag{7.51}$$

Next the result of Lemma 7.5 gives that there exist $s_0 \in (0,1)$, $0 < \widetilde{C}_1 < \infty$, $\alpha > 0$ such that

$$\mathbb{E}[|G(k,l;z)|^{s_0}] \leq \widetilde{C}^{(1)} \kappa(t, s_0, \mu) \sum_{i,j=0}^{1} e^{-\alpha|2n-2m+2i+2j|}. \tag{7.52}$$

Using the definition of m, n this yields (3.15). ■

8 Neumann Boundary Conditions

In this section we study in detail the properties of one of the finite volume restrictions of the "free" unitary operator S introduced in Section 7.1. This particular restricted operator will share many of the properties of selfadjoint Neumann Laplacians. We will therefore think of them as restrictions of S to finite volume with Neumann boundary conditions.

8.1 Neumann Boundary Conditions for $d = 1$

Let $e^{i\eta} = r + it$, $L \in \mathbb{N}$, and on $l^2([0, 2L-1])$ define

$$S_N^{[0,2L-1]} = \begin{pmatrix} re^{i\eta} & rt & -t^2 & & & & & & \\ -te^{i\eta} & r^2 & -rt & & & & & & \\ & rt & r^2 & rt & -t^2 & & & & \\ & -t^2 & -rt & r^2 & -rt & & & & \\ & & & & \ddots & & & & \\ & & & & & rt & r^2 & rt & -t^2 \\ & & & & & -t^2 & -rt & r^2 & -rt \\ & & & & & & & rt & r^2 & te^{i\eta} \\ & & & & & & & -t^2 & -rt & re^{i\eta} \end{pmatrix}, \tag{8.1}$$

which is a special case of the restrictions of S_0 to finite intervals defined in Section 7.1. To characterize the spectrum of $S_N^{[0,2L-1]}$ define $\lambda_k = \arccos(r^2 - t^2\cos(k\pi/L))$, $k = 0, 1, \ldots, L$, i.e. in particular $\lambda_L = 0$ and $\lambda_0 = \arccos(r^2 - t^2)$, the latter coinciding with λ_0 as introduced in Section 2 and giving the band edges of S_0 as $e^{\pm i\lambda_0}$.

Proposition 8.1. *The $2L$ eigenvalues of $S_N^{[0,2L-1]}$ are non degenerate and given by*

$$\sigma(S_N^{[0,2L-1]}) = \{e^{i\lambda_0}, e^{i\lambda_L} = 1\} \cup \{e^{\pm i\lambda_k} : k = 1, .., L-1\}. \tag{8.2}$$

In particular, $\sigma(S_N^{[0,2L-1]}) \subset \sigma(S)$. Moreover,

$$\varphi_0 = (i, 1, i, 1, \cdots, i, 1)^t \tag{8.3}$$

is an eigenvector associated with $e^{i\lambda_0}$ and

$$\varphi_L = (i, 1, -i, -1, \cdots, (-1)^{L+1}i, (-1)^{L+1})^t \tag{8.4}$$

is an eigenvector associated with 1.

To prove Proposition 8.1 we will use the transfer matrix formalism, i.e. solutions of $U_\omega \psi = z\psi$ are characterized by the relations

$$\begin{pmatrix} \psi_{2k+1} \\ \psi_{2k+2} \end{pmatrix} = T_z(\theta^\omega_{2k}, \theta^\omega_{2k+1}) \begin{pmatrix} \psi_{2k-1} \\ \psi_{2k} \end{pmatrix}, \tag{8.5}$$

for all $k \in \mathbb{Z}$, where the transfer matrices $T_z : \mathbb{T}^2 \to GL(2, \mathbb{C})$ are defined by (7.3). In the "free" case $S_0\psi = e^{i\lambda}\psi$, the transfer matrix takes the simple form

$$T(\lambda) := T_{e^{i\lambda}}(0,0) = \begin{pmatrix} -e^{-i\lambda} & \frac{r}{t}\left(1 - e^{-i\lambda}\right) \\ \frac{r}{t}\left(1 - e^{-i\lambda}\right) & -\frac{e^{i\lambda}}{t^2} + \frac{r^2}{t^2}\left(2 - e^{-i\lambda}\right) \end{pmatrix}. \tag{8.6}$$

The following lemma will be used in the proof of Proposition 8.1.

Lemma 8.1. *The vector $(i,1)^t$ is an eigenvector of $T(\lambda)^L$ if and only if $\lambda \in \{0, \arccos(r^2 - t^2)\} \cup \{\lambda : \cos(\lambda) = r^2 - t^2\cos(k\pi/L), k = 1, .., L-1\}$.*

Proof: In order to simplify the analysis we distinguish between two cases:

(i) $(i,1)^t$ is an eigenvector of $T(\lambda)$. A straightforward calculation shows that this is only true for $\lambda \in \{0, \arccos(r^2 - t^2)\}$ with corresponding eigenvalues $\{-1, 1\}$ respectively.

(ii) The second case is when $(i,1)^t$ is an eigenvector of $T(\lambda)^L$ but not of $T(\lambda)$, i.e.

$$T(\lambda)^L \begin{pmatrix} i \\ 1 \end{pmatrix} = a \begin{pmatrix} i \\ 1 \end{pmatrix}, \tag{8.7}$$

while $v = T(\lambda)(i,1)^t$ is linearly independent of $(i,1)^t$. Then it follows that

$$T(\lambda)^L v = T(\lambda)^{L+1} \begin{pmatrix} i \\ 1 \end{pmatrix} = aT(\lambda) \begin{pmatrix} i \\ 1 \end{pmatrix} = av. \tag{8.8}$$

Also since v and $(i,1)^t$ are linearly independent it follows that $T(\lambda)^L = aI$. Finally using that the determinant of $T(\lambda)$ is one, we see that $a^2 = 1$.

Since the eigenvalues of $T(\lambda)$ are given by $e^{\pm i \arccos((r^2 - cos(\lambda))/t^2)}$, we have that a is an eigenvalue of $T(\lambda)^L$ if and only if $e^{\pm i 2L \arccos((r^2 - cos(\lambda))t^2)} = 1$, i.e.

$$\begin{aligned} \lambda &\in \{\cos(\lambda) = r^2 - t^2\cos(k\pi/L), k = 1, .., 2L-1\} \\ &= \{\cos(\lambda) = r^2 - t^2\cos(k\pi/L), k = 1, .., L\}. \end{aligned} \tag{8.9}$$

Combining the two cases gives the result. ∎

Proof of Proposition 8.1: In light of the previous Lemma, it suffices to show that $e^{i\lambda} \in \sigma(S_N^{[0,2L-1]})$ if and only if $(i,1)^t$ is an eigenvector of $T(\lambda)^L$.

First it is not hard to see that $e^{i\lambda} \in \sigma(S_N^{[0,2L-1]})$ means the existence of $\psi \in l^2([0, 2L-1])$ such that

$$\begin{pmatrix} \psi_{2m+1} \\ \psi_{2m+2} \end{pmatrix} = T(\lambda) \begin{pmatrix} \psi_{2m-1} \\ \psi_{2m} \end{pmatrix}, \tag{8.10}$$

for $m \in \{1, .., L-2\}$, while

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_0 \begin{pmatrix} \frac{1}{t}(r - e^{i(\eta-\lambda)}) \\ \frac{1}{t^2}(r^2 - re^{i(\eta-\lambda)} - e^{i\lambda} + re^{i\eta}) \end{pmatrix}, \tag{8.11}$$

and

$$\begin{pmatrix} \psi_{2L-3} \\ \psi_{2L-2} \end{pmatrix} = \psi_{2L-1} \begin{pmatrix} \frac{1}{t^2}(r^2 - re^{i(\eta-\lambda)} - e^{i\lambda} + re^{i\eta}) \\ \frac{-1}{t}(r - e^{i(\eta-\lambda)}) \end{pmatrix}. \tag{8.12}$$

Define $\tilde{\psi}_{-1}$ such that

$$T(\lambda)^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_{-1} \\ \psi_0 \end{pmatrix}. \tag{8.13}$$

Using (8.11) and that $e^{i\eta} = r + it$, one can see that this definition is equivalent to having $\tilde{\psi}_{-1} = i\psi_0$. Similarly, defining $\tilde{\psi}_{2L}$ such that

$$\begin{pmatrix} \psi_{2L-1} \\ \tilde{\psi}_{2L} \end{pmatrix} = T(\lambda) \begin{pmatrix} \psi_{2L-3} \\ \psi_{2L-2} \end{pmatrix}, \tag{8.14}$$

we see that $\tilde{\psi}_{2L} = -i\psi_{2L-1}$. Also by definition,

$$T(\lambda)^L \begin{pmatrix} \tilde{\psi}_{-1} \\ \psi_0 \end{pmatrix} = \begin{pmatrix} \psi_{2L-1} \\ \tilde{\psi}_{2L} \end{pmatrix} \tag{8.15}$$

$$T(\lambda)^L \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{-i\psi_{2L-1}}{\psi_0} \begin{pmatrix} i \\ 1 \end{pmatrix}. \tag{8.16}$$

which shows the required assertion. The last two claims of Proposition 8.1 follow from the above together with $T(0)(1,i)^t = -(1,i)^t$ and $T(\lambda_0)(1,i)^t = (1,i)^t$. ∎

We conclude this subsection with several remarks:

(i) $e^{i\arccos(r^2-t^2)} \in \sigma(S_N^{[0,2L-1]})$, while $e^{-i\arccos(r^2-t^2)}$ is not.

(ii) For $\lambda_0 = \arccos(r^2 - t^2) = \arccos(1 - 2t^2)$, $S_N^{[0,2L-1]}\varphi_0 = e^{i\lambda_0}\varphi_0$ has solutions with $|\varphi_0(k)| = 1$ for all $k \in [0, 2L-1]$.

(iii) There is a gap between the upper edge of the spectrum of $S_N^{[0,2L-1]}$, given by $e^{i\arccos(r^2-t^2)}$, and the next closest eigenvalue. In particular, for any $k \in \{1, \cdots, L-1\}$ we have

$$\begin{aligned} |e^{i\arccos(r^2-t^2)} - e^{i\arccos(r^2-t^2\cos(\pi k/L))}| &> t^2(1-\cos(\pi k/L)) = 2t^2\sin^2(\pi k/2L) \\ &\geq t^2 \left(\frac{\pi k}{L}\right)^2 \frac{(4-\pi)^2}{32}, \end{aligned} \tag{8.17}$$

using the property $\sin(x) \geq x(1 - \pi/4)$ if $x \in (0, \pi/2)$.

(iv) The previous remarks as well as the "Neumann-bracketing" property to be found in Section 10 below will make the operators $S_N^{[0,2L-1]}$ a suitable tool when studying properties of finite volume restrictions of the unitary Anderson model near the upper edge $e^{i\lambda_0}$ of the spectrum of S_0. To get the corresponding results also at the lower edge $e^{-i\lambda_0}$ of the spectrum of S_0 one needs to modify the definition of $S_N^{[0,2L-1]}$ by setting $e^{i\eta} = r - it$ in (8.1). In this case we use, in particular, that the vector $(-i,1)^t$ is an eigenvector of $T(-\lambda_0)$, leading to $e^{-i\lambda_0}$ becoming an eigenvalue of the restricted operator. One gets properties similar to Proposition 8.1 and Remarks (i), (ii) and (iii) above.

