

# What is the total Betti number of a random real hypersurface?

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**Abstract.** We bound from above the expected total Betti number of a high degree random real hypersurface in a smooth real projective manifold. This upper bound is deduced from the equidistribution of critical points of a real Lefschetz pencil restricted to the complex domain of such a random hypersurface, equidistribution which we first establish. Our proofs involve Hörmander’s theory of peak sections as well as the formula of Poincaré–Martinelli.

## Introduction

The topology of real projective manifolds is under study since the nineteenth century, when Axel Harnack and Felix Klein discovered that the number of connected components of the real locus of a smooth real projective curve is bounded from above by the sum of its genus and the number of connected components of its complex domain, see [9, 14], while David Hilbert has devoted his sixteenth problem to such a study. Recall that by definition, a real projective manifold  $X$  is the vanishing locus in some complex projective space of a collection of homogeneous polynomials with real coefficients. It inherits an antiholomorphic involution  $c_X$  from the ambient complex conjugation. The real locus  $\mathbb{R}X$  is the set of real solutions of the polynomial equations, that is, the fixed point set of  $c_X$ . René Thom [24] later observed as a consequence of Smith’s theory in equivariant homology, that the total Betti number of the real locus of a smooth real projective manifold is actually always bounded from above by the total Betti number of its complex locus, extending Harnack–Klein’s inequality, see Theorem 2. On the other hand, John Nash proved that every closed smooth manifold can be realized as a component of the real locus of a smooth real projective manifold.

Real projective manifolds achieving the upper bound given by Harnack–Klein or Smith–Thom’s inequalities are called maximal. Real maximal curves in smooth real projective surfaces appear to be exponentially rare in their linear system as their degree grows, see [7]. What is then the expected topology of real hypersurfaces in a given smooth real projective manifold  $X$ ? We tackle this question here, measuring the topology of hypersurfaces by the total Betti numbers of their real loci. The answer to this question indeed turns out to be only known for the real projective line thanks to Mark Kac [13], Michael Shub and Stephen Smale [23] or Alan Edelman and Eric Kostlan [6]. It follows from the latter works that the expected number of real roots of a random real polynomial in one variable and degree  $d$  is  $\sqrt{d}$ , while for the former

it follows that it is asymptotically  $\frac{2}{\pi} \log d$ , for a different choice of the probability measure though, see Section 3.1. We establish here general upper bounds for the expected total Betti numbers of real hypersurfaces in real projective manifolds. More precisely, let  $X$  be a smooth real projective manifold of positive dimension  $n$  equipped with a real ample line bundle  $L$ . The growth of the total Betti number of complex loci of hypersurfaces linearly equivalent to  $L^d$  is polynomial in  $d$  of degree  $n$ , see Lemma 3. We prove the following, see Theorem 4 and 5.

**Theorem 1.** *Let  $(X, c_X)$  be a smooth real projective manifold of dimension  $n$  greater than one equipped with a Hermitian real line bundle  $(L, c_L)$  of positive curvature. Then, the expected total Betti number of real loci of hypersurfaces linearly equivalent to  $L^d$  is  $o(d^n)$ . If  $n = 2$  or if  $X$  is a product of smooth real projective curves, then it is even  $O(d^{\frac{n}{2}}(\log d)^n)$ .*

The probability measure that we consider on the complete linear system of real divisors associated to  $L^d$  is the Fubini–Study measure arising from the  $L^2$ -scalar product induced by the Hermitian metric of positive curvature fixed on  $L$ , see (1), (6) and Section 3.1. When  $X$  is one-dimensional, upper bounds as the ones given by Theorem 1 can already be deduced from our work [7].

