

Landauer’s principle for trajectories of repeated interaction systems

Eric P. Hanson¹, Alain Joye², Yan Pautrat³, and Renaud Raquépas⁴

¹Dept. Applied Math and Theoretical Physics, Centre for Mathematical Sciences,
University of Cambridge, Cambridge, CB3 0WA, United Kingdom

²Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38 000 Grenoble, France

³Laboratoire de Mathématiques d’Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay,
91405 Orsay, France

⁴Dept. of Mathematics and Statistics, McGill University, 1005–805 rue Sherbrooke O.,
Montréal (Québec) H3A 0B9

March 12, 2018

Abstract

We analyze Landauer’s principle for *repeated interaction systems* consisting of a reference quantum system \mathcal{S} in contact with an environment \mathcal{E} which is a chain of independent quantum probes. The system \mathcal{S} interacts with each probe sequentially, for a given duration, and the Landauer principle relates the energy variation of \mathcal{E} and the decrease of entropy of \mathcal{S} by the entropy production of the dynamical process. We consider refinements of the Landauer bound at the level of the *full statistics* (FS) associated to a two-time measurement protocol of, essentially, the energy of \mathcal{E} . The emphasis is put on the *adiabatic regime* where the environment, consisting of $T \gg 1$ probes, displays variations of order T^{-1} between the successive probes, and the measurements take place initially and after T interactions. We prove a large deviation principle and a central limit theorem as $T \rightarrow \infty$ for the classical random variable describing the entropy production of the process, with respect to the FS measure. In a special case, related to a detailed balance condition, we obtain an explicit limiting distribution of this random variable without rescaling. At the technical level, we obtain a non-unitary adiabatic theorem generalizing that of [HJPR17] and analyze the spectrum of complex deformations of families of irreducible completely positive trace-preserving maps.

1 Introduction

The present paper studies a refinement of Landauer’s principle in terms of a two-time measurement protocol (better known as “full counting statistics”) for repeated interaction systems, in an adiabatic regime. We describe shortly the various elements we study.

Landauer’s principle is a universal principle commonly formulated as a lower bound for the energetic cost of erasing a bit of information in a fixed system \mathcal{S} by interaction with an environment \mathcal{E} initially at thermal equilibrium. It was first stated by Landauer in [Lan61]. A recent, mathematically sound derivation (in [RW14], later extended to the case of infinitely extended systems in [JP14]) is based on the entropy balance equation, given by $\Delta S_{\mathcal{S}} + \sigma = \beta \Delta Q_{\mathcal{E}}$ where $\Delta S_{\mathcal{S}}$ is the average decrease in entropy of \mathcal{S} during the process, $\Delta Q_{\mathcal{E}}$ the average increase in energy of \mathcal{E} , and β is the inverse temperature of the environment¹. The term σ is called the entropy production of the

¹we will always set the Boltzmann constant to 1, so that $\beta = 1/\Theta$, Θ the temperature.

process. As it can be written as a relative entropy, the entropy production is non-negative which yields the inequality $\Delta S_{\mathcal{S}} \leq \beta \Delta Q_{\mathcal{E}}$. One of the questions of interest regarding Landauer’s principle concerns the saturation of that identity, i.e. the vanishing of σ . It is a general physical principle that when the system–environment coupling is described by a time-dependent Hamiltonian, the entropy production σ vanishes in the adiabatic limit, that is, when the coupling between \mathcal{S} and \mathcal{E} is a slowly varying time-dependent function. More precisely, if the typical time scale of the coupling is T , one considers the regime $T \rightarrow \infty$.

A repeated interaction system (or RIS) is a system where the environment consists of a sequence of “probes” \mathcal{E}_k , $k = 1, \dots, T$, initially in a thermal state at inverse temperature β_k , and \mathcal{S} interacts with \mathcal{E}_k (and only \mathcal{E}_k) during the time interval $[k\tau, (k+1)\tau)$. In such a situation, the entropy balance equation becomes $\sum_{k=1}^T \Delta S_{\mathcal{S}} + \sum_{k=1}^T \sigma_k = \sum_{k=1}^T \beta_k \Delta Q_{\mathcal{E},k}$, where each term with index k corresponds to the interaction between \mathcal{S} and \mathcal{E}_k . We describe the repeated interaction system as an “adiabatic RIS” when the various parameters of the probes are sampled from sufficiently smooth functions on $[0, 1]$ as the values at times k/T , $k = 1, \dots, T$. This is the setup that was studied in [HJPR17]; there we showed that the total entropy production $\lim_{T \rightarrow \infty} \sum_{k=1}^T \sigma_k$ was finite only under the condition $X(s) = 0$ for all $s \in [0, 1]$, where $X(s)$ is a quantity depending on the probe parameters at time $s \in [0, 1]$ which we discuss below, see (18). The proof of this result relied mostly on a new discrete, non-unitary adiabatic theorem that allowed us to control a product of T slowly varying completely positive, trace-preserving (CPTP) maps that represent the reduced dynamics acting on \mathcal{S} .

A refinement of the above formulation of Landauer’s principle is however possible using the so-called full counting statistics. Full counting statistics were first introduced in the study of charge transport, and have met with success in the study of fluctuation relations and work in quantum mechanics (see Kurchan [Kur00] and Tasaki [Tas00]). An example of their use in improving Landauer’s principle was given in [BFJP17, GCG⁺17]. In the present situation, the formulation of Landauer’s principle in terms of full counting statistics can be stated by defining random variables $\Delta s_{\mathcal{S}}$ and $\Delta q_{\mathcal{E}_k}$ which are outcomes of simple physical experiments, which we now describe. In such an experiment, one initially measures the quantity $-\log \rho_{\mathcal{S}}$ ($\rho_{\mathcal{S}}$ is the state of the small system) and the energies $h_{\mathcal{E}_k}$ for each k ($h_{\mathcal{E}_k}$ is the free Hamiltonian of \mathcal{E}_k), then lets the system interact with the chain of probes, then measures again the same quantities. With the right sign conventions, the changes in these quantities are random variables which we denote $\Delta s_{\mathcal{S}}$ and $\Delta q_{\mathcal{E}_k}$. Our refinement discusses the connections between the probability distributions of $\Delta s_{\mathcal{S}}$ and $\sum_k \beta_k \Delta q_{\mathcal{E}_k}$. One can show that the expectations of these distributions are $\Delta S_{\mathcal{S}}$ and $\sum_k \beta_k \Delta Q_{\mathcal{E}_k}$ respectively; there is, therefore, more information in these distributions than in the previously considered scalar quantities.

We consider an adiabatic repeated interaction system and study the limiting distributions of the above random variables as $T \rightarrow \infty$. Again, we show that in the case $X(s) \equiv 0$ we have the expected refinement of Landauer’s principle, which is essentially that when $T \rightarrow \infty$, one has $\Delta s_{\mathcal{S}} = \sum_k \beta_k \Delta q_{\mathcal{E}_k}$ almost-surely. In the case $X(s) \not\equiv 0$, we show that $\sum_k \beta_k \Delta q_{\mathcal{E}_k}$ satisfies a law of large numbers, a central limit theorem, and a large deviation principle, all of these for the time scale T , and with explicit parameters. In particular, $\sum_k \beta_k \Delta q_{\mathcal{E}_k}$ is of order T , whereas $\Delta s_{\mathcal{S}}$ is a bounded quantity. All results in the case $X(s) \not\equiv 0$ can actually be extended to the case where the probe observables measured at each step k are not simply $\beta_k h_{\mathcal{E}_k}$ but a more general observable, or when the system observables are not $-\log \rho_{\mathcal{S}}$.

We show in addition that the random variable $\varsigma_T = \sum_k \beta_k \Delta q_{\mathcal{E}_k} - \Delta s_{\mathcal{S}}$ can be expressed as a relative information random variable between the probability measure describing the experiment outcomes, and the probability measure corresponding to a backwards experiment. Since we obtain

a full large deviation principle for this random variable as $T \rightarrow \infty$, this connects these results with the appearance of the arrow of time (see [ABL64, BJPP17]). We discuss in particular the appearance of symmetries in the moment generating functions, and their implications in terms of Gallavotti–Cohen type symmetries.

To study the limiting distributions, we relate their moment generating functions to products of deformations of the completely positive, trace-preserving maps representing the reduced dynamics. We study the peripheral spectrum and associated spectral projector of these deformed dynamics. However, because little can be said about the spectral data of those deformed maps, studying the asymptotics of these quantities requires an improvement of the adiabatic theorem of [HJPR17]. These technical results, concerning the spectral study of deformations of CPTP maps, and the improved discrete non-unitary adiabatic theorem, are of independent interest, and we describe them in wider generality than required for our present endeavor.

This approach gives an improvement over [HJPR17] in various aspects. First of all, Theorem 4.2 (in the case $X(s) \equiv 0$) and Theorem 5.6 together with Corollary 5.8 and Theorem 5.10 (in the general case) characterize the limiting distributions of relevant random variables, whereas in [HJPR17] we only derived information about the behaviour of their expectations. We recover our former results (and more) about these expectations, as Theorem 4.2 implies in particular the convergence of $\lim_{T \rightarrow \infty} \sum_{k=1}^T \sigma_k$ to an explicit quantity when $X(s) \equiv 0$, and Theorem 5.10 gives the divergence of the same quantity under generic assumptions when $X(s)$ does not vanish identically. In addition, Corollary 3.14 gives an expression for the adiabatic evolution of any initial state. Most of all, our adiabatic theorem can be applied to a wider range of situations, as illustrated here by its application to deformed dynamics.

The structure of the present paper is as follows: in Section 2 we describe our general framework and notation, and more precisely we describe repeated interaction systems, Landauer’s principle (for unitary evolutions), and full counting statistics. We describe our full counting statistics for probe observables Y_k more general than just $\beta_k h_{\mathcal{E}_k}$, leading to random variables $\Delta y_T^{\text{tot}} = \sum_k \Delta y_k$, and we generalize $\Delta s_{S,T}$ (emphasising the T dependence in the notation), to random variables Δa_T as well. In Section 3 we discuss the various properties of the full statistics random variables: we give an entropy balance equation “at the level of trajectories”, i.e. almost-sure identities between the different random variables, relate the moment generating functions of e.g. Δy_T^{tot} to deformations of reduced dynamics, and give a general adiabatic result for products of these deformations. In Section 4 we describe the limiting distribution of the pair $(\Delta y_T^{\text{tot}}, \Delta s_{S,T})$ as $T \rightarrow \infty$ in the case $Y = \beta h_{\mathcal{E}}$ when $X(s) \equiv 0$. In Section 5 we derive a large deviation principle for Δy_T^{tot} in the general case, which in turn implies a law of large numbers and a central limit theorem. Our technical results regarding the peripheral spectrum and associated spectral projectors of deformations of completely positive, trace-preserving maps are given in Appendix A. Our improved discrete, non-unitary adiabatic theorem is given in Appendix B. Various proofs are collected in Appendix C.

Acknowledgements The research of E.H. was partly supported by ANR contract ANR-14-CE25-0003-0 and the Cantab Capital Institute for the Mathematics of Information (CCIMI) and he would like to thank Cambyse Rouzé for informative discussions. The research of Y.P. was supported by ANR contract ANR-14-CE25-0003-0. The research of R.R. was partly supported by NSERC, FRQNT and ANR project RMTQIT ANR-12-IS01-0001-01. R.R. would like to thank the Institut Fourier, where part of this research was carried, for its support and hospitality. E.H. and R.R. would like to also thank the organizers of the *Stochastic Methods in Quantum Mechanics* summer school (Autrans, July 2017) for the informative and hospitable event. All four authors would like to thank Vojkan Jakšić for stimulating discussions regarding this project.

2 General framework

In this section we will introduce our general framework. We will use the following notational conventions: for \mathcal{X} a Banach space, we denote by $\mathcal{B}(\mathcal{X})$ the space of bounded linear operators on \mathcal{X} and by Id the identity on \mathcal{X} . For \mathcal{H} a Hilbert space, we denote by $\mathcal{I}_1(\mathcal{H})$ the space of trace-class, linear operators on \mathcal{H} , and by $\mathcal{D}(\mathcal{H})$ the set of density matrices on \mathcal{H} , i.e. elements of $\mathcal{I}_1(\mathcal{H})$ which are non-negative operators with trace one. We will freely use the word “state” for an element $\rho \in \mathcal{D}(\mathcal{H})$, therefore identifying the density matrix and the linear map $\mathcal{B}(\mathcal{H}) \ni A \mapsto \rho(A)$. We say that a state is faithful if the density matrix ρ is positive-definite. Scalar products will generally be denoted by $\langle \phi, \psi \rangle$ and are respectively linear and antilinear in the right and left variable. We denote by $|\psi\rangle\langle\phi|$ the map on the Hilbert space defined by $\kappa \mapsto \langle \phi, \kappa \rangle \psi$.

2.1 Repeated interaction systems

A quantum repeated interaction system (RIS) consists of a system \mathcal{S} interacting sequentially with a chain $\mathcal{E}_1, \mathcal{E}_2, \dots$ of probes (or environments). This physical model can describe for example an electromagnetic cavity which undergoes repeated indirect measurement by probes; its physical archetype is the one-atom maser (see [MWM85]). For more detail we refer the reader to the review [BJM08].

We will describe the quantum system \mathcal{S} by a finite-dimensional Hilbert space $\mathcal{H}_{\mathcal{S}}$, a (time-independent) Hamiltonian $h_{\mathcal{S}} = h_{\mathcal{S}}^* \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$, and an initial state $\rho^i \in \mathcal{D}(\mathcal{H}_{\mathcal{S}})$. Likewise, the k th quantum probe \mathcal{E}_k will be described by a finite dimensional Hilbert space $\mathcal{H}_{\mathcal{E},k}$, Hamiltonian $h_{\mathcal{E}_k} = h_{\mathcal{E}_k}^* \in \mathcal{B}(\mathcal{H}_{\mathcal{E},k})$, and initial state $\xi_k^i \in \mathcal{D}(\mathcal{H}_{\mathcal{E},k})$. We will assume the probe Hilbert spaces $\mathcal{H}_{\mathcal{E},k}$ are all identical, $\mathcal{H}_{\mathcal{E},k} \equiv \mathcal{H}_{\mathcal{E}}$, and that the initial state of each probe is a Gibbs state at inverse temperature $\beta_k > 0$:

$$\xi_k^i = \frac{e^{-\beta_k h_{\mathcal{E}_k}}}{\text{Tr}(e^{-\beta_k h_{\mathcal{E}_k}})}.$$

We will at times use $Z_{\beta,k}$ to denote the trace $\text{Tr}(e^{-\beta_k h_{\mathcal{E}_k}})$.

The state of the system \mathcal{S} evolves by interacting with each probe, one at a time, as follows. Assume that after interacting with the first $k-1$ probes the state of the system is ρ_{k-1} . Then the system and the k th probe, with joint initial state $\rho_{k-1} \otimes \xi_k^i$, evolve for a time τ via the free Hamiltonian plus interaction v_k according to the unitary operator

$$U_k := \exp(-i\tau(h_{\mathcal{S}} \otimes \text{Id} + \text{Id} \otimes h_{\mathcal{E}_k} + v_k)),$$

yielding a joint final state $U_k(\rho_{k-1} \otimes \xi_k^i)U_k^*$. The probe \mathcal{E}_k is traced out, resulting in the system state

$$\rho_k := \text{Tr}_{\mathcal{E}}(U_k(\rho_{k-1} \otimes \xi_k^i)U_k^*),$$

where $\text{Tr}_{\mathcal{E}}$ is the partial trace over $\mathcal{H}_{\mathcal{E}}$, mapping $\mathcal{I}_1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$ to $\mathcal{I}_1(\mathcal{H}_{\mathcal{S}})$, with $\text{Tr}_{\mathcal{E}}(X \otimes Y) = \text{Tr}(Y)X$. We define similarly $\text{Tr}_{\mathcal{S}}$, the partial trace over $\mathcal{H}_{\mathcal{S}}$ and, for later use, also introduce $\xi_k^f := \text{Tr}_{\mathcal{S}}(U_k(\rho_{k-1} \otimes \xi_k^i)U_k^*)$. The evolution of the system \mathcal{S} during the k th step is given by the *reduced dynamics*

$$\begin{aligned} \mathcal{L}_k : \mathcal{I}_1(\mathcal{H}_{\mathcal{S}}) &\rightarrow \mathcal{I}_1(\mathcal{H}_{\mathcal{S}}) \\ \eta &\mapsto \text{Tr}_{\mathcal{E}}(U_k(\eta \otimes \xi_k^i)U_k^*), \end{aligned} \quad (1)$$

that is $\rho_k = \mathcal{L}_k(\rho_{k-1})$; remark that \mathcal{L}_k maps $\mathcal{D}(\mathcal{H}_{\mathcal{S}})$ to $\mathcal{D}(\mathcal{H}_{\mathcal{S}})$. By iterating this evolution, we find that the state of the system \mathcal{S} after k steps is given by the composition

$$\rho_k = (\mathcal{L}_k \circ \dots \circ \mathcal{L}_1)(\rho^i). \quad (2)$$

We will often omit the parentheses and composition symbols. For more details about the dynamics of RIS processes in various regimes, see [BJM14, HJPR17]. We now turn to energetic and entropic considerations on RIS, at the root of Landauer’s principle.

2.2 Landauer’s principle and the adiabatic limit

In what follows, for η, ζ in $\mathcal{D}(\mathcal{H})$, $S(\eta)$ denotes the von Neumann entropy of the state η and $S(\eta|\zeta)$ denotes the relative entropy between the states η and ζ :

$$S(\eta) = -\text{Tr}(\eta \log \eta), \quad S(\eta|\zeta) = \text{Tr}(\eta(\log \eta - \log \zeta)). \quad (3)$$

We recall that $S(\eta) \geq 0$ and $S(\eta|\zeta) \geq 0$. For each step k of the RIS process, we define the quantities

$$\begin{aligned} \Delta S_k &:= S(\rho_{k-1}) - S(\rho_k), \\ \Delta Q_k &:= \text{Tr}_{\mathcal{E}}(h_{\mathcal{E}_k} \xi_k^f) - \text{Tr}_{\mathcal{E}}(h_{\mathcal{E}_k} \xi_k^i), \end{aligned}$$

that represent the decrease in entropy of the small system, and the increase in energy of probe k , respectively, and

$$\sigma_k := S(U_k(\rho_{k-1} \otimes \xi_k^i)U_k^*|\rho_k \otimes \xi_k^i), \quad (4)$$

the *entropy production* of step k . For notational simplicity, we at times omit the “i” superscript in ξ_k^i . Also, we omit tensored identities for operators acting trivially on the environment or on the system, whenever the context is clear.

These quantities are related through the *entropy balance equation*

$$\Delta S_k + \sigma_k = \beta_k \Delta Q_k, \quad (5)$$

(see e.g. [RW14] for this computation). This equation, together with $\sigma_k \geq 0$, i.e. the nonnegativity of the entropy production term, encapsulates the more general *Landauer principle*: when a system undergoes a state transformation by interacting with a thermal bath, the average increase in energy of the bath is bounded below by β^{-1} times the average decrease in entropy of the system. This principle was first presented in 1961 by Landauer [Lan61] and its saturation in quantum systems has more recently been investigated by Reeb and Wolf [RW14] and Jakšić and Pillet [JP14], the latter providing a treatment of the case of infinitely extended quantum systems.

If we consider a RIS with T steps, then summing (5) over $k = 1, \dots, T$ yields the *total entropy balance equation*

$$\Delta S_{\mathcal{S},T} + \sigma_T^{\text{tot}} = \sum_{k=1}^T \beta_k \Delta Q_k, \quad (6)$$

where $\Delta S_{\mathcal{S},T} = S(\rho^i) - S(\rho^f)$ and $\rho^f = \rho_T$ is the state of \mathcal{S} after the final step of the RIS process (see (2)) and

$$\sigma_T^{\text{tot}} := \sum_{k=1}^T \sigma_k, \quad (7)$$

is the expected *total entropy production*.

In [HJPR17], the present authors analyzed the Landauer principle and its saturation in the framework of an adiabatic limit of RIS that we briefly recall here. We introduce the *adiabatic parameter* $T \in \mathbb{N}$ and consider a repeated interaction process with T probes, such that the parameters governing the k th probe and its interaction with \mathcal{S} , namely $(h_{\mathcal{E}_k}, \beta_k, v_k)$, are chosen by sampling sufficiently smooth functions as described by the following assumption. Below, we say that a function f is C^2 on $[0, 1]$ if it is C^2 on $(0, 1)$, and its first two derivatives admit limits at 0^+ , 1^- .

ADRIS We are given a family of RIS processes indexed by an adiabatic parameter $T \in \mathbb{N}$ such that there exist C^2 functions $s \mapsto h_{\mathcal{E}}(s), \beta(s), v(s)$ on $[0, 1]$ for which

$$h_{\mathcal{E}_k} = h_{\mathcal{E}}\left(\frac{k}{T}\right), \quad \beta_k = \beta\left(\frac{k}{T}\right), \quad v_k = v\left(\frac{k}{T}\right)$$

for all $k = 1, \dots, T$ when the adiabatic parameter has value T .

In this case, we may define

$$\begin{aligned} U(s) &= \exp\left(-i\tau(h_{\mathcal{E}}(s) + h_{\mathcal{E}}(s) + v(s))\right), \\ \mathcal{L}(s) &= \text{Tr}_{\mathcal{E}}\left(U(s)(\cdot \otimes \xi(s))U(s)^*\right), \end{aligned} \tag{8}$$

where $\xi(s)$ is the Gibbs state at inverse temperature $\beta(s)$ for the Hamiltonian $h_{\mathcal{E}}(s)$ and τ is kept constant. Then, $[0, 1] \ni s \mapsto \mathcal{L}(s)$ is a $\mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}))$ -valued C^2 function, and $\mathcal{L}_k = \mathcal{L}\left(\frac{k}{T}\right)$ when the adiabatic parameter has value T . Note that for each $s \in [0, 1]$, the map $\mathcal{L}(s)$ is completely positive (CP) and trace preserving (TP). For some results, we will need to make some extra hypotheses on the family $(\mathcal{L}(s))_{s \in [0, 1]}$. We introduce such conditions:

Irr For each $s \in [0, 1]$, the map $\mathcal{L}(s)$ is *irreducible*, meaning that it has (up to a multiplicative constant) a unique invariant, which is a faithful state.

Prim For each $s \in [0, 1]$, the map $\mathcal{L}(s)$ is *primitive*, meaning that it is irreducible and 1 is its only eigenvalue of modulus one.

We recall in Appendix A equivalent definitions and implications of these assumptions. We recall in particular that the peripheral spectrum of an irreducible completely positive, trace-preserving map is a subgroup of the unit circle. We denote by $z(s)$ the order of that subgroup for $\mathcal{L}(s)$.

In [HJPR17], the present authors used a suitable adiabatic theorem to characterize the large T behaviour of the total entropy production term (7), which monitors the saturation of the Landauer bound in the adiabatic limit (note that the terms in the sum (7) are T -dependent through ADRIS). Briefly, under suitable assumptions, convergence of σ_T^{tot} is characterized by the fact that the term $X(s)$ defined in (18) below vanishes identically.

2.3 Full statistics of two-time measurement protocols

We now describe a two-time measurement protocol for repeated interaction systems with T probes. The outcome of this protocol is random, and we will relate its expectation to the quantities involved in the balance equation (6). Note that a similar protocol was considered in [HP13] (see also [BJPP17, BCJP18]).

For the purpose of defining the full statistics measure for an RIS, we will consider observables to be measured on both the system \mathcal{S} and the probes \mathcal{E}_k , $k \in \mathbb{N}$.

First, we assume we are given two observables A^i and A^f in $\mathcal{B}(\mathcal{H}_{\mathcal{S}})$ with spectral decomposition

$$A^i = \sum_{a^i} a^i \pi_{a^i}^i, \quad A^f = \sum_{a^f} a^f \pi_{a^f}^f$$

where a^i, a^f run over the distinct eigenvalues of A^i, A^f respectively, and $\pi_{a^i}^i, \pi_{a^f}^f$ denote the corresponding spectral projectors. When we consider increasing the number of probes T , we assume the observable A^i is independent of T (as we measure it on \mathcal{S} before the system interacts with any number of probes), but allow A^f to depend on T , as long as the family $(A^f)_{T=1}^{\infty}$ is uniformly bounded in T .

On the chain, we consider probe observables $Y_k \in \mathcal{B}(\mathcal{H}_{\mathcal{E}})$ to be measured on the probe \mathcal{E}_k . We require that each observable commutes with the corresponding probe Hamiltonian:

$$[Y_k, h_{\mathcal{E}_k}] = 0.$$

We write the spectral decomposition of each Y_k as

$$Y_k = \sum_{i_k} y_{i_k} \Pi_{i_k}^{(k)}.$$

If the k th probe is initially in the state ξ , a measurement of Y_k before the time evolution will yield y_{i_k} with probability $\text{Tr}(\xi \Pi_{i_k}^{(k)})$.

When assuming **ADRS** and discussing measured observables Y , we will always assume

Comm There is a twice continuously differentiable $\mathcal{B}(\mathcal{H}_{\mathcal{E}})$ -valued function $s \mapsto Y(s)$ on $[0, 1]$ such that $[Y(s), h_{\mathcal{E}}(s)] = 0$ at all $s \in [0, 1]$ for which, when the adiabatic parameter has value T ,

$$Y_k = Y\left(\frac{k}{T}\right), \quad k = 1, \dots, T.$$

The family of probe Hamiltonians themselves $Y(s) = h_{\mathcal{E}}(s)$ are suitable, but in our applications to Landauer's principle, we will be particularly interested in $Y(s) = \beta(s)h_{\mathcal{E}}(s)$.

Associated to the observables A^i , A^f and $(Y_k)_{k=1}^T$ and the state ρ^i , we define two processes: the *forward process*, and the *backward process*.