8.2 Neumann Boundary Conditions for $d > 1$

For a box $\Lambda := [2l_1, 2m_1 - 1] \times \ldots \times [2l_d, 2m_d - 1] \subset \mathbb{Z}^d$ define

$$S_N^\Lambda = \otimes_{j=1}^d S_N^{[2l_j,2m_j-1]} \quad \text{on} \quad \otimes_{j=1}^d l^2([l_j, 2m_j - 1]) = l^2(\Lambda). \tag{8.18}$$

Note here that the discussion of Section 8.1 applies with obvious modifications to intervals of the form $[2l_j, 2m_j - 1]$ and integers $l_j < m_j$. We will be particularly interested in the case of cubic boxes $\Lambda_L := [-2L, 2L+1]^d$ for $L \in \mathbb{N}$. The spectrum of $S_N^{\Lambda_L}$ is given by the $|\Lambda_L| = (4L+2)^d$ eigenvalues

$$\sigma(S_N^{\Lambda_L}) = \left\{ \Pi_{j=1}^d e^{i\sigma_j \lambda_{k_j}} \right\} \tag{8.19}$$

where

$$\begin{aligned} k_j \in \{0,1,2,\cdots,2L+1\}^d \ \text{ for } \ j = 1,\ldots,d, \\ \sigma_j \in \{+1,-1\} \ \text{ for } \ k_j = 1,\ldots,2L, \quad \sigma_j = 1 \ \text{ for } \ k_j \in \{0, 2L+1\}. \end{aligned} \tag{8.20}$$

Under the assumption $d \arccos(r^2 - t^2) < \pi$, the upper edge of the spectrum of S, $e^{id\arccos(r^2-t^2)} = e^{id\lambda_0}$, belongs to $\sigma(S_N^\Lambda)$ and is non degenerate. An eigenvector corresponding to $e^{id\lambda_0}$ is $\varphi_0^{(d)} = \otimes_1^d \varphi_0$, whose components all have modulus one.

Moreover, there exists c_0, a numerical constant, such that

$$\text{dist}\,(e^{id\lambda_0}, \sigma(S_N^{\Lambda_L}) \setminus \{e^{id\lambda_0}\}) > \frac{c_0 t^2}{|\Lambda_L|^{2/d}}. \tag{8.21}$$

Indeed, the closest eigenvalue to $e^{id\lambda_0}$ is $e^{i((d-1)\lambda_0 + \arccos(r^2 - t^2\cos(\pi/(2L+1))))}$, which is d-fold degenerate, so that the distance (8.21) equals

$$\begin{aligned} |e^{id\lambda_0} - e^{i((d-1)\lambda_0 + \arccos(r^2-t^2\cos(\pi/(2L+1))))}| &= |e^{i\lambda_0} - e^{i\arccos(r^2-t^2\cos(\pi/(2L+1)))}| \\ &> \frac{t^2}{(4L+2)^2} c_0 = \frac{c_0 t^2}{|\Lambda_L|^{2/d}}, \end{aligned} \tag{8.22}$$

where $c_0 = (\pi(4-\pi))^2/8$, see (8.17).

For later study of spectral properties near the lower edge $e^{-id\lambda_0}$ of the spectrum of S, we use the modified version of $S_N^{[2l_j,2m_j-1]}$ from the fourth remark at the end of the previous subsection in the definition (8.18)

9 The Feynman-Hellmann Formula

The Feynmann-Hellmann formula provides, on the level of first order perturbation theory, the change of an isolated simple eigenvalue of a selfadjoint operator under an additive perturbation. Here we will need a corresponding result for multiplicative perturbations of unitary operators. We prove such a formula in an analytic framework, which will suffice for our purpose.

Proposition 9.1. *Let $I \subset \mathbb{R}$ be an open interval containing zero, $\mathcal{H}$ be a separable Hilbert space and $I \ni \alpha \mapsto U(\alpha)$ an analytic map with values in the set of unitary operators on $\mathcal{H}$. Assume $\beta(0) \in \mathbb{S}$ is an isolated simple eigenvalue of $U(0)$ with normalized corresponding eigenvector $\varphi(0) \in \mathcal{H}$. Then, there exists an open disc centered at 0 of radius $\alpha_0 > 0$, $D(0, \alpha_0) \subset \mathbb{C}$, and two analytic maps $D(0, \alpha_0) \ni \alpha \mapsto \beta(\alpha) \in \mathbb{C}$ and $D(0, \alpha_0) \ni \alpha \mapsto \varphi(\alpha) \in \mathcal{H}$ such that*

$$\begin{aligned} &U(\alpha)\varphi(\alpha) = \beta(\alpha)\varphi(\alpha) \quad \forall \alpha \in D(0, \alpha_0), \\ &dist\,(\beta(\alpha), \sigma(U(\alpha) \setminus \{\beta(\alpha)\}) > 0 \\ &\|\varphi(\alpha)\| = 1 \quad if\ \alpha \in D(0, \alpha_0) \cap I. \end{aligned} \tag{9.1}$$

Moreover, for all $\alpha \in D(0, \alpha_0) \cap I$,

$$\beta'(\alpha) = \langle \varphi(\alpha) | U'(\alpha) \varphi(\alpha) \rangle. \tag{9.2}$$

Remark 9.1. *i) For a given α, the last formula is of course true for any choice of normalized eigenvector of $U(\alpha)$, corresponding to $\beta(\alpha)$.*

ii) If $I \ni \alpha \mapsto U(\alpha)$ is analytic and takes its values in the set of unitary finite *matrices, all its eigenvalues and spectral projectors admit analytic extensions in a complex neighborhood of I, even at the values of α where eigenvalues of $U(\alpha)$ may cross, see [42]. Consequently, an analytic choice of normalized eigenvectors can be made for all $\alpha \in I$.*

Proof: By the general theory of analytic perturbations of operators, see e.g. [42], the operator $U(\alpha)$ admits an isolated simple eigenvalue $\beta(\alpha)$, for small enough values of $|\alpha|$, say in $D(0, \alpha_0)$. Also, the analytic rank one spectral projector on $\beta(\alpha)$, $P(\alpha)$, given by the Riesz formula is analytic for all $\alpha \in D(0, \alpha_0)$.

By definition, for all $\alpha \in D(0, \alpha_0)$,

$$P(\alpha)U(\alpha) = U(\alpha)P(\alpha) = \beta(\alpha)P(\alpha), \tag{9.3}$$

and since $U(\alpha)$ is unitary on the real axis, $P(\alpha)$ is self-adjoint for real α's. Now define the analytic operator $W(\alpha)$ as the unique solution to the ODE

$$W'(\alpha) = [P'(\alpha), P(\alpha)]W(\alpha), \quad W(0) = I, \quad \alpha \in D(0, \alpha_0). \tag{9.4}$$

It is a well known property ([42]) that the following intertwining property holds for all $\alpha \in D(0, \alpha_0)$,

$$P(\alpha)W(\alpha) = W(\alpha)P(0). \tag{9.5}$$

Note that $W(\alpha)$ is unitary on the real axis, since its generator is easily seen to be anti self-adjoint there. We define an analytic vector by

$$\varphi(\alpha) = W(\alpha)\varphi(0). \tag{9.6}$$

Identities (9.5) and (9.3) show that

$$U(\alpha)\varphi(\alpha) = \beta(\alpha)\varphi(\alpha) \tag{9.7}$$

and φ is normalized on the real axis, since W is unitary there. By differentiation of the previous identity and application of $P(\alpha)$ to the result, we obtain

$$P(\alpha)U'(\alpha)\varphi(\alpha) + P(\alpha)U(\alpha)\varphi'(\alpha) = \beta'(\alpha)P(\alpha)\varphi(\alpha) + \beta(\alpha)P(\alpha)\varphi'(\alpha) \tag{9.8}$$

which reduces to

$$P(\alpha)U'(\alpha)\varphi(\alpha) = \beta'(\alpha)\varphi(\alpha) \tag{9.9}$$

due to (9.3) and (9.5). Since for all $\alpha \in I \cap D(0, \alpha_0)$ we can write

$$P(\alpha) = |\varphi(\alpha)\rangle\langle\varphi(\alpha)| = W(\alpha)|\varphi(0)\rangle\langle\varphi(0)|W^{-1}(\alpha), \tag{9.10}$$

the result follows. ■

As a specific application, let us consider the family of analytic unitary matrices

$$U^\Lambda(\alpha) = D(\alpha)S_N^\Lambda = \text{diag}\{e^{-i\alpha\theta_k}\}S_N^\Lambda, \tag{9.11}$$

where S_N^Λ is the Neumann restriction of S to a d-dimensional box Λ introduced in Section 8, $\alpha \in \mathbb{R}$, and $\theta_k \in \mathbb{T}$ for all $k \in \Lambda$. $U^\Lambda(\alpha)$ interpolates between S_N^Λ and $\text{diag}\{e^{-i\theta_k}\}S_N^\Lambda$, at $\alpha = 0$ and $\alpha = 1$, respectively. Introducing the self-adjoint matrix $H^\Lambda = \sum_{k\in\Lambda} \theta_k\, |e_k\rangle\langle e_k|$ on $l^2(\Lambda)$, we can rewrite

$$U^\Lambda(\alpha) = e^{-i\alpha H^\Lambda} S_N^\Lambda, \quad \alpha \in \mathbb{R}. \tag{9.12}$$

Lemma 9.1. *If $e^{i\lambda(0)}$ is a discrete non-degenerate eigenvalue of S_N^Λ with normalized eigenstate $\varphi(0)$, then, for all $\alpha \in \mathbb{R}$, there exist analytic eigenvalues $e^{i\lambda(\alpha)}$ of $U^\Lambda(\alpha)$ with analytic normalized eigenvectors $\varphi(\alpha)$ such that*

$$\frac{d}{d\alpha} e^{i\lambda(\alpha)} = -ie^{i\lambda(\alpha)} \sum_{k\in\Lambda} \theta_k |\langle e_k|\varphi(\alpha)\rangle|^2. \tag{9.13}$$

In particular, for all $\alpha \in \mathbb{R}$, $\lambda'(\alpha) = -\sum_{k\in\Lambda} \theta_k\, |\langle e_k|\varphi(\alpha)\rangle|^2$.

Proof:

The existence of analytic eigenvalues $e^{i\lambda(\alpha)}$ and analytic eigenvectors $\varphi(\alpha)$ of $U^\Lambda(\alpha)$, $\alpha \in \mathbb{R}$, follows from Proposition 9.1 and the remark following it.