In order to prove Theorem 1, we first fix a real Lefschetz pencil on  $X$ , which restricts to a Lefschetz pencil on every generic hypersurface of  $X$ . The number of critical points of such a restriction has the same asymptotic as the total Betti number of the hypersurface, see Section 1.2. We then prove that these critical points get uniformly distributed in  $X$  when the degree increases and more precisely that the expected normalized counting measure supported by these critical points converges to the volume form induced by the curvature of the Hermitian bundle  $L$ , see Theorem 6. The latter weak convergence proved in Theorem 6 is established outside of the critical locus of the original pencil when  $n > 2$  and away from the real locus of  $X$ . Note that we first prove this equidistribution result over the complex numbers, see Theorem 3. In order to deduce Theorem 1 from this equidistribution result, we observe that the total Betti number of real loci of hypersurfaces is bounded from above by the number of critical points of the restricted Lefschetz pencil in a neighborhood of the real locus, which we choose of size  $\log d / \sqrt{d}$  thanks to the theory of Hörmander’s peak sections, see Section 2.3. Indeed, we actually cannot estimate directly the number of critical points on the real locus. Nevertheless, we can estimate from above the number of critical points in the  $\log d / \sqrt{d}$  neighborhood as the difference between the total number of critical points computed in Proposition 2 and the number of critical points on its complement computed by Theorem 6. The bound  $d^{\frac{n}{2}}(\log d)^n$  of Theorem 1 indeed appears to be the volume of a  $\log d / \sqrt{d}$  neighborhood of the real locus for the metric induced by the curvature form of  $L^d$ .

Theorems 3 and 6 on equidistribution of critical points are independent of Theorem 1 which motivated this work. Note also that the expected Euler characteristic of the real locus of such random hypersurfaces has been computed in [3, 12], while many similar equidistribution results using similar methods have been obtained by Bernard Shiffman and Steve Zelditch (see [1, 4, 20]). Finally, while we were writing this paper in June 2011, Peter Sarnak informed us that he and Igor Wigman are able to prove that, for the same measure than ours, the expected number of connected components of real curves of degree  $d$  in  $\mathbb{R}P^2$  is even  $O(d)$ .

Our paper is organized as follows. In Section 1, we recall some results about total Betti numbers of real projective manifolds, critical points of Lefschetz pencils and their asymptotics. Section 2 is devoted to Theorem 3 about equidistribution of critical points of Lefschetz pencils restricted to complex random hypersurfaces. The theory of peak sections of Hörmander

and Poincaré–Martinelli’s formula play a crucial rôle in the proof, see Sections 2.2.2 and 2.3. Finally in Section 3, we first establish the real analogue of Theorem 3, see Theorem 6. Then, we deduce Theorem 1 from it, that is, upper bounds for the expected total Betti numbers of the real locus of random real hypersurfaces, see Theorems 4 and 5.

**Acknowledgement.** The research leading to these results has received funding from the European Community’s Seventh Framework Programme (FP7/2007-2013, FP7/2007-2011) under grant agreement no. 258204, as well as from the French Agence nationale de la recherche, ANR-08-BLAN-0291-02. We are grateful to the referee for fruitful comments on the paper.

## 1. Betti numbers and critical points of Lefschetz pencils

This first section is devoted to Lefschetz pencils, total Betti numbers and their asymptotics.

**1.1. Real Lefschetz pencils and Betti numbers.** Let  $X$  be a smooth complex projective manifold of positive dimension  $n$ .

**Definition 1.** A Lefschetz pencil on  $X$  is a rational map  $p : X \dashrightarrow \mathbb{C}P^1$  having only non-degenerated critical points and defined by two sections of a holomorphic line bundle with smooth and transverse vanishing loci.

We denote by  $B$  the base locus of a Lefschetz pencil  $p$  given by Definition 1, that is, the codimension two submanifold of  $X$  where  $p$  is not defined. A Lefschetz pencil without base locus is called a Lefschetz fibration. Blowing up once the base locus of a Lefschetz pencil turns it into a Lefschetz fibration. When the dimension  $n$  of  $X$  equals one, the base locus is always empty and a Lefschetz fibration is nothing but a branched cover with simple ramifications. Hence, the following Proposition 1 extends to Lefschetz fibrations the classical Riemann–Hurwitz formula.

**Proposition 1.** *Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz fibration  $p : X \rightarrow \mathbb{C}P^1$  and let  $F$  be a regular fiber of  $p$ . Then, the Euler characteristics of  $X$  and  $F$  satisfy the relation*

$$\chi(X) = 2\chi(F) + (-1)^n \# \text{Crit}(p),$$

where  $\text{Crit}(p)$  denotes the set of critical points of  $p$ .