The forward process The system \mathcal{S} starts in some initial state $\rho^i \in \mathcal{D}(\mathcal{H}_{\mathcal{S}})$ and the probe \mathcal{E}_k starts in the initial Gibbs state $\xi_k \in \mathcal{D}(\mathcal{H}_{\mathcal{E}_k})$; we write the state of the chain of T probes $\Xi = \bigotimes_{k=1}^T \xi_k$. We measure A^i on \mathcal{S} and measure Y_k on \mathcal{E}_k for each $k = 1, \dots, T$. We obtain results a^i and $\vec{i} = (i_k)_{k=1}^T$ with probability

$$\text{Tr}((\rho^i \otimes \Xi)(\pi_{a^i}^i \otimes \Pi_{\vec{i}})),$$

where $\Pi_{\vec{i}} := \bigotimes_{k=1}^T \Pi_{i_k}^{(k)}$. Then the system interacts with each probe, one at a time, starting at $k = 1$ until $k = T$, via the time evolution

$$U_k := \exp(-i\tau(h_{\mathcal{S}} + h_{\mathcal{E}_k} + v_k)).$$

Next, we measure A^f on the system and measure Y_k on \mathcal{E}_k for each $k = 1, \dots, T$, yielding outcomes a^f and $\vec{j} = (j_k)_{k=1}^T$. Using the rules of measurement in quantum mechanics and conditional probabilities, the quantum mechanical probability of measuring the sequence $(a^i, a^f, \vec{i}, \vec{j})$ of outcomes is given by

$$\text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}})(\rho^i \otimes \Xi)(\pi_{a^i}^i \otimes \Pi_{\vec{i}})U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}})).$$

We emphasize that the outcomes are labelled by $(a^i, a^f, \vec{i}, \vec{j})$ which refers to the eigenprojectors of the operators involved, but not to the corresponding eigenvalues which only need to be distinct. Also, we may write the second measurement projector $\pi_{a^f}^f \otimes \Pi_{\vec{j}}$ only once by cyclicity of the trace.

The backward process The system starts in state

$$\rho_T^f := \text{Tr}_{\mathcal{E}}(U_T \cdots U_1(\rho^i \otimes \Xi)U_1^* \cdots U_T^*),$$

and the probe \mathcal{E}_k starts in the state ξ_k . We measure observable A^f on \mathcal{S} and Y_k on \mathcal{E}_k for each $k = 1, \dots, T$, yielding outcomes a^f and $(j_k)_{k=1}^T$. Then the system interacts with each probe, one at a time, starting with $k = T$ until $k = 1$, via the time evolution

$$U_k^* = \exp(i\tau(h_{\mathcal{S}} + h_{\mathcal{E}_k} + v_k)).$$

Then we measure A^i on \mathcal{S} and Y_k on \mathcal{E}_k for each $k = 1, \dots, T$, yielding outcomes a^i and $(i_k)_{k=1}^T$. The probability of these outcomes is given by

$$\text{Tr}(U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}})(\rho_T^f \otimes \Xi)(\pi_{a^f}^f \otimes \Pi_{\vec{j}})U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}})).$$

The full statistics associated to the two-step measurement process For notational simplicity, we assume that the cardinality of $\text{sp } Y(s)$ does not depend on k . We can therefore use the same index set \mathcal{J} for all eigenvalue sets: $\text{sp } Y_k = (y_{i_k})_{i_k \in \mathcal{J}}$ for all $k = 1, \dots, T$. We define the space

$$\Omega_T := \text{sp } A^i \times \text{sp } A^f \times \mathcal{J}^T \times \mathcal{J}^T$$

and equip it with the maximal σ -algebra $\mathcal{P}(\Omega_T)$. We will refer to elements $(a^i, a^f, \vec{i}, \vec{j})$ of Ω_T as *trajectories*, and denote them by the letter ω .

Definition 2.1. On Ω_T , we call the law of the outcomes for the forward process,

$$\mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j}) := \text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}})(\rho^i \otimes \Xi)(\pi_{a^i}^i \otimes \Pi_{\vec{i}})U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}})), \quad (9)$$

the forward *full statistics measure*. We denote by \mathbb{E}_T the expectation with respect to \mathbb{P}_T^F . We also consider the backward full statistics measure

$$\mathbb{P}_T^B(a^i, a^f, \vec{i}, \vec{j}) := \text{Tr}(U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}})(\rho_T^f \otimes \Xi)(\pi_{a^f}^f \otimes \Pi_{\vec{j}})U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}})) \quad (10)$$

which is the law of the outcomes for the backward process. Let us emphasize here that \mathbb{P}_T^F and \mathbb{P}_T^B depend on the spectral projectors $(\pi_{a^i}^i)_{a^i}$ of A^i , $(\pi_{a^f}^f)_{a^f}$ of A^f , and $(\Pi_{\vec{i}})$ of the $(Y_k)_k$, and not on the spectral values of these operators. In particular, the probabilities \mathbb{P}_T^F and \mathbb{P}_T^B associated with two families of observables $(Y(s))_{s \in [0,1]}$, $(Y'(s))_{s \in [0,1]}$ that have the same spectral projectors (as e.g. $Y(s) = \beta(s)h_{\mathcal{E}}(s)$ and $Y'(s) = h_{\mathcal{E}}(s)$) will be the same.

To $(Y_k)_{k=1}^T$, A^i , and A^f , we associate two generic classical random variables on $(\Omega_T, \mathcal{P}(\Omega_T))$:

$$\Delta a_T(a^i, a^f, \vec{i}, \vec{j}) := a^i - a^f, \quad (11)$$

$$\Delta y_T^{\text{tot}}(a^i, a^f, \vec{i}, \vec{j}) := \sum_{k=1}^T (y_{j_k}^{(k)} - y_{i_k}^{(k)}). \quad (12)$$

Note that the choice of defining Δa_T as $a^i - a^f$, i.e. as the decrease of the quantity a , is consistent with the standard formulation of Landauer's principle as given in Section 2.2. Additionally, the assumption that $(A^f)_{T=1}^\infty$ has uniformly bounded norm yields that the random variable Δa_T has L^∞ norm uniformly bounded in T .

Remark 2.2. When we work with an **ADRS**, the dependence in T of the U_k (remember that in this case U_k is of the form $U(k/T)$) prevents the family $(\mathbb{P}_T^F)_T$ from being consistent. The \mathbb{P}_T^F are therefore a priori not the restrictions of a probability \mathbb{P}^F on the space Ω_∞ , as is the case in [BJPP17] where the environments \mathcal{E}_k and the parameters $h_{\mathcal{E}_k}$, β_k and v_k do not depend on k .

3 Properties of the full statistics

In the present section we obtain a relation between classical random variables arising from the protocol defined in Subsection 2.3, and the quantity (6). We study the relevant properties of the distributions \mathbb{P}_T^F and \mathbb{P}_T^B , their relative information random variable, and its moment generating function.

3.1 Entropy production and entropy balance on the level of trajectories

We turn to obtaining an analogue of (6) for random variables on the probability space $(\Omega_T, \mathcal{P}(\Omega_T), \mathbb{P}_T^F)$. Remark first that $\mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j})$ and $\mathbb{P}_T^B(a^i, a^f, \vec{i}, \vec{j})$ are of the form $\mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j}) = \text{Tr}((\rho^i \otimes \Xi) S^* S)$ and $\mathbb{P}_T^B(a^i, a^f, \vec{i}, \vec{j}) = \text{Tr}((\rho_T^f \otimes \Xi) S S^*)$. Under the assumption that ρ^i and ρ_T^f are faithful we therefore have

$$\mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j}) = 0 \text{ if and only if } \mathbb{P}_T^B(a^i, a^f, \vec{i}, \vec{j}) = 0.$$

Since the image of a faithful state by an irreducible CPTP map is faithful (see the discussion following Definition A.1 below), ρ^i and ρ_T^f will be faithful as soon as ρ^i is faithful and assumption **Irr** holds.

This allows us to give the following definition:

Definition 3.1. If ρ^i and ρ_T^f are faithful, we define the classical random variable

$$\varsigma_T(a^i, a^f, \vec{i}, \vec{j}) := \log \frac{\mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j})}{\mathbb{P}_T^B(a^i, a^f, \vec{i}, \vec{j})},$$

on $(\Omega_T, \mathcal{P}(\Omega_T), \mathbb{P}_T^F)$, which we call the entropy production of the repeated interaction system associated to the trajectory $\omega = (a^i, a^f, \vec{i}, \vec{j})$.

Note that the random variable ς_T is the logarithm of the ratio of likelihoods, also known as the relative information random variable between \mathbb{P}_T^F and \mathbb{P}_T^B (see e.g. [CT06]). It is well-known that the distribution of such a random variable is related to the distinguishability of the two distributions (here \mathbb{P}_T^F and \mathbb{P}_T^B): see e.g. [BD15]. Distinguishing between \mathbb{P}_T^F and \mathbb{P}_T^B amounts to testing the arrow of time; we refer the reader to [JOPS12, BJPP17] for a further discussion of this idea.

We have the following result, essentially present in [HP13], whose proof is given in Appendix C.

Lemma 3.2. *Assume ρ^i and ρ_T^f are faithful. If*

$$i. \pi_{a^i}^i \rho^i \pi_{a^i}^i = \frac{\text{Tr}(\rho^i \pi_{a^i}^i)}{\dim \pi_{a^i}^i} \pi_{a^i}^i \text{ for each } a^i,$$

$$ii. \pi_{a^f}^f \rho_T^f \pi_{a^f}^f = \frac{\text{Tr}(\rho_T^f \pi_{a^f}^f)}{\dim \pi_{a^f}^f} \pi_{a^f}^f \text{ for each } a^f,$$

iii. *for each $k = 1, \dots, T$, the state ξ_k (or equivalently $h_{\mathcal{E}_k}$) is a function of Y_k ,*

then

$$\varsigma_T(a^i, a^f, \vec{i}, \vec{j}) = \log \left(\frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\text{Tr}(\pi_{a^f}^f \rho_T^f)} \frac{\dim \pi_{a^f}^f}{\dim \pi_{a^i}^i} \right) + \sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}), \quad (13)$$

where $E_{i_k}^{(k)} = \frac{\text{Tr}(h_{\mathcal{E}_k} \Pi_{i_k}^{(k)})}{\dim \Pi_{i_k}^{(k)}}$ are the energy levels of the k th probe.

Remark 3.3. The first two hypotheses are automatically satisfied if, for example, A^i and A^f are non-degenerate (all their spectral projectors are rank-one). All three hypotheses are automatically satisfied if, for example, ρ^i , ρ_T^f and ξ_k can be written as functions of A^i , A^f and Y_k (for each $k = 1, \dots, T$) respectively.

Again, ς_T depends on the spectral projectors of the observables A^i , A^f and $(Y_k)_{k=1}^T$, but not on their eigenvalues. However, with the choices $A^i = -\log \rho^i$, $A^f = -\log \rho_T^f$ and $Y(s) = \beta(s)h_{\mathcal{E}}(s)$, and writing the spectral decompositions $\rho^i = \sum r_a^i \pi_a^i$, $\rho_T^f = \sum r_a^f \pi_a^f$, the relation (13) takes the simpler form of a sum of differences of the obtained eigenvalues (measurement results):

$$\varsigma_T(\omega) = (-\log r_{a^f}^f) - (-\log r_{a^i}^i) + \sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}),$$

which is the random variable introduced earlier as $-\Delta a_T + \Delta y_T^{\text{tot}}(\omega)$ (again, in the case $Y = \beta h_{\mathcal{E}}$). In this case, $\Delta a_T = (-\log r_{a^i}^i) - (-\log r_{a^f}^f)$ is a classical random variable that is the difference of measurements of entropy observables on the system \mathcal{S} , which we call $\Delta s_{\mathcal{S},T}(\omega)$. On the other hand, $\sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)})$ is a classical random variable that encapsulates Clausius' notion of the entropy increase of the chain $(\mathcal{E}_k)_{k=1}^T$ on the level of trajectories, which we call $\Delta s_{\mathcal{E},T}(\omega)$. Then,

$$\Delta s_{\mathcal{S},T}(\omega) + \varsigma_T(\omega) = \Delta s_{\mathcal{E},T}(\omega), \quad (14)$$

and $\varsigma_T(\omega)$ measures the difference between these two entropy variations, on the trajectory ω .

Moreover, Proposition 3.4 below, whose proof is also left for the Appendix, links expression (13) to the entropy balance equation (6). Indeed, by showing that under suitable hypotheses the two terms on the right hand side of (13) average to the corresponding terms in (6), we show that $\mathbb{E}_T(\varsigma_T) = \sigma_T^{\text{tot}}$. In other words, σ_T^{tot} coincides with the relative entropy or Kullback–Leibler divergence $D(\mathbb{P}_T^F \parallel \mathbb{P}_T^B)$ between the classical distributions \mathbb{P}_T^F and \mathbb{P}_T^B . Recall that $D(\mathbb{P}_T^F \parallel \mathbb{P}_T^B) = 0$ if and only if $\mathbb{P}_T^F = \mathbb{P}_T^B$. Hence, we will refer to (13) as the *entropy balance equation on the level of trajectories*.

Proposition 3.4. *Assume that ρ^i is faithful and a function of A^i , that ρ_T^f is faithful and a function of A^f , and the state ξ_k (or equivalently $h_{\mathcal{E}_k}$) is a function of Y_k for each $k = 1, \dots, T$, then*

$$\mathbb{E}_T \left(\log \left(\frac{\text{Tr}(\pi_{a^i}^i \rho^i) \dim \pi_{a^f}^f}{\text{Tr}(\pi_{a^f}^f \rho^f) \dim \pi_{a^i}^i} \right) \right) = -\mathbb{E}_T(\Delta s_{\mathcal{S},T}) = S(\rho^f) - S(\rho^i) \quad (15)$$

and

$$\mathbb{E}_T \left(\sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}) \right) = \mathbb{E}_T(\Delta s_{\mathcal{E},T}) = \sum_{k=1}^T \beta_k \Delta Q_k. \quad (16)$$

Therefore,

$$\mathbb{E}_T(\varsigma_T) = \sigma_T^{\text{tot}}, \quad (17)$$

and relation (13) reduces to the entropy balance equation (6) upon taking expectation with respect to \mathbb{P}_T^F .

Before we move on with our program, let us make a number of remarks on the choice of $A^i = -\log \rho^i$ and $A^f = -\log \rho_T^f$.

Remarks 3.5.

- We made the assumption above that the operator A^f was uniformly bounded in T . This is true for $A^f = -\log \rho_T^f$, as mentioned in Remark 3.15 below.
- The observable $-\log \rho^i$ is the analogue of the information random variable in classical information theory.
- The observable $-\log \rho^f$ has the same interpretation and is the initial condition for the backward process in [EHM09, HP13] but it might seem odd that the observer is expected to have access to $\rho^f = \mathcal{L}_T \circ \dots \circ \mathcal{L}_1(\rho^i)$. However, one can see that the reduced state of the probe after the forward experiment is random with \mathbb{P}_T^F -expectation equal to ρ^f . In addition, ζ_T is a relative information random variable, and as such is relevant only to an observer who knows both distributions (here \mathbb{P}_T^F and \mathbb{P}_T^B). Such an observer, knowing the possible outcomes for the random states after the experiment, and their distribution, would necessarily know their average ρ^f .

We are interested in the full statistics of the random variables $\zeta_T(\omega)$ that we will address through its cumulant generating functions in the limit $T \rightarrow \infty$. We will consider two cases: $\lim_{T \rightarrow \infty} \sigma_T^{\text{tot}} < \infty$, and $\lim_{T \rightarrow \infty} \sigma_T^{\text{tot}} = \infty$. The behaviour of this averaged quantity was investigated in [HJPR17]. For a RIS satisfying the assumptions **ADRS** and **Prim**, the condition

$$\limsup_{T \rightarrow \infty} \sigma_T^{\text{tot}} < \infty$$

can be shown to be equivalent to the identity $X(s) \equiv 0$, where

$$X(s) := U(s)(\rho^{\text{inv}}(s) \otimes \xi^i(s))U(s)^* - \rho^{\text{inv}}(s) \otimes \xi^i(s), \quad (18)$$

and $\rho^{\text{inv}}(s)$ is the unique invariant state of $\mathcal{L}(s)$. If the assumption $X(s) \equiv 0$ does not hold, then $\lim_{T \rightarrow \infty} \sigma_T^{\text{tot}} = \infty$. It was proven in [HJPR17] that the condition $X(s) \equiv 0$ is equivalent to the existence of a family $(k_{\mathcal{S}}(s))_{s \in [0,1]}$ of observables on $\mathcal{H}_{\mathcal{S}}$ such that $[k_{\mathcal{S}}(s) + h_{\mathcal{E}}(s), U(s)] \equiv 0$.

We will consider the case $X(s) \equiv 0$ in Section 4, and the other case, $\sup_{s \in [0,1]} \|X(s)\|_1 > 0$, in Section 5. In either case, our main object of interest will be the moment generating function of the variables Δy_T^{tot} and Δa_T , which we can relate to deformations $\mathcal{L}^{(\alpha)}(s)$ of $\mathcal{L}(s)$.

3.2 Moment generating functions and deformed CP maps

We recall that the quantities Δa_T and Δy_T^{tot} are defined in (11) and (12). We also recall that the *moment generating function* (MGF) of a real-valued random variable V (with respect to the probability distribution \mathbb{P}_T^F , which will always be implicit in the present paper) is defined as the map $M_V : \alpha \mapsto \mathbb{E}_T(e^{\alpha V})$, and the MGF of a pair (V_1, V_2) as the map $M_{(V_1, V_2)} : (\alpha_1, \alpha_2) \mapsto \mathbb{E}_T(e^{\alpha_1 V_1 + \alpha_2 V_2})$. When V or (V_1, V_2) are given by the random variables Δy_T^{tot} , Δa_T , the above functions M_V (resp. $M_{(V_1, V_2)}$) are defined for all $\alpha \in \mathbb{C}$ (resp. for all $(\alpha_1, \alpha_2) \in \mathbb{C}^2$). For relevant properties of moment generating functions we refer the reader to Sections 21 and 30 of [Bil95].

Our main tool to study these moment generating functions is the following proposition:

Proposition 3.6. *For $\alpha \in \mathbb{C}$, define an analytic deformation of $\mathcal{L}(s)$ by the complex parameter α corresponding to the observable $Y(s)$:*

$$\mathcal{L}_Y^{(\alpha)}(s) : \mathcal{I}_1(\mathcal{H}_{\mathcal{S}}) \rightarrow \mathcal{I}_1(\mathcal{H}_{\mathcal{S}})$$

$$\eta \mapsto \text{Tr}_{\mathcal{E}} (e^{\alpha Y(s)} U(s) (\eta \otimes \xi(s)) e^{-\alpha Y(s)} U(s)^*). \quad (19)$$

Under assumption **Comm**, the moment generating function of Δy_T^{tot} is given by

$$M_{\Delta y_T^{\text{tot}}}(\alpha) = \text{Tr}_{\mathcal{S}} (\mathcal{L}_Y^{(\alpha)}(\frac{T}{T}) \cdots \mathcal{L}_Y^{(\alpha)}(\frac{1}{T}) (\sum_{a^i} \pi_{a^i}^i \rho^i \pi_{a^i}^i)).$$

If in addition $[A^i, \rho^i] = 0$, then the moment generating function of the pair $(\Delta y_T^{\text{tot}}, \Delta a_T)$ is given by

$$M_{(\Delta y_T^{\text{tot}}, \Delta a_T)}(\alpha_1, \alpha_2) = \text{Tr} (e^{-\alpha_2 A^f} \mathcal{L}_Y^{(\alpha_1)}(\frac{T}{T}) \cdots \mathcal{L}_Y^{(\alpha_1)}(\frac{1}{T}) (e^{+\alpha_2 A^i} \rho^i)).$$

so that in particular the moment generating function of $-\Delta a_T + \Delta y_T^{\text{tot}}$ is given by

$$M_{-\Delta a_T + \Delta y_T^{\text{tot}}}(\alpha) = \text{Tr}_{\mathcal{S}} (e^{+\alpha A^f} \mathcal{L}_Y^{(\alpha)}(\frac{T}{T}) \cdots \mathcal{L}_Y^{(\alpha)}(\frac{1}{T}) (e^{-\alpha A^i} \rho^i)).$$

See Appendix **C** for the proof. In Section **5** we will analyze the above moment generating functions, with the help of an adiabatic theorem for the non-unitary discrete time operators $\mathcal{L}_Y^{(\alpha)}$. The case $Y(s) = \beta(s) h_{\mathcal{E}}(s)$ plays a particular role for the analysis of Landauer's principle. The complex deformation of the map $\mathcal{L}(s)$ we consider is similar to the deformations introduced in [HMO07] for hypothesis testing on spin chains, and to the complex deformation of Lindblad operators introduced in [JPW14] suited to the study of entropy fluctuations for continuous time evolution.

Dropping the s -dependence from the notation, below, we first provide the expression for the adjoint of the deformation of $\mathcal{L}(s)$ with respect to the duality bracket on $\mathcal{B}(\mathcal{H}_{\mathcal{S}})$, $\langle C_1, C_2 \rangle = \text{Tr}_{\mathcal{S}}(C_1^* C_2)$. We temporarily make explicit the dependency of $\mathcal{L}_Y^{(\alpha)}$ in τ by denoting it $\mathcal{L}_Y^{(\alpha; \tau)}$; in particular, $\mathcal{L}_Y^{(\alpha; -\tau)}$ is obtained by replacing the unitary $U(s)$ with its adjoint $U^*(s)$.

Lemma 3.7. *The adjoint of the operator $\mathcal{L}_Y^{(\alpha)}$ is given by*

$$\mathcal{L}_Y^{(\alpha)*} : \eta \mapsto \text{Tr}_{\mathcal{E}} (e^{-(\bar{\alpha} Y + \beta h_{\mathcal{E}})} U^*(\eta \otimes \xi) e^{(\bar{\alpha} Y + \beta h_{\mathcal{E}})} U). \quad (20)$$

In particular, for $Y = \beta h_{\mathcal{E}}$ we have $\mathcal{L}_{\beta h_{\mathcal{E}}}^{(\alpha; \tau)*} = \mathcal{L}_{\beta h_{\mathcal{E}}}^{(-\bar{\alpha} - 1; -\tau)}$.

Proof. First note the identity $\text{Tr}_{\mathcal{E}}((\text{Id} \otimes C)D) = \text{Tr}_{\mathcal{E}}(D(\text{Id} \otimes C))$ for any operators C and D on $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$, respectively. Let $C_1, C_2 \in \mathcal{B}(\mathcal{H}_{\mathcal{S}})$. The straightforward computation

$$\begin{aligned} \langle \mathcal{L}_Y^{(\alpha; \tau)}(C_1), C_2 \rangle &= \text{Tr}_{\mathcal{S}} ((\text{Tr}_{\mathcal{E}}(e^{\alpha Y} U(\tau)(C_1 \otimes \xi) e^{-\alpha Y} U(-\tau)))^* C_2) \\ &= \text{Tr}_{\mathcal{S}} (\text{Tr}_{\mathcal{E}}(U(\tau) e^{-\bar{\alpha} Y} (C_1^* \otimes \xi) U(-\tau) e^{\bar{\alpha} Y}) C_2) \\ &= Z_{\beta}^{-1} \text{Tr} ((e^{\bar{\alpha} Y} U(\tau)(C_1^* \otimes e^{-\beta h_{\mathcal{E}}}) e^{-\bar{\alpha} Y} U(-\tau))(C_2 \otimes \text{Id})) \\ &= Z_{\beta}^{-1} \text{Tr} ((C_1^* \otimes \text{Id}) e^{-(\bar{\alpha} Y + \beta h_{\mathcal{E}})} U(-\tau) \\ &\quad (C_2 \otimes e^{-\beta h_{\mathcal{E}}}) e^{(\bar{\alpha} Y + \beta h_{\mathcal{E}})} U(\tau)) \\ &= \text{Tr}_{\mathcal{S}} (C_1^* \text{Tr}_{\mathcal{E}}(e^{-(\bar{\alpha} Y + \beta h_{\mathcal{E}})} U(-\tau)(C_2 \otimes \xi) e^{(\bar{\alpha} Y + \beta h_{\mathcal{E}})} U(\tau))) \end{aligned}$$

directly yields the result. □

We now consider the Kraus form of this deformation.

Lemma 3.8. Assume that **Comm** holds. For all $\alpha \in \mathbb{R}$, $\mathcal{L}_Y^{(\alpha)}$ is a completely positive map. In addition, there exists a Kraus decomposition

$$\mathcal{L}_Y(\eta) = \sum_{i,j} K_{i,j} \eta K_{i,j}^*$$

of \mathcal{L}_Y such that $\mathcal{L}_Y^{(\alpha)}$ admits the Kraus decomposition

$$\mathcal{L}_Y^{(\alpha)}(\eta) = \sum_{i,j} e^{\alpha(y_j - y_i)} K_{i,j} \eta K_{i,j}^*.$$

Proof. Let $\{\psi_m\}_{m=1}^{\dim \mathcal{H}_\mathcal{E}}$ be an orthonormal basis of eigenvectors of Y . We introduce

$$\begin{aligned} \text{Id} \otimes |\psi_m\rangle : \mathcal{H}_\mathcal{S} &\rightarrow \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E} \quad \text{defined by } \varphi \mapsto \varphi \otimes \psi_m \\ \text{Id} \otimes \langle \psi_m| : \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E} &\rightarrow \mathcal{H}_\mathcal{S} \quad \text{defined by } \varphi \otimes \psi \mapsto \langle \psi_m, \psi \rangle \varphi, \end{aligned}$$

so that $(\text{Id} \otimes |\psi_m\rangle)^* = \text{Id} \otimes \langle \psi_m|$. We observe that using $[Y, \xi] = 0$ and $\text{Tr}_\mathcal{E}((\text{Id} \otimes C)D) = \text{Tr}_\mathcal{E}(D(\text{Id} \otimes C))$ for any operators C and D on $\mathcal{H}_\mathcal{E}$ and $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$, respectively, we can write

$$\begin{aligned} \mathcal{L}_Y^{(\alpha)}(\eta) &= \text{Tr}_\mathcal{E} \left((\text{Id} \otimes e^{\alpha Y/2}) U (\text{Id} \otimes e^{-\alpha Y/2} \xi^{1/2}) \right. \\ &\quad \left. (\eta \otimes \text{Id}) (\text{Id} \otimes \xi^{1/2} e^{-\alpha Y/2}) U^* (\text{Id} \otimes e^{\alpha Y/2}) \right). \end{aligned} \quad (21)$$

This shows that $\mathcal{L}_Y^{(\alpha)}$ is a completely positive map. Then we express the partial trace on $\mathcal{H}_\mathcal{E}$ using the orthonormal basis $\{\psi_m\}_{m=1}^{\dim \mathcal{H}_\mathcal{E}}$ by means of the set of operators on $\mathcal{H}_\mathcal{S}$

$$K_{i,j}^{(\alpha)} := (\text{Id} \otimes \langle \psi_j|) (\text{Id} \otimes e^{\alpha Y/2}) U (\text{Id} \otimes e^{-\alpha Y/2}) (\text{Id} \otimes \xi^{1/2} |\psi_i\rangle) \quad (22)$$

(again $K_{i,j}^{(\alpha)}$ depends on the choice of Y). Thus for any $\eta \in \mathcal{I}_1(\mathcal{H}_\mathcal{S})$, and all $\alpha \in \mathbb{R}$,

$$\mathcal{L}_Y^{(\alpha)}(\eta) = \sum_{i,j} K_{i,j}^{(\alpha)} \eta (K_{i,j}^{(\alpha)})^*. \quad (23)$$

This yields the Kraus decomposition of $\mathcal{L}_Y^{(\alpha)}$. Moreover, we note that

$$K_{i,j}^{(\alpha)} = e^{\alpha(y_j - y_i)/2} K_{i,j}^{(0)},$$

and letting $K_{i,j} := K_{i,j}^{(0)}$ gives our final statement. \square

Lemma 3.8 proves in particular that $\mathcal{L}_Y^{(\alpha)}$ is a deformation of \mathcal{L} in the sense of Appendix A. Let us now address the regularity of $\mathcal{L}_Y^{(\alpha)}(s)$ in (s, α) .