We compute

$$(U^\Lambda)'(\alpha) = -iH_\Lambda U^\Lambda(\alpha), \quad U^\Lambda(0) = S_N^\Lambda \tag{9.14}$$

and

$$\langle\varphi(\alpha)| - iH^\Lambda U^\Lambda(\alpha)\varphi(\alpha)\rangle = -i \sum_{k\in\Lambda} \theta_k\, |\langle e_k|\varphi(\alpha)\rangle|^2 e^{i\lambda(\alpha)} \tag{9.15}$$

and apply (9.2). ■

10 Splitting Boxes by Neumann Boundary Conditions

Throughout this section we will assume that boxes $\Lambda \subset \mathbb{Z}^d$ are compatible with Neumann boundary conditions as defined in Section 8, S_N^Λ is given by (8.18) and

$$U^\Lambda = DS_N^\Lambda = \text{diag}(e^{-i\theta_k})S_N^\Lambda. \tag{10.1}$$

For notational simplicity we will assume in this section that the box Λ has a vertex at the origin, which does not cause a restriction. We first deal with dimension $d = 1$.

Consider a one dimensional box Λ_0 consisting of two disjoint adjacent boxes Λ_1 and Λ_2:

$$\Lambda_1 = [0, 2l-1], \quad \Lambda_2 = [2l, 2(l+n)-1], \quad \Lambda_0 = [0, 2(l+n)-1], \tag{10.2}$$

with $n, l \geq 2$ (to avoid the special case $S_N^{[0,1]}$). We note that U^{Λ_0} and $U^{\Lambda_1} \oplus U^{\Lambda_2}$ are both defined on $l^2([0, 2(l+n)-1])$.

We want to show that the eigenvalues of $U^{\Lambda_1} \oplus U^{\Lambda_2}$ are closer to the upper band edge $e^{i(\lambda_0+a)}$ of the almost sure spectrum Σ of U than those of U^{Λ_0}. Recall here that in Section 3.2.3 we have assumed that $|\theta_k| \leq a$ and $\lambda_0 + a < \pi$. This is the analog of the well known property $H^{\Lambda_1} \oplus H^{\Lambda_2} \leq H^{\Lambda_0}$, where H^Λ is the Neumann restriction to a box Λ of the discrete Schrödinger operator.

The following simple observation is the starting point of the analysis. Splitting a box by imposing Neumann boundary conditions is a rank one perturbation:

Lemma 10.1. *Let $S_N^{\Lambda_j}$, $j = 0, 1, 2$ be defined as above. Then,*

$$S_N^{\Lambda_0} = S_N^{\Lambda_1} \oplus S_N^{\Lambda_2} + |\psi\rangle\langle\varphi| \tag{10.3}$$

where

$$\begin{aligned} \psi &= -te_{2l-2} - re_{2l-1} - ire_{2l} + ite_{2l+1} \\ \varphi &= t(-ie_{2l-1} + e_{2l}) \end{aligned} \tag{10.4}$$

The proof is an easy computation. This leads us to using the following fact about rank one perturbations of unitary operators which return a unitary operator.

Lemma 10.2. *Let U a unitary operator on a Hilbert space $\mathcal{H}$ and $f, g \in \mathcal{H} \setminus \{0\}$. If*

$$V = U + |f\rangle\langle g| \tag{10.5}$$

is unitary, then there exists $\beta \in (-\pi, \pi]$ such that $e^{i\beta} = 1 + \langle Ug|f\rangle$ and

$$V = e^{i\beta|\hat{f}\rangle\langle\hat{f}|}U, \tag{10.6}$$

where $\hat{f} = f/\|f\|$.

Proof:
The identity $VV^* = \mathbb{I}$ implies that

$$|Ug\rangle\langle f| + |f\rangle\langle Ug| + \|g\|^2|f\rangle\langle f| = 0. \tag{10.7}$$

Applying this to f shows that Ug is proportional to f, so that

$$Ug = \frac{\langle f|Ug\rangle}{\|f\|^2} f. \tag{10.8}$$

With this it follows from (10.5) that

$$\begin{aligned} V &= (I + |f\rangle\langle Ug|)U \\ &= (I + \langle Ug, f\rangle|\hat{f}\rangle\langle\hat{f}|)U \\ &= (I - |\hat{f}\rangle\langle\hat{f}| + \mu|\hat{f}\rangle\langle\hat{f}|)U, \end{aligned} \tag{10.9}$$

where $\mu = 1 + \langle Ug|f\rangle$. Thus $I - |\hat{f}\rangle\langle\hat{f}| + \mu|\hat{f}\rangle\langle\hat{f}| = VU^*$ is unitary, which shows that $|\mu| = 1$, i.e. $\mu = e^{i\beta}$ for $\beta \in (-\pi, \pi]$, and $I - |\hat{f}\rangle\langle\hat{f}| + \mu|\hat{f}\rangle\langle\hat{f}| = e^{i\beta|\hat{f}\rangle\langle\hat{f}|}$. ∎

Taking into account the random phases, we apply the previous lemma to our case with

$$U^{\Lambda_0} = U^{\Lambda_1} \oplus U^{\Lambda_2} + |D\psi\rangle\langle\varphi| \tag{10.10}$$

and ψ, φ from (10.4). We compute $\|D\psi\| = \|\psi\| = \sqrt{2}$, i.e. $\hat{\psi} = \psi/\|\psi\| = \psi/\sqrt{2}$, and

$$e^{i\beta} = 1 + \langle(U^{\Lambda_1} \oplus U^{\Lambda_2})\varphi|D\psi\rangle = 1 + \langle(S_N^{\Lambda_1} \oplus S_N^{\Lambda_2})\varphi|\psi\rangle = e^{-i\arccos(r^2-t^2)} = e^{-i\lambda_0} \tag{10.11}$$

so that

$$U^{\Lambda_0} = e^{-i\lambda_0|D\hat{\psi}\rangle\langle D\hat{\psi}|}\, U^{\Lambda_1} \oplus U^{\Lambda_2} = De^{-i\lambda_0|\hat{\psi}\rangle\langle\hat{\psi}|}\, S_N^{\Lambda_1} \oplus S_N^{\Lambda_2}. \tag{10.12}$$

Let us introduce an analytic family of unitary operators defined in I, a complex neighborhood of $[0,1]$, by

$$I \ni \alpha \mapsto U(\alpha) = e^{-i\alpha\lambda_0|D\hat{\psi}\rangle\langle D\hat{\psi}|}\, U^{\Lambda_1} \oplus U^{\Lambda_2}, \tag{10.13}$$

such that $U(0) = U^{\Lambda_1} \oplus U^{\Lambda_2}$ and $U(1) = U^{\Lambda_0}$. By Lemma 9.1 we immediately get the following

Proposition 10.1. *Let $\alpha \in I$ and $e^{i\lambda(\alpha)}$ denote any analytic eigenvalue of $U(\alpha)$, which is isolated except at a finite set of values of α. Then*

$$\arg(\lambda(1)) \leq \arg(\lambda(0)). \tag{10.14}$$

Remark 10.1. *In other words, when a Neumann boundary condition is introduced to split Λ_0 into $\Lambda_1 \cup \Lambda_2$, the eigenvalues of $U_{\Lambda_1} \oplus U_{\Lambda_2}$ are closer to $e^{i(\lambda_0+a)}$ than those of U_{Λ_0}.*

Let us generalize now to dimension $d \geq 1$. Consider a box Λ_0 of the form

$$\Lambda_0 = \Lambda_0(1) \times \Lambda(2) \times \dots \Lambda(d) \tag{10.15}$$

where

$$\Lambda_0(1) = [0, 2(l+n)-1], \quad \Lambda(j) = [0, 2l_j - 1], \ j = 2, \cdots, d, \tag{10.16}$$

which we split by a Neumann boundary condition perpendicular to the first axis as $\Lambda_0 = \Lambda_1 \cup \Lambda_2$ with

$$\Lambda_k = \Lambda_k(1) \times \Lambda(2) \times \dots \Lambda(d), \ k = 1, 2 \tag{10.17}$$

and

$$\Lambda_0(1) = \Lambda_1(1) \cup \Lambda_2(1) = [0, 2l-1] \cup [2l, 2(l+n)-1]. \tag{10.18}$$

By the previous results, the corresponding operators U^{Λ_k}, $k = 0, 1, 2$ are related by

$$U^{\Lambda_0} = e^{-i\lambda_0 |D\hat{\psi}\rangle\langle D\hat{\psi}| \otimes I \otimes \cdots \otimes I} \, U^{\Lambda_1} \oplus U^{\Lambda_2}. \tag{10.19}$$

Applying Lemma 9.1 again, with H_Λ replaced by the non negative operator $|D\hat{\psi}\rangle\langle D\hat{\psi}| \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$, shows that the spectra of U^{Λ_0} and $U^{\Lambda_1} \oplus U^{\Lambda_2}$ are related in the same way as in the one dimensional case, e.g. by (10.14). Clearly, the splitting by Neumann boundary conditions can be done perpendicular to any of the d coordinate axes and can also be iterated. Thus we get the above form of spectral monotonicity also, for example, when spitting U^{Λ_L} over the cube $[-2L, 2L+1]^d$ into a direct sum of U_{Λ_i} for $((2L+1)/l)^d$ cubes Λ_i of sidelength $2l$.

11 A Combes-Thomas Estimate

Combes-Thomas bounds, originating from [19], have become the standard tool in Schrödinger operator theory to show exponential decay of eigenfunctions to eigenvalues which lie outside of the essential spectrum. They also provide a key step in localization proofs for random Schrödinger operators in the band edge regime, see e.g. [56]. Here we provide a Combes-Thomas type estimate for unitary operators with band structure.

Let U be unitary on $l^2(\mathbb{Z}^d)$. We say that U has band structure of width $w > 0$, if it can be written as

$$U = D + O \ \text{ with } \ \langle e_k | D e_j \rangle = \delta_{kj} \langle e_k | D e_k \rangle \ \text{ and } \ \langle e_k | O e_j \rangle = 0 \ \text{ if } \ |j-k| > w. \tag{11.1}$$

Proposition 11.1 (Combes-Thomas type estimate)**.** *For a unitary operator U on $l^2(\mathbb{Z}^d)$ with band structure of width w, there exist $0 < B < \infty$ which depends on U only, such that*

$$|\langle e_j | (U-z)^{-1} e_k \rangle| \leq \frac{2}{\operatorname{dist}(z, \sigma(U))} e^{-\operatorname{dist}(z,\sigma(U))|j-k|B}. \tag{11.2}$$

Remark 11.1. *i) The same result holds for U defined on a finite dimensional Hilbert space $l^2(\Lambda)$, $\Lambda \subset \mathbb{Z}^d$, with constants independent of Λ.*

ii) Actually, our proof works more generally for bounded normal operators U with band structure. Results of this type are known in the literature, e.g. [25].