*Proof.* Denote by  $\infty = p(F) \in \mathbb{C}P^1$  and by  $F_0$  the fiber of  $p$  associated to a regular value  $0 \in \mathbb{C}P^1 \setminus \{\infty\}$ . Let  $U_0$  (resp.  $U_\infty$ ) be a neighborhood of  $0$  (resp.  $\infty$ ) in  $\mathbb{C}P^1$ , without any critical value of  $p$ . Since  $U_\infty$  (resp.  $U_0$ ) retracts on  $F$  (resp.  $F_0$ ), we know that  $\chi(p^{-1}(U_0)) = \chi(p^{-1}(U_\infty)) = \chi(F)$ , whereas from additivity of the Euler characteristic,  $\chi(\bar{X}) = \chi(X) - 2\chi(F)$ , where  $\bar{X}$  denotes the complement  $X \setminus p^{-1}(U_0 \cup U_\infty)$ . Without loss of generality, we may assume that in an affine chart  $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$ ,  $0$  corresponds to the origin,  $U_0$  to a ball centered at the origin and  $U_\infty$  to a ball centered at  $\infty$ . The manifold  $\bar{X}$  comes then equipped with a function  $f : x \in \bar{X} \mapsto |p(x)|^2 \in \mathbb{R}_+^* \subset \mathbb{C}$ , taking values in a compact

interval  $[a, b]$  of  $\mathbb{R}_+^*$ . This function  $f$  is Morse and has the same critical points as  $p$ , all being of index  $n$ . Indeed, the differential of  $f$  reads  $df = p\bar{\partial}p + \bar{p}\partial p$  and vanishes at  $x \in \bar{X}$  if and only if  $\partial p|_x$  vanishes. Moreover, its second differential is the composition of the differential of the norm  $|\cdot|^2$  with the second differential of  $p$ . The multiplication by  $i$  exchanges stable and unstable spaces of these critical points which are non-degenerated. Hence,  $\bar{X}$  is equipped with a Morse function  $f : X \rightarrow [a, b]$  having  $\#\text{Crit}(p)$  critical points, all of index  $n$ . By the Morse lemma (see [17]), the topology of  $f^{-1}([a, a + \epsilon])$  changes, as  $\epsilon$  grows, only at the critical points, where a handle  $D^n \times D^n$  of index  $n$  is glued on a submanifold diffeomorphic to  $D^n \times S^{n-1}$ . From this Morse theory we deduce

$$\chi(\bar{X}) = \#\text{Crit}(p)(1 - \chi(S^{n-1})) = (-1)^n \#\text{Crit}(p)$$

and the result. □

Recall that a complex projective manifold  $X \subset \mathbb{C}P^N$  is said to be real when it is defined over the reals, as the vanishing locus of a system of polynomial equations with real coefficients. It inherits then an antiholomorphic involution  $c_X : X \rightarrow X$ , which is the restriction of the complex conjugation  $\text{conj} : (z_0 : \cdots : z_n) \in \mathbb{C}P^n \rightarrow (\bar{z}_0 : \cdots : \bar{z}_n) \in \mathbb{C}P^n$ . Its fixed point set  $\mathbb{R}X \subset \mathbb{R}P^n$  is called the real locus of  $X$ . When  $X$  is smooth, the latter is either empty or half-dimensional.

**Definition 2.** Let  $(X, c_X)$  be a smooth real projective manifold of positive dimension  $n$ . A Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  is said to be real iff it satisfies  $p \circ c_X = \text{conj} \circ p$ .

Such a real Lefschetz pencil given by Definition 2 is then defined by two real sections  $\sigma_0, \sigma_1$  of a holomorphic real line bundle  $\pi : (N, c_N) \rightarrow (X, c_X)$ , where  $\pi \circ c_N = c_X \circ \pi$ .

Now, if  $M$  is a smooth manifold of positive dimension  $n$ , we denote by

$$b_*(M; \mathbb{Z}/2\mathbb{Z}) = \sum_{i=0}^n \dim H_i(M; \mathbb{Z}/2\mathbb{Z})$$

its total Betti number with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients.

**Lemma 1.** Let  $M$  be a smooth manifold equipped with a smooth fibration  $p : M \rightarrow \mathbb{R}P^1$  and  $F$  be a regular fiber of  $p$ . Then, the total Betti numbers of  $M$  and  $F$  satisfy

$$b_*(M; \mathbb{Z}/2\mathbb{Z}) \leq 4b_*(F; \mathbb{Z}/2\mathbb{Z}) + \#\text{Crit}(p).$$

This relation also holds when  $M$  is the real locus of a smooth real projective manifold and  $p$  the restriction of a real Lefschetz pencil.