Lemma 3.9. Assume **ADRIS** and suppose $s \mapsto Y(s) \in C^2([0, 1], \mathcal{B}(\mathcal{H}_\mathcal{S}))$. Then, the map

$$[0, 1] \times \mathbb{C} \ni (s, \alpha) \mapsto \mathcal{L}_Y^{(\alpha)}(s) \in \mathcal{B}(\mathcal{H}_\mathcal{S})$$

is of class C^2 .

Proof. First observe that since the dimensions of \mathcal{H}_S and \mathcal{H}_E are finite, it is enough to check regularity of the matrix elements of $\mathcal{L}_Y^{(\alpha)}(s)$. From the Kraus decomposition above, if $\{\varphi_k\}_{k=1}^{\dim \mathcal{H}_S}$ and $\{\psi_i\}_{i=1}^{\dim \mathcal{H}_E}$ are fixed orthonormal bases of \mathcal{H}_S and \mathcal{H}_E , it is enough to check regularity of the \mathbb{C} valued functions

$$\langle \varphi_k | K_{i,j} \varphi_l \rangle = \langle \varphi_k \otimes \psi_j | e^{\alpha Y(s)} U(s) e^{-\alpha Y(s)} \xi(s)^{1/2} \varphi_l \otimes \psi_i \rangle.$$

By **ADRS** and the explicit dependence in α of the matrices involved, one gets immediately the result. \square

Remark 3.10. In case the regularity assumption in s in **ADRS** and that of Y are understood in the operator norm sense, and $h_E(s)$ is such that $\xi(s)$ is C^2 in the trace norm sense on \mathcal{H}_E , the map $(s, \alpha) \mapsto \mathcal{L}_Y^{(\alpha)}(s) \in \mathcal{B}(\mathcal{I}_1(\mathcal{H}_S))$ is C^2 in the norm sense, irrespectively of the dimensions of \mathcal{H}_S and \mathcal{H}_E ; see Appendix C.

We conclude this section with a discussion of the effect of time-reversal on the operator $K_{i,j}$, and therefore on the operator $\mathcal{L}_Y^{(\alpha)}$. A relevant assumption will be the following:

TRI We say that an **ADRS** satisfies *time-reversal invariance* if for every $s \in [0, 1]$ there exist two antiunitary involutions $C_S(s) : \mathcal{H}_S \rightarrow \mathcal{H}_S$ and $C_E(s) : \mathcal{H}_E \rightarrow \mathcal{H}_E$ such that if $C(s) = C_S(s) \otimes C_E(s)$ one has for all $s \in [0, 1]$

$$[h_S, C_S(s)] = 0, \quad [h_E(s), C_E(s)] = 0, \quad [v(s), C(s)] = 0.$$

This holds for example if each h_S, h_E and v are real valued matrices in the same basis, and C_S, C_E are complex conjugation in the corresponding basis.

In the following result we denote by $K_{i,j}^{(\tau)}(s)$ the operator $K_{i,j}(s)$ associated with the unitary $U(s)$ as defined in (8). The operator $K_{i,j}^{(-\tau)}(s)$ is therefore associated in the same way with the unitary $U^*(s)$.

Lemma 3.11. *Assume that an **ADRS** satisfies **Comm** and **TRI**. Then for all i, j and all $s \in [0, 1]$ one has*

$$C_S(s) K_{i,j}^{(\tau)}(s) C_S(s) = K_{i,j}^{(-\tau)}(s). \quad (24)$$

This implies in particular that for $Y = \beta h_E$ and all $s \in [0, 1]$,

$$\lambda^{(\alpha)}(s) = \lambda^{(-1-\alpha)}(s), \quad (25)$$

where $\lambda^{(\alpha)}(s)$ is the spectral radius of $\mathcal{L}_Y^{(\alpha)}(s)$. Equivalently the function $\alpha \mapsto \lambda^{(\alpha)}(s)$ is symmetric about $\alpha = -1/2$ for all $s \in [0, 1]$.

Proof. Once again we drop the s variable. Using (22), $K_{i,j}^{(\tau)}$ we have

$$\begin{aligned} \langle \varphi_1, C_S K_{i,j}^{(\tau)} C_S \varphi_2 \rangle &= \overline{\langle C_S \varphi_1, K_{i,j}^{(\tau)} C_S \varphi_2 \rangle} \\ &= \overline{\langle C_S \varphi_1 \otimes \psi_j, U \xi^{1/2} C_S \varphi_2 \otimes \psi_i \rangle} \end{aligned}$$

and by **Comm** we can choose the basis $(\psi_i)_i$ such that $C_E \psi_i = \psi_i$, $C_E \psi_j = \psi_j$, so that

$$\begin{aligned} &= \overline{\langle C(\varphi_1 \otimes \psi_j), U \xi^{1/2} C(\varphi_2 \otimes \psi_i) \rangle} \\ &= \langle \varphi_1 \otimes \psi_j, C U C \xi^{1/2} \varphi_2 \otimes \psi_i \rangle \\ &= \langle \varphi_1 \otimes \psi_j, U^* \xi^{1/2} \varphi_2 \otimes \psi_i \rangle \end{aligned}$$

and this proves relation (24). If we now denote $K_S = C_S \cdot C_S$ the map on $\mathcal{B}(\mathcal{H}_S)$, then for $\alpha \in \mathbb{R}$ this implies $K_S \circ \mathcal{L}^{(\alpha, \tau)} \circ K_S = \mathcal{L}^{(\alpha, -\tau)}$ for any Y satisfying **Comm**. By Lemma 3.7, for $Y = \beta h_E$ this implies $K_S \circ \mathcal{L}^{(\alpha, \tau)} \circ K_S = \mathcal{L}^{(-\alpha-1; \tau)^*}$ which in turn implies $\lambda^{(\alpha)} = \lambda^{(-\alpha-1)}$. \square

3.3 A general adiabatic result

The moment generating functions of Δy_T^{tot} and Δa_T have been related in Proposition 3.6 to products of T operators $\mathcal{L}_Y^{(\alpha)}(s)$ that differ little from each other, locally, by an amount of order $1/T$. The general result stated below will allow us to discuss the asymptotic behaviour of Δy_T^{tot} as $T \rightarrow \infty$.

Let $s \mapsto \mathcal{L}(s)$ be a family of CPTP maps satisfying **Irr**, and define $\mathcal{L}_Y^{(\alpha)}(s)$ by (19). For each $s \in [0, 1]$, the map $\mathcal{L}_Y^{(\alpha)}(s)$ satisfies (23), and is therefore a deformation of $\mathcal{L}(s)$ in the sense of Appendix A. From Proposition A.3, there exist maps $\lambda_Y^{(\alpha)}(s)$, $\mathbf{I}_Y^{(\alpha)}(s)$ and $\rho_Y^{(\alpha)}(s)$ from $[0, 1] \times \mathbb{R} \ni (s, \alpha)$ to, respectively, \mathbb{R}_+^* , the set of positive-definite operators, and the set of faithful states of \mathcal{H}_S ; and maps $z(s), u(s)$ from $[0, 1]$ to, respectively, \mathbb{N} and the set of unitary operators, with the following properties:

- the identities $[u(s), \mathbf{I}_Y^{(\alpha)}(s)] = [u(s), \rho_Y^{(\alpha)}(s)] = 0$, and $u(s)^{z(s)} = \text{Id}$ hold;
- the peripheral spectrum of $\mathcal{L}_Y^{(\alpha)}(s)$ is $\lambda_Y^{(\alpha)}(s) S_{z(s)}$, where $S_z = \{\theta^m \mid \theta = e^{2i\pi/z}, m = 0, \dots, z-1\}$;
- the spectral decomposition $u(s) = \sum_{m=1}^{z(s)} e^{2i\pi m/z(s)} p_m(s)$ holds;
- the map $\eta \mapsto \text{Tr}(\mathbf{I}_Y^{(\alpha)}(s) u(s)^{-m} \eta) \rho_Y^{(\alpha)}(s) u(s)^m$ is the spectral projector of $\mathcal{L}_Y^{(\alpha)}(s)$ associated with $\lambda_Y^{(\alpha)}(s) e^{2i\pi m/z(s)}$;
- the unitary $u(s)$ and cardinal $z(s)$ of the peripheral spectrum of $\mathcal{L}_Y^{(\alpha)}(s)$ do not depend on α or Y .

Note that we have $\lambda_Y^{(0)}(s) = 1$, $\mathbf{I}_Y^{(0)}(s) = \text{Id}$ and $\rho_Y^{(0)}(s) = \rho^{\text{inv}}(s)$ for all Y and s . As mentioned above, the case $Y = \beta h_{\mathcal{E}}$ will be particularly relevant to the discussion of the Landauer principle. We therefore drop the indices Y , and simply denote by $\lambda^{(\alpha)}(s)$, $\mathbf{I}^{(\alpha)}(s)$ and $\rho^{(\alpha)}(s)$ the above quantities, in the case where $Y = \beta h_{\mathcal{E}}$. We define

$$\tilde{\mathcal{L}}^{(\alpha)}(s) = (\lambda^{(\alpha)}(s))^{-1} \mathcal{L}^{(\alpha)}(s).$$

The following result will be our main technical tool.

Proposition 3.12. *Consider an **ADRIS** with the family $(\mathcal{L}(s))_{s \in [0,1]}$ satisfying **Irr** with $z(s) \equiv z$. Then, there exist continuous functions $\mathbb{R} \ni \alpha \mapsto \ell'(\alpha) \in (0, 1)$ and $\mathbb{R} \ni \alpha \mapsto C(\alpha) \in \mathbb{R}_+$, and a function $\alpha \mapsto T_0(\alpha) \in \mathbb{N}$ that is bounded on any compact set of \mathbb{R} , such that for all $\alpha \in \mathbb{R}$, $T \geq T_0(\alpha)$, and $k \leq T$,*

$$\begin{aligned} \left\| \tilde{\mathcal{L}}^{(\alpha)}\left(\frac{k}{T}\right) \dots \tilde{\mathcal{L}}^{(\alpha)}\left(\frac{1}{T}\right) \rho^i - z e^{-\vartheta_Y^{(\alpha)}} \sum_{m=0}^{z-1} \text{Tr}(\mathbf{I}^{(\alpha)}(0) p_m(0) \rho^i) \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) p_{m-k}\left(\frac{k}{T}\right) \right\| \\ \leq \frac{C(\alpha)}{T(1-\ell'(\alpha))} + C(\alpha) \ell'(\alpha)^k. \end{aligned}$$

where the index of the spectral projector $p_{m-k}(\frac{k}{T})$ is interpreted modulo z , and

$$\vartheta_Y^{(\alpha)} := \int_0^{k/T} \text{Tr}(\mathbf{I}_Y^{(\alpha)}(s) \frac{\partial}{\partial s} \rho_Y^{(\alpha)}(s)) ds.$$

Proof. By expression (23), the map $\mathbb{R} \times [0, 1] \ni (\alpha, s) \mapsto \mathcal{L}_Y^{(\alpha)}(s)$ is real analytic in α and C^2 in s . We know from Proposition A.3 that the spectral radius $\lambda_Y^{(\alpha)}(s)$ of $\mathcal{L}_Y^{(\alpha)}(s)$ is a simple eigenvalue for $\mathcal{L}_Y^{(\alpha)}(s)$ with eigenvector $\rho_Y^{(\alpha)}(s)$, and for $\mathcal{L}_Y^{(\alpha)}(s)^*$ with eigenvector $\mathbf{I}_Y^{(\alpha)}(s)$. By standard perturbation theory, the maps $(\alpha, s) \mapsto \lambda_Y^{(\alpha)}(s), \mathbf{I}_Y^{(\alpha)}(s), \rho_Y^{(\alpha)}(s)$ are C^2 functions of $s \in [0, 1]$ and real analytic functions of $\alpha \in \mathbb{R}$. The unitary $u(s)$ is an eigenvector for the isolated eigenvalue $\theta = e^{2i\pi/z}$ of $\mathcal{L}^*(s)$, and is therefore a C^2 function of s . The peripheral spectrum of $\tilde{\mathcal{L}}_Y^{(\alpha)}(s)$ is the set $S_z = \{\theta^m \mid m = 0, \dots, z-1\}$, each peripheral eigenvalue θ^m is simple, and the associated peripheral projector $\eta \mapsto P_m^{(\alpha)}(s) = \text{Tr}(\mathbf{I}_Y^{(\alpha)}(s)u(s)^{-m}\eta)u(s)^m\rho_Y^{(\alpha)}(s)$ is therefore a C^2 function of s and a real analytic function of α . In addition, denoting $Q^{(\alpha)}(s) = \text{Id} - \sum_{m=1}^z P_m^{(\alpha)}(s)$ the quantity $\ell(\alpha) = \sup_{s \in [0,1]} \text{spr} \tilde{\mathcal{L}}_Y^{(\alpha)}(s)Q_\alpha(s) < 1$ is a continuous function of α .

From the above discussion, the family $s \mapsto \tilde{\mathcal{L}}_Y^{(\alpha)}(s)$ satisfies **Hyp0–Hyp4** and is therefore admissible, in the sense of Appendix B, with simple peripheral eigenvalues. We can therefore apply Corollary B.9. Denote by $\vartheta_{Y,m}^{(\alpha)}$ the integral appearing in the exponential factor:

$$\vartheta_{Y,m}^{(\alpha)} = \int_0^{k/T} \text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(t)u(t)^{-m} \frac{\partial}{\partial t} (u^m(t)\rho_Y^{(\alpha)}(t)) \right) dt. \quad (26)$$

We can prove that $\vartheta_{Y,m}^{(\alpha)}$ does not depend on m :

Lemma 3.13. *We have for $m = 0, \dots, z-1$*

$$\vartheta_{Y,m}^{(\alpha)} = \vartheta_Y^{(\alpha)} := \int_0^{k/T} \text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(t) \frac{\partial}{\partial t} \rho_Y^{(\alpha)}(t) \right) dt.$$

Proof. The proof follows from a simple expansion of $\frac{\partial}{\partial t} (u^m(t)\rho_Y^{(\alpha)}(t))$ and commutation properties:

$$\begin{aligned} \vartheta_{Y,m}^{(\alpha)} &= \vartheta_Y^{(\alpha)} + \int_0^{k/T} \text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(t)u^{-m}(t) \sum_{k=0}^{m-1} u^k(t) \frac{\partial u}{\partial t}(t)u^{m-k-1}(t)\rho_Y^{(\alpha)}(t) \right) dt \\ &= \vartheta_Y^{(\alpha)} + m \int_0^{k/T} \text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(t)u^{-1}(t) \frac{\partial u}{\partial t}(t)\rho_Y^{(\alpha)}(t) \right) dt. \end{aligned}$$

However, as $u(t)^z = \text{Id}$, plugging $m = 0$ or $m = z$ in the right-hand side of expression (26) gives the same expression $\vartheta_Y^{(\alpha)}$, so that necessarily

$$\int_0^{k/T} \text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(t)u^{-1}(t) \frac{\partial u}{\partial t}(t)\rho_Y^{(\alpha)}(t) \right) dt = 0$$

and the conclusion follows. \square

Lemma 3.13 and Corollary B.9 therefore imply that for any $\ell'(\alpha) \in (\ell(\alpha), 1)$, there exist $T_0(\alpha) \in \mathbb{N}$, and $C(\alpha) > 0$ which is a (fixed) continuous function of

$$c_P(\alpha) = \sup_{s \in [0,1]} \max_{m=1, \dots, z} \max \left(\|\phi_{Y,m}^{(\alpha)}(s)\|, \|\phi_{Y,m}'^{(\alpha)}(s)\|, \|\psi_{Y,m}^{(\alpha)}(s)\|, \|\psi_{Y,m}'^{(\alpha)}(s)\| \right)$$

with

$$\phi_{Y,m}^{(\alpha)}(s) = \rho_Y^{(\alpha)}(s)u(s)^m \quad \psi_{Y,m}^{(\alpha)}(s) = \mathbf{I}_Y^{(\alpha)}(s)u(s)^m,$$

such that for any $T \geq T_0(\alpha)$,

$$\begin{aligned} \left\| \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{k}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right) \rho^i - e^{-\vartheta_Y^{(\alpha)}} \sum_{m=0}^{z-1} \theta^{mk} \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) u^m\left(\frac{k}{T}\right) \text{Tr}\left(\mathbf{I}^{(\alpha)}(0) u^{-m}(0) \rho^i\right) \right\| \\ \leq \frac{C(\alpha)}{T(1-\ell'(\alpha))} + C(\alpha) \ell'(\alpha)^k. \end{aligned} \quad (27)$$

In addition, $T_0(\alpha)$ can be chosen depending on $c_P(\alpha)$ and $\ell'(\alpha)$ alone.

Recall that we have the spectral decomposition $u = \sum_{m=0}^{z-1} \theta^m p_m$. Then, with all sums from 0 to $z-1$ understood modulo z , we have by a discrete Fourier-type computation

$$\begin{aligned} \sum_m \theta^{mk} \text{Tr}\left(\mathbf{I}^{(\alpha)}(0) u^{-m}(0) \rho^i\right) \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) u^m\left(\frac{k}{T}\right) \\ = \sum_m \theta^{mk} \sum_n \theta^{-nm} \text{Tr}\left(\mathbf{I}^{(\alpha)}(0) p_n(0) \rho^i\right) \sum_\ell \theta^{\ell m} \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) p_\ell\left(\frac{k}{T}\right) \\ = \sum_{n,\ell} \text{Tr}\left(\mathbf{I}^{(\alpha)}(0) p_n(0) \rho^i\right) \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) p_\ell\left(\frac{k}{T}\right) \sum_m \theta^{m(k-n+\ell)} \\ = \sum_{n,\ell} \text{Tr}\left(\mathbf{I}^{(\alpha)}(0) p_n(0) \rho^i\right) \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) p_\ell\left(\frac{k}{T}\right) z \mathbb{1}_{\ell=n-k} \\ = z \sum_n \text{Tr}\left(\mathbf{I}^{(\alpha)}(0) p_n(0) \rho^i\right) \rho_Y^{(\alpha)}\left(\frac{k}{T}\right) p_{n-k}\left(\frac{k}{T}\right). \end{aligned}$$

This expression along with (27) yields the result. \square

By taking $\alpha = 0$, this result allows adiabatic approximation of the state of \mathcal{S} under the physical evolution $\mathcal{L}\left(\frac{k}{T}\right) \cdots \mathcal{L}\left(\frac{1}{T}\right)$ after k steps of an irreducible RIS. This corresponds to a generalization of the results of [HJPR17], which could only treat the primitive case, i.e. $z = 1$.

Corollary 3.14. *Consider an **ADRS** with the family $(\mathcal{L}(s))_{s \in [0,1]}$ satisfying **Irr** with $z(s) \equiv z$. Then, there exists $\ell' < 1$, $C > 0$, and $T_0 > 0$ such that for all $T \geq T_0$, and $k \leq T$,*

$$\left\| \mathcal{L}\left(\frac{k}{T}\right) \cdots \mathcal{L}\left(\frac{1}{T}\right) \rho^i - \rho_{\text{adiab}}(k, T) \right\| \leq \frac{C}{T(1-\ell')} + C \ell'^k$$

where

$$\rho_{\text{adiab}}(k, T) := z \sum_{n=0}^{z-1} \text{Tr}\left(p_n(0) \rho^i\right) \rho^{\text{inv}}\left(\frac{k}{T}\right) p_{n-k}\left(\frac{k}{T}\right) \quad (28)$$

is a state, and the index of the spectral projector $p_{n-k}\left(\frac{k}{T}\right)$ is interpreted modulo z . Moreover, if ρ^i is faithful, we have the uniform bound

$$\inf_{T > 1} \inf_{k \leq T} \inf \text{sp} \rho_{\text{adiab}}(k, T) \geq z \left(\min_{1 \leq j \leq z} \text{Tr}\left(p_j(0) \rho^i\right) \right) \inf_{s \in [0,1]} \inf \text{sp} \rho^{\text{inv}}(s) > 0.$$

Proof. We apply Proposition 3.12 for $\alpha = 0$, and use that $I^{(0)}(s) \equiv \text{Id}$, $\rho_Y^{(0)} = \rho^{\text{inv}}$, and $\vartheta_Y^{(0)} = 0$ which follows from $\text{Tr} \rho_Y^{(0)}(s) \equiv 1$. Next, we check the formula $\text{Tr}(\rho^{\text{inv}}\left(\frac{k}{T}\right) p_\ell\left(\frac{k}{T}\right)) = \frac{1}{z}$ for each $\ell = 0, \dots, z-1$. We drop the argument $\frac{k}{T}$ in what follows, and write $\mathcal{L}_{k/T}(\cdot) = \sum_i V_i \cdot V_i^*$ the Kraus decomposition. Recalling that $p_\ell V_i = V_i p_{\ell+1}$ for all i and ℓ as discussed in Appendix A,

$$\text{Tr}(\rho^{\text{inv}} p_\ell) = \text{Tr}(\mathcal{L}(\rho^{\text{inv}}) p_\ell) = \sum_i \text{Tr}(V_i \rho^{\text{inv}} V_i^* p_\ell)$$

$$= \sum_i \text{Tr}(V_i \rho^{\text{inv}} p_{\ell+1} V_i^*) = \text{Tr}(\mathcal{L}(\rho^{\text{inv}} p_{\ell+1})),$$

so $\text{Tr}(\rho^{\text{inv}} p_\ell) = \text{Tr}(\rho^{\text{inv}} p_{\ell+1})$ using that \mathcal{L} is trace-preserving. As $\sum_\ell \text{Tr}(\rho^{\text{inv}} p_\ell) = \text{Tr} \rho^{\text{inv}} = 1$, we must have $\text{Tr}(\rho^{\text{inv}} p_\ell) = \frac{1}{z}$. Therefore,

$$\begin{aligned} \text{Tr}(\rho_{\text{adiab}}(k, T)) &= z \sum_n \text{Tr}(p_n(0) \rho^i) \text{Tr}(\rho^{\text{inv}}(\frac{k}{T}) p_{n-k}(\frac{k}{T})) = \sum_n \text{Tr}(p_n(0) \rho^i) \\ &= \text{Tr} \rho^i = 1. \end{aligned}$$

Moreover, given a normalized vector $\psi \in \mathcal{H}$, we have

$$\begin{aligned} \langle \psi, \rho_{\text{adiab}}(k, T) \psi \rangle &= z \sum_n \text{Tr}(p_n(0) \rho^i) \langle \psi, \rho^{\text{inv}}(\frac{k}{T}) p_{n-k}(\frac{k}{T}) \psi \rangle \\ &= z \sum_n \text{Tr}(p_n(0) \rho^i) \langle p_{n-k}(\frac{k}{T}) \psi, \rho^{\text{inv}}(\frac{k}{T}) p_{n-k}(\frac{k}{T}) \psi \rangle. \end{aligned}$$

using $[\rho^{\text{inv}}(\frac{k}{T}), p_{n-k}(\frac{k}{T})] = 0$. Since $\rho^{\text{inv}}(\frac{k}{T}) > 0$ and $\text{Tr}(p_n(0) \rho^i) > 0$, each term in the sum is non-negative, and we have

$$\begin{aligned} \langle \psi, \rho_{\text{adiab}}(k, T) \psi \rangle &\geq z \left(\min_{1 \leq j \leq z} \text{Tr}(p_j(0) \rho^i) \right) \sum_n \langle \psi, \rho^{\text{inv}}(\frac{k}{T}) p_{n-k}(\frac{k}{T}) \psi \rangle \\ &= z \left(\min_{1 \leq j \leq z} \text{Tr}(p_j(0) \rho^i) \right) \langle \psi, \rho^{\text{inv}}(\frac{k}{T}) \psi \rangle \\ &\geq z \left(\min_{1 \leq j \leq z} \text{Tr}(p_j(0) \rho^i) \right) \inf_{s \in [0, 1]} \inf \text{sp} \rho^{\text{inv}}(s). \quad \square \end{aligned}$$

Remarks 3.15.

- Given an **ADRS** the family $(\mathcal{L}(s))_{s \in [0, 1]}$ satisfying **Irr**, for faithful ρ^i the state $\rho_T^f = \mathcal{L}_T \cdots \mathcal{L}_1 \rho^i$ is faithful for each $T > 1$ (see the remark after Definition A.1). Corollary 3.14 and Weyl's inequalities (see Section III.2 in [Bha97]) give the stronger result $\inf_{T > 1} \inf \text{sp} \rho_T^f > 0$. In particular, we may make the choice $A^f = -\log \rho_T^f$ which is bounded uniformly in T .
- If we assume, in the notation of [HJPR17], that $\rho^i = (P_0^1 + Q_0) \rho^i$ (i.e. ρ^i has no components corresponding to the peripheral eigenvalues of $\mathcal{L}(0)$ other than 1), then one can check that $\rho_{\text{adiab}}(k, T) = \rho^{\text{inv}}(\frac{k}{T})$.