Proof:

Let $x = (x_1, \cdots, x_d)$, where x_n is the self-adjoint multiplication operator acting on e_k, with $k = (k_1, \dots, k_d)$, as $x_n e_k = k_n e_k$ and defined on its natural domain. We introduce the vector $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ and construct the self-adjoint operator

$$e^{\alpha x} \text{ acting as } e^{\alpha x} e_k = e^{\alpha k} e_k \tag{11.3}$$

on

$$\mathcal{D}_\alpha = \{\psi \in l^2(\mathbb{Z}^d) \text{ s.t. } \sum_{k \in \mathbb{Z}^d} |\langle e_k|\psi\rangle|^2 e^{\alpha k} < \infty\}. \tag{11.4}$$

Here $\alpha k = \sum_n \alpha_n k_n$. Consider the operator

$$U_\alpha := e^{\alpha x} U e^{-\alpha x} = e^{\alpha x} D e^{-\alpha x} + e^{\alpha x} O e^{-\alpha x} = D_\alpha + O_\alpha \tag{11.5}$$

defined *a priori* on the dense set

$$c_0 = \{\psi \in l^2(\mathbb{Z}^d) \text{ s.t. } \langle e_k|\psi\rangle = 0 \text{ for } |k| \text{ large enough}\}. \tag{11.6}$$

The operator U_α is bounded because for any $\psi \in c_0$

$$O_\alpha \psi = \sum_{j \in F} \sum_{\substack{k \neq j \\ |k-j| \leq w}} e^{\alpha(k-j)} \langle e_k|O e_j\rangle \langle e_j|\psi\rangle e_k \tag{11.7}$$

where the set F is finite and $e^{\alpha(k-j)} \leq e^{|\alpha| w}$. Moreover, $D_\alpha = D$ which shows that $\|U_\alpha\| \leq C_1(\alpha) \leq C_1 < \infty$ on c_0, for α in a bounded set. Similarly,

$$\|U - U_\alpha\| = \|O - O_\alpha\| \leq C_2(\alpha) \leq C_3|\alpha|, \tag{11.8}$$

for $|\alpha|$ small enough. From the resolvent identity, if $z \in \rho(U)$ and $C_3|\alpha|/\text{dist}\,(z, \sigma(U)) < 1/2$,

$$(U_\alpha - z)^{-1} = (U - z)^{-1}(\mathbb{I} + (U_\alpha - U)(U - z)^{-1})^{-1} \tag{11.9}$$

with

$$\|(U_\alpha - z)^{-1}\| \leq 2\|(U - z)^{-1}\| \leq 2/\text{dist}\,(z, \sigma(U)). \tag{11.10}$$

Finally, by the formula

$$(U_\alpha - z)^{-1} = e^{\alpha x}(U - z)^{-1} e^{-\alpha x}, \tag{11.11}$$

we derive

$$\langle j|e^{\alpha x}(U-z)^{-1}e^{-\alpha x}k\rangle = e^{-\alpha(k-j)}\langle j|(U-z)^{-1}k\rangle = \langle j|(U_\alpha - z)^{-1}k\rangle, \tag{11.12}$$

from which we get

$$|\langle j|(U-z)^{-1}k\rangle| \leq 2\frac{e^{+\alpha(k-j)}}{\text{dist}\,(z,\sigma(U))}. \tag{11.13}$$

Choosing the components of α and their sign in such a way that $|\alpha_n| = |\alpha| > 0$ and

$$\alpha(j-k) = \sum_{n=1}^{d} \alpha_n (j-k)_n \geq |\alpha||j-k|, \tag{11.14}$$

we obtain the result, with $|\alpha| = \text{dist}\,(z,\sigma(U))/4C_3$, and $B = 1/4C_3$. ■

12 The Genesis of Lifshits Tails

After having introduced some tools in the previous two sections we will now start with the actual proof of Theorem 3.5. Throughout this proof we will focus on localization at the upper band edge $e^{i(d\lambda_0+a)}$ of the almost sure spectrum Σ of U_ω. The proof at the lower band edge is completely analogous. It uses the alternate form of Neumann boundary conditions discussed in Section 8 (setting $e^{i\eta} = r - it$ in (8.1) rather than $r + it$) and a corresponding adjustment of the results on splitting boxes in Section 10.

We find it convenient to rotate the upper band edge of Σ to be identical with $e^{id\lambda_0}$, the upper band edge of S. This is achieved by replacing the original U_ω by $e^{-ia}U_\omega$. In other words, setting $\theta_M = 2a$ we now assume

$$\text{supp}\,\mu \subset [0,\theta_M] \quad \text{with} \quad 0 \in \text{supp}\,\mu \quad \text{and} \quad 2d\lambda_0 + \theta_M < 2\pi. \tag{12.1}$$

The latter means that Σ has the gap $\{e^{i\vartheta} : d\lambda_0 < \vartheta < 2\pi - (d\lambda_0 + \theta_M)\}$.

As in earlier sections we will frequently drop the subscript ω from our notation.

We will first establish a Lifshits tail estimate for the spectrum near the band edge $e^{id\lambda_0}$. At the root of this is the following proposition which we prove by following the steps of Stollmann [56]. As in Section 8, for $L \in \mathbb{N}$ we set $\Lambda_L = [-2L, 2L+1]^d$.

Proposition 12.1. *Let $e^{i\lambda(U^{\Lambda_L})}$, respectively $e^{id\lambda_0}$, be the eigenvalue of largest argument of U^{Λ_L}, respectively $S_N^{\Lambda_L}$. Then $\lambda(U^{\Lambda_L}) \leq d\lambda_0$ and there exist $b > 0$ and $\gamma > 0$, independent of L and d, such that*

$$\mathbb{P}\left(|e^{i\lambda(U^{\Lambda_L})} - e^{id\lambda_0}| \leq \frac{b}{L^2}\right) \leq e^{-\gamma L^d}, \tag{12.2}$$

for L large enough.

Let us first give an easy Corollary of Lemma 9.1. Recall that $U^{\Lambda_L}(\alpha)$ is defined by (9.11).

Lemma 12.1. *Consider a fixed realization of $U^{\Lambda_L}(\alpha)$ in dimension $d \geq 1$. Then the analytic continuation of any eigenvalue $e^{i\lambda(\alpha)}$ of $U^{\Lambda_L}(\alpha)$ is such that $\lambda(\alpha)$ is non increasing. Consequently, $\left|e^{i\lambda(\alpha)} - e^{i\lambda(0)}\right|$ is a non decreasing function of $\alpha \geq 0$, as long as $\lambda(0) - \lambda(\alpha) < \pi$.*

Moreover, the eigenvalue $e^{id\lambda_0}$ of $S_N^{\Lambda_L}$ and its analytic continuation $e^{i\lambda_0(\alpha)}$ satisfy

$$\frac{d}{d\alpha}\lambda_0(\alpha)|_{\alpha=0} = -\frac{1}{(4L+2)^d}\sum_{k\in\Lambda_L}\theta_k. \tag{12.3}$$

Proof:

The first statement follows from Lemma 9.1 and from

$$\frac{d}{d\alpha}\left|e^{i\lambda(\alpha)} - e^{i\lambda(0)}\right|^2 = 2\sin(\lambda(\alpha)-\lambda(0))\lambda'(\alpha). \tag{12.4}$$

The second statement makes use of the fact that the components of the eigenvector $\varphi_0^{(d)}$ all have equal modulus. ■

Recall the following standard large deviation estimate whose proof can be found, e.g., in Lemma 2.1.1 of [56].

Lemma 12.2. *For non-trivial and non-negative i.i.d. random variables θ_k and $s_0 = \gamma_0 = -\frac{1}{2}\ln(\mathbb{E}(e^{-\theta_0})) > 0$, we have*

$$\mathbb{P}\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\theta_i \leq s_0\right) \leq e^{-\gamma_0|\Lambda|}. \tag{12.5}$$

Let us consider the small α behavior of $e^{i\lambda_0(\alpha)}$.

Lemma 12.3. *There exist $c_1 > 0$ and $c_2 > 0$, independent of d and L, such that for L sufficiently large*

$$\left|e^{i\lambda_0(\alpha)} - e^{id\lambda_0} - \alpha\left(\frac{d}{d\alpha}e^{i\lambda_0(\alpha)}\right)_{\alpha=0}\right| \leq c_1\alpha^2L^2,\ 0 \leq \alpha \leq \frac{c_2}{L^2}. \tag{12.6}$$

Proof:

Expanding $e^{id\lambda_0(\alpha)}$ in terms of $\alpha\in\mathbb{R}$, we get that

$$e^{i\lambda_0(\alpha)} - e^{id\lambda_0} - \alpha\frac{d}{d\alpha}e^{i\lambda_0(0)} = \frac{\alpha^2}{2}\frac{d^2}{d^2\alpha}e^{i\lambda_0(\tilde\alpha)} \tag{12.7}$$

for some $0 < \tilde\alpha < \alpha$. Next we use Cauchy's integral formula to bound

$$\frac{d^2}{d^2\alpha}e^{i\lambda_0(\tilde\alpha)} = \frac{2!}{2\pi i}\int_{|z-\tilde\alpha|=r}\frac{e^{i\lambda_0(z)}}{(z-\tilde\alpha)^3}dz = \frac{2!}{2\pi i}\int_{|z-\tilde\alpha|=r}\frac{e^{i\lambda_0(z)}-e^{id\lambda_0}}{(z-\tilde\alpha)^3}dz \tag{12.8}$$

for $\tilde{\alpha}$ small enough and suitable $r > 0$. Thus we need to control $e^{i\lambda_0(z)}$ for z complex. Since

$$U^{\Lambda_L}(\alpha) - U^{\Lambda_L}(0) = (e^{-i\alpha H^{\Lambda_L}} - I)S_N^{\Lambda}, \tag{12.9}$$

where

$$\|e^{-i\alpha H^{\Lambda_L}} - I\| \leq e^{|\alpha|\theta_M} - 1, \tag{12.10}$$

we have, by the second resolvent identity,

$$(U^{\Lambda_L}(\alpha) - z)^{-1} = (S_N^{\Lambda_L} - z)^{-1}(I - (e^{-i\alpha H^{\Lambda_L}} - I)S_N^{\Lambda}(U^{\Lambda_L}(\alpha) - z)^{-1}) \tag{12.11}$$

for $z \notin \sigma(U^{\Lambda_L}(\alpha)) \cup \sigma(S_N^{\Lambda_L})$. Hence,

$$\text{dist}\,(z, \sigma(S_N^{\Lambda_L})) > e^{|\alpha|\theta_M} - 1 \quad \Longrightarrow \quad z \in \rho(U^{\Lambda_L}(\alpha)). \tag{12.12}$$

Now (8.21) says

$$\text{dist}\,(e^{id\lambda_0}, \sigma(S_N^{\Lambda_L}) \setminus \{e^{id\lambda_0}\}) > \delta = \frac{t^2 c_0}{|\Lambda_L|^{2/d}}. \tag{12.13}$$

Thus, if $|\alpha| < \alpha_0 := \frac{\ln(1+\delta/2)}{\theta_M}$,

$$\{z \mid |z - e^{id\lambda_0}| = \delta/2\} \subset \rho(U^{\Lambda_L}(\alpha)). \tag{12.14}$$

We now take $\alpha \in (-\alpha_0/2, \alpha_0/2)$ so that $\tilde{\alpha} < \alpha_0/2$ and $r = \alpha_0/2$ so that

$$\{z \mid |z - \tilde{\alpha}| = \alpha_0/2\} \subset \{z \mid |z| \leq \alpha_0\} \tag{12.15}$$

and for such z's $|e^{i\lambda_0(z)} - e^{id\lambda_0}| < \delta/2$. Using $\delta/4 < \ln(1+\delta/2) < \delta/2$ if $\delta < 1/2$, one gets that if $0 \leq \alpha < \frac{\delta}{8\theta_M}$,

$$\left|\frac{\alpha^2}{2}\frac{d^2}{d^2\alpha}e^{i\lambda_0(\tilde{\alpha})}\right| \leq \alpha^2 \frac{\delta}{2}\left(\frac{2}{\alpha_0}\right)^2 \leq \alpha^2 32 \frac{\theta_M^2}{\delta}. \tag{12.16}$$