*Proof.* Denote by  $\infty = p(F) \in \mathbb{R}P^1$  and by  $F_0$  the fiber of  $p$  associated to a regular value  $0 \in \mathbb{R}P^1 \setminus \{\infty\}$ . Let  $I_0$  (resp.  $I_\infty$ ) be a neighborhood of  $0$  (resp.  $\infty$ ) in  $\mathbb{R}P^1$ , so that  $I_0$  and  $I_\infty$  cover  $\mathbb{R}P^1$ , such that  $I_\infty$  contains only regular values of  $p$ . Now, set  $U_0 = p^{-1}(I_0)$  and  $U_\infty = p^{-1}(I_\infty)$ , so that  $U_0 \cup U_\infty = M$ . From the Mayer–Vietoris formula it follows that

$$b_*(M) \leq b_*(U_0) + b_*(U_\infty) + b_*(U_0 \cap U_\infty) \leq b_*(U_0) + 3b_*(F),$$

since we may assume that  $U_0 \cap U_\infty$  retracts onto two fibers of  $p$ . Now, the restriction of  $p$  to  $U_0$  is a Morse function taking values in  $I_0$  and having the same critical points as  $p$ . By the Morse lemma,  $b_*(U_0) \leq b_*(F) + \#\text{Crit}(p|_{\mathbb{R}X})$ . This proves the first part of Lemma 1.

If  $p : X \dashrightarrow \mathbb{C}P^1$  is a real Lefschetz pencil with base locus  $B$ , we denote by  $\tilde{X} \rightarrow X$  the blow-up of  $B$  in  $X$  and by  $\tilde{p} : \tilde{X} \rightarrow \mathbb{C}P^1$  the induced Lefschetz fibration. From what has just been proved, we know that

$$b_*(\mathbb{R}\tilde{X}; \mathbb{Z}/2\mathbb{Z}) \leq 4b_*(\mathbb{R}\tilde{F}; \mathbb{Z}/2\mathbb{Z}) + \#\text{Crit}(\tilde{p}|_{\mathbb{R}\tilde{X}}),$$

where  $\tilde{F}$  denotes the fiber of  $\tilde{p}$  associated to  $F$ . Moreover, the morphism

$$H_*(\mathbb{R}\tilde{X}; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_*(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$$

is onto, since every element of  $H_*(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$  has a representative transverse to  $\mathbb{R}B$  and a proper transform in  $\mathbb{R}\tilde{X}$ . The inequality  $b_*(\mathbb{R}X) \leq b_*(\mathbb{R}\tilde{X})$  follows, whereas the projection  $\mathbb{R}\tilde{F} \rightarrow \mathbb{R}F$  is a diffeomorphism.  $\square$

Finally, we recall the following theorem proved by R. Thom in [24], as a consequence of Smith's exact sequence in equivariant homology.

**Theorem 2.** *Let  $(X, c_X)$  be a smooth real projective manifold with real locus  $\mathbb{R}X$ . Then, the total Betti numbers of  $X$  and  $\mathbb{R}X$  satisfy  $b_*(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) \leq b_*(X; \mathbb{Z}/2\mathbb{Z})$ .*

The manifolds for which equality holds in Theorem 2 are called maximal. For instance, real projective spaces are maximal. When  $X$  is one-dimensional and irreducible, Smith–Thom's inequality given by Theorem 2 reduces to the Harnack–Klein inequality, up to which the number of connected components of  $\mathbb{R}X$  is bounded from above by  $g(X) + 1$ , where  $g(X)$  denotes the genus of the curve  $X$ , see [26] and the references therein. Real maximal curves in real projective surfaces turn out to become exponentially rare in their linear system as their degree grows, see [7].

**1.2. Asymptotics.** Given a holomorphic line bundle  $L$  over a smooth complex projective manifold  $X$ , we denote, for every non-trivial section  $\sigma$  of  $L$ , by  $C_\sigma$  its vanishing locus.