4 Special case: bounded adiabatic entropy production

In this section we consider $(\mathcal{L}(s))_{s \in [0, 1]}$ satisfying **ADRS** and the primitivity assumption **Prim**, with $X(s) \equiv 0$, where $X(s)$ is defined in (18). We specialize to the case $Y(s) = \beta(s) h_{\mathcal{E}}(s)$ for all $s \in [0, 1]$, and thus drop the subscript Y in the notation. We recall that when $X(s) \equiv 0$, there exists a family $(k_{\mathcal{S}}(s))_{s \in [0, 1]}$ of observables on $\mathcal{H}_{\mathcal{S}}$ satisfying $[k_{\mathcal{S}}(s) + h_{\mathcal{E}}(s), U(s)] \equiv 0$. We claim that for any $\alpha \in \mathbb{C}$,

$$e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \tag{29}$$

is an invariant for $\mathcal{L}^{(\alpha)}(s)$, the deformation of $\mathcal{L}(s)$ corresponding to $Y(s) = \beta(s) h_{\mathcal{E}}(s)$. This follows from the straightforward computation

$$\mathcal{L}^{(\alpha)}(s) (e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)})$$

$$\begin{aligned}
&= \text{Tr}_{\mathcal{E}} \left(e^{\alpha\beta(s)h_{\mathcal{E}}(s)} U(s) \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \otimes \frac{e^{-\beta(s)h_{\mathcal{E}}(s)}}{Z_{\beta}(s)} \right) e^{-\alpha\beta(s)h_{\mathcal{E}}(s)} U(s)^* \right) \\
&= e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)}.
\end{aligned}$$

Since $\mathcal{L}^{(\alpha)}(s)$ is completely positive and irreducible for $\alpha \in \mathbb{R}$ (see Appendix A for details), and $e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)}$ is positive-definite, 1 is necessarily the spectral radius of $\mathcal{L}^{(\alpha)}(s)$. We therefore have $\lambda^{(\alpha)}(s) = 1$, for all s and $\alpha \in \mathbb{R}$, and in addition,

$$\rho^{(\alpha)}(s) = \frac{e^{-(1+\alpha)\beta(s)k_{\mathcal{S}}(s)}}{\text{Tr} \left(e^{-(1+\alpha)\beta(s)k_{\mathcal{S}}(s)} \right)}, \quad (30)$$

with $\rho^{(0)}(s) = \rho^{\text{inv}}(s)$, the invariant state of $\mathcal{L}(s)$. Similarly, using Lemma 3.7, $e^{\bar{\alpha}\beta(s)k_{\mathcal{S}}(s)}$ is an invariant for $\mathcal{L}^{(\alpha)}(s)^*$, so that for $\alpha \in \mathbb{R}$

$$\mathbb{I}^{(\alpha)}(s) = \frac{\text{Tr} \left(e^{-(1+\alpha)\beta(s)k_{\mathcal{S}}(s)} \right)}{\text{Tr} \left(e^{-\beta(s)k_{\mathcal{S}}(s)} \right)} e^{\alpha\beta(s)k_{\mathcal{S}}(s)} \quad (31)$$

and satisfies the normalization condition $\text{Tr} \left(\mathbb{I}^{(\alpha)}(s) \rho^{(\alpha)}(s) \right) \equiv 1$.

Lemma 4.1. *Under the assumptions **ADRS** and **Prim**, and with $X(s) \equiv 0$, and $\ell'(\alpha) \in (0, 1)$, $C(\alpha) \in \mathbb{R}_+$, $T_0(\alpha) \in \mathbb{N}$ as in Proposition 3.12, for all $\alpha \in \mathbb{R}$ and $T \geq T_0(\alpha)$, all $\eta \in \mathcal{I}_1(\mathcal{H})$,*

$$\left\| \mathcal{L}^{(\alpha)}\left(\frac{T}{T}\right) \cdots \mathcal{L}^{(\alpha)}\left(\frac{1}{T}\right)\eta - \text{Tr}(\rho^{\text{inv}}(0)^{-\alpha}\eta) \rho^{\text{inv}}(1)^{1+\alpha} \right\| \leq \frac{C(\alpha)}{T(1-\ell'(\alpha))} + C(\alpha)(\ell'(\alpha))^T,$$

Proof. First note that in the primitive case, $z \equiv 1$ so that $\theta = 1$ and only the term with $m = 0$ is present. In addition, as we have proved above, for $X(s) \equiv 0$ one has $\lambda^{(\alpha)}(s) \equiv 1$ and therefore $\tilde{\mathcal{L}}^{(\alpha)}(s) = \mathcal{L}^{(\alpha)}(s)$. Proposition 3.12 together with expressions (30) and (31) then yield

$$\begin{aligned}
&\left\| \mathcal{L}^{(\alpha)}\left(\frac{T}{T}\right) \cdots \mathcal{L}^{(\alpha)}\left(\frac{1}{T}\right)\eta \right. \\
&\quad \left. - e^{-\vartheta(\alpha)} \frac{\text{Tr} \left(e^{-(1+\alpha)\beta(0)k_{\mathcal{S}}(0)} \right) \text{Tr} \left(e^{\alpha\beta(0)k_{\mathcal{S}}(0)}\eta \right)}{\text{Tr} \left(e^{-(1+\alpha)\beta(1)k_{\mathcal{S}}(1)} \right) \text{Tr} \left(e^{-\beta(0)k_{\mathcal{S}}(0)} \right)} e^{-(1+\alpha)\beta(1)k_{\mathcal{S}}(1)} \right\| \\
&\leq \frac{C(\alpha)}{T(1-\ell'(\alpha))} + C(\alpha)(\ell'(\alpha))^T.
\end{aligned}$$

with

$$\begin{aligned}
\vartheta(\alpha) &= \int_0^1 \text{Tr} \left(\frac{\text{Tr} \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right)}{\text{Tr} \left(e^{-\beta(s)k_{\mathcal{S}}(s)} \right)} e^{\alpha\beta(s)k_{\mathcal{S}}(s)} \frac{d}{ds} \frac{e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)}}{\text{Tr} \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right)} \right) ds \\
&= \int_0^1 \frac{\text{Tr} \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right)}{\text{Tr} \left(e^{-\beta(s)k_{\mathcal{S}}(s)} \right)} \text{Tr} \left(e^{\alpha\beta(s)k_{\mathcal{S}}(s)} \frac{d}{ds} \frac{e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)}}{\text{Tr} \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right)} \right) ds \\
&= \int_0^1 \left(\text{Tr} \left(e^{-\beta(s)k_{\mathcal{S}}(s)} \right) \right)^{-1} \text{Tr} \left(e^{\alpha\beta(s)k_{\mathcal{S}}(s)} \frac{d}{ds} e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right) ds \\
&\quad - \int_0^1 \left(\text{Tr} \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right) \right)^{-1} \frac{d}{ds} \text{Tr} \left(e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right) ds.
\end{aligned}$$

Thanks to the general formula $\frac{d}{ds} e^{A(s)} = \int_0^1 e^{xA(s)} \frac{d}{ds} A(s) e^{(1-x)A(s)} dx$ and to the cyclicity of the trace, we have

$$\text{Tr} \left(e^{\alpha\beta(s)k_{\mathcal{S}}(s)} \frac{d}{ds} e^{-\beta(s)(1+\alpha)k_{\mathcal{S}}(s)} \right) = -(1+\alpha) \text{Tr} \left(\frac{d}{ds} \left(\beta(s)k_{\mathcal{S}}(s) \right) e^{-\beta(s)k_{\mathcal{S}}(s)} \right)$$

$$= (1 + \alpha) \frac{d}{ds} \text{Tr}(e^{-\beta(s)k_S(s)}),$$

so that

$$\vartheta^{(\alpha)} = (1 + \alpha) \int_0^1 \frac{d}{ds} \log \text{Tr}(e^{-\beta(s)k_S(s)}) ds - \int_0^1 \frac{d}{ds} \log \text{Tr}(e^{-(1+\alpha)\beta(s)k_S(s)}) ds$$

and

$$e^{-\vartheta^{(\alpha)}} = \frac{(\text{Tr}(e^{-\beta(0)k_S(0)})^{1+\alpha} \text{Tr}(e^{-(1+\alpha)\beta(1)k_S(1)})}{(\text{Tr}(e^{-\beta(1)k_S(1)})^{1+\alpha} \text{Tr}(e^{-(1+\alpha)\beta(0)k_S(0)})}.$$

The rest of the proof is obtained by direct computation. \square

Recall that for ρ^i, ρ^f two faithful states, $A^i = -\log \rho^i$, $A^f = -\log \rho^f$ and $Y(s) = \beta(s)h_{\mathcal{E}}(s)$, we have the decomposition $\varsigma_T = -\Delta_{S_{\mathcal{S}},T} + \Delta_{S_{\mathcal{E}},T}$ (see (14)).

Theorem 4.2. *Under the assumptions **ADRIS** and **Prim**, and with $X(s) \equiv 0$, the distribution of the pair $(\Delta_{S_{\mathcal{S}},T}, \Delta_{S_{\mathcal{E}},T})$ converges weakly to a probability measure characterized by its moment generating function*

$$M_{(\Delta_{S_{\mathcal{E}},T}, \Delta_{S_{\mathcal{S}},T})}(\alpha_1, \alpha_2) = \text{Tr}(\rho^{\text{inv}}(0)^{-\alpha_1} (\rho^i)^{1-\alpha_2}) \text{Tr}(\rho^{\text{inv}}(1)^{1+\alpha_1+\alpha_2}).$$

This probability measure has finite support, contained in the set

$$\begin{aligned} & (\log \text{sp} \rho^{\text{inv}}(1) - \log \text{sp} \rho^{\text{inv}}(0)) \times (\log \text{sp} \rho^{\text{inv}}(1) - \log \text{sp} \rho^i) = \\ & \{(\log r_1 - \log r_0, \log r'_1 - \log r^i) \mid r_1, r'_1 \in \text{sp} \rho^{\text{inv}}(1), r_0 \in \text{sp} \rho^{\text{inv}}(0), r^i \in \text{sp} \rho^i\}. \end{aligned} \quad (32)$$

In particular, the limiting moment generating function $M_{\varsigma}(\alpha) := \lim_{T \rightarrow \infty} M_{\varsigma_T}(\alpha)$ satisfies

$$\log M_{\varsigma}(\alpha) = S_{-\alpha}(\rho^{\text{inv}}(0) | \rho^i),$$

where S_{α} denotes the (unnormalized) Rényi relative entropy

$$S_{\alpha}(\eta | \zeta) := \log \text{Tr}(\eta^{\alpha} \zeta^{1-\alpha}).$$

Proof. By a direct application of Proposition 3.6, for all $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$M_{(\Delta_{S_{\mathcal{E}},T}, \Delta_{S_{\mathcal{S}},T})}(\alpha_1, \alpha_2) = \text{Tr}(e^{-\alpha_2 A^f} \mathcal{L}^{(\alpha_1)}(\frac{T}{T}) \dots \mathcal{L}^{(\alpha_1)}(\frac{1}{T})(e^{+\alpha_2 A^i} \rho^i)),$$

so that by Lemma 4.1

$$|M_{(\Delta_{S_{\mathcal{E}},T}, \Delta_{S_{\mathcal{S}},T})}(\alpha_1, \alpha_2) - \text{Tr}(\rho^{\text{inv}}(0)^{-\alpha_1} (\rho^i)^{1-\alpha_2}) \text{Tr}(\rho^{\text{inv}}(1)^{1+\alpha_1} (\rho^f)^{\alpha_2})|$$

converges to 0 as $T \rightarrow \infty$ and again from Lemma 4.1 with $\alpha = 0$, $\lim_{T \rightarrow \infty} \rho^f = \rho^{\text{inv}}(1)$ and the latter state is faithful. This shows that the moment generating function converges as $T \rightarrow \infty$ for all (α_1, α_2) , to the desired identity. By the results in Section 30 of [Bil95], this shows the convergence in distribution of the pair $(\Delta_{S_{\mathcal{E}},T}, \Delta_{S_{\mathcal{S}},T})$. \square

Remark 4.3. Relation (17), and the fact that the derivative of $S_{-\alpha}(\eta | \zeta)$ is the relative entropy $S(\eta | \zeta) = \text{Tr}(\eta(\log \eta - \log \zeta))$ imply in particular (again see Section 30 of [Bil95]) that under the assumptions **ADRIS** and **Prim**, and with $X(s) \equiv 0$,

$$\lim_{T \rightarrow \infty} \sigma_T^{\text{tot}} = S(\rho^{\text{inv}}(0) | \rho^i).$$

Theorem 4.2 therefore gives us a refinement of the results of [HJPR17], where an explicit expression of the limit was missing. Remark also that the quantity $S_{-\alpha}(\rho^{\text{inv}}(0) | \rho^i)$ can be expressed as the cumulant generating function of an explicit distribution related to the relative modular operator for $\rho^{\text{inv}}(0)$ and ρ^i (see e.g. Chapter 2 in [JOPP12]).

Corollary 4.4. *Under the assumptions of Theorem 4.2, if in addition $\rho^i = \rho^{\text{inv}}(0)$ then the limiting distribution for $(\Delta_{s_{\mathcal{E},T}, \Delta_{s_{\mathcal{S},T}})$ has support on the diagonal and equivalently the limiting distribution for ς_T is a Dirac measure at zero. If we write the spectral decomposition of $\rho^{\text{inv}}(0)$, $\rho^{\text{inv}}(1)$ as*

$$\rho^{\text{inv}}(0) = \sum_j \tau_j(0) \pi_j(0) \quad \rho^{\text{inv}}(1) = \sum_j \tau_j(1) \pi_j(1)$$

then the limiting distribution for $\Delta_{s_{\mathcal{E},T}$ gives the following weight to $s \in \mathbb{R}$

$$\sum_{k,j} \text{Tr}(\rho^{\text{inv}}(0) \pi_j(0)) \text{Tr}(\rho^{\text{inv}}(1) \pi_k(1)) \mathbb{1}_s(\log \tau_k(1) - \log \tau_j(0)),$$

where $\mathbb{1}_s(t) = 1$ if $s = t$ and 0 otherwise.

Proof. If $\rho^i = \rho^{\text{inv}}(0)$ then with the notation of Theorem 4.2, one has $\log M_\varsigma(\alpha) \equiv 0$, so that the limiting distribution for ς_T is a Dirac measure at zero. In addition, the limiting moment generating function for $\Delta_{s_{\mathcal{E},T}$ is

$$\text{Tr}(\rho^{\text{inv}}(0)^{1-\alpha}) \text{Tr}(\rho^{\text{inv}}(1)^{1+\alpha})$$

and the expression of the corresponding distribution follows by inspection. \square

Example 4.5. Let us recall the simplest non-trivial RIS, which is considered in [HJPR17, Example 6.1], for which the system and probes are 2-level systems, with $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{E}} = \mathbb{C}^2$, along with Hamiltonians $h_{\mathcal{S}} := Ea^*a$ and $h_{\mathcal{E}_k} \equiv h_{\mathcal{E}} := E_0b^*b$ where a/a^* (resp. b/b^*) are the Fermionic annihilation/creation operators for \mathcal{S} (resp. \mathcal{E}), with $E, E_0 > 0$ constants with units of energy. As matrices in the (ground state, excited state) bases $\{|0\rangle, |1\rangle\}$ for \mathcal{S} and \mathcal{E} , we write

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^*a = b^*b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider a constant potential $v_{\text{RW}} \in \mathcal{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$,

$$v_{\text{RW}} = \frac{\mu_1}{2}(a^* \otimes b + a \otimes b^*)$$

where $\mu_1 = 1$ with units of energy. Given $s \mapsto \beta(s) \in [0, 1]$ a C^2 curve of inverse probe temperatures, an interaction time $\tau > 0$ and coupling constant $\lambda > 0$, we let

$$U = \exp(-i\tau(h_{\mathcal{S}} + h_{\mathcal{E}} + \lambda v_{\text{RW}})).$$

Then $\text{sp } \mathcal{L}(s)$ is independent of s , with 1 as a simple eigenvalue with eigenvector

$$\rho^{\text{inv}}(s) = \exp(-\beta^*(s)h_{\mathcal{S}}) / \text{Tr}(\exp(-\beta^*(s)h_{\mathcal{S}}))$$

for $\beta^*(s) = \frac{E_0}{E}\beta(s)$.

With $\nu := \sqrt{(E - E_0)^2 + \lambda^2}$, the assumption $\nu\tau \notin 2\pi\mathbb{Z}$ yields that $\mathcal{L}(s)$ is primitive, and moreover, the fact that $[v_{\text{RW}}, a^*a + b^*b] = 0$ yields $X(s) \equiv 0$. Here, we may take $k_{\mathcal{S}} \equiv \frac{E_0}{E}h_{\mathcal{S}}$ independently of s , which satisfies $[k_{\mathcal{S}} + h_{\mathcal{E}}, U] \equiv 0$.

We choose an initial system state $\rho^i > 0$, and set $Y(s) := \beta(s)h_{\mathcal{E}}$, $A^i := \log \rho^i$, $A^f := \log \rho_T^f$, for $\rho_T^f := \mathcal{L}(\frac{T}{T}) \cdots \mathcal{L}(\frac{1}{T})\rho^i$. By considering the forward and backward processes of Section 2.3, we define the forward (resp. backward) probability distribution \mathbb{P}_T^F (resp. \mathbb{P}_T^B), and the entropy production $\varsigma_T = \log \frac{\mathbb{P}_T^F}{\mathbb{P}_T^B}$ on $\Omega = \text{sp } \rho^i \times \text{sp } \rho^f \times \{0, 1\}^T \times \{0, 1\}^T$. Then Theorem 4.2 yields the asymptotic

moment generating function of ς_T : let $\rho^i = r_0|v_0\rangle\langle v_0| + r_1|v_1\rangle\langle v_1| > 0$ be the spectral decomposition of the initial state, then

$$\begin{aligned} \lim_{T \rightarrow \infty} M_{\varsigma_T}(\alpha) &= (1 + e^{-\beta(0)E_0})^\alpha \left(r_0^{1+\alpha} (|\langle 0|v_0\rangle|^2 + |\langle 1|v_0\rangle|^2 e^{\alpha\beta(0)E_0}) \right. \\ &\quad \left. + r_1^{1+\alpha} (|\langle 0|v_1\rangle|^2 + |\langle 1|v_1\rangle|^2 e^{\alpha\beta(0)E_0}) \right). \end{aligned} \quad (33)$$

5 General case: large deviations and the central limit theorem

In this section we consider a general observable $Y(s)$ with $[Y(s), h_{\mathcal{E}}(s)] \equiv 0$. We will prove a large deviation principle, and essentially deduce from it a law of large numbers and a central limit theorem.

Our main technical tool will be Proposition 3.12, together with the following result:

Lemma 5.1. *Under the assumptions **ADRS** and **Irr** with $z(s) \equiv z$, for any faithful initial state $\rho^i > 0$, for any α in \mathbb{R} we have*

$$0 < \liminf_{T \rightarrow \infty} \text{Tr} \left(\tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right) \rho^i \right) \limsup_{T \rightarrow \infty} \text{Tr} \left(\tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right) \rho^i \right) < \infty.$$

Proof. By Proposition 3.12,

$$\begin{aligned} \text{Tr} \left(\tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right) \rho^i \right) &= z e^{-\vartheta_Y^{(\alpha)}} \sum_{n=0}^{z-1} \text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(0) p_n(0) \rho^i \right) \text{Tr} \left(\rho_Y^{(\alpha)}(1) p_{n-T}(1) \right) \\ &\quad + O\left(\frac{C}{T(1-\ell')}\right) + O(C\ell'^T). \end{aligned} \quad (34)$$

Because $\mathbf{I}_Y^{(\alpha)}(0)$, ρ^i , are strictly positive matrices we have $\text{Tr} \left(\mathbf{I}_Y^{(\alpha)}(0) p_n(0) \rho^i \right) > 0$ for all n , and because $\rho_Y^{(\alpha)}(1)$ is a trace one non-negative matrix,

$$\text{Tr} \left(\rho_Y^{(\alpha)}(1) p_{n-T}(1) \right) > 0$$

for some n . By strict positivity of $e^{-\vartheta_Y^{(\alpha)}}$, the leading term in (34) is therefore strictly positive. This proves the lower bound, whereas the upper bound follows from continuity of the operator-valued maps $(\alpha, s) \mapsto \mathbf{I}_Y^{(\alpha)}(s)$, $\rho_Y^{(\alpha)}(s)$. \square

Lemma 5.2. *Under the assumptions **ADRS** and **Irr** with $z(s) \equiv z$, for any faithful initial state $\rho^i > 0$, for any $\alpha \in \mathbb{R}$, the moment generating function of the random variable Δy_T^{tot} with respect to \mathbb{P}_T^F satisfies*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log M_{\Delta y_T^{\text{tot}}}(\alpha) = \int_0^1 \log \lambda_Y^{(\alpha)}(s) ds =: \Lambda_Y(\alpha). \quad (35)$$

Proof. By Lemma 5.1,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \text{Tr} \left(\tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right) (\rho^i) \right) = 0.$$

But the moment generating function reads

$$M_{\Delta y_T^{\text{tot}}} = \text{Tr} \left(\mathcal{L}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \mathcal{L}_Y^{(\alpha)}\left(\frac{1}{T}\right) \rho^i \right) = \left(\prod_{k=1}^T \lambda_Y^{(\alpha)}\left(\frac{k}{T}\right) \right) \text{Tr} \left(\tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right) (\rho^i) \right)$$

by definition of $\tilde{\mathcal{L}}_Y^{(\alpha)}(s)$. Hence, the result follows from the Riemann sum convergence

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \log \left(\prod_{k=1}^T \lambda_Y^{(\alpha)}\left(\frac{k}{T}\right) \right) &= \lim_{T \rightarrow \infty} \sum_{k=1}^T \left(\frac{k}{T} - \frac{k-1}{T}\right) \log \lambda_Y^{(\alpha)}\left(\frac{k}{T}\right) \\ &= \int_0^1 \log \lambda_Y^{(\alpha)}(s) ds. \end{aligned} \quad \square$$

Remark 5.3. Lemma 5.2 also holds for e.g. the random variable $\Delta a_T(\omega) + \Delta y_T^{\text{tot}}$ in place of Δy_T^{tot} because Lemma 5.1 holds with additional factors of $e^{-\alpha A^i}$ and $e^{\alpha A^f}$ inside the trace. Alternatively, one may remark that $\Delta y_T^{\text{tot}}(\omega)$ and $-\Delta a_T + \Delta y_T^{\text{tot}}(\omega)$ only differ by a uniformly bounded term $\Delta a_T(\omega)$.

The regularity of Λ , and the value of its first and second derivatives at zero, are relevant to the asymptotic behaviour of Δy_T^{tot} . We therefore give the following simple lemma.

Lemma 5.4. *Assume **ADRS** and **Irr**. Then the function Λ_Y is twice continuously differentiable on \mathbb{R} , with*

$$\begin{aligned} \Lambda_Y'(0) &= \int_0^1 \frac{\partial \lambda_Y^{(\alpha)}}{\partial \alpha}(s)|_{\alpha=0} ds, \\ \Lambda_Y''(0) &= \int_0^1 \left(\frac{\partial^2 \lambda_Y^{(\alpha)}}{\partial \alpha^2}(s)|_{\alpha=0} - \left(\frac{\partial \lambda_Y^{(\alpha)}}{\partial \alpha}(s)|_{\alpha=0} \right)^2 \right) ds, \end{aligned} \quad (36)$$

and

$$\begin{aligned} \frac{\partial \lambda_Y^{(\alpha)}}{\partial \alpha}(s)|_{\alpha=0} &= \sum_{i,j} (y_j - y_i) \text{Tr} \left(K_{i,j}(s) \rho^{\text{inv}}(s) K_{i,j}^*(s) \right), \\ \frac{\partial^2 \lambda_Y^{(\alpha)}}{\partial \alpha^2}(s)|_{\alpha=0} &= \sum_{i,j} (y_j - y_i)^2 \text{Tr} \left(K_{i,j}(s) \rho^{\text{inv}}(s) K_{i,j}^*(s) \right) \\ &\quad + 2 \sum_{i,j} (y_j - y_i) \text{Tr} \left(K_{i,j}(s) \eta(s) K_{i,j}^*(s) \right) \end{aligned} \quad (37)$$

where $\eta(s)$ is the unique solution with zero trace of

$$\begin{aligned} (\text{Id} - \mathcal{L}(s))(\eta) &= \sum_{i,j} (y_j - y_i) K_{i,j}(s) \rho^{\text{inv}}(s) K_{i,j}^*(s) \\ &\quad - \sum_{i,j} (y_j - y_i) \text{Tr} \left(K_{i,j}(s) \rho^{\text{inv}}(s) K_{i,j}^*(s) \right) \rho^{\text{inv}}(s). \end{aligned}$$

In particular, for $Y(s) = \beta(s) h_{\mathcal{E}}(s)$, one has

$$\frac{\partial \lambda^{(\alpha)}}{\partial \alpha}(s)|_{\alpha=0} = \beta(s) \text{Tr} \left(X(s) (\text{Id} \otimes h_{\mathcal{E}}(s)) \right). \quad (38)$$

Proof. That Λ_Y is twice continuously differentiable is clear from the expression (35) and the fact that $(\alpha, s) \mapsto \lambda_Y^{(\alpha)}(s)$ is C^2 in s and analytic in α , bounded and bounded away from zero. The expressions (36) follow from the dominated convergence theorem. The expressions (37) are obtained by an explicit expansion to second order in α of the relation $\mathcal{L}_Y^{(\alpha)}(\rho_Y^{(\alpha)}) = \lambda_Y^{(\alpha)} \rho_Y^{(\alpha)}$, together with the fact that $\text{Tr}(\rho_Y^{(\alpha)}) \equiv 1$. Last, remark that $\eta(s)$ is uniquely determined as 1 is a simple eigenvalue of $\mathcal{L}(s)$, and the associated eigenvectors have nonzero trace. \square

The values of the derivatives of Λ_Y at $\pm\infty$ will also be relevant. We have:

Lemma 5.5. *Assume **ADRS** and **Irr**. Denote by*

$$\begin{aligned}\nu_{Y,+}(s) &= \max\{y_j(s) - y_i(s) \mid K_{i,j}(s) \neq 0\}, \\ \nu_{Y,-}(s) &= \min\{y_j(s) - y_i(s) \mid K_{i,j}(s) \neq 0\}.\end{aligned}$$

Then

$$\lim_{\alpha \rightarrow \pm\infty} \Lambda'_Y(\alpha) = \nu_{Y,\pm} := \int_0^1 \nu_{Y,\pm}(s) \, ds.$$

Proof. Because Λ_Y is convex and everywhere differentiable, $\lim_{\alpha \rightarrow \pm\infty} \Lambda'_Y(\alpha) = \lim_{\alpha \rightarrow \pm\infty} \frac{1}{\alpha} \Lambda_Y(\alpha)$. Besides, it follows immediately from Proposition A.9 that

$$\lim_{\alpha \rightarrow \pm\infty} \frac{1}{\alpha} \log \lambda_Y^{(\alpha)}(s) = \nu_{Y,\pm}(s). \quad \square$$

The above technical results allow us to give a large deviation principle for $\Delta s_{\mathcal{E},T}$ or, equivalently, for ζ_T . In the statement below we denote, for E a subset of \mathbb{R} , by $\text{int } E$ and $\text{cl } E$ its interior and closure respectively.