We get the announced result with

$$c_1 = \frac{128\theta_M^2}{t^2 c_0}, \quad c_2 = \frac{t^2 c_0}{32\theta_M}, \tag{12.17}$$

provided $|\Lambda_L|^{2/d} = (4L+2)^2 > 2t^2 c_0$, i.e. for L sufficiently large. ■

Proof of Proposition 12.1: Assume that $|e^{i\lambda(U^{\Lambda_L})} - e^{id\lambda_0}| \leq b/L^2$, with b to be determined later. Using the monotony in α (Lemma 12.1) and Lemma 12.3, we have for $0 \leq \alpha \leq c_2/L^2$ and L large enough

$$\begin{aligned}
\left|\alpha\left(\frac{d}{d\alpha}e^{i\lambda_0(\alpha)}\right)_{\alpha=0}\right| &\leq \left|e^{i\lambda_0(\alpha)} - e^{id\lambda_0}\right| + c_1\alpha^2 L^2 \\
&\leq \left|e^{i\lambda(U^{\Lambda})} - e^{id\lambda_0}\right| + c_1\alpha^2 L^2 \\
&\leq \frac{b}{L^2} + c_1\alpha^2 L^2.
\end{aligned} \tag{12.18}$$

Dividing by α and then choosing $\alpha = c_4/L^2$ such that $c_4 \le c_2$ and $c_1 c_4 \le s_0/2$, we obtain that

$$\left| \left(\frac{d}{d\alpha} e^{i\lambda_0(\alpha)} \right)_{\alpha=0} \right| \le b/c_4 + s_0/2. \tag{12.19}$$

Next we choose b such that $b/c_4 \le s_0/2$ to get that

$$\left| \left(\frac{d}{d\alpha} e^{i\lambda_0(\alpha)} \right)_{\alpha=0} \right| \le s_0. \tag{12.20}$$

Note that b is thus independent of d and L. On the other hand we have from (12.3) that

$$\left| \left(\frac{d}{d\alpha} e^{i\lambda_0(\alpha)} \right)_{\alpha=0} \right| = \frac{1}{(4L+2)^d} \sum_{k \in \Lambda_L} \theta_k. \tag{12.21}$$

In probabilistic terms

$$\left\{ \omega \mid |e^{i\lambda(U^{\Lambda_L})} - e^{id\lambda_0}| \le b/L^2 \right\} \subset \left\{ \omega \mid \frac{1}{(4L+2)^d} \sum_{k \in \Lambda_L} \theta_k \le s_0 \right\}. \tag{12.22}$$

Finally an application of Lemma 12.2 ends the proof with $\gamma = 2^d \gamma_0$. ■

The Lifshits tail estimate of Proposition 12.1 and the properties of the Neumann boundary conditions, Proposition 10.1, allow to prove the following result, which is based on an equivalent result for Schrödinger operators provided in [56].

Proposition 12.2. *Let $\beta \in (0,1)$. There exist finite positive constants $\bar{\gamma}$, C and a sequence of positive integers L_k with $L_k \to \infty$ such that for any k and any $z \in \mathbb{C}$ with $1 < |z| < 2$ and $d\lambda_0 - 1/L_k^\beta \le \arg z \le d\lambda_0$,*

$$\mathbb{P}(\text{dist } (z, \sigma(U^{\Lambda_{L_k}})) \le 1/L_k^\beta) \le C L_k^{d(1-\beta/2)} e^{-\bar{\gamma} L_k^{d\beta/2}}. \tag{12.23}$$

Proof: Let $\beta \in (0,1)$ and $b > 0$ the constant found in Proposition 12.1. Fix a constant $C > 1$.

We claim that for each sufficiently large $k \in \mathbb{N}$ there exists $L_k \in \mathbb{N}$ which is a multiple of k and such that

$$\frac{b}{Ck^2} \le \frac{2}{L_k^\beta} \le \frac{b}{k^2}. \tag{12.24}$$

To see this, note that (12.24) is equivalent to

$$L_k \in \left[(2k^2/b)^{1/\beta}, (2k^2/b)^{1/\beta} C^{1/\beta} \right]. \tag{12.25}$$

As $\beta < 1$, for k sufficiently large, this interval has length larger than k, allowing for a choice of L_k as required.

We now show that (12.24) holds for these L_k. Split the box $\Lambda_{L_k} = [-2L_k, 2L_k+1]^d$ into $M = |\Lambda_{L_k}|/|\Lambda_k| = (L_k/k)^d$ disjoint boxes as

$$\Lambda_{L_k} = \cup_{j=1}^M \Lambda_k(j), \tag{12.26}$$

where $\Lambda_k(j)$ denotes the suitably translated box $\Lambda_k + c(j)$. Consider now

$$U_N = U^{\Lambda_k(1)} \oplus U^{\Lambda_k(2)} \oplus \cdots \oplus U^{\Lambda_k(M)} \tag{12.27}$$

on $l^2(\Lambda_{L_k}) = \oplus_{j=1}^M l^2(\Lambda_k(j))$, where each $U^{\Lambda_k(j)}$ is provided with Neumann boundary conditions. Proposition 10.1 shows that passing from $U^{\Lambda_{L_k}}$ to U_N by introducing Neumann boundary conditions makes the eigenvalues come closer to the upper band edge, i.e.

$$\lambda(U^{\Lambda_{L_k}}) \leq \lambda(U^{\Lambda_k(j_0)}) \text{ for some } j_0 \in \{1, 2, \cdots, M\}, \tag{12.28}$$

where $e^{i\lambda(U)}$ denotes the eigenvalue of largest argument of U. As a consequence, taking the stochastic independence of the $(U^{\Lambda_k(j)})$ into account together with the relation (12.24)

$$\begin{aligned} \mathbb{P}\left(|e^{i\lambda(U^{\Lambda_{L_k}})} - e^{id\lambda_0}| \leq 2/L_k^\beta\right) &\leq \mathbb{P}\left(|e^{i\lambda(U^{(\Lambda_k(j_0))})} - e^{id\lambda_0}| \leq b/k^2 \text{ for some } j_0\right) \\ &\leq M\mathbb{P}\left(|e^{i\lambda(U^{(\Lambda_k(1))})} - e^{id\lambda_0}| \leq b/k^2\right). \end{aligned} \tag{12.29}$$

By applying Proposition 12.1 to the box $\Lambda_k(1)$ we see that the latter is bounded by

$$\frac{L_k^d}{k^d} e^{-\gamma k^d} \leq C L_k^{d(1-\beta/2)} e^{-\bar{\gamma} L_k^{d\beta/2}}. \tag{12.30}$$

Finally, since $\mathrm{dist}(z, \sigma(U^{\Lambda_{L_k}})) \leq 1/L_k^\beta$ for z such that $|z - e^{id\lambda_0}| \leq 1/L_k^\beta$ implies $|e^{i\lambda(U^{\Lambda_{L_k}})} - e^{id\lambda_0}| \leq 2/L_k^\beta$, we get the result. ■

13 Towards an Iterative Proof of Exponential Decay

The proof of Theorem 3.5, to be completed in Section 14, will proceed as follows: To prove exponential decay of $\mathbb{E}(|G(k,l;z)|^s)$ we will join the two sites k and l by a chain of boxes of side length L. For a suitable choice of L and $\arg z$ close to the edge of Σ, the Lifshits tail and Combes-Thomas estimates will show that the fractional moment of the finite volume Green function $G^{(L)}(k,j;z)$ is small (think "less than one" even if this is only true up to some factors which can be controlled). Here $G^{(L)}$ is the resolvent of a restriction of U to a box of side length L centered at k and j is a boundary site of this box. To turn this into a proof of exponential decay of the infinite volume Green function, we need two more tools: (i) a factorization of the infinite volume Green function into finite volume factors, often referred to as a *geometric resolvent identity*, (ii) a *decoupling argument* which allows to factorize the fractional moments in the geometric resolvent identity. These two remaining tools will be provided in this section.

As explained at the beginning of Section 12 we will continue to focus on the localization proof at the upper band edge and continue to assume (12.1), so that the upper edge of Σ is $e^{id\lambda_0}$.

13.1 A Geometric Resolvent Identity

Due to the specific structure of our operators (in particular their ergodicity with respect to translations by two) it is of advantage to cut up $\mathbb{Z}^d$ into cubes of side length two. Thus, for $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ let

$$C_n := [2n_1, 2n_1 + 1] \times \dots \times [2n_d, 2n_d + 1] \tag{13.1}$$

and $\chi_n := \chi_{C_n}$ the characteristic function of C_n. For $L \in \mathbb{N}$ let

$$\Lambda_L = \bigcup_{|n| \le L} C_n = [-2L, 2L+1]^d. \tag{13.2}$$

We will work with restrictions $U_\omega^{\Lambda_L}$ and $U_\omega^{\Lambda_L^c}$ of U_ω to Λ_L and its complement $\Lambda_L^c = \mathbb{Z}^d \setminus \Lambda_L$. We choose $U_\omega^{\Lambda_L} = D_\omega S_N^{\Lambda_L}$, where $S_N^{\Lambda_L}$ is the unitary Laplacian with Neumann boundary conditions from (8.18). In fact, the choice of boundary conditions is rather irrelevant as long as matrix elements are only affected near the boundary, e.g. we have

$$U_\omega^{\Lambda_L}(j,k) = U_\omega(j,k) \quad \text{if } j,k \in \Lambda_{L-1}. \tag{13.3}$$

Our definition of Neumann operators from Section 8 does not directly extend to operators on exterior domains such as Λ_L^c, where the unitary Laplacian can not be defined as a tensor product of one-dimensional Laplacians. While it is possible to define Neumann boundary conditions directly for the d-dimensional operator, we choose a more simplistic approach and define

$$U_\omega^{\Lambda_L^c} = P^{\Lambda_L^c} U_\omega P^{\Lambda_L^c}, \tag{13.4}$$

viewed as an operator on $\ell^2(\Lambda_L^c)$. Here $P^{\Lambda_L^c}$ denotes the orthogonal projection onto $\ell^2(\Lambda_L^c)$. The price for our simplemindedness is that $U_\omega^{\Lambda_L^c}$ is not unitary. However, it is a contraction, i.e. $\|U_\omega^{(\Lambda_L^c)}\| \le 1$ and therefore $\sigma(U_\omega(\Lambda_L^c)) \subset \{z \in \mathbb{C} : |z| \le 1\}$, and it remains a band matrix whose entries satisfy, by definition,

$$U_\omega^{\Lambda_L^c}(j,k) = U_\omega(j,k) \quad \text{if } j,k \in \Lambda_L^c. \tag{13.5}$$

These properties will suffice for what we need in Section 14.