Theorem 5.6. *Assume **ADRS** and **Irr**, and that the initial state ρ^i is faithful. Let Λ_Y be defined by relation (35) and denote by Λ_Y^* the Fenchel–Legendre transform of Λ_Y , i.e. for $x \in \mathbb{R}$ let*

$$\Lambda_Y^*(x) = \sup_{\alpha \in \mathbb{R}} (\alpha x - \Lambda_Y(\alpha)).$$

Then $\Lambda_Y^*(x) = +\infty$ for $x \notin [\nu_{Y,-}, \nu_{Y,+}]$, and for any Borel set E of \mathbb{R} one has

$$\begin{aligned}- \inf_{x \in \text{int } E} \Lambda_Y^*(x) &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}_T^F \left(\frac{\Delta y_T^{\text{tot}}}{T} \in \text{int } E \right) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}_T^F \left(\frac{\Delta y_T^{\text{tot}}}{T} \in \text{cl } E \right) \leq - \inf_{x \in \text{cl } E} \Lambda_Y^*(x).\end{aligned}$$

The same statement holds with $-\Delta a_T + \Delta y_T^{\text{tot}}$ in place of Δy_T^{tot} . In particular, for $Y = \beta h_{\mathcal{E}}$, one has

$$\begin{aligned}- \inf_{x \in \text{int } E} \Lambda^*(x) &\leq \liminf_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}_T^F \left(\frac{S_T}{T} \in \text{int } E \right) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}_T^F \left(\frac{S_T}{T} \in \text{cl } E \right) \leq - \inf_{x \in \text{cl } E} \Lambda^*(x)\end{aligned}$$

and the same statement holds with $\Delta s_{\mathcal{E},T}$ in place of ζ_T .

Proof. From Lemma 5.4, Λ_Y is continuously differentiable on \mathbb{R} and its derivative takes values in $[\nu_{Y,-}, \nu_{Y,+}]$. The definition of Λ_Y^* implies that it is $+\infty$ outside this interval. The rest of the statement follows from the Gärtner–Ellis theorem (Theorem 2.3.6 in [DZ10]) because Λ_Y , being differentiable everywhere, is (in the language of [DZ10]) essentially smooth. That the same statement holds with $-\Delta a_T + \Delta y_T^{\text{tot}}$ in place of Δy_T^{tot} follows from Remark 5.3. \square

Remarks 5.7.

- In the case of $Y(s) = \beta(s)h_{\mathcal{E}}(s)$, and under the assumption (TRI), the symmetry (25) in Lemma 3.11 is at the very heart of the Gallavotti–Cohen Theorem relating in a parameter free formulation the probabilities to observe opposite signs entropies. Indeed, it implies that Λ is symmetric about the $\alpha = -1/2$ axis. A direct computation shows that

$$\Lambda^*(x) = x + \Lambda^*(-x). \quad (39)$$

A consequence of Theorem 5.6 together with this equality is that if e.g. $\Lambda''(0) \neq 0$,

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{T} \log \frac{\mathbb{P}(+\Delta s_{\mathcal{E},T} \in [s - \delta, s + \delta])}{\mathbb{P}(-\Delta s_{\mathcal{E},T} \in [s - \delta, s + \delta])} = -s.$$

This is obtained by observing that in the present case, Λ is analytic in a neighbourhood of the real axis, and so is Λ^* if Λ is strictly convex. See e.g. [EHM09, JOPP12, CJPS17] for more information on the role of symmetries such as (39).

- In the case of $Y(s) = \beta(s)h_{\mathcal{E}}(s)$, and under the assumption that $X(s) \equiv 0$, we have observed in Section 4 that $\lambda^{(\alpha)}(s) = 1$ for all $\alpha \in \mathbb{R}$ and $s \in [0, 1]$. In that case $\Lambda(\alpha) \equiv 0$ and

$$\Lambda^*(x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The above large deviation statement therefore gives a concentration of $\frac{1}{T}\zeta_T$ or $\frac{1}{T}\Delta s_{\mathcal{E},T}$ at zero which is faster than exponential.

A first consequence is a result similar to a law of large numbers for Δy_T^{tot} :

Corollary 5.8. *Under the same assumptions as in Theorem 5.6, for all $\epsilon > 0$ there exists $r_\epsilon > 0$ such that for T large enough*

$$\mathbb{P}_T^F(|\frac{1}{T} \Delta y_T^{\text{tot}} - \Lambda'_Y(0)| > \epsilon) \leq \exp -r_\epsilon T. \quad (40)$$

Proof. See e.g. Theorem II.6.3 in [Ell85]. □

Remark 5.9. Such a result is sometimes called exponential convergence. If one could replace \mathbb{P}_T^F by a T -independent probability measure \mathbb{P}^F in (40) (see Remark 2.2) then the Borel–Cantelli lemma would imply that $\frac{1}{T}\Delta y_T^{\text{tot}}$ converges \mathbb{P}^F -almost-surely to $\Lambda'_Y(0)$. It implies, however, that $\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}(\Delta y_T^{\text{tot}}) = \Lambda'_Y(0)$. In the case $Y = \beta h_{\mathcal{E}}$, the positivity of σ_T^{tot} implies $\Lambda'(0) \geq 0$. Formula (38) shows that $\Lambda'(0) = 0$ if $X(s) \equiv 0$. The proof of Corollary 6.4 in [HJPR17] shows $\Lambda'(0) > 0$ if $X(s) \not\equiv 0$.

We also obtain a central limit-type result by a slight improvement of the results in Theorem 5.6.

Theorem 5.10. *Under the same assumptions as in Theorem 5.6 we have*

$$\frac{1}{\sqrt{T}}(\Delta y_T^{\text{tot}} - T \Lambda'_Y(0)) \xrightarrow{T \rightarrow \infty} \mathcal{N}(0, \Lambda''_Y(0))$$

in distribution.

Proof. From Corollary A.8, for fixed $s \in [0, 1]$, there exists a complex neighbourhood $N(s)$ of the origin such that for $\alpha \in N(s)$ the peripheral spectrum of $\mathcal{L}_Y^{(\alpha)}(s)$ is of the form $\{\lambda_Y^{(\alpha)}(s)\theta^m \mid m = 0, \dots, z-1\}$, and each $\lambda_Y^{(\alpha)}(s)\theta^m$ is a simple eigenvalue. Denote by $\phi_m^{(\alpha)}(s)\psi_m^{(\alpha)*}(s)$ the corresponding spectral projector, parameterized so that $(\alpha, s) \mapsto \phi_m^{(\alpha)}(s), \psi_m^{(\alpha)}(s)$ are C^2 functions. By compactness of $[0, 1]$, we can find a complex neighbourhood N of the origin containing $\bigcap_{s \in [0, 1]} N(s)$ such that the following holds: for $\alpha \in N$, the family $(\tilde{\mathcal{L}}^{(\alpha)}(s))_{s \in [0, 1]}$ satisfies **Hyp0–Hyp4**. Moreover,

$$\begin{aligned} \sup_{\alpha \in N} \sup_{s \in [0, 1]} |1 - \lambda_Y^{(\alpha)}(s)| &< 1/2, \\ \sup_{\alpha \in N} \sup_{s \in [0, 1]} \max(\|\phi_m^{(\alpha)}(s)\|, \|\phi_m^{(\alpha)'}(s)\|, \|\psi_m^{(\alpha)}(s)\|, \|\psi_m^{(\alpha)'}(s)\|) &< \infty, \\ \sup_{\alpha \in N} \left| \int_0^1 \psi_m^{(\alpha)*}(\phi_m^{(\alpha)'}(t)) dt \right| &< \infty. \end{aligned}$$

We therefore have

$$\sup_{\alpha \in N} \sup_{T \in \mathbb{N}} \left| \frac{1}{T} \sum_{k=1}^T \log \lambda_Y^{(\alpha)}\left(\frac{k}{T}\right) \right| < \infty$$

and by Corollary B.9,

$$\sup_{\alpha \in N} \sup_{T \in \mathbb{N}} \left| \frac{1}{T} \log \text{Tr} \left(\tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{T}{T}\right) \cdots \tilde{\mathcal{L}}_Y^{(\alpha)}\left(\frac{1}{T}\right)(\rho^i) \right) \right| < \infty.$$

This implies

$$\sup_{\alpha \in N} \sup_{T \in \mathbb{N}} \left| \frac{1}{T} \log M_{\Delta y_T^{\text{tot}}}(\alpha) \right| < \infty.$$

In addition, from Lemma 5.2, $\frac{1}{T} \log M_{\Delta y_T^{\text{tot}}}(\alpha)$ converges as $T \rightarrow \infty$ for $\alpha \in N \cap \mathbb{R}$. By Bryc's theorem [Bry93] (see also Appendix A.4 in [JOPP12]) as $T \rightarrow \infty$, $\frac{1}{\sqrt{T}}(\Delta y_T^{\text{tot}} - T \Lambda_Y'(0))$ converges in distribution to $\mathcal{N}(0, \Lambda_Y''(0))$. \square

Remark 5.11. In the case $Y = \beta h_{\mathcal{E}}$, Remark 5.9 and Theorem 5.10 show that, if $\Lambda''(0) \neq 0$ (which is generically expected) then $\sigma_T^{\text{tot}} \rightarrow \infty$ as $T \rightarrow \infty$.

Example 5.12. Let us consider the setup of Example 4.5 using the full-dipole interaction potential $v_{\text{FD}} \in \mathcal{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$,

$$v_{\text{FD}} = \frac{\mu_1}{2}(a + a^*) \otimes (b + b^*),$$

instead of v_{RW} . This example was considered in [HJPR17, Section 7.1], where it was shown that **Prim** is satisfied, and that $\sigma_T \rightarrow \infty$ with a finite and nonzero rate $\lim_{T \rightarrow \infty} \frac{1}{T} \sigma_T$ for generic choices of parameters $\{E, E_0, \tau\}$.

We take $Y(s) = \beta(s)h_{\mathcal{E}}$ as in Example 4.5. Introducing the shorthand $\eta := \sqrt{(E_0 + E)^2 + \lambda^2}$, we compute a matrix expression for $\mathcal{L}_s^{(\alpha)}$ by identifying $\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}) \cong \text{Mat}_{2 \times 2}(\mathbb{C}) \cong \mathbb{C}^4$ via $(\begin{smallmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{smallmatrix}) \mapsto \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$. Working in the (ground state, excited state) basis for \mathcal{S} , we obtain

$$\mathcal{L}^{(\alpha)}(s) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & e & 0 \\ f & 0 & 0 & g \end{pmatrix},$$

where

$$\begin{aligned}
a &= \frac{(2(E_0 + E)^2 + \lambda^2 + \lambda^2 \cos(\eta\tau))}{2(1 + e^{E_0\beta(s)})\eta^2 e^{-E_0\beta(s)}} + \frac{2(E_0 - E)^2 + \lambda^2 + \lambda^2 \cos(\nu\tau)}{2(1 + e^{E_0\beta(s)})\nu^2}, \\
d &= \lambda^2 \left(-\frac{2e^{-E_0\alpha\beta(s)}(\cos(\eta\tau) - 1)}{4(1 + e^{E_0\beta(s)})\eta^2} - \frac{2e^{E_0(\alpha+1)\beta(s)}(\cos(\nu\tau) - 1)}{4(1 + e^{E_0\beta(s)})\nu^2} \right), \\
c &= \frac{\lambda^2 \cosh(E_0\beta(s)(\frac{1}{2} + \alpha)) \operatorname{sech}\left(\frac{E_0\beta(s)}{2}\right) \sin\left(\frac{\eta\tau}{2}\right) \sin\left(\frac{\nu\tau}{2}\right)}{\sqrt{E_0^4 + 2(\lambda^2 - E^2)E_0^2 + (E^2 + \lambda^2)^2}}, \\
b &= \frac{(i\eta \cos\left(\frac{\eta\tau}{2}\right) + (E_0 + E) \sin\left(\frac{\eta\tau}{2}\right)) ((E_0 - E) \sin\left(\frac{\nu\tau}{2}\right) - i\nu \cos\left(\frac{\nu\tau}{2}\right))}{\sqrt{E_0^4 + 2(\lambda^2 - E^2)E_0^2 + (E^2 + \lambda^2)^2}}, \\
e &= \frac{(-e^{i\nu\tau}E_0 + E_0 - E + \nu + e^{i\nu\tau}(E + \nu)) (\eta \cos\left(\frac{\eta\tau}{2}\right) + i(E_0 + E) \sin\left(\frac{\eta\tau}{2}\right))}{2\eta\nu e^{\frac{1}{2}i\nu\tau}}, \\
f &= \frac{e^{-E_0\alpha\beta(s)}\lambda^2}{4(1 + e^{E_0\beta(s)})} \left(\frac{2 - 2\cos(\nu\tau)}{\nu^2} - \frac{2e^{E_0(2\alpha+1)\beta(s)}(\cos(\eta\tau) - 1)}{\eta^2} \right), \\
g &= \frac{(2(E_0 + E)^2 + \lambda^2 + \lambda^2 \cos(\eta\tau))}{2(1 + e^{-E_0\beta(s)})\eta^2 e^{E_0\beta(s)}} + \frac{2(E_0 - E)^2 + \lambda^2 + \lambda^2 \cos(\nu\tau)}{2(1 + e^{-E_0\beta(s)})\nu^2},
\end{aligned}$$

which depend on s through $\beta(s)$. The computation was performed with Mathematica, using [Cub09]. We make a particular choice of parameters, $\lambda = 2$, $\tau = 0.5$, $E_0 = 0.8$, $E = 0.9$, and two choices of $[0, 1] \ni s \mapsto \beta(s)$:

$$\beta_1(s) = \frac{2(3 + 4 \tanh(2s))}{3 + 2 \log(\cosh(2))} \quad (41)$$

and

$$\beta_2(s) = a_1 \tanh(2s) - a_2 \tanh\left(\frac{s}{2}\right) - a_3 s^3 + a_4 s^2 - a_5 s + a_6 \quad (42)$$

for $a_1 = 35.483$, $a_2 = 141.929$, $a_3 = 42.945$, $a_4 = 93.5$, $a_5 = 17.808$, $a_6 = 1.061$. We have $\beta_1(0) = \beta_2(0) = 1.06$, and $\beta_1(1) = \beta_2(1) = 2.43$, as well as $\int_0^1 \beta_1(s) ds = \int_0^1 \beta_2(s) ds = 2$. These are plotted in Figure 1.

We compute numerically the function $\Lambda(\alpha)$ for each choice of $s \mapsto \beta(s)$, as shown in Figure 2. Figures 3 and 4 shows the convergence described by Theorem 5.10 by simulating 2,000 instances of this repeated interaction system at four values of T .

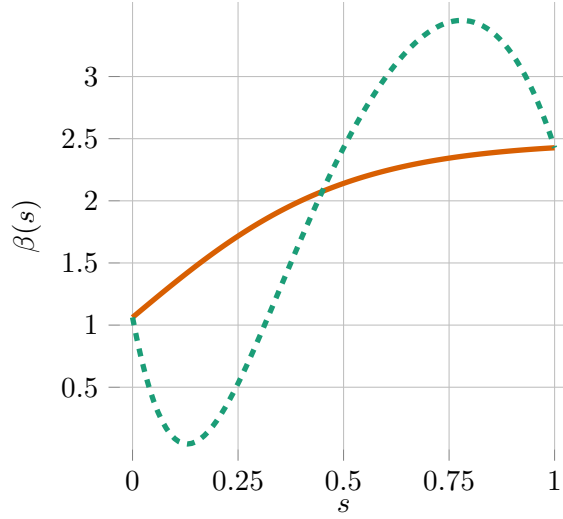


Figure 1: Two choices of curves $s \mapsto \beta(s)$. In the solid orange line, $\beta(s) = \beta_1(s)$, given by (41), and in dashed green line, $\beta(s) = \beta_2(s)$, given by (42).

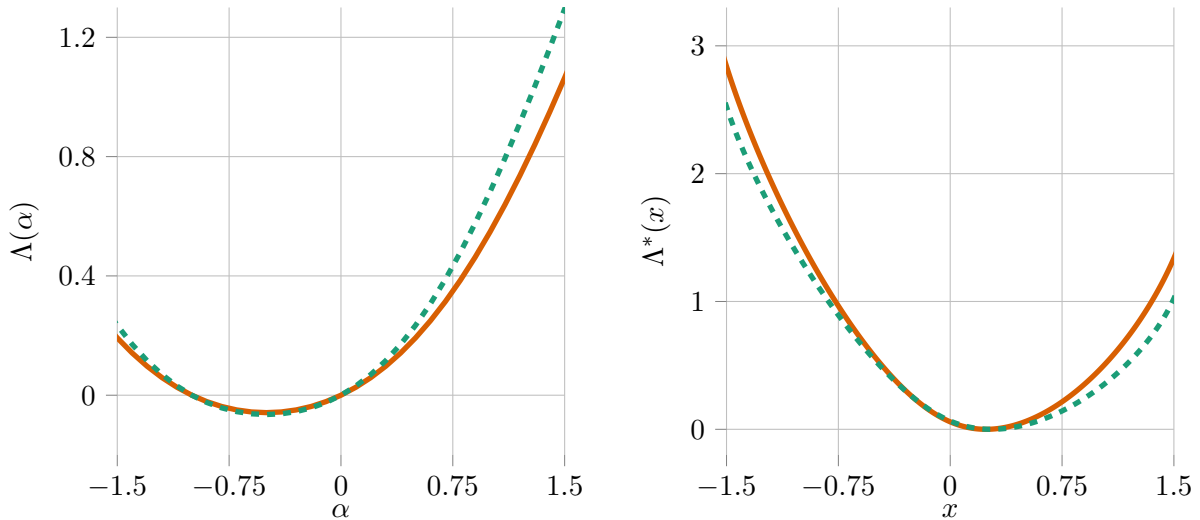


Figure 2: Left: The function $\Lambda(\alpha)$ for $Y = \beta h_{\mathcal{E}}$ in the system of Example 5.12, with $\lambda = 2$, $\tau = 0.5$, $E_0 = 0.8$, $E = 0.9$, plotted for each choice of $\beta(s)$. Right: The rate function $\Lambda^*(\alpha)$, for the same setup. In each plot, the solid orange line corresponds to the choice $\beta(s) = \beta_1(s)$, defined in (41), and the dashed green line corresponds to $\beta(s) = \beta_2(s)$, defined in (42).

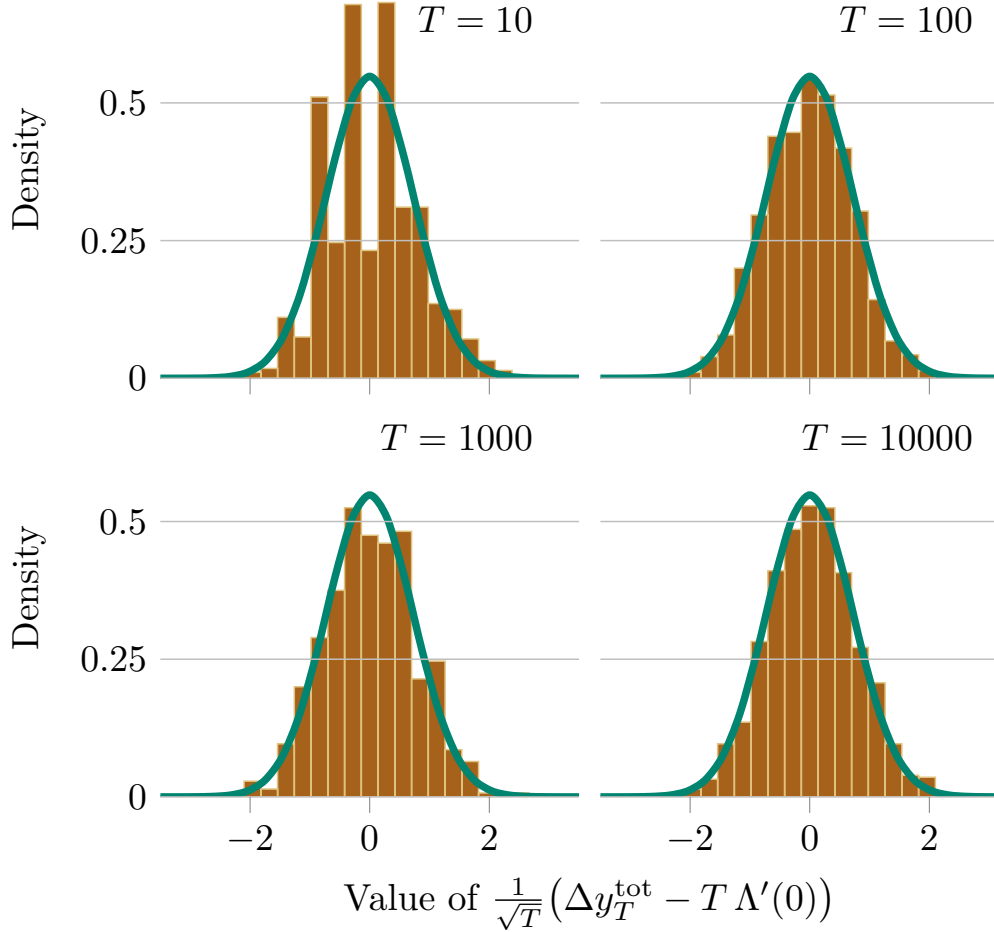


Figure 3: Convergence of $\frac{1}{\sqrt{T}}(\Delta y_T^{\text{tot}} - T \Lambda'(0))$ to a normal distribution, where $\Lambda'(0) \approx 0.240$, with $\beta(s) = \beta_1(s)$ given by (41). Each plot was generated by simulating the two-time measurement protocol in 2,000 instances of the repeated interaction system described in Example 5.12. The value of $\frac{1}{\sqrt{T}}(\Delta y_T^{\text{tot}} - T \Lambda'(0))$ was calculated for each instance and plotted in a histogram in orange, with bar heights normalized to yield total mass 1. In green, the probability density function of $\mathcal{N}(0, \Lambda''(0))$ is plotted, where $\Lambda''(0) \approx 0.530$. As T increases, one sees qualitatively the convergence of $\frac{1}{\sqrt{T}}(\Delta y_T^{\text{tot}} - T \Lambda'(0))$ to the normal distribution, as guaranteed by Theorem 5.10.

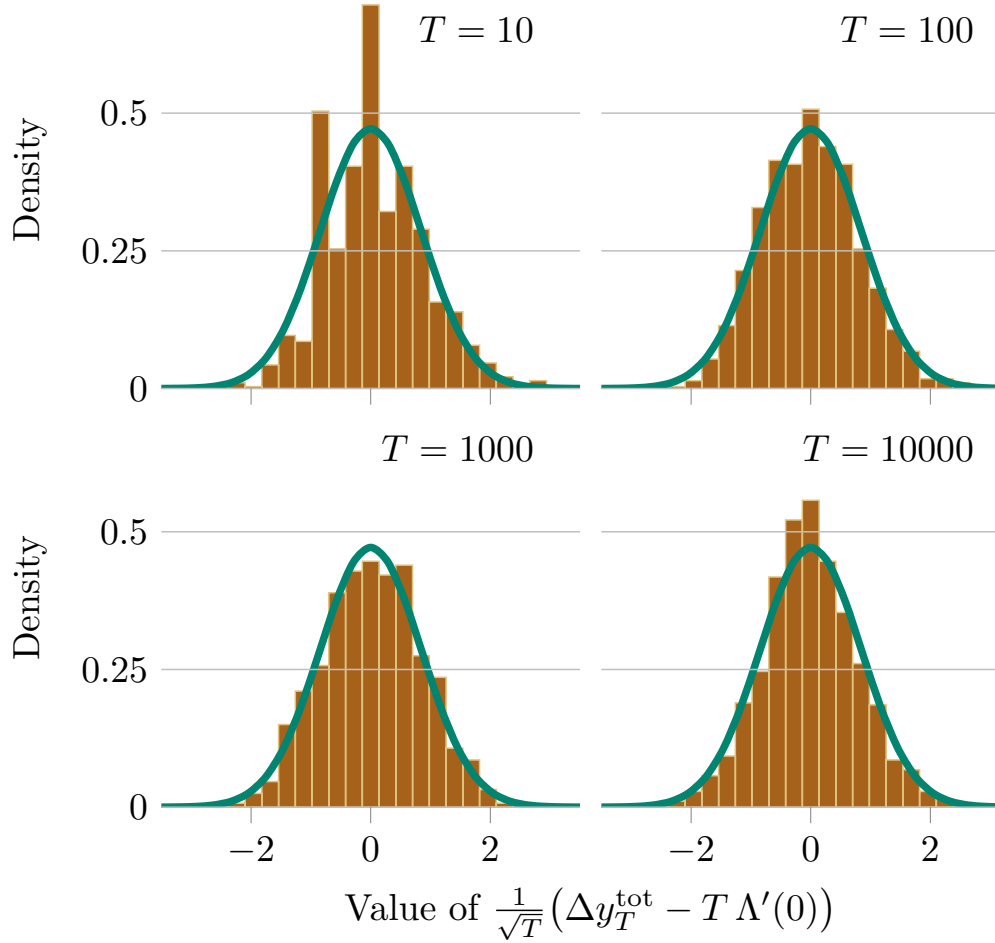


Figure 4: The same setup as Figure 3, with $\beta(s) = \beta_2(s)$ given by (42). Here, $\Lambda'(0) \approx 0.275$, and $\Lambda''(0) \approx 0.716$.

A Peripheral spectrum of CPTP maps and their deformations

In this section we discuss a full study of the peripheral spectrum, and associated spectral projectors, of CPTP maps and their deformations. This will in particular apply to the deformed reduced dynamical operators $\mathcal{L}_Y^{(\alpha)}$.

We start by collecting various results from the seminal paper [EHK78]. Let us therefore consider a finite-dimensional Hilbert space \mathcal{H} , and Φ a completely positive, not necessarily trace-preserving map Φ on $\mathcal{I}_1(\mathcal{H})$. Since \mathcal{H} is finite-dimensional, we can identify $\mathcal{I}_1(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$, so that all definitions below apply to either Φ or Φ^* . Any completely positive map on $\mathcal{B}(\mathcal{H})$ (with finite-dimensional \mathcal{H}) admits a *Kraus decomposition*, i.e. there exist maps $V_i \in \mathcal{B}(\mathcal{H})$ for i in a finite set I , such that $\Phi(\rho) = \sum_{i \in I} V_i \rho V_i^*$ for all ρ .

Definition A.1. If the completely positive map Φ satisfies either of the following equivalent properties

- the only self-adjoint projectors P on \mathcal{H} satisfying $\Phi(P\mathcal{I}_1(\mathcal{H})P) \subseteq P\mathcal{I}_1(\mathcal{H})P$ are Id and 0,
- the only subspaces E of \mathcal{H} such that $V_i E \subset E$ for all $i \in I$ are $\{0\}$ and \mathcal{H} ,

we say that Φ is *irreducible*. If for any nonzero self-adjoint projector P on \mathcal{H} , there exists n such that the map $\Phi^n(P)$ is positive-definite, we say that Φ is *primitive*.

Clearly, if Φ is primitive then it is irreducible. In addition, it is immediate to see from the above equivalences that Φ is irreducible (resp. primitive) if and only if Φ^* is irreducible (resp. primitive). Remark also that an irreducible completely positive map Φ will map a faithful state ρ to a positive-definite operator, as otherwise the support projector P of $\Phi(\rho)$ will satisfy $P \leq c\rho$, and therefore $\Phi(P) \leq c\Phi(\rho) \leq c'P$, for some $c, c' > 0$, and therefore contradict the definition of irreducibility above.

It is shown in [EHK78] that, if Φ is irreducible, then its spectral radius λ is a simple eigenvalue and the associated spectral subspace is generated by a positive-definite operator. An immediate consequence is that any positive-definite eigenvector of Φ must be an eigenvector for λ .

If Φ is CPTP then necessarily $\lambda = 1$. It is also shown in [EHK78] that, if Φ is completely positive, irreducible, and trace-preserving, then

- the peripheral spectrum of Φ is a subgroup $S_z = \{\theta^m \mid m = 0, \dots, z-1\}$ of the unit circle, where $\theta = e^{2i\pi/z}$, and each θ^m is a simple eigenvalue,
- there exist a faithful state ρ^{inv} , and a unitary operator u (called a Perron–Frobenius unitary of Φ) satisfying $[\rho^{\text{inv}}, u] = 0$, $u^z = \text{Id}$ and $u^k \neq \text{Id}$ for $k = 0, \dots, z-1$, such that

$$\begin{aligned} \Phi(\rho u) &= \theta \Phi(\rho) u \quad \forall \rho \in \mathcal{I}_1(\mathcal{H}) \\ \Phi^*(uX) &= \bar{\theta} u \Phi^*(X) \quad \forall X \in \mathcal{B}(\mathcal{H}). \end{aligned} \tag{43}$$

A consequence of the above is that the (unique up to a multiplicative constant) eigenvector of Φ (resp. Φ^*) associated with the eigenvalue θ^m is $\rho^{\text{inv}} u^m$ (resp. u^{-m}), and the spectral projector of Φ associated with θ^m is $\eta \mapsto \text{Tr}(u^{-m} \eta) \rho^{\text{inv}} u^m$.

Remark that relations (43) are equivalent to $V_i u = \theta u V_i$ (see [FP09]). Last, a CPTP map is primitive if and only if it is irreducible with $z = 1$, or equivalently if and only if Φ^n is irreducible for any $n \in \mathbb{N}$. Conversely, a CPTP map that admits a faithful state as a unique (up to a multiplicative constant) invariant is irreducible. If, in addition, 1 is the only eigenvalue of modulus one, then Φ is

primitive. In particular, our description of assumptions **Irr** and **Prim** are consistent with the above definitions.

In addition, the spectral decomposition of u is of the form $u = \sum_{m=0}^{z-1} \theta^m p_m$, where the projectors p_m satisfy $p_m V_i = V_i p_{m+1}$ for all i and m (here and below, $m+1$ means $m+1 \pmod z$ whenever it appears as the index of a projector p and we adopt the same convention for $m-1$). Each subspace $\mathcal{B}(\text{Ran } p_m)$ of $\mathcal{B}(\mathcal{H})$ is therefore invariant by Φ^{*z} and the restriction of Φ^{*z} to that subspace is primitive.

We now define deformations of CPTP maps, or, rather, of their Kraus decompositions. For this, fix a finite set I . We call a family $V = (V_i)_{i \in I}$ of operators on $\mathcal{B}(\mathcal{H})$ an *irreducible Kraus family* (indexed by I) if $\sum_{i \in I} V_i^* V_i = \text{Id}$, and the only subspaces E of \mathcal{H} such that $V_i E \subset E$ for all $i \in I$ are $\{0\}$ and \mathcal{H} . We fix a set I and denote by \mathfrak{K}_I the set of irreducible Kraus maps indexed by I . From the above discussions, any irreducible Kraus family $(V_i)_{i \in I}$ defines an irreducible CPTP map Φ by $\Phi(\rho) = \sum V_i \rho V_i^*$.

Remark A.2. Conversely, any irreducible CPTP admits an irreducible Kraus decomposition indexed by $I = \{1, \dots, (\dim \mathcal{H})^2\}$ (possibly with $V_i = 0$ for some i). However, in applications of the present results in Section 3.3, where $\mathcal{H} = \mathcal{H}_S$, our model yields a Kraus family indexed by pairs $(i, j) \in \text{sp } Y \times \text{sp } Y$ where Y is an operator acting on a Hilbert space \mathcal{H}_E unrelated to \mathcal{H}_S . We therefore need to consider Kraus families indexed by an arbitrary set I .

Now fix $v = (v_i)_{i \in I}$ a family of strictly positive real numbers. For $(V_i)_{i \in I}$ an irreducible Kraus family and $\alpha \in \mathbb{R}$ we define a map $\Phi^{(\alpha)}$ on $\mathcal{I}_1(\mathcal{H})$ by

$$\Phi^{(\alpha)}(\rho) = \sum_i v_i^\alpha V_i \rho V_i^*.$$

This map $\Phi^{(\alpha)}$ is a completely positive map, and since $\Phi^{(0)} = \Phi$, it can be viewed as a deformation of Φ . We will prove the following result about the peripheral spectrum of $\Phi^{(\alpha)}$.

Proposition A.3. *Let $(V_i)_{i \in I}$ an irreducible Kraus family, $v = (v_i)_{i \in I}$ a family of strictly positive real numbers, and define $\Phi, \Phi^{(\alpha)}$ as above. Let u be a Perron–Frobenius unitary for Φ , and denote by $p_m, m = 0, \dots, z-1$ its spectral projectors. There exist three smooth maps $\alpha \mapsto \lambda^{(\alpha)}, \mathbf{I}^{(\alpha)}, \rho^{(\alpha)}$ from \mathbb{R} to, respectively, \mathbb{R}_+^* , the set of positive-definite operators, and the set of faithful states, such that for all α in \mathbb{R} ,*

- the peripheral spectrum of $\Phi^{(\alpha)}$ is $\lambda^{(\alpha)} S_z = \{\lambda^{(\alpha)} \theta^k \mid k = 0, \dots, z-1\}$,
- one has the commutation relations $[\mathbf{I}^{(\alpha)}, u] = 0$, and $[\rho^{(\alpha)}, u] = 0$,
- one has $\text{Tr}(\rho^{(\alpha)} \mathbf{I}^{(\alpha)}) = 1$ for all $\alpha \in \mathbb{R}$,
- the (unique up to a multiplicative constant) eigenvector of $\Phi^{(\alpha)}$ (resp. $\Phi^{(\alpha)*}$) associated with the eigenvalue $\lambda^{(\alpha)} \theta^m$ is $\rho^{(\alpha)} u^m$ (resp. $\mathbf{I}^{(\alpha)} u^{-m}$), and the spectral projector of $\Phi^{(\alpha)}$ associated with $\lambda^{(\alpha)} \theta^m$ is

$$\eta \mapsto \text{Tr}(\mathbf{I}^{(\alpha)} u^{-m} \eta) \rho^{(\alpha)} u^m.$$

Remark A.4. For $\alpha = 0$ we have $\lambda^{(\alpha)} = 1$, $\mathbf{I}^{(\alpha)} = \text{Id}$ and $\rho^{(\alpha)} = \rho^{\text{inv}}$. Note also that $\lambda^{(\alpha)}, \mathbf{I}^{(\alpha)}, \rho^{(\alpha)}$ depend on the choice of $v = (v_i)_{i \in I}$.

Proof. By the criterion on irreducibility cited above, the map $\Phi^{(\alpha)*}$ is completely positive and irreducible. Therefore, its spectral radius $\lambda^{(\alpha)} > 0$ is a simple eigenvalue, which is locally isolated,

with positive-definite eigenvector $I^{(\alpha)}$. Moreover, recall that $I^{(\alpha)}$ is the unique positive-definite eigenvector (up to a positive constant) associated to a positive eigenvalue. By standard perturbation theory we can parameterize the map $\alpha \mapsto I^{(\alpha)}$ to be analytic in a neighbourhood of the origin. This $I^{(\alpha)}$ is defined up to a multiplicative constant, which we will specify later on. We define a map $\widehat{\Phi}^{(\alpha)}$ and its adjoint $\widehat{\Phi}^{(\alpha)*}$ by

$$\begin{aligned}\widehat{\Phi}^{(\alpha)}(\eta) &= (\lambda^{(\alpha)})^{-1} (I^{(\alpha)})^{1/2} \Phi^{(\alpha)} \left((I^{(\alpha)})^{-1/2} \eta (I^{(\alpha)})^{-1/2} \right) (I^{(\alpha)})^{1/2} \\ \widehat{\Phi}^{(\alpha)*}(X) &= (\lambda^{(\alpha)})^{-1} (I^{(\alpha)})^{-1/2} \Phi^{(\alpha)*} \left((I^{(\alpha)})^{1/2} X (I^{(\alpha)})^{1/2} \right) (I^{(\alpha)})^{-1/2}.\end{aligned}\tag{44}$$

Note that $\widehat{\Phi}^{(\alpha)}$ writes $\widehat{\Phi}^{(\alpha)}(\rho) = \sum_{i \in I} \widehat{V}_i(\alpha) \rho \widehat{V}_i(\alpha)^*$, with

$$\widehat{V}_i(\alpha) = (v_i^\alpha / \lambda^{(\alpha)})^{1/2} (I^{(\alpha)})^{+1/2} V_i (I^{(\alpha)})^{-1/2}.\tag{45}$$

The application $\widehat{\Phi}^{(\alpha)*}$ is completely positive, irreducible since $I^{(\alpha)}$ entering in the definition of its Kraus operators is invertible, and satisfies $\widehat{\Phi}^{(\alpha)*}(\text{Id}) = \text{Id}$. Hence the map $\widehat{\Phi}^{(\alpha)}$ is irreducible, completely positive and trace-preserving, so that $\widehat{V} = (\widehat{V}_i(\alpha))_{i \in I} \in \mathfrak{K}_I$. We can therefore define a map $T_v^{(\alpha)}$ on \mathfrak{K}_I by $T_v^{(\alpha)} : V \mapsto \widehat{V}(\alpha)$. Note that $\widehat{V}_i(0) = V_i$, so that \widehat{V} is a deformation of V . We have the following easy result:

Lemma A.5. *With the above notation (and fixed $v = (v_i)_{i \in I}$), for any $\alpha \in \mathbb{R}$ the map $T_v^{(\alpha)}$ is invertible with inverse $T_v^{(-\alpha)}$.*

Proof of Lemma A.5. For $\rho \in \mathcal{I}_1(\mathcal{H})$ consider

$$\rho \mapsto \sum_i v_i^{-\alpha} \widehat{V}_i(\alpha) \rho \widehat{V}_i(\alpha)^* = (\lambda^{(\alpha)})^{-1} \sum_i (I^{(\alpha)})^{+1/2} V_i (I^{(\alpha)})^{-1/2} \rho (I^{(\alpha)})^{-1/2} V_i^* (I^{(\alpha)})^{+1/2}.$$

The dual of this map is

$$X \mapsto (\lambda^{(\alpha)})^{-1} \sum_i (I^{(\alpha)})^{-1/2} V_i^* (I^{(\alpha)})^{+1/2} X (I^{(\alpha)})^{+1/2} V_i (I^{(\alpha)})^{-1/2},$$

which admits $(I^{(\alpha)})^{-1}$ as an eigenvector for $(\lambda^{(\alpha)})^{-1}$. Since $(I^{(\alpha)})^{-1}$ is positive-definite, $(\lambda^{(\alpha)})^{-1}$ is the spectral radius of this map, with associated eigenvector $(I^{(\alpha)})^{-1}$. Applying the above definition of $T_v^{(-\alpha)}$ therefore shows that $T_v^{(-\alpha)}(\widehat{V})$ consists of maps $(v_i^{-\alpha} / \lambda^{(\alpha)-1})^{1/2} (I^{(\alpha)})^{-1/2} \widehat{V}_i(\alpha) (I^{(\alpha)})^{+1/2} = V_i$. \square

As mentioned above, $\widehat{\Phi}^{(\alpha)}$ is an irreducible CPTP map. From the results recalled above, its peripheral spectrum is of the form $S_{z^{(\alpha)}}$, all peripheral eigenvalues are simple, and an eigenvector associated with $\theta^{(\alpha)} = e^{2ij\pi/z^{(\alpha)}}$ is of the form $\widehat{\rho}^{(\alpha)}(u^{(\alpha)})^m$ with $\widehat{\rho}^{(\alpha)} \in \mathcal{D}(\mathcal{H})$ positive-definite and $u^{(\alpha)}$ unitary. Remark already that since $\widehat{\rho}^{(\alpha)}$ is associated with the simple, isolated eigenvalue 1, we can parameterize $\alpha \mapsto \widehat{\rho}^{(\alpha)}$ to be analytic in a neighbourhood of the origin. In addition, an operator $\eta \in \mathcal{I}_1(\mathcal{H})$ is an eigenvector of $\Phi^{(\alpha)}$ for the eigenvalue μ if and only if $(I^{(\alpha)})^{+1/2} \eta (I^{(\alpha)})^{+1/2}$ is an eigenvector of $\widehat{\Phi}^{(\alpha)}$ for the eigenvalue $(\lambda^{(\alpha)})^{-1} \mu$. Therefore, $\rho^{(\alpha)} = (I^{(\alpha)})^{-1/2} \widehat{\rho}^{(\alpha)} (I^{(\alpha)})^{-1/2}$ is an eigenvector of $\Phi^{(\alpha)}$ associated with $\lambda^{(\alpha)}$, the peripheral spectrum of $\Phi^{(\alpha)}$ is $\lambda^{(\alpha)} S_{z^{(\alpha)}}$, and the peripheral eigenvalues are simple. Because the definition of $\widehat{\Phi}^{(\alpha)}$ does not depend on the free multiplicative constant in $I^{(\alpha)}$, this $\widehat{\rho}^{(\alpha)}$ is uniquely defined; we can therefore fix the constant in $I^{(\alpha)}$ so that $\rho^{(\alpha)}$ has trace one. We now prove that $z^{(\alpha)}$ is independent of α , and that $u^{(\alpha)}$ can be chosen to be constant equal to u .

Lemma A.6. *With the above notations we have $z^{(\alpha)} = z$ for all $\alpha \in \mathbb{R}$.*

Proof of Lemma A.6. We have $\Phi^{(\alpha)*}(p_j X p_j) = p_{j+1} \Phi^{(\alpha)*}(X) p_{j+1}$ for all X in $\mathcal{B}(\mathcal{H})$ from the commutation relations for p_j and V_i . In addition, each $p_j \mathbb{I}^{(\alpha)} p_j$ is nonzero since $\mathbb{I}^{(\alpha)}$ is positive-definite, and satisfies $\Phi^{(\alpha)*}(p_j \mathbb{I}^{(\alpha)} p_j) = \lambda^{(\alpha)} p_{j+1} \mathbb{I}^{(\alpha)} p_{j+1}$. This implies by a direct computation that for any $n = 0, \dots, z-1$, the non-zero operator $\sum_{j=0}^{z-1} e^{2i\pi j n/z} p_j \mathbb{I}^{(\alpha)} p_j$ is an eigenvector of $\Phi^{(\alpha)*}$ for the eigenvalue $\lambda^{(\alpha)} e^{-2i\pi n/z}$, so that $\Phi^{(\alpha)*}$ has at least z peripheral eigenvalues, and $z \leq z^{(\alpha)}$.

Lemma A.5 shows that this same inequality applied to $\widehat{\Phi}^{(\alpha)}$ in place of Φ and $-\alpha$ in place of α gives $z^{(\alpha)} \leq z$. We therefore have $z = z^{(\alpha)}$. \square

This implies in turn that $u^{(\alpha)}$ is an eigenvector of $\widehat{\Phi}^{(\alpha)*}$ for the simple isolated eigenvalue $\bar{\theta}$, so that we can parameterize $\alpha \mapsto u^{(\alpha)}$ to be analytic in a neighbourhood of the origin.

Lemma A.7. *With the above notation, we have $u^{(\alpha)} = u$ and $[\mathbb{I}^{(\alpha)}, u] = 0$, $[\rho^{(\alpha)}, u] = 0$ for all $\alpha \in \mathbb{R}$.*

Proof of Lemma A.7. Consider the simple eigenvalue $\bar{\theta}$ of $\widehat{\Phi}^{(\alpha)*}$. The associated eigenspace is one-dimensional and contains $u^{(\alpha)}$. We also show that $(\mathbb{I}^{(\alpha)})^{+1/2} u (\mathbb{I}^{(\alpha)})^{-1/2}$ is another eigenvector of $\bar{\theta}$ for $\widehat{\Phi}^{(\alpha)*}$:

$$\begin{aligned} \widehat{\Phi}^{(\alpha)*}((\mathbb{I}^{(\alpha)})^{+1/2} u (\mathbb{I}^{(\alpha)})^{-1/2}) &= \sum_i \frac{v_i^\alpha}{\lambda^{(\alpha)}} (\mathbb{I}^{(\alpha)})^{-1/2} V_i^* \mathbb{I}^{(\alpha)} u V_i (\mathbb{I}^{(\alpha)})^{-1/2} \\ &= \bar{\theta} \sum_i \frac{v_i^\alpha}{\lambda^{(\alpha)}} (\mathbb{I}^{(\alpha)})^{-1/2} V_i^* \mathbb{I}^{(\alpha)} V_i u (\mathbb{I}^{(\alpha)})^{-1/2} \\ &= \frac{\bar{\theta}}{\lambda^{(\alpha)}} (\mathbb{I}^{(\alpha)})^{-1/2} \left(\sum_i v_i^\alpha V_i^* \mathbb{I}^{(\alpha)} V_i \right) u (\mathbb{I}^{(\alpha)})^{-1/2} \\ &= \bar{\theta} (\mathbb{I}^{(\alpha)})^{+1/2} u (\mathbb{I}^{(\alpha)})^{-1/2}. \end{aligned}$$

We therefore have $(\mathbb{I}^{(\alpha)})^{+1/2} u (\mathbb{I}^{(\alpha)})^{-1/2} = \gamma^{(\alpha)} u^{(\alpha)}$ for some $\gamma^{(\alpha)} \in \mathbb{C}$, and the relation $u^z = (u^{(\alpha)})^z = \text{Id}$ requires that $\gamma^{(\alpha)}$ is a z th root of unity. Now, $(u^{(\alpha)})^* u^{(\alpha)} = \text{Id}$ implies that $u^* \mathbb{I}^{(\alpha)} u = \mathbb{I}^{(\alpha)}$, so that $[\mathbb{I}^{(\alpha)}, u] = 0$. This finally gives us $u = \gamma^{(\alpha)} u^{(\alpha)}$ and since we chose $u^{(\alpha)}$ to be analytic in α , the phase $\gamma^{(\alpha)}$ is necessarily 1. Last, $[\widehat{\rho}^{(\alpha)}, u^{(\alpha)}] = 0$ and this implies $[\rho^{(\alpha)}, u] = 0$. \square

We can now conclude the proof of Proposition A.3. The validity of our parameterizations rely only on the fact that the peripheral eigenvalues for $\Phi^{(\alpha)}$, $\widehat{\Phi}^{(\alpha)}$ and $\widehat{\Phi}^{(\alpha)*}$ are isolated. Since the peripheral spectra for these maps are, respectively, $\lambda^{(\alpha)} S_z$, S_z and S_z , all peripheral eigenvalues are isolated uniformly for α in any compact set containing the origin. This allows us to extend all parameterizations to be analytic on \mathbb{R} . Last, the eigenvector of $\Phi^{(\alpha)}$ (resp. $\Phi^{(\alpha)*}$) associated with the eigenvalue $\lambda^{(\alpha)} \theta^m$ is $\rho^{(\alpha)} u^m$ (resp. $\mathbb{I}^{(\alpha)} u^{-m}$), and this gives the form of the corresponding spectral projectors. \square

The preceding results also give some information about the peripheral spectrum of $\Phi^{(\alpha)}$ for complex α , as the following corollary shows:

Corollary A.8. *Let $(V_i)_{i \in I}$ and $v = (v_i)_{i \in I}$ be as in Proposition A.3. For any α_0 in \mathbb{R} , there exists a neighbourhood N_{α_0} of α_0 in \mathbb{C} , such that for α in N_{α_0} the peripheral spectrum of $\Phi^{(\alpha)}$ is of the form $\{\lambda^{(\alpha)} \theta^m \mid m = 0, \dots, z-1\}$ for some $\lambda^{(\alpha)}$ in \mathbb{C} .*

Proof. By Proposition A.3, the peripheral spectrum of $\Phi^{(\alpha_0)}$ is $\{\lambda^{(\alpha_0)}\theta^m \mid m = 0, \dots, z-1\}$. By standard perturbation theory, for $m = 0, \dots, z-1$ there exist analytic functions $\alpha \mapsto \lambda_m^{(\alpha)}$ defined on a neighbourhood of α_0 , such that $\lambda_m^{(\alpha_0)} = \lambda^{(\alpha_0)}\theta^m$ and the $\lambda_m^{(\alpha)}$ are eigenvalues of $\Phi^{(\alpha)}$. In particular, there exists a (complex) neighbourhood N_{α_0} of α_0 such that for α in N_{α_0} , any eigenvalue of $\Phi^{(\alpha)}$ of maximum modulus is one of the $\lambda_m^{(\alpha)}$. Denote (consistently with the above notation) by $I^{(\alpha)}$ an eigenvector of $\Phi^{(\alpha)*}$ for $\bar{\lambda}_0^{(\alpha)}$. Since $u^{-1}V_i = \theta V_i u^{-1}$ we have $\Phi^{(\alpha)*}(I^{(\alpha)}u^{-m}) = \bar{\lambda}_0^{(\alpha)}\theta^m I^{(\alpha)}u^{-m}$, so that $\bar{\lambda}_0^{(\alpha)}\theta^m$ is an eigenvalue of $\Phi^{(\alpha)*}$ for $m = 0, \dots, z-1$. Since all such $\lambda_0^{(\alpha)}\theta^m$ have the same modulus, one has necessarily $\lambda_m^{(\alpha)} = \lambda_0^{(\alpha)}\theta^m$ for $m = 0, \dots, z-1$. The conclusion follows by letting $\lambda^{(\alpha)} := \lambda_0^{(\alpha)}$. \square

The next result gives the asymptotics of the spectral radius of $\Phi^{(\alpha)}$ as $\alpha \rightarrow \pm\infty$.

Proposition A.9. *Let $(V_i)_{i \in I}$ and $v = (v_i)_{i \in I}$ be as in Proposition A.3 and assume that $V_i \neq 0$ for all $i \in I$. Define $v_+ = \max_{i \in I} v_i$, $v_- = \min_{i \in I} v_i$, and*

$$I_{\pm} = \{i \in I \mid v_i = v_{\pm}\}, \quad \Phi_{\pm} = \sum_{i \in I_{\pm}} V_i \cdot V_i^*, \quad \lambda_{\pm} = \text{spr } \Phi_{\pm}.$$

Then

$$\text{spr } \Phi^{(\alpha)} = v_{\pm}^{\alpha}(\lambda_{\pm} + o(1)) \quad \text{for } \alpha \rightarrow \pm\infty.$$

Proof. Remark that $\lambda_{\pm} \neq 0$. The statement follows immediately from $\Phi^{(\alpha)} = v_{\pm}^{\alpha}(\Phi_{\pm} + o(1))$ and standard perturbation theory. \square

B Adiabatic theorem for discrete non-unitary evolutions

We devote this section to elements of adiabatic theory that are suitable for discrete non-unitary time evolution. To be precise, the theory is applicable to discrete dynamics arising from a family $(F(s))_{s \in [0,1]}$ of maps from a Banach space X to itself satisfying

Hyp0 The mapping $s \mapsto F(s)$ is a continuous $\mathcal{B}(X)$ -valued function of $s \in [0, 1]$;

Hyp1 For all $s \in [0, 1]$, $\text{spr } F(s) = 1$;

Hyp2 The peripheral spectrum of $F(s)$ consists of finitely many isolated semi-simple eigenvalues for all $s \in [0, 1]$;

Hyp3 With $P(s)$ the spectral projector of $F(s)$ onto the peripheral eigenvalues, the map $s \mapsto F^P(s) := F(s)P(s)$ is a C^2 $\mathcal{B}(X)$ -valued function of $s \in [0, 1]$;

Hyp4 With $Q(s) := \text{Id} - P(s)$,

$$\ell := \sup_{s \in [0,1]} \text{spr } F(s)Q(s) < 1.$$

We call such a family *admissible* for our adiabatic theorems. We emphasize that hypotheses are stated in terms of spectral radii, and not of norms as was the case for the hypotheses [HJPR17], which we recall here (adapting slightly the notation for coherence) for comparison:

H1. For all $s \in [0, 1]$, $\|F(s)\| \leq 1$, i.e. $F(s)$ is a contraction;

H2. There is a uniform gap $\epsilon > 0$ such that, for $s \in [0, 1]$, each peripheral eigenvalue $e^j(s) \in \text{sp } F(s) \cap S^1$ is simple, and $|e^j(s) - e^i(s)| > 2\epsilon$ for any $e^j(s) \neq e^i(s)$ in $\text{sp } F(s) \cap S^1$;

- H3.** Let $P^m(s)$ be the spectral projector associated with $e^m(s) \in \text{sp } \mathcal{L}(s) \cap S^1$, and $P(s) = \sum_m P^m(s)$ the *peripheral spectral projector*. The map $s \mapsto F^P(s) := F(s)P(s)$ is C^2 on $[0, 1]$;
- H4.** With $Q(s) := \text{Id} - P(s)$,

$$\ell := \sup_{s \in [0,1]} \|F(s)Q(s)\| < 1.$$

In applications, the Banach space X is again $\mathcal{I}_1(\mathcal{H}_S)$ equipped with the trace norm, and the role of $F(s)$ is played by appropriate deformations of the reduced dynamics $\mathcal{L}(s)$ arising from a repeated interaction system satisfying **ADRS**.