We will use what is often referred to as a geometric resolvent identity, relating the resolvents of U_ω, $U_\omega^{\Lambda_L}$ and $U_\omega^{\Lambda_L^c}$. Following an argument which for the selfadjoint Anderson model is used in [5], we start by defining the boundary operator $T_\omega^{(L)}$ through

$$U_\omega = U_\omega^{\Lambda_L} \oplus U_\omega^{\Lambda_L^c} + T_\omega^{(L)}. \tag{13.6}$$

By the above construction of $U_\omega^{\Lambda_L}$ and $U_\omega^{\Lambda_L^c}$, in particular (13.3) and (13.5), the operator $T_\omega^{(L)}$ has non-vanishing matrix-elements only near the boundary of Λ_L, more specifically

$$T_\omega^{(L)} \chi_x = \chi_x T_\omega^{(L)} = 0 \quad \text{if } |x| \le L-1 \text{ or } |x| \ge L+2 \tag{13.7}$$

as well as

$$\chi_x T_\omega^{(L)} \chi_y = 0 \quad \text{if } |x-y| \geq 2. \tag{13.8}$$

Also, the matrix-elements of $T_\omega^{(L)}$ are uniformly bounded in L and ω.

To keep the length of the following equations under control we will drop the arguments ω and z and for the rest of this section write

$$G := (U_\omega - z)^{-1} \tag{13.9}$$

and

$$G^{(L)} := (U_\omega^{\Lambda_L} \oplus U_\omega^{\Lambda_L^c} - z)^{-1} = (U_\omega^{\Lambda_L} - z)^{-1} \oplus (U_\omega^{\Lambda_L^c} - z)^{-1}. \tag{13.10}$$

We do a double-decoupling, once on Λ_L and once on Λ_{L+1}. Using the resolvent identity twice gives

$$\begin{aligned} G &= G^{(L)} - G^{(L)}T^{(L)}G \\ &= G^{(L)} - G^{(L)}T^{(L)}G^{(L+1)} + G^{(L)}T^{(L)}GT^{(L+1)}G^{(L+1)}. \end{aligned} \tag{13.11}$$

Observe that for $y \in \mathbb{Z}^d$ with $|y| \geq L+2$ one has $\chi_0 G^{(L)} \chi_y = 0$ and $\chi_0 G^{(L)} T^{(L)} G^{(L+1)} \chi_y = 0$. Thus

$$\chi_0 G \chi_y = \chi_0 G^{(L)} T^{(L)} G T^{(L+1)} G^{(L+1)} \chi_y, \tag{13.12}$$

which is the geometric resolvent identity to be used below.

13.2 Decoupling of Fractional Moments

The next result says that the fractional moment $\mathbb{E}(\|\chi_0 G \chi_y\|^s)$ can be decoupled along the boundary of Λ_L.

Proposition 13.1. *For every $s \in (0,1/3)$ there exists a constant $C = C(s) < \infty$ such that*

$$\mathbb{E}(\|\chi_0 G \chi_y\|^s) \leq C \sum_{u:\,|u|=L} \mathbb{E}(\|\chi_0 G^{(L)} \chi_u\|^s) \sum_{v':\,|v'|=L+2} \mathbb{E}(\|\chi_{v'} G^{(L+1)} \chi_y\|^s) \tag{13.13}$$

uniformly in z with $1 < |z| < 2$, $L \in \mathbb{N}$ and $y \in \mathbb{Z}^d$ with $|y| \geq L+2$.

Proof of Proposition 13.1: From here on, the symbol C will denote a generic constant which may change from line to line but which depends on inessential quantities only.

Define the boundary of Λ_L by

$$\begin{aligned} \partial\Lambda_L &:= \{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \chi_x T^{(L)} \chi_y \neq 0\} \\ &\subset \{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d : L \leq |x| \leq L+1, L \leq |y| \leq L+1, |x-y| \leq 1\}. \end{aligned} \tag{13.14}$$

Expanding (13.12) over the boundaries of Λ_L and Λ_{L+1} gives

$$\chi_0 G \chi_y = \sum_{\substack{(u,u') \in \partial\Lambda_L \\ (v,v') \in \partial\Lambda_{L+1}}} \chi_0 G^{(L)} \chi_u T^{(L)} \chi_{u'} G \chi_v T^{(L+1)} \chi_{v'} G^{(L+1)} \chi_y. \tag{13.15}$$

Taking fractional moments and also using that $T^{(L)}$ and $T^{(L+1)}$ have uniformly bounded matrix-elements we get

$$\mathbb{E}(\|\chi_0 G \chi_y\|^s) \leq C \sum_{\substack{(u,u') \in \partial\Lambda_L \\ (v,v') \in \partial\Lambda_{L+1}}} \mathbb{E}\left(\|\chi_0 G^{(L)} \chi_u\|^s \|\chi_{u'} G \chi_v\|^s \|\chi_{v'} G^{(L+1)} \chi_y\|^s\right). \tag{13.16}$$

Notice that the first and third factor in the expectation on the right are independent. Unfortunately, they are correlated via the middle factor. In order to decouple the factors we use a *re-sampling argument*, following a strategy developed in [3] and [14] as a tool in the fractional moments approach to continuum Anderson-type models. For this, fix two pairs $(u,u') \in \partial\Lambda_L$ and $(v,v') \in \partial\Lambda_{L+1}$. Let $\mathcal{J} := C_u \cup C_{u'} \cup C_v \cup C_{v'}$. In the resolvents $G^{(L)}$ and $G^{(L+1)}$ we will re-sample the random variables θ_n, $n \in \mathcal{J}$. For this choose i.i.d. random variables $\{\hat\theta_n\}_{n\in\mathcal{J}}$ with the same distribution as the θ_n but independent from them.

Noting that $D_\omega = \sum_{n\in\mathbb{Z}^d} e^{-i\theta_n} P_n$ (where $P_n = \langle e_n, \cdot\rangle e_n$ is the projection onto the canonical basis vector e_n) we define the re-sampled $D_{\omega,\hat\omega} := D_\omega - \hat D$, where

$$\hat D := \sum_{n\in\mathcal{J}} (e^{-i\theta_n} - e^{-i\hat\theta_n}) P_n, \tag{13.17}$$

i.e. the variables $\{\theta_n\}_{n\in\mathcal{J}}$ are replaced by the corresponding $\hat\theta_n$. Also define

$$U^{(L)}_{\omega,\hat\omega} := D_{\omega,\hat\omega} S_N^{\Lambda_L} = U_\omega^{\Lambda_L} - \hat D S_N^{\Lambda_L} \tag{13.18}$$

where $U^{(L)}_\omega = U^{\Lambda_L}_\omega$, $U_0^{(L)} = S_N^{\Lambda_L}$ and

$$\hat G^{(L)} := (U^{(L)}_{\omega,\hat\omega} - z)^{-1}. \tag{13.19}$$

The resolvent identity yields

$$G^{(L)} = \hat G^{(L)} - \hat G^{(L)} \hat D S_N^{\Lambda_L} G^{(L)} \tag{13.20}$$

and

$$G^{(L+1)} = \hat G^{(L+1)} - G^{(L+1)} \hat D S_N^{\Lambda_{L+1}} \hat G^{(L+1)}. \tag{13.21}$$

We use this to bound the terms on the right of (13.16) by

$$\begin{aligned} & \mathbb{E}\left(\|\chi_0 G^{(L)} \chi_u\|^s \|\chi_{u'} G \chi_v\|^s \|\chi_{v'} G^{(L+1)} \chi_y\|^s\right) \\ \leq\ & \hat{\mathbb{E}}\mathbb{E}\Big[(\|\chi_0 \hat G^{(L)} \chi_u\|^s + \|\chi_0 \hat G^{(L)} \hat D S_N^{\Lambda_L} G^{(L)} \chi_u\|^s) \|\chi_{u'} G \chi_v\|^s \\ & (\|\chi_{v'} \hat G^{(L+1)} \chi_y\|^s + \|\chi_{v'} G^{(L+1)} \hat D S_N^{\Lambda_{L+1}} \hat G^{(L+1)} \chi_y\|^s)\Big] \\ =:\ & A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{13.22}$$

Here we have argued that the above bound holds for arbitrary fixed values of the $\hat{\theta}_n$. Thus it also holds after the average over these variables, denoted by $\hat{\mathbb{E}}$, is taken. Of the four terms $A_1, \ldots, A_4$, found by expanding the two sums in (13.22), we will now find bounds for A_1, the one most easily handled, and A_4, the most complicated one. Corresponding bounds for the two mixed terms A_2 and A_3 can then be found by "interpolating" the provided arguments.

Let $\mathbb{E}(\ldots|\mathcal{J})$ denote the conditional expectation with respect to the σ-field generated by the family $\{\theta_k\}_{k\notin\mathcal{J}}$. Due to independence this means that

$$\mathbb{E}(X|\mathcal{J}) = \int \ldots \int X \prod_{n\in\mathcal{J}} \tau(\theta_n) d\theta_n. \tag{13.23}$$

The re-sampled resolvents $\hat{G}^{(L)}$ and $\hat{G}^{(L+1)}$ are independent of the variables $(\theta_n)_{n\in\mathcal{J}}$. Thus, by the rule

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\mathcal{J})) \tag{13.24}$$

for conditional expectations,

$$\begin{aligned} A_1 &:= \hat{\mathbb{E}}\mathbb{E}\left[\|\chi_0\hat{G}^{(L)}\chi_u\|^s\|\chi_{u'}G\chi_v\|^s\|\chi_{v'}\hat{G}^{(L+1)}\chi_y\|^s\right] \\ &= \hat{\mathbb{E}}\mathbb{E}\left[\|\chi_0\hat{G}^{(L)}\chi_u\|^s\mathbb{E}(\|\chi_{u'}G\chi_v\|^s|\mathcal{J})\|\chi_{v'}\hat{G}^{(L+1)}\chi_y\|^s\right] \\ &\leq C\mathbb{E}(\|\chi_0G^{(L)}\chi_u\|^s)\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_y\|^s). \end{aligned} \tag{13.25}$$

In the last estimate we have used the bound $\mathbb{E}(\|\chi_{u'}G\chi_v\|^s|\mathcal{J}) \leq C$, e.g. Theorem 3.1, that the distribution of $(\hat{\theta}_n)_{n\in\mathcal{J}}$ is identical to the distribution of $(\theta_n)_{n\in\mathcal{J}}$, and that $\chi_0G^{(L)}\chi_u$ and $\chi_{v'}G^{(L+1)}\chi_y$ are stochastically independent. This bound for A_1 is of the form required in Proposition 13.1.

We continue with A_4, where we use (13.24) again and then apply Hölder's inequality to the conditional expectation:

$$\begin{aligned} A_4 &:= \hat{\mathbb{E}}\mathbb{E}\left[\|\chi_0\hat{G}^{(L)}\hat{D}S_N^{\Lambda_L}G^{(L)}\chi_u\|^s\|\chi_{u'}G\chi_v\|^s\|\chi_{v'}G^{(L+1)}\hat{D}S_N^{\Lambda_{L+1}}\hat{G}^{(L+1)}\chi_y\|^s\right] \\ &= \hat{\mathbb{E}}\mathbb{E}\left[\mathbb{E}(\|\ldots\|^s\|\ldots\|^s\|\ldots\|^s|\mathcal{J})\right] \\ &\leq \hat{\mathbb{E}}\mathbb{E}\Big[\mathbb{E}(\|\chi_0\hat{G}^{(L)}\hat{D}S_N^{\Lambda_L}G^{(L)}\chi_u\|^{3s}|\mathcal{J})^{1/3} \\ &\quad \times \mathbb{E}(\|\chi_{u'}G\chi_v\|^{3s}|\mathcal{J})^{1/3} \\ &\quad \times \mathbb{E}(\|\chi_{v'}G^{(L+1)}\hat{D}S_N^{\Lambda_{L+1}}\hat{G}^{(L+1)}\chi_y\|^{3s}|\mathcal{J})^{1/3}\Big]. \end{aligned} \tag{13.26}$$

We will now bound the three conditional expectations on the right separately. As $s < 1/3$, we can use Theorem 3.1 to bound the second factor by

$$\mathbb{E}(\|\chi_{u'}G\chi_v\|^{3s}|\mathcal{J})^{1/3} \leq C < \infty \tag{13.27}$$

uniformly in $(\theta_k)_{k\notin\mathcal{J}}$ and z.

To bound the first factor, we start from the definition of $\hat{D}$ to get

$$\|\chi_0 \hat{G}^{(L)} \hat{D} S_N^{\Lambda_L} G^{(L)} \chi_u\|^{3s} \le C \sum_{\ell \in \mathcal{J} \cap \Lambda_L} \|\chi_0 \hat{G}^{(L)} P_\ell\|^{3s} \|P_\ell S_N^{\Lambda_L} G^{(L)} \chi_u\|^{3s}, \tag{13.28}$$

note here that $\chi_0 \hat{G}^{(L)} P_\ell = 0$ for $\ell \notin \Lambda_L$. The unitary diagonal operator D_ω commutes with P_ℓ and thus

$$\begin{aligned} \|P_\ell S_N^{\Lambda_L} G^{(L)} \chi_u\| &= \|P_\ell S_N^{\Lambda_L} (D_\omega S_N^{\Lambda_L} - z)^{-1} \chi_u\| \\ &= \|P_\ell D_\omega S_N^{\Lambda_L} (D_\omega S_N^{\Lambda_L} - z)^{-1} \chi_u\| \\ &= \|P_\ell \chi_u + z P_\ell G^{(L)} \chi_u\| \\ &\le 1 + |z| \|P_\ell G^{(L)} \chi_u\|. \end{aligned} \tag{13.29}$$

The re-sampled resolvent $\hat{G}^{(L)}$ does not depend on the variables $(\theta_n)_{n \in \mathcal{J}}$. Thus, using Theorem 3.1 again, we get

$$\begin{aligned} &\mathbb{E}(\|\chi_0 \hat{G}^{(L)} \hat{D} S_N^{\Lambda_L} G^{(L)} \chi_u\|^{3s} | \mathcal{J}) \\ &\quad \le C \sum_{\ell \in \mathcal{J} \cap \Lambda_L} \|\chi_0 \hat{G}^{(L)} P_\ell\|^{3s} \mathbb{E}((1 + |z| \|P_\ell G^{(L)} \chi_u\|)^{3s} | \mathcal{J}) \\ &\quad \le C \sum_{\ell \in \mathcal{J} \cap \Lambda_L} \|\chi_0 \hat{G}^{(L)} P_\ell\|^{3s}. \end{aligned} \tag{13.30}$$

In a similar way we find

$$\mathbb{E}(\|\chi_{v'} G^{(L+1)} \hat{D} S_N^{\Lambda_{L+1}} \hat{G}^{(L+1)} \chi_y\|^{3s} | \mathcal{J}) \le C \sum_{\ell \in \mathcal{J} \cap \Lambda_{L+1}^c} \|P_\ell S_N^{\Lambda_{L+1}} \hat{G}^{(L+1)} \chi_y\|^{3s}. \tag{13.31}$$

By a calculation as in (13.29) we have $\|P_\ell S_N^{\Lambda_{L+1}} \hat{G}^{(L+1)} \chi_y\| = \|P_\ell \chi_y + z P_\ell \hat{G}^{(L+1)} \chi_y\| = |z| \|P_\ell \hat{G}^{(L+1)} \chi_y\|$. We conclude

$$\mathbb{E}(\|\chi_{v'} G^{(L+1)} \hat{D} S_N^{\Lambda_{L+1}} \hat{G}^{(L+1)} \chi_y\|^{3s} | \mathcal{J}) \le C \sum_{\ell \in \mathcal{J} \cap \Lambda_{L+1}^c} \|P_\ell \hat{G}^{(L+1)} \chi_y\|^{3s}. \tag{13.32}$$

Combining the bounds (13.27), (13.30) and (13.32) into (13.26) we arrive at

$$\begin{aligned} A_4 &\le C \sum_{\ell \in \mathcal{J} \cap \Lambda_L,\, \ell' \in \mathcal{J} \cap \Lambda_{L+1}^c} \hat{\mathbb{E}} \mathbb{E}(\|\chi_0 \hat{G}^{(L)} P_\ell\|^s \|P_{\ell'} \hat{G}^{(L+1)} \chi_y\|^s) \\ &= C \sum_{\ell \in \mathcal{J} \cap \Lambda_L} \mathbb{E}(\|\chi_0 G^{(L)} P_\ell\|^s) \sum_{\ell' \in \mathcal{J} \cap \Lambda_{L+1}^c} \mathbb{E}(\|P_{\ell'} G^{(L+1)} \chi_y\|^s). \end{aligned} \tag{13.33}$$

Here it was also used that $\mathcal{J}$ has a fixed finite number of elements and thus $(\sum_{j \in \mathcal{J}} x_j^{3s})^{1/3} \le C \sum_{j \in \mathcal{J}} x_j^s$. The last identity uses that $(\hat{\theta}_n)$ and (θ_n) are identically distributed and that $\chi_0 G^{(L)} P_\ell$ and $P_{\ell'} G^{(L+1)} \chi_y$ are stochastically independent.

The bounds (13.25), (13.33) and related bounds for the mixed terms A_2 and A_3 combine via (13.22) and (13.16) to prove Proposition 13.1. ■

13.3 The Start of an Iteration

We plan to use (13.13) as the first step in an iterative argument, where the next step consists of applying (13.13) again, but this time with $\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_y\|^s)$ on the left hand side with v' as the new origin. However, before doing this we need to replace $G^{(L+1)}$ with the original G, which can be done by reasoning similar to the decoupling argument of the previous section.

Proposition 13.2. *For every $s \in (0, 1/3)$ there exists a constant $C = C(s) < \infty$ such that*

$$\mathbb{E}(\|\chi_0 G\chi_y\|^s) \leq CL^{d-1} \sum_{|u|=L} \mathbb{E}(\|\chi_0 G^{(L)}\chi_u\|^s) \sum_{|x'|\in\{L+1,L+2\}} \mathbb{E}(\|\chi_{x'}G\chi_y\|^s) \tag{13.34}$$

uniformly in z with $1 < |z| < 2$, $L \in \mathbb{N}$ and $y \in \mathbb{Z}^d$ with $|y| \geq L + 2$.

Proof: According to Theorem 13.1 we need a bound for $\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_y\|^s)$ in terms of fractional moments of the full Green function G for each fixed v' with $\|v'\|_\infty = L + 2$. We start from the resolvent identity

$$G^{(L+1)} = G + G^{(L+1)}T^{(L+1)}G \tag{13.35}$$

and expand

$$T^{(L+1)} = \sum_{(w,w')\in\partial\Lambda_{L+1}} \chi_w T^{(L+1)}\chi_{w'}. \tag{13.36}$$

Combining both yields

$$\begin{aligned}\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_y\|^s) \quad &\leq \quad \mathbb{E}(\|\chi_{v'}G\chi_y\|^s) \\ &\qquad + C \sum_{(w,w')\in\partial\Lambda_{L+1}} \mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^s\|\chi_{w'}G\chi_y\|^s). \end{aligned}\tag{13.37}$$

With the goal of factorizing the expectation on the right we fix $(w, w') \in \partial\Lambda_{L+1}$ and resample over the variables θ_n for $n \in \tilde{\mathfrak{J}} := C_{v'} \cup C_w \cup C_{w'}$. With independent random variables $(\tilde{\theta}_n)_{n\in\tilde{\mathfrak{J}}}$ independent from the θ_n, but with identical distribution, define

$$\begin{aligned}\tilde{D} \quad &:= \quad \sum_{n\in\tilde{\mathfrak{J}}}(e^{-i\theta_n} - e^{-i\tilde{\theta}_n})P_n, &(13.38)\\ D_{\omega,\tilde{\omega}} \quad &:= \quad D_\omega - \tilde{D}, &(13.39)\\ U_{\omega,\tilde{\omega}} \quad &:= \quad D_{\omega,\tilde{\omega}}S = U_\omega - \tilde{D}S, &(13.40)\\ \tilde{G} \quad &:= \quad (U_{\omega,\tilde{\omega}} - z)^{-1}. &(13.41)\end{aligned}$$

The resolvent identity

$$G = \tilde{G} - G\tilde{D}S\tilde{G} \tag{13.42}$$

yields

$$\begin{aligned}
&\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^s\|\chi_{w'}G\chi_y\|^s) \\
&\quad\leq \quad \tilde{\mathbb{E}}\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^s\|\chi_{w'}\tilde{G}\chi_y\|^s) \\
&\qquad\quad + \tilde{\mathbb{E}}\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^s\|\chi_{w'}G\tilde{D}S\tilde{G}\chi_y\|^s) \\
&\quad =: \quad B_1 + B_2,
\end{aligned} \tag{13.43}$$

where $\tilde{\mathbb{E}}$ denotes averaging over the variables $\tilde{\theta}_n$. Also writing $\mathbb{E}(\dots|\tilde{\mathfrak{J}})$ for the conditional expectation with respect to the σ-field generated by the variables $(\theta_n)_{n\notin\tilde{\mathfrak{J}}}$ and arguing as in the previous section, one has

$$\begin{aligned}
B_1 &= \tilde{\mathbb{E}}\mathbb{E}\left(\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^s|\tilde{\mathfrak{J}})\|\chi_{w'}\tilde{G}\chi_y\|^s\right) \\
&\leq C\mathbb{E}(\|\chi_{w'}G\chi_y\|^s)
\end{aligned} \tag{13.44}$$

Hölder's inequality yields

$$\begin{aligned}
B_2 &\leq \tilde{\mathbb{E}}\mathbb{E}\Big(\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^{2s}|\tilde{\mathfrak{J}})^{1/2} \\
&\qquad \times \mathbb{E}(\|\chi_{w'}G\tilde{D}S\tilde{G}\chi_y\|^{2s}|\tilde{\mathfrak{J}})^{1/2}\Big).
\end{aligned} \tag{13.45}$$

We have $\mathbb{E}(\|\chi_{v'}G^{(L+1)}\chi_w\|^{2s}|\tilde{\mathfrak{J}}) \leq C$ and, by an argument as above,

$$\mathbb{E}(\|\chi_{w'}G\tilde{D}S\tilde{G}\chi_y\|^{2s}|\tilde{\mathfrak{J}}) \leq C\sum_{\ell\in\tilde{\mathfrak{J}}}\|P_\ell\tilde{G}\chi_y\|^{2s}. \tag{13.46}$$

This leads to the bound

$$\begin{aligned}
B_2 &\leq C\tilde{\mathbb{E}}\mathbb{E}\left(\sum_{\ell\in\tilde{\mathfrak{J}}}\|P_\ell\tilde{G}\chi_y\|^s\right) \\
&\leq C\sum_{x':\,C_{x'}\subset\tilde{\mathfrak{J}}}\mathbb{E}(\|\chi_{x'}G\chi_y\|^s).
\end{aligned} \tag{13.47}$$

Collecting (13.43), (13.44) and (13.47) into (13.37), using that $\partial\Lambda_{L+1}$ has CL^{d-1} elements, and ultimately applying Proposition 13.1 completes the proof of Proposition 13.2. ■

14 Proof of Band Edge Localization

We finally have reached the point where all the main results of the previous three sections can be put together to prove Theorem 3.5. Specifically, we will use the Combes-Thomas-type bound of Proposition 11.1, the Lifshits tail estimate of Proposition 12.2 and the decoupling estimate in the form provided in Proposition 13.2. Also frequently enter will the a priori boundedness of fractional moments established in Theorem 3.1.