B.1 Adiabatic theorem for products of projectors

We start with a result about products of projectors.

Definition B.1. Let $(P^m(s))_{s \in [0,1]}$, $m = 1, \dots, z$ be C^1 families of projector-valued operators in a Banach space X , satisfying $\sum_{m=1}^z P^m(s) = \text{Id}$ for all $s \in [0, 1]$. Let $W : [0, 1] \rightarrow \mathcal{B}(X)$ be the family of *intertwining operators* given by

$$W'(s) = \sum_{m=1}^z P^{m'}(s)P^m(s)W(s), \quad W(0) = \text{Id}, \quad (46)$$

where $P^{m'}(s)$ is the derivative of $s \mapsto P^m(s)$.

Standard results (see e.g. Section II.5 in [Kat76]) imply that

$$W(s)P^m(0) = P^m(s)W(s) \quad (47)$$

for all $s \in [0, 1]$ and $m = 1, \dots, z$, and that $W(s)$ is invertible, with inverse $W^{-1}(s)$ solution to

$$V'(s) = - \sum_{m=1}^z V(s)P^{m'}(s)P^m(s).$$

Note that, if we are given a single C^1 family $(P(s))_{s \in [0,1]}$ of operators then we can apply the above to $P^1(s) = P(s)$, $P^2(s) = \text{Id} - P(s)$.

Remark that we have the immediate relations

$$P^m(s)P^{m'}(s)P^m(s) = 0 \quad (48)$$

for all $s \in [0, 1]$ and $m = 1, \dots, z$. For any family $(P^m(s))_{s \in [0,1]}$ as above we will denote by C_P the quantity

$$C_P = \sup_{s \in [0,1]} \sup_{m=1, \dots, z} \max(\|P^m(s)\|, \|P^{m'}(s)\|). \quad (49)$$

Proposition B.2. Let $(P^m(s))_{s \in [0,1]}$ and W be as in Definition B.1. Then there exists $C > 0$ and $T_0 \in \mathbb{N}$ such that for $T \geq T_0$ and $\{s_k\}_{k \in \{0,1,2, \dots, T\}} \subset [0, 1]$ with $|s_k - s_{k-1}| = 1/T$ with $k \leq T$ one has

$$\|P^m(s_k)P^m(s_{k-1}) \cdots P^m(s_0) - W(s_k)P^m(0)W^{-1}(s_0)\| \leq C/T \quad (50)$$

where C and T_0 depend on C_P defined by (49) only, and C is a continuous function of C_P .

Remark B.3. Before we prove this, let us mention

- i. For simplicity, differentiability is understood in the norm sense in case $\dim(X) = \infty$.
- ii. It is enough that the maps $s \mapsto P^m(s)$ be C^1 for this proposition to hold.
- iii. The s_k 's need not be distinct, except those with consecutive indices.
- iv. The norms $\|P^m(s)\|$ are in general larger than one.

Proof. For any $s, s' \in [0, 1]$, we have

$$P^m(s)P^m(s') = W(s)P^m(0)W^{-1}(s)W(s')P^m(0)W^{-1}(s'),$$

where, using the shorthand $K(s) = \sum_{k=1}^z P^{k'}(s)P^k(s)$,

$$\begin{aligned} W^{-1}(s)W(s') &= \text{Id} + W^{-1}(s)(W(s') - W(s)) \\ &= \text{Id} + W^{-1}(s) \int_s^{s'} K(t)W(t) dt \\ &= \text{Id} + \int_s^{s'} (W^{-1}(s)W(t))W^{-1}(t)K(t)W(t) dt \\ &= \text{Id} + \int_s^{s'} W^{-1}(t)K(t)W(t) dt \\ &\quad + \int_s^{s'} \int_s^t W^{-1}(s)K(u)W(u) du W^{-1}(t)K(t)W(t) dt. \end{aligned}$$

In addition, by relation (48), $P^m(t)P^{m'}(t)P^m(t) \equiv 0$, so that

$$P^m(0)W^{-1}(t)K(t)W(t)P^m(0) \equiv 0 \tag{51}$$

and

$$P^m(s)P^m(s') = W(s)P^m(0) \left(\text{Id} + \int_s^{s'} \int_s^t W^{-1}(s)K(u)W(u)W^{-1}(t)K(t)W(t) du dt \right) P^m(0)W^{-1}(s').$$

Denote by $J(s, s')$ the integral term:

$$J(s, s') = \int_s^{s'} \int_s^t W^{-1}(s)K(u)W(u)W^{-1}(t)K(t)W(t) du dt$$

and

$$\tilde{J}(s, s') = J(s, s') + [P^m(0), J(s, s')].$$

We have

$$P^m(0)J(s, s')P^m(0) = \tilde{J}(s, s')P^m(0),$$

with

$$\max(\|J(s, s')\|, \|\tilde{J}(s, s')\|) \leq c(s - s')^2,$$

for some $c > 0$ which is a continuous function of C_P , (49). Using these considerations iteratively on the product (50), we get

$$\begin{aligned} W(s_k)P^m(0)W(s_k)^{-1}W(s_{k-1})P^m(0)W^{-1}(s_{k-1}) \cdots W(s_0)P^m(0)W^{-1}(s_0) \\ = W(s_k)P^m(0)(\text{Id} + J(s_k, s_{k-1}))P^m(0)(\text{Id} + J(s_{k-1}, s_{k-2}))P^m(0) \end{aligned}$$

$$\begin{aligned}
& \cdots P^m(0)(\text{Id} + J(s_1, s_0))P^m(0)W^{-1}(s_0) \\
&= W(s_k)(\text{Id} + \tilde{J}(s_k, s_{k-1}))(\text{Id} + \tilde{J}(s_{k-1}, s_{k-2})) \\
& \cdots (\text{Id} + \tilde{J}(s_1, s_0))P^m(0)W^{-1}(s_0).
\end{aligned}$$

We denote by $\text{Id} + R_1$ the product

$$\text{Id} + R_1 = (\text{Id} + \tilde{J}(s_k, s_{k-1}))(\text{Id} + \tilde{J}(s_{k-1}, s_{k-2})) \cdots (\text{Id} + \tilde{J}(s_1, s_0)).$$

With c as above, by a standard combinatorics argument, we get

$$\begin{aligned}
\|R_1\| &\leq \sum_{k=1}^k (c/T^2)^k \binom{n}{k} = (1 + c/T^2)^k - 1 \\
&\leq (1 + c/T^2)^T - 1 = e^{T \ln(1+c/T^2)} - 1,
\end{aligned}$$

so that $\|R_1\| \leq C'/T$, for T larger than some T_0 (which depends only on c), where C' has the required properties. This yields the result with a C as stated, since $\sup_{s \in [0,1]} \|W^{\pm 1}(s)\|$ and $\|P^m(0)\|$ satisfy the requirements as well. \square

If the projectors $P(s)$ are rank one, they write $P(s) = \phi(s)\psi^*(s)$ with $\phi(s) \in X$ and $\psi(s)^* \in X^*$ such that $\psi^*(s)(\phi(s)) = 1$. In applications, we will consider linear forms associated to the inner product $(M, N) \mapsto \text{Tr}(M^*N)$ for M and N in $\mathcal{B}(\mathcal{H}_S)$. We then have the following result.

Corollary B.4. *Let $(P^m(s))_{s \in [0,1]}$, $m = 1, \dots, z$ be C^1 families of rank one projectors, i.e. $P^m(s) = \phi_m(s)\psi_m^*(s)$, $s \in [0, 1]$, where the maps $s \mapsto \phi_m(s)$ and $s \mapsto \psi_m^*(s)$ are C^1 and non-vanishing for $m = 1, \dots, z$. Then there exist $C > 0$ and $T_0 \in \mathbb{N}$ such that for $T \geq T_0$ one has for any $k \leq T$*

$$\sup_{m=1, \dots, z} \left\| P^m\left(\frac{k}{T}\right)P^m\left(\frac{k-1}{T}\right) \cdots P^m\left(\frac{1}{T}\right)P^m(0) - e^{-\int_0^{k/T} \psi_m^*(t)(\phi_m'(t)) dt} \phi_m(1)\psi_m^*(0) \right\| \leq C/T$$

where C and T_0 depend on

$$c_P = \sup_{s \in [0,1]} \max_{m=1, \dots, z} (\|\phi_m(s)\|, \|\phi_m'(s)\|, \|\psi_m^*(s)\|, \|\psi_m^{*'}(s)\|) \quad (52)$$

only, and C is a continuous function of c_P .

Proof. We apply Proposition B.2 with $s_k = k/T$. Since $s_0 = 0$, and $\psi_j^*(0)$ is a linear form, it is enough to compute $\tilde{\phi}_m(s) := W(s)\phi_m(0)$. By the intertwining property, $\tilde{\phi}_m(s) \in \text{Ran } P^m(s)$, i.e. $\tilde{\phi}_m(s) = v_m(s)\phi_m(s)$, where $v_m(s) \in \mathbb{C}$. Because of the differential equation (46) and the identity (48), $\tilde{\phi}_m(s)$ satisfies

$$\begin{aligned}
0 &\equiv P^m(s)\tilde{\phi}_m'(s) = \phi_m(s)\psi_m^*(s)(v_m'(s)\phi_m(s) + v_m(s)\phi_m'(s)) \\
&= (v_m'(s) + v_m(s)\psi_m^*(s)(\phi_m'(s)))\phi_m(s),
\end{aligned}$$

so that $v_m(s) = e^{-\int_0^s \psi_m^*(t)(\phi_m'(t)) dt}$. This concludes the proof. \square

B.2 Main result

We now turn to our final adiabatic theorem. We recall that an admissible family $(F(s))_{s \in [0,1]}$ is one that satisfies **Hyp0–Hyp4**. We denote by $\lambda^m(s)$ and $P^m(s)$, $m = 1, \dots, z$ the peripheral eigenvalues and associated spectral projectors of $F(s)$; assumptions **Hyp2**, **Hyp3** and standard perturbation theory ensure that one can parameterize eigenvalues such that both $s \mapsto \lambda^m(s)$ and $s \mapsto P^m(s)$ are C^2 functions. We denote by C_P and $W(s)$ the constant (49) and the family of intertwining operators in Definition B.1.

Theorem B.5. *If the family $(F(s))_{s \in [0,1]}$ is admissible, then for any $\ell' \in (\ell, 1)$ there exist $C > 0$ and $T_0 \in \mathbb{N}$ such that for $T \geq T_0$ one has*

$$\sup_{k=1, \dots, T} \left\| F\left(\frac{k}{T}\right) \cdots F\left(\frac{1}{T}\right) - \sum_{m=1}^z \left(\prod_{n=1}^k \lambda^m\left(\frac{n}{T}\right) \right) W\left(\frac{k}{T}\right) P^m(0) - F^Q\left(\frac{k}{T}\right) \cdots F^Q\left(\frac{1}{T}\right) Q(0) \right\| \leq \frac{C}{T(1-\ell')}$$

for all $T \geq T_0$, where $F^Q(s)$ denotes $Q(s)F(s)$. Moreover,

$$\|F^Q\left(\frac{k}{T}\right) \cdots F^Q\left(\frac{1}{T}\right) Q(0)\| \leq C\ell'^k,$$

and C depends on C_P only, is a continuous function of C_P , and T_0 depends on C_P and ℓ' only.

Proof. The proof consists of revisiting the proof of Theorem 4.4 in [HJPR17], relaxing the hypotheses made there to **Hyp0–Hyp4**. We first focus on the combinatorial part of the proof stated as Proposition 4.5 in [HJPR17], borrowing freely the notation used there. We denote in particular $F^P(s) = P(s)F(s)$ and $F_k^\# = F\left(\frac{k}{T}\right)^\#$, where $\# \in \{P, Q\}$. Under our hypotheses on the spectral radii instead of the assumptions on the norm used in [HJPR17], we will see below that the starting estimates

$$\left\| \prod_{a \in A_n} F_a^Q \right\| \leq \ell^{|A_n|}, \quad \left\| \prod_{b \in B_n} F_b^P \right\| \leq 1. \quad (\text{A9 of [HJPR17]})$$

used in Proposition 4.5 in [HJPR17] are replaced by the following bounds. For some constants $D, D' > 1$, and $\ell' < 1$,

$$\left\| \prod_{a \in A_n} F_a^Q \right\| \leq D(\ell')^{|A_n|}, \quad \left\| \prod_{b \in B_n} F_b^P \right\| \leq D', \quad (\text{NewBounds})$$

where D, D' are independent of the number of terms in the products, and satisfy the required dependence. Following Proposition 4.5 in [HJPR17], we need to bound the norms of terms of the following forms:

$$\left(\prod_{a \in A_d} F_a^Q \right) \left(\prod_{b \in B_d} F_b^P \right) \cdots \left(\prod_{a \in A_1} F_a^Q \right) \left(\prod_{b \in B_1} F_b^P \right) P_0, \quad (53)$$

$$\left(\prod_{b \in B_{d+1}} F_b^P \right) \left(\prod_{a \in A_d} F_a^Q \right) \left(\prod_{b \in B_d} F_b^P \right) \cdots \left(\prod_{a \in A_1} F_a^Q \right) \left(\prod_{b \in B_1} F_b^P \right) P_0, \quad (54)$$

$$\left(\prod_{a \in A_{d+1}} F_a^Q \right) \left(\prod_{b \in B_{d+1}} F_b^P \right) \cdots \left(\prod_{a \in A_2} F_a^Q \right) \left(\prod_{b \in B_2} F_b^P \right) \left(\prod_{a \in A_1} F_a^Q \right) P_0, \quad (55)$$

$$\left(\prod_{b \in B_{d+1}} F_b^P \right) \left(\prod_{a \in A_d} F_a^Q \right) \left(\prod_{b \in B_{d-1}} F_b^P \right) \cdots \left(\prod_{a \in A_2} F_a^Q \right) \left(\prod_{b \in B_2} F_b^P \right) \left(\prod_{a \in A_1} F_a^Q \right) P_0. \quad (56)$$

In [HJPR17], we obtain the bounds

$$\begin{aligned} \|(53)\| &\leq (c/T)^{2d-1} \ell^{\sum_n |A_n|}, & \|(54)\| &\leq (c/T)^{2d} \ell^{\sum_n |A_n|}, \\ \|(55)\| &\leq (c/T)^{2d+1} \ell^{\sum_n |A_n|}, & \|(56)\| &\leq (c/T)^{2d} \ell^{\sum_n |A_n|}, \end{aligned} \quad (\text{Bounds from HJPR15})$$

via (A9 of [HJPR17]). Instead, if we use (NewBounds), we obtain the bounds

$$\begin{aligned} \|(53)\| &\leq (c/T)^{2d-1} D^d \ell^{\sum_n |A_n|} D'^d, & \|(54)\| &\leq (c/T)^{2d} D^{d+1} \ell^{\sum_n |A_n|} D'^d, \\ \|(55)\| &\leq (c/T)^{2d+1} D^{d+1} \ell^{\sum_n |A_n|} D'^{d+1}, & \|(56)\| &\leq (c/T)^{2d} D^{d+1} \ell^{\sum_n |A_n|} D'^d. \end{aligned}$$

For the bound on form (53), for example, since we assumed $D, D' > 1$, then we may simply change $c \rightarrow cDD'$ and obtain essentially the same bound as in [HJPR17]. Thus, the rest of the combinatorial argument consisting in counting the number of such terms for each d , multiplying by this bound, and summing over d , yields the same final bounds as in Proposition 4.5 in [HJPR17], up to modified constants.

We now turn to find D, D', ℓ' such that we have (NewBounds).

Lemma B.6. *Under the assumptions Hyp0–Hyp4, there exist $T_0 \in \mathbb{N}$ and $D' > 0$, both depending on C_P only, and D' being a continuous function of C_P , such that for all $T \geq T_0$ and $n_0 < n \leq T$, we have*

$$\|F_n^P \cdots F_{n_0}^P\| < D'.$$

Proof. For each $k \in \{n_0, \dots, n\}$, write $F_k^P = \sum_{m=1}^z \lambda_k^m P_k^m$ using semisimplicity of peripheral eigenvalues. Recall that for each m , $\lambda^m(s)$ and $P^m(s)$ are C^2 in $[0, 1]$. Then,

$$\begin{aligned} \prod_{k=n_0}^n F_k^P &= \prod_{k=n_0}^n \sum_{m=1}^z \lambda_k^m P_k^m \\ &= \sum_{m=1}^z \left(\prod_{k=n_0}^n \lambda_k^m \right) P_n^m \cdots P_{n_0}^m + \sum_{\substack{i_{n_0}, \dots, i_n=1, \dots, z \\ \text{not all equal}}} \left(\prod_{k=n_0}^n \lambda_k^{i_k} \right) P_n^{i_n} \cdots P_{n_0}^{i_{n_0}}. \end{aligned}$$

For each m and all $k' \leq k$, Proposition B.2 gives for $T \geq T_0$

$$\|P_k^m \cdots P_{k'}^m\| \leq C/T + \sup_{1 \geq s > s' \geq 0} \|P^m(s)W(s)W(s')^{-1}\| \leq C(1 + 1/T), \quad (57)$$

for some constant C which depends continuously on C_P . Again we can bound this $C(1 + 1/T)$ by a new constant C with the same properties as the original C . Taking $k = n$ and $k' = n_0$ in (57) bounds the first sum by zC , since the eigenvalues are on the unit circle. For the second sum, we know that $P_k^i P_{k-1}^m \leq C_P/T$ if $i \neq m$, as a consequence of the relation $P^i(s)P^m(s) = 0$. Bounding the terms in the second sum is again done by a simple combinatorial argument: let d be the number of transitions $P_k^m P_{k-1}^i$ where $m \neq i$. In each term in the second sum, there is at least one such transition by design. If we have $n - n_0$ stars representing projectors, and d bars representing transitions, then there are $n - n_0 - 1$ gaps between the stars, from which we need to choose d to put a bar. So $\binom{n-n_0-1}{d}$ is the number of ways to divide the projectors into groupings. For each grouping, we have at most z choices of which projector it should be. So in total, there are at most $\binom{n-n_0-1}{d} z^{d+1}$ terms with d transitions. Each such term has norm bounded by $C^{d+1} C_P^d / T^d$ via (57), and using that each transition yields a factor C_P/T , and that all the eigenvalues have modulus one.

Lastly, there cannot be more than $n - n_0 - 1$ transitions (in fact, fewer than $(n - n_0)/z$). So, in total, we may bound the second sum by

$$\begin{aligned} \sum_{d=1}^{n-n_0-1} zC \binom{n-n_0-1}{d} \frac{(zCC_P)^d}{T^d} &\leq zC \left(1 + \frac{zCC_P}{T}\right)^{n-n_0-1} \\ &\leq zC \left(1 + \frac{zCC_P}{T}\right)^T \leq zC \exp(zCC_P). \end{aligned}$$

In total then, we have

$$\|F_n^P \cdots F_{n_0}^P\| \leq zC(1 + \exp(zCC_P)) =: D'. \quad \square$$

Let us turn now to compositions of \mathcal{L}^Q 's.

Lemma B.7. *Under the assumptions **Hyp0**–**Hyp4**, for all $\ell' \in (\ell, 1)$, there exists $D > 0$, $T_0 \in \mathbb{N}$ depending on C_P only, and D a continuous function of C_P , such that for any $T \geq T_0$ and $n_0 < n \leq T$, we have*

$$\|F_n^Q \cdots F_{n_0}^Q\| \leq D\ell'^{n-n_0}.$$

Proof. Let $\ell' \in (\ell, 1)$. As shown in Lemma 4.3 of [HJPR17], the spectral hypothesis **Hyp4** and the regularity assumptions **Hyp0**, **Hyp3** imply that given $\epsilon > 0$, we may choose $m \in \mathbb{N}$ uniformly in s so that

$$\|F^Q(s)^m\| < (\ell + \epsilon)^m. \quad (58)$$

Choose ϵ so that $\ell + \epsilon < \ell'$.

We now deduce by induction that for all $\mathbb{N} \ni m < T$, $\|F_{k+m}^Q \cdots F_k^Q - (F_k^Q)^{m+1}\| \leq \delta_m(T)$ for some map $\delta_m : \mathbb{R}_*^+ \rightarrow \mathbb{R}_*^+$, independent of k , such that $\lim_{T \rightarrow \infty} \delta_m(T) = 0$. This is trivially true for $m = 0$ and for $m \geq 1$, we have

$$\begin{aligned} F_{k+m}^Q \cdots F_k^Q - (F_k^Q)^{m+1} &= (F_{k+m}^Q - F_k^Q)(F_{k+m-1}^Q \cdots F_k^Q) \\ &\quad + F_k^Q((F_{k+m-1}^Q \cdots F_k^Q) - (F_k^Q)^{m-1}). \end{aligned}$$

With $F_j^Q = F^Q(s = \frac{j}{T})$, the first term is bounded above by

$$\begin{aligned} &\|F^Q(\frac{k+m}{T}) - F^Q(\frac{m}{T})\| \left(\sup_{s \in [0,1]} \|F^Q(s)\| \right)^m \\ &\leq \sup_{\substack{s \in [0,1] \\ s + \frac{m}{T} \in [0,1]}} \|F^Q(s + \frac{m}{T}) - F^Q(\frac{m}{T})\| \left(\sup_{s \in [0,1]} \|F^Q(s)\| \right)^m. \end{aligned} \quad (59)$$

This expression goes to zero as $T \rightarrow \infty$ by uniform continuity of F^Q on $[0, 1]$, and depends parametrically on m only. The second term is bounded above by $\sup_{s \in [0,1]} \|F^Q(s)\| \delta_{m-1}(T)$, by induction hypothesis, hence the claim is proved with $\delta_m(T) = (59) + \sup_{s \in [0,1]} \|F^Q(s)\| \delta_{m-1}(T)$.

Thus, given ϵ and m chosen so that (58) above holds, for $\ell = (\ell + \epsilon)^m + \delta_m(T)$, we have $\|F_{k+m}^Q \cdots F_k^Q\| < \tilde{\ell}$, where $\tilde{\ell} < 1$ if $T > T(m, \epsilon)$, for some $T(m, \epsilon)$ large enough.

Any integer $p \geq m$ can be partitioned as

$$p = \underbrace{m + m + \cdots + m}_{[p/m] \text{ times}} + \underbrace{(p - m[p/m])}_{< m},$$

so that for $c = \sup_{h=1,2,\dots,m-1} \left(\sup_{s \in [0,1]} \|F^Q(s)\| \right)^h$, the previous bound yields,

$$\|F_{k+p}^Q \cdots F_k^Q\| \leq c \tilde{\ell}^{\lfloor p/m \rfloor} \leq c \tilde{\ell}^{(p/m)-1} = \frac{c}{\tilde{\ell}} \left(\tilde{\ell}^{1/m} \right)^p.$$

Note that here, in fact, we do not need $p \geq m$. If $p < m$, then $\|F_{k+p}^Q \cdots F_k^Q\| \leq c$, $\frac{1}{\tilde{\ell}} \tilde{\ell}^{p/m} > 1$, and the bound still holds. Finally, set $\ell'' = (\tilde{\ell})^{1/m}$. Since $(\ell + \epsilon)^m + \delta_m(T) \leq ((\ell + \epsilon) + \delta_m(T)^{1/m})^m$, we have $\ell'' < \ell + \epsilon + \delta_m(T)^{1/m}$. For $T > T'(m, \epsilon)$ for some $T'(m, \epsilon)$ large enough, we have $\delta_m(T)^{1/m} < \ell' - (\ell + \epsilon)$, and therefore $\ell'' < \ell'$. Thus, for $D = c/\tilde{\ell}$, we have

$$\|F_n^Q \cdots F_{n_0}^Q\| \leq D \ell''^{m-n_0} < D \ell'^{m-n_0}. \quad \square$$

Our next ingredient is to approximate the composition $F_k^P \cdots F_0^P P(0)$, i.e. to show the equivalent of Proposition 4.6 in [HJPR17].