We will show the following fact which is equivalent to Theorem 3.5 (now in the normalization introduced at the beginning of Section 12 and only stated for the upper band edge): For $0 < s < 1/3$ there exist $\delta > 0$, $\alpha > 0$ and $C < \infty$ such that

$$\mathbb{E}(\|\chi_0 G(z)\chi_y\|^s) \leq Ce^{-\alpha|y|} \tag{14.1}$$

for all $y \in \mathbb{Z}^d$ and all $z \in \mathbb{C}$ with $1/2 < |z| < 2$, $|z| \neq 1$ and $\arg z \in [d\lambda_0 - \delta, d\lambda_0]$. Here χ_0 and χ_y are the characteristic functions of the cubes C_0 and C_y introduced in Section 13.1. Making the z-dependence explicit we write $G(z) = (U_\omega - z)^{-1}$ and $G^{(L)}(z) = (U_\omega^{\Lambda_L} - z)^{-1}$ in this section. The bound (14.1) implies $\mathbb{E}(\|\chi_x G(z)\chi_y\|^s) \leq Ce^{-\mu\|x-y\|_\infty}$ for arbitrary $x, y \in \mathbb{Z}^d$ due to ergodicity.

It suffices to prove (14.1) only for those z which in addition satisfy $|z| > 1$. To see this, use the identity

$$G^*(z) = -U_\omega(U_\omega - \bar{z}^{-1})/\bar{z} = -U_\omega G(1/\bar{z})/\bar{z}, \tag{14.2}$$

which implies

$$\|\chi_x G(z)\chi_y\| = \|\chi_y G^*(z)\chi_x\| = \|\chi_y U G(1/\bar{z})\chi_x\|/|z|. \tag{14.3}$$

Inserting the partition $\sum_{y'} \chi_{y'}$ and using that $\chi_y U \chi_{y'} = 0$ for $|y' - y| > 1$ we conclude

$$\|\chi_x G(z)\chi_y\| \leq \frac{1}{|z|} \sum_{|y'-y|\leq 1} \|\chi_{y'} G(1/\bar{z})\chi_x\|. \tag{14.4}$$

This shows that (14.1) carries over to z with $1/2 < |z| < 1$ once it has been proven for $1 < |z| < 2$, which is assumed for the remainder of this section.

Proposition 12.2 shows that the probability that $U^{\Lambda_{L_k}}$ has an eigenvalue close to $e^{id\lambda_0}$ is small for the sequence L_k found there. Combined with the Combes-Thomas bound Proposition 11.1 this can be used to show smallness of the fractional moments $\mathbb{E}(\|\chi_0 G^{(L_k)}(z)\chi_u\|^s)$ on the right hand side of (13.13) for values of z close to $e^{id\lambda_0}$.

Proposition 14.1. *For any $s \in (0, 1/3)$ there exist a sequence of integers L_k with $L_k \to \infty$, $g > 0$ and $C < \infty$ such that*

$$\mathbb{E}(\|\chi_0 G^{(L_k)}(z)\chi_u\|^s) \leq Ce^{-gL_k^{d/(d+2)}} \tag{14.5}$$

for all k sufficiently large, any $z \in \mathbb{C}$ such that $1 < |z| < 2$ and $\arg z \in [d\lambda_0 - L_k^{-2/(2+d)}, d\lambda_0]$ and any $u \in \mathbb{Z}^d$ with $|u| = L_k$.

Proof:
Let $\delta_L > 0$, to be specified later. The Combes-Thomas estimate Proposition 11.1 states that there exists $B > 0$ independent of L such that

$$\|\chi_0 G^{(L)}(z)\chi_u\| \leq \frac{2}{\delta_L} e^{-BL\delta_L} \tag{14.6}$$

for all z with $\mathrm{dist}(z,\sigma(U_\omega^{\Lambda_L}))>\delta_L$ and all $u\in\mathbb{Z}^d$ with $|u|=L$.

This takes care of the realizations ω such that the values of z are far enough from $\sigma(U_\omega^{\Lambda_L})$. The Lifshits tail estimate takes care of the realizations where this is not the case, in the sense that such instances are very unlikely.

We set

$$\Omega_G=\{\omega \mid \text{dist }(z,\sigma(U_\omega^{\Lambda_L}))>\delta_L\} \text{ and } \Omega_B=\Omega_G^C=\{\omega \mid \text{dist }(z,\sigma(U_\omega^{\Lambda_L}))\leq\delta_L\}. \quad (14.7)$$

Making use of (14.6), we can write by means of Hölder's inequality

$$\begin{aligned}\mathbb{E}(\|\chi_0G^{(L)}(z)\chi_u\|^s) &= \mathbb{E}(\|\chi_0G^{(L)}(z)\chi_u\|^s1_{\{\omega\in\Omega_G\}})+\mathbb{E}(\|\chi_0G^{(L)}(z)\chi_u\|^s1_{\{\omega\in\Omega_B\}})\\ &\leq \frac{2^s}{\delta_L^s}e^{-sBL\delta_L}\mathbb{E}(1_{\{\omega\in\Omega_G\}}))\\ &\quad+(\mathbb{E}(\|\chi_0G^{(L)}(z)\chi_u\|^{st})^{1/t}(\mathbb{E}(1_{\{\omega\in\Omega_B\}}))^{1/t'}, \end{aligned} \quad (14.8)$$

with $1<t<1/s$ and $1/t+1/t'=1$. Since $\mathbb{E}(1_{\{\omega\in\Omega\}})=\mathbb{P}(\Omega)$ and $\mathbb{E}(\|\chi_0G^{(L)}(z)\chi_u\|^{st})\leq C$ for $st<1$, by Theorem 3.1, we get

$$\mathbb{E}(\|\chi_0G^{(L)}(z)\chi_u\|^s)\leq C\frac{e^{-sBL\delta_L}}{\delta_L^s}+C(\mathbb{P}(\text{dist }(z,\sigma(U^{\Lambda_L}))\leq\delta_L))^{1/t'}. \quad (14.9)$$

To be useful for our purpose, this last quantity need to decay as $L\to\infty$, which requires $L\delta_L\to\infty$. On the other hand, we need $\delta_L\to 0$ for the probability that z is a distance δ_L only away from $\sigma(U^{\Lambda_L})$ to be very small, for suitable z in a neighborhood of the band edge $e^{id\lambda_0}$. In particular, this holds for the choice $\delta_L=1/L^\beta$ and any $\beta\in(0,1)$.

More specifically, with this choice of δ_L Proposition 12.2 yields the existence of a sequence L_k with $L_k\to\infty$ and positive $\bar\gamma$ and C such that

$$\mathbb{E}(\|\chi_0G^{(L_k)}(z)\chi_u\|^s)\leq Ce^{-sBL_k^{1-\beta}}L_k^{\beta s}+CL_k^{d(1-\beta/2)/t'}e^{-\frac{\bar\gamma}{t'}L_k^{d\beta/2}} \quad (14.10)$$

for all k, $1<|z|<2$, $d\lambda_0-1/L_k^\beta\leq\arg z\leq d\lambda_0$ and $|u|=L_k$. The choice $\beta=2/(2+d)$ leads to equal exponents of L_k in the two exponentials on the right hand side of (14.10). Choosing $g<\min(sB,\bar\gamma/t')$ and requiring k to be sufficiently large we can absorb the power terms in (14.10) into the exponentials and arrive at (14.5) ■

We proceed with the proof of (14.1) by fixing $s\in(0,1/3)$ and choosing the sequence L_k and g as in Proposition 14.1. We now also choose $\delta_k=L_k^{-2/(2+d)}$.

Proposition 13.2 says that

$$\mathbb{E}(\|\chi_0G(z)\chi_y\|^s)\leq CL_k^{d-1}\sum_{|u|=L_k}\mathbb{E}(\|\chi_0G^{(L_k)}(z)\chi_u\|^s)\sum_{|x'|\in\{L_k+1,L_k+2\}}\mathbb{E}(\|\chi_{x'}G(z)\chi_y\|^s) \quad (14.11)$$

if $|y|\geq L_k+2$. Let $1<|z|<2$ with $\arg z\in[d\lambda_0-\delta_k,d\lambda_0]$. This along with Proposition 14.1 imply, for k sufficiently large,

$$\mathbb{E}(\|\chi_0G(z)\chi_y\|^s)\leq CL_k^{2(d-1)}e^{-gL_k^{2/(2+d)}}\sum_{|x'|\in\{L_k+1,L_k+2\}}\mathbb{E}(\|\chi_{x'}G(z)\chi_y\|^s). \quad (14.12)$$

With the constant C from (14.12), fix $L = L_k$ for k sufficiently large such that

$$b := CL^{2(d-1)} e^{-gL^{2/(2+d)}} \#\{x' \in \mathbb{Z}^d : L+1 \leq |x'| \leq L+2\} < 1 \tag{14.13}$$

and get from (14.12) that

$$\mathbb{E}(\|\chi_0 G(z)\chi_y\|^s) \leq b \max_{\|x'\|_\infty \in \{L+1, L+2\}} \mathbb{E}(\|\chi_{x'} G(z)\chi_y\|^s). \tag{14.14}$$

Note that $\mathbb{E}(\|\chi_{x'} G(z)\chi_y\|^s) = \mathbb{E}(\|\chi_0 G(z)\chi_{y-x'}\|^s)$, which allows to iterate (14.14). If $x', x^{(2)}, x^{(3)}, \ldots$ is one of the chains of sites obtained in this way, then the iteration may be continued as long as $|x^{(j)} - y| \geq L+2$, i.e. at least $\frac{|y|}{L+2} - 1$ times. For the last entry in the chain we use Theorem 3.1 to bound $\mathbb{E}(\|\chi_{x^{(j)}} G(z)\chi_y\|^s)$ by $\tilde{C}$. In (14.14) this leads to the bound

$$\mathbb{E}(\|\chi_0 G(z)\chi_y\|^s) \leq \tilde{C} b^{\frac{|y|}{L+2}-1} = \frac{\tilde{C}}{b} e^{\frac{\log b}{L+2}|y|}. \tag{14.15}$$

Thus we have proven (14.1) with $C = \frac{\tilde{C}}{b}$ and $\alpha = \frac{|\log b|}{L+2}$.

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