Lemma B.8. *Under the assumptions and notation of Proposition B.5, there exist T_0 and $C > 0$ with the same properties as in that Proposition, such that*

$$\sup_{k=1,\dots,T} \left\| F^P\left(\frac{k}{T}\right) \cdots F^P\left(\frac{1}{T}\right) P(0) - \sum_{m=1}^z \left(\prod_{n=1}^k \lambda_n^m \right) W\left(\frac{k}{T}\right) P^m(0) \right\| \leq C/T. \quad (60)$$

Proof. We define for all $k \leq T$

$$\mathcal{K}_k = W\left(\frac{k}{T}\right) P(0), \quad \mathcal{K}_k^\dagger = P(0) W^{-1}\left(\frac{k}{T}\right), \quad (61)$$

$$\Phi_k = \sum_j \left(\prod_{n=1}^k \lambda_n^m \right) P^m(0), \quad \Phi_k^\dagger = \sum_j \left(\prod_{n=1}^k \bar{\lambda}_n^m \right) P^m(0), \quad (62)$$

$$A_k = \mathcal{K}_k \Phi_k, \quad A_k^\dagger = \Phi_k^\dagger \mathcal{K}_k^\dagger. \quad (63)$$

The above expression for A_k gives in particular that

$$A_k = \sum_j \left(\prod_{n=1}^k \lambda_n^m \right) W\left(\frac{k}{T}\right) P^m(0). \quad (64)$$

We have the following identities which are consequences of the properties of the intertwining operators W :

$$\mathcal{K}_k \mathcal{K}_k^\dagger = P_k, \quad \mathcal{K}_k^\dagger \mathcal{K}_k = P(0), \quad \mathcal{K}_k P^m(0) = P_k^m \mathcal{K}_k, \quad \mathcal{K}_k^\dagger P_k^m = P^m(0) \mathcal{K}_k^\dagger, \quad (65)$$

$$\Phi_k P^m(0) = P^m(0) \Phi_k, \quad \Phi_k^\dagger P^m(0) = P^m(0) \Phi_k^\dagger, \quad \Phi_k \Phi_k^\dagger = P(0) = \Phi_k^\dagger \Phi_k, \quad (66)$$

and

$$\mathcal{K}_k = \mathcal{W}_k \mathcal{K}_{k-1}, \quad \mathcal{K}_k^\dagger = \mathcal{K}_{k-1}^\dagger \mathcal{W}_k^\dagger, \quad (67)$$

where

$$\mathcal{W}_k = W\left(\frac{k}{T}\right) W^{-1}\left(\frac{k-1}{T}\right) P\left(\frac{k-1}{T}\right), \quad \mathcal{W}_k^\dagger = W\left(\frac{k-1}{T}\right) W^{-1}\left(\frac{k}{T}\right) P\left(\frac{k}{T}\right). \quad (68)$$

Hence A_k is uniformly bounded in T and $k \leq T$, since \mathcal{K}_k and Φ_k are, and satisfies relevant intertwining properties. The notation is chosen to be close to that used in the proof of Proposition 4.6 in [HJPR17], and one gets that all steps of that of Proposition 4.6 in [HJPR17] go through, which ends the proof. \square

This concludes the proof of Theorem B.5. \square

Combining Theorem B.5 with the proof of Corollary B.4 immediately implies the following:

Corollary B.9. *If the family $(F(s))_{s \in [0,1]}$ is admissible, and its peripheral eigenvalues are simple, with associated projectors $P^m(s) = \phi_m(s)\psi_m^*(s)$ then for any $\ell' \in (\ell, 1)$ there exist $C > 0$ and $T_0 \in \mathbb{N}$, such that for $T \geq T_0$ and $k \leq T$,*

$$\|F(\frac{k}{T}) \cdots F(\frac{1}{T}) - \sum_{m=1}^z \left(\prod_{n=1}^k \lambda_n^m \right) e^{-\int_0^{k/T} \psi_m^*(t) (\phi_m^t(t)) dt} \phi_m(1) \psi_m^*(0) - F^Q(\frac{k}{T}) \cdots F^Q(\frac{1}{T}) Q(0)\| \leq \frac{C}{T(1-\ell')}.$$

Moreover,

$$\|F^Q(\frac{k}{T}) \cdots F^Q(\frac{1}{T}) Q(0)\| \leq C \ell'^k$$

and C depends on c_P defined in (52) only, is a continuous function of c_P , and T_0 depends on c_P and ℓ' only.

C Proofs for Section 3

Proof of Lemma 3.2. By definition

$$\varsigma_T(a^i, a^f, \vec{i}, \vec{j}) = \log \frac{\text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}})(\rho^i \otimes \Xi)(\pi_{a^i}^i \otimes \Pi_{\vec{i}}) U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}))}{\text{Tr}(U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}})(\rho_{a^f}^f \otimes \Xi)(\pi_{a^f}^f \otimes \Pi_{\vec{j}}) U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}))}$$

using assumptions i., ii. and iii.,

$$\begin{aligned} & \frac{\text{Tr}(\rho^i \pi_{a^i}^i)}{\dim \pi_{a^i}^i} \prod_{k=1}^T \frac{\text{Tr}(\xi_k \Pi_{i_k}^{(k)})}{\dim \Pi_{i_k}^{(k)}} \text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}) U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}})) \\ &= \log \frac{\text{Tr}(\rho^i \pi_{a^i}^i) \prod_{k=1}^T \frac{\text{Tr}(\xi_k \Pi_{i_k}^{(k)})}{\dim \Pi_{i_k}^{(k)}} \text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}) U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}))}{\text{Tr}(\rho^f \pi_{a^f}^f) \prod_{k=1}^T \frac{\text{Tr}(\xi_k \Pi_{j_k}^{(k)})}{\dim \Pi_{j_k}^{(k)}} \text{Tr}(U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}) U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}))} \\ &= \log \frac{\text{Tr}(\rho^i \pi_{a^i}^i) \dim \pi_{a^f}^f}{\text{Tr}(\rho^f \pi_{a^f}^f) \dim \pi_{a^i}^i} + \log \frac{\text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}) U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}))}{\text{Tr}(U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}) U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}))} \\ & \quad + \log \prod_{k=1}^T \frac{\text{Tr}(e^{-\beta_k h \varepsilon_k \Pi_{i_k}^{(k)}})}{Z_k \dim \Pi_{i_k}^{(k)}} - \log \prod_{k=1}^T \frac{\text{Tr}(e^{-\beta_k h \varepsilon_k \Pi_{j_k}^{(k)}})}{Z_k \dim \Pi_{j_k}^{(k)}} \end{aligned}$$

and because $\frac{\text{Tr}(e^{-\beta_k h \varepsilon_k \Pi_{i_k}^{(k)}})}{\dim \Pi_{i_k}^{(k)}} = \exp\left(-\beta_k \frac{\text{Tr}(e^{-\beta_k h \varepsilon_k \Pi_{i_k}^{(k)}})}{\dim \Pi_{i_k}^{(k)}}\right)$ by assumption iii. again,

$$\begin{aligned} &= \log \frac{\text{Tr}(\rho^i \pi_{a^i}^i) \dim \pi_{a^f}^f}{\text{Tr}(\rho^f \pi_{a^f}^f) \dim \pi_{a^i}^i} + \log \frac{\text{Tr}(U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}) U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}))}{\text{Tr}(U_1^* \cdots U_T^*(\pi_{a^f}^f \otimes \Pi_{\vec{j}}) U_T \cdots U_1(\pi_{a^i}^i \otimes \Pi_{\vec{i}}))} \\ & \quad + \sum_{k=1}^T \frac{\text{Tr}(-\beta_k h \varepsilon_k \Pi_{i_k}^{(k)})}{\dim \Pi_{i_k}^{(k)}} - \sum_{k=1}^T \frac{\text{Tr}(-\beta_k h \varepsilon_k \Pi_{j_k}^{(k)})}{\dim \Pi_{j_k}^{(k)}} \end{aligned}$$

and the last term vanishes by cyclicity of the trace. \square

Proof of Proposition 3.4. We start with the relation (15). On one hand,

$$\begin{aligned}
\mathbb{E}_T \left(\log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \right) &= \sum_{a^i} \log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \sum_{a^f, \vec{i}, \vec{j}} \mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j}) \\
&= \sum_{a^i} \log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \\
&\quad \sum_{a^f, \vec{i}, \vec{j}} \text{Tr} (U_T \cdots U_1 (\pi_{a^i}^i \rho^i \pi_{a^i}^i \otimes \Pi_{\vec{i}} \Xi \Pi_{\vec{i}}) U_1^* \cdots U_T^* (\pi_{a^f}^f \otimes \Pi_{\vec{j}})) \\
&= \sum_{a^i} \log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \\
&\quad \sum_{\vec{i}} \text{Tr} (U_T \cdots U_1 (\pi_{a^i}^i \rho^i \pi_{a^i}^i \otimes \Pi_{\vec{i}} \Xi \Pi_{\vec{i}}) U_1^* \cdots U_T^* (\text{Id} \otimes \text{Id})).
\end{aligned}$$

Using that each ξ_k is a function of Y_k , we have $\sum_{\vec{i}} \Pi_{\vec{i}} \Xi \Pi_{\vec{i}} = \Xi$, and

$$\mathbb{E}_T \left(\log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \right) = \sum_{a^i} \log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \text{Tr} (U_T \cdots U_1 (\pi_{a^i}^i \rho^i \pi_{a^i}^i \otimes \Xi) U_1^* \cdots U_T^*).$$

As ρ^i is a function of A^i , and using the cyclicity of the trace, we have

$$\begin{aligned}
\mathbb{E}_T \left(\log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \right) &= \sum_{a^i} \log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i} \text{Tr} (\pi_{a^i}^i \otimes \Xi) \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i}, \\
&= \sum_{a^i} \text{Tr}(\pi_{a^i}^i \rho^i) \log \frac{\text{Tr}(\pi_{a^i}^i \rho^i)}{\dim \pi_{a^i}^i}.
\end{aligned}$$

Using the commutation relation $[\pi_{a^i}^i, \rho^i] = 0$, the right-hand side is precisely $-S(\rho^i)$. The term involving ρ^f is treated similarly.

We turn to (16):

$$\begin{aligned}
&\mathbb{E}_T \left(\sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}) \right) \\
&= \sum_{\vec{i}, \vec{j}} \sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}) \sum_{a^i, a^f} \mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j})
\end{aligned}$$

using $[\rho^i, A^i] = 0$,

$$\begin{aligned}
&= \sum_{\vec{i}, \vec{j}} \sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}) \text{Tr} (U_T \cdots U_1 (\rho^i \otimes \Pi_{\vec{i}} \Xi \Pi_{\vec{i}}) U_1^* \cdots U_T^* (\text{Id} \otimes \Pi_{\vec{j}})) \\
&= \sum_{\vec{j}} \sum_{k=1}^T \beta_k E_{j_k}^{(k)} \text{Tr} (U_T \cdots U_1 (\rho^i \otimes \Xi) U_1^* \cdots U_T^* (\text{Id} \otimes \Pi_{\vec{j}})) \\
&\quad - \sum_{\vec{i}} \sum_{k=1}^T \beta_k E_{i_k}^{(k)} \text{Tr} (U_T \cdots U_1 (\rho^i \otimes \Pi_{\vec{i}} \Xi \Pi_{\vec{i}}) U_1^* \cdots U_T^* (\text{Id} \otimes \text{Id}))
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^T \sum_{j_k} \beta_k \operatorname{Tr} (U_T \cdots U_1 (\rho^i \otimes \Xi) U_1^* \cdots U_T^* (\operatorname{Id} \otimes \Pi_{j_k}^{(k)} E_{j_k}^{(k)})) \\
&\quad - \sum_{k=1}^T \sum_{i_k} \beta_k \operatorname{Tr} (\rho^i \otimes \Xi \Pi_{i_k}^{(k)} E_{i_k}^{(k)}).
\end{aligned}$$

Using that each ξ_k is a function of Y_k , we have $\sum_{i_k} \Pi_{i_k}^{(k)} E_{i_k}^{(k)} = h_{\varepsilon_k}$ and thus

$$\mathbb{E}_T \left(\sum_{k=1}^T \beta_k (E_{j_k}^{(k)} - E_{i_k}^{(k)}) \right) = \sum_{k=1}^T \beta_k \operatorname{Tr} (\xi_k^f h_{\varepsilon_k}) - \sum_{k=1}^T \beta_k \operatorname{Tr} (\xi_k^i h_{\varepsilon_k}). \quad \square$$

Proof of Proposition 3.6.

Assume that $[Y(s), \xi(s)] = 0$ for all s . Then by definition and from expression (9)

$$\begin{aligned}
\mathbb{E}(e^{\alpha \Delta y_T^{\text{tot}}}) &= \sum_{\vec{i}, \vec{j}} \sum_{a^i, a^f} \exp \left(\alpha \sum_{k=1}^Y (y_{j_k}^{(k)} - y_{i_k}^{(k)}) \right) \mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j}) \\
&= \sum_{\vec{i}, \vec{j}} \operatorname{Tr} \left(U_T \cdots U_1 \left(\sum_{a^i} \pi_{a^i}^i \rho^i \pi_{a^i}^i \otimes \prod_{k=1}^T e^{-\alpha y_{i_k}^{(k)}} \Pi_{i_k} \Xi \right) \right. \\
&\quad \left. U_1^* \cdots U_T^* \left(\sum_b \pi_{a^f}^f \otimes \prod_{k=1}^T e^{\alpha y_{j_k}^{(k)}} \Pi_{j_k} \right) \right) \\
&= \operatorname{Tr} \left(U_T \cdots U_1 \left(\sum_{a^i} \pi_{a^i}^i \rho^i \pi_{a^i}^i \otimes \prod_{k=1}^T e^{-\alpha Y_k} \Xi \right) \right. \\
&\quad \left. U_1^* \cdots U_T^* \left(\operatorname{Id} \otimes \prod_{k=1}^T e^{\alpha Y_k} \right) \right) \\
&= \operatorname{Tr} \left(\mathcal{L}_Y^{(\alpha)} \left(\frac{T}{T} \right) \circ \dots \circ \mathcal{L}_Y^{(\alpha)} \left(\frac{1}{T} \right) \left(\sum_{a^i} \pi_{a^i}^i \rho^i \pi_{a^i}^i \right) \right).
\end{aligned}$$

Assume in addition that $[A^i, \rho^i] = 0$. Then similarly, for α_1, α_2 in \mathbb{R} ,

$$\begin{aligned}
\mathbb{E}(e^{\alpha_1 \Delta y_T^{\text{tot}} + \alpha_2 \Delta a_T}) &= \sum_{\vec{i}, \vec{j}} \sum_{a^i, a^f} \exp \left(\alpha_1 \sum_{k=1}^Y (y_{j_k}^{(k)} - y_{i_k}^{(k)}) \right) e^{\alpha_2 (a^i - a^f)} \mathbb{P}_T^F(a^i, a^f, \vec{i}, \vec{j}) \\
&= \sum_{\vec{i}, \vec{j}} \operatorname{Tr} \left(U_T \cdots U_1 \left(\sum_{a^i} e^{+\alpha_2 a^i} \pi_{a^i}^i \rho^i \otimes \prod_{k=1}^T e^{-\alpha_1 y_{i_k}^{(k)}} \Pi_{i_k} \Xi \right) \right. \\
&\quad \left. U_1^* \cdots U_T^* \left(\sum_b e^{-\alpha_2 a^f} \pi_{a^f}^f \otimes \prod_{k=1}^T e^{\alpha_1 y_{j_k}^{(k)}} \Pi_{j_k} \right) \right) \\
&= \operatorname{Tr} \left(U_T \cdots U_1 (e^{+\alpha_2 A^i} \rho^i \otimes \prod_{k=1}^T e^{-\alpha_1 Y_k} \Xi) U_1^* \cdots U_T^* (e^{-\alpha_2 A^f} \otimes \prod_{k=1}^T e^{\alpha_1 Y_k}) \right) \\
&= \operatorname{Tr} (e^{-\alpha_2 A^f} \mathcal{L}_Y^{(\alpha_1)} \left(\frac{T}{T} \right) \circ \dots \circ \mathcal{L}_Y^{(\alpha_1)} \left(\frac{1}{T} \right) (e^{+\alpha_2 A^i} \rho^i)).
\end{aligned}$$

□

Proof of Remark 3.10. The dependence in s will be treated last, since it fixes the overall regularity, as we will see. We consider the applications between the different Banach spaces involved in the definition of $\mathcal{L}_Y^{(\alpha)} = \text{Tr}_{\mathcal{E}}(e^{\alpha Y} U^{(\tau)}(\cdot \otimes \xi) e^{-\alpha Y} U^{(-\tau)})$. We will denote the trace norm by $\|\cdot\|_1$ when the underlying Hilbert space is determined by the context, and we use the shorthand $\mathcal{H}_{\mathcal{T}} := \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$ for the total Hilbert space.

1. The map P such that $\xi \mapsto P_{\xi} = \cdot \otimes \xi$ mapping $\mathcal{I}_1(\mathcal{H}_{\mathcal{E}})$ to $\mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{I}_1(\mathcal{H}_{\mathcal{T}}))$ is a linear isometry, hence a C^{∞} map. Indeed, linearity is immediate. For all $\eta \in \mathcal{I}_1(\mathcal{H}_{\mathcal{S}})$, $\|P_{\xi}(\eta)\|_{\mathcal{I}_1(\mathcal{H}_{\mathcal{T}})} = \|\eta \otimes \xi\|_1 = \|\eta\|_1 \|\xi\|_1$, which shows $\|P_{\xi}\|_{\mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{I}_1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}))} = \|\xi\|_1$.
2. For any Hilbert spaces \mathcal{H} , the maps $(\alpha, A) \mapsto \alpha A : \mathbb{C} \times \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ and $(A, B) \mapsto AB : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ are bilinear, thus C^{∞} , and the map $A \mapsto e^A : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is C^{∞} as well.
3. The map from $\mathcal{B}(\mathcal{H}_{\mathcal{T}}) \times \mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{E}}), \mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{B}(\mathcal{H}_{\mathcal{T}}))) \times \mathcal{B}(\mathcal{H}_{\mathcal{T}}) \rightarrow \mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{E}}), \mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{B}(\mathcal{H}_{\mathcal{T}})))$ such that $(A, P, B) \mapsto APB$ is well defined and trilinear, which makes it C^{∞} . Indeed, for any $(\eta, \xi) \in \mathcal{I}_1(\mathcal{H}_{\mathcal{S}}) \times \mathcal{I}_1(\mathcal{H}_{\mathcal{E}})$, we have $APB(\xi) = AP_{\xi}B : \eta \mapsto AP_{\xi}(\eta)B$. The trace norm of the latter in $\mathcal{H}_{\mathcal{T}}$ is

$$\begin{aligned} \|AP_{\xi}(\eta)B\|_1 &\leq \|A\|_{\mathcal{B}(\mathcal{H}_{\mathcal{T}})} \|P_{\xi}(\eta)\|_{\mathcal{I}_1(\mathcal{H}_{\mathcal{T}})} \|B\|_{\mathcal{B}(\mathcal{H}_{\mathcal{T}})} \\ &= \|A\|_{\mathcal{B}(\mathcal{H}_{\mathcal{T}})} \|P_{\xi}(\cdot)\|_{\mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{I}_1(\mathcal{H}_{\mathcal{T}}))} \|B\|_{\mathcal{B}(\mathcal{H}_{\mathcal{T}})} \|\eta\|_1. \end{aligned}$$

This yields

$$\begin{aligned} \|APB\|_{\mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{E}}), \mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{I}_1(\mathcal{H}_{\mathcal{T}})))} \\ \leq \|A\|_{\mathcal{B}(\mathcal{H}_{\mathcal{T}})} \|P(\cdot)\|_{\mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{E}}), \mathcal{B}(\mathcal{I}_1(\mathcal{H}_{\mathcal{S}}), \mathcal{I}_1(\mathcal{H}_{\mathcal{T}})))} \|B\|_{\mathcal{B}(\mathcal{H}_{\mathcal{T}})} \end{aligned}$$

and boundedness of the trilinear map.

4. We saw that the map $\text{Tr}_{\mathcal{E}} : \mathcal{I}_1(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}) \rightarrow \mathcal{I}_1(\mathcal{H}_{\mathcal{S}})$ is a linear contraction, hence it is C^{∞} .

Consequently, we get that $(\alpha, Y, U^{(\tau)}, \xi) \mapsto \text{Tr}_{\mathcal{E}}(e^{\alpha Y} U^{(\tau)}(\cdot \otimes \xi) e^{-\alpha Y} U^{(-\tau)})$ is a C^{∞} map from $\mathbb{C} \times \mathcal{B}(\mathcal{H}_{\mathcal{T}}) \times \mathcal{B}(\mathcal{H}_{\mathcal{T}}) \times \mathcal{I}_1(\mathcal{H}_{\mathcal{E}})$ to $\mathcal{B}(\mathcal{I}(\mathcal{H}_{\mathcal{S}}))$. The hypotheses made on the s -dependence of Y , $U^{(\tau)}$ and ξ yield the result. \square

References

- [ABL64] Yakir Aharonov, Peter G. Bergmann, and Joel L. Lebowitz, *Time symmetry in the quantum process of measurement*, Phys. Rev. (2) **134** (1964), B1410–B1416.
- [BCJP18] Tristan Benoist, Noé Cuneo, Vojkan Jakšić, and Claude-Alain Pillet, *On entropy production of repeated quantum measurements II. Examples*, in preparation (2018).
- [BD15] Peter J. Bickel and Kjell A. Doksum, *Mathematical statistics—basic ideas and selected topics. Vol. 1*, second ed., Texts in Statistical Science Series, CRC Press, Boca Raton, FL, 2015.
- [BFJP17] Tristan Benoist, Martin Fraas, Vojkan Jakšić, and Claude-Alain Pillet, *Full statistics of erasure processes: Isothermal adiabatic theory and a statistical Landauer principle*, Rev. Roumaine Math. Pures Appl (2017), no. 62, 259–286.

- [Bha97] Rajendra Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997.
- [Bil95] Patrick Billingsley, *Probability and measure*, third ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1995, A Wiley-Interscience Publication.
- [BJM08] Laurent Bruneau, Alain Joye, and Marco Merkli, *Random repeated interaction quantum systems*, Comm. Math. Phys. **284** (2008), no. 2, 553–581.
- [BJM14] ———, *Repeated interactions in open quantum systems*, J. Math. Phys. **55** (2014), no. 7, 075204.
- [BJPP17] Tristan Benoist, Vojkan Jakšić, Yan Pautrat, and Claude-Alain Pillet, *On entropy production of repeated quantum measurements I. General theory*, Communications in Mathematical Physics (2017).
- [Bry93] Włodzimierz Bryc, *A remark on the connection between the large deviation principle and the central limit theorem*, Statist. Probab. Lett. **18** (1993), no. 4, 253–256.
- [CJPS17] Noé Cuneo, Vojkan Jakšić, Claude-Alain Pillet, and Armen Shirikyan, *Fluctuation Theorem and Thermodynamic Formalism*, arXiv preprint (2017), arXiv:1712.05167.
- [CT06] Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, second ed., Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2006.
- [Cub09] Toby Cubitt, *Bi/tripartite partial trace*, http://www.dr-qubit.org/Matlab_code.html, 2009, Mathematica package.
- [DZ10] Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, Stochastic Modelling and Applied Probability, vol. 38, Springer-Verlag, Berlin, 2010, Corrected reprint of the second (1998) edition.
- [EHK78] David E. Evans and Raphael Høegh-Krohn, *Spectral properties of positive maps on C^* -algebras*, J. London Math. Soc. (2) **17** (1978), no. 2, 345–355.
- [EHM09] Massimiliano Esposito, Upendra Harbola, and Shaul Mukamel, *Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems*, Rev. Mod. Phys. **81** (2009), 1665–1702.
- [Ell85] Richard S. Ellis, *Entropy, large deviations, and statistical mechanics*, Grundlehren der mathematischen Wissenschaften 271, Springer New York, 1985.
- [FP09] Franco Fagnola and Rely Pellicer, *Irreducible and periodic positive maps*, Commun. Stoch. Anal. **3** (2009), no. 3, 407–418.
- [GCG⁺17] Giacomo Guarneri, Steve Campbell, John Goold, Simon Pigeon, Bassano Vacchini, and Mauro Paternostro, *Full counting statistics approach to the quantum non-equilibrium landauer bound*, New Journal of Physics **19** (2017), no. 10, 103038.
- [HJPR17] Eric P. Hanson, Alain Joye, Yan Pautrat, and Renaud Raquépas, *Landauer’s principle in repeated interaction systems*, Comm. Math. Phys. **349** (2017), no. 1, 285–327.

- [HMO07] Fumio Hiai, Milan Mosonyi, and Tomohiro Ogawa, *Large deviations and Chernoff bound for certain correlated states on a spin chain*, Journal of Mathematical Physics **48** (2007), no. 12, 123301.
- [HP13] Jordan M. Horowitz and Juan M. R. Parrondo, *Entropy production along nonequilibrium quantum jump trajectories*, New Journal of Physics **15** (2013), no. 8, 085028.
- [JOPP12] Vojkan Jakšić, Yoshiko Ogata, Yan Pautrat, and Claude-Alain Pillet, *Entropic fluctuations in quantum statistical mechanics. an introduction*, Quantum Theory from Small to Large Scales, Lecture Notes of the Les Houches Summer School **95** (2012), no. 978-0-19-965249-5, pp.213–410.
- [JOPS12] Vojkan Jakšić, Yoshiko Ogata, Claude-Alain Pillet, and Robert Seiringer, *Quantum hypothesis testing and non-equilibrium statistical mechanics*, Rev. Math. Phys. **24** (2012), no. 6, 1230002, 67.
- [JP14] Vojkan Jakšić and Claude-Alain Pillet, *A note on the Landauer principle in quantum statistical mechanics*, J. Math. Phys. **55** (2014), no. 7, 075210–75210:21.
- [JPW14] Vojkan Jakšić, Claude-Alain Pillet, and Matthias Westrich, *Entropic fluctuations of quantum dynamical semigroups*, Journal of Statistical Physics **154** (2014), no. 1, 153–187.
- [Kat76] Tosio Kato, *Perturbation theory for linear operators*, Classics in mathematics, Springer, 1976.
- [Kur00] Jorge Kurchan, *A quantum fluctuation theorem*, arXiv preprint (2000), cond-mat/0007360.
- [Lan61] Rolf Landauer, *Irreversibility and heat generation in the computing process*, IBM Journal of Research and Development **5** (1961), 183–191.
- [MWM85] D. Meschede, H. Walther, and G. Müller, *One-atom maser*, Phys. Rev. Lett. **54** (1985), 551–554.
- [RW14] David Reeb and Michael M. Wolf, *An improved Landauer principle with finite-size corrections*, New Journal of Physics **16** (2014), no. 10, 103011.
- [Tas00] Hal Tasaki, *Jarzynski relations for quantum systems and some applications*, arXiv preprint (2000), cond-mat/0009244.