**Lemma 2.** *Let  $L$  be a holomorphic line bundle over a smooth complex projective manifold  $X$  of positive dimension  $n$ . For every section  $\sigma$  of  $L$  which vanishes transversally, the Chern classes of its vanishing locus  $C_\sigma$  read*

$$\forall j \in \{1, \dots, n-1\}, \quad c_j(C_\sigma) = \sum_{k=0}^j (-1)^k c_1(L)^k \wedge c_{j-k}(X)|_{C_\sigma} \in H^{2j}(C_\sigma; \mathbb{Z}).$$

In particular, if  $(\sigma_d)_{d>0}$  is a sequence of sections of  $L^d$  given by Lemma 2, the Euler characteristic of  $C_{\sigma_d}$  is a polynomial of degree  $n$  in  $d$  with leading coefficient

$$(-1)^{n-1} \int_X c_1(L)^n.$$

*Proof.* The adjunction formula for  $X$ ,  $C_\sigma$  and  $L$  reads  $c(X)|_{C_\sigma} = c(C_\sigma) \wedge c(L)|_{C_\sigma}$ , since the restriction of  $L$  to  $C_\sigma$  is isomorphic to the normal bundle of  $C_\sigma$  in  $X$ . As a consequence,  $c_1(X)|_{C_\sigma} = c_1(C_\sigma) + c_1(L)|_{C_\sigma}$  and for every  $j \in \{2, \dots, n-1\}$ ,

$$c_j(X)|_{C_\sigma} = c_j(C_\sigma) + c_{j-1}(C_\sigma) \wedge c_1(L)|_{C_\sigma}.$$

Summing up, we get the result.  $\square$

**Lemma 3.** *Let  $L$  be an ample line bundle over a smooth complex projective manifold of positive dimension  $n$ . Let  $(\sigma_d)_{d>0}$  be a sequence of sections of  $L^d$  vanishing transversally. Then,*

$$b_*(C_{\sigma_d}; \mathbb{Z}/2\mathbb{Z}) = (-1)^{n-1} \chi(C_{\sigma_d}) + O(1) = \left( \int_X c_1(L)^n \right) d^n + O(d^{n-1}).$$

*Proof.* When  $d$  is large enough,  $L^d$  is very ample and we choose an embedding of  $X$  in  $\mathbb{C}P^N$ ,  $N > 0$ , such that  $L^d$  coincides with the restriction of  $\mathcal{O}_{\mathbb{C}P^N}(1)$  to  $X$ . Then,  $C_{\sigma_d}$  reads  $X \cap H$  where  $H$  is a hyperplane of  $\mathbb{C}P^N$ . By Lefschetz's theorem of hyperplane sections, for  $0 \leq i \leq n-1$ ,  $\dim H_i(C_{\sigma_d}; \mathbb{Z}/2\mathbb{Z}) = \dim H_i(X; \mathbb{Z}/2\mathbb{Z})$  and then by Poincaré duality,  $\dim H_{2n-2-i}(C_{\sigma_d}; \mathbb{Z}/2\mathbb{Z}) = \dim H_{2n-i}(X; \mathbb{Z}/2\mathbb{Z})$ . Hence

$$\begin{aligned} b_*(C_{\sigma_d}; \mathbb{Z}/2\mathbb{Z}) &= \dim H_{n-1}(C_{\sigma_d}; \mathbb{Z}/2\mathbb{Z}) + O(1) \\ &= (-1)^{n-1} \chi(C_{\sigma_d}) + O(1). \end{aligned}$$

The result now follows from Lemma 2.  $\square$

**Proposition 2.** *Let  $X$  be a smooth complex projective manifold of dimension  $n$  greater than one equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$ . Let  $L \rightarrow X$  be a holomorphic line bundle and  $\sigma_d$  be a section of  $L^d$  which vanishes transversally, where  $d > 0$ . Assume that the restriction of  $p$  to  $C_{\sigma_d}$  is Lefschetz. Then, the number of critical points of the restriction  $p|_{C_{\sigma_d}}$  equals*

$$\left( \int_X c_1(L)^n \right) d^n + O(d^{n-1}).$$

*Proof.* Denote by  $\tilde{X}$  (resp.  $\tilde{C}_{\sigma_d}$ ) the blow-up of the base locus  $B$  (resp.  $B \cap C_{\sigma_d}$ ) of  $p$  (resp.  $p|_{C_{\sigma_d}}$ ), so that  $\tilde{X}$  (resp.  $\tilde{C}_{\sigma_d}$ ) is equipped with a Lefschetz fibration induced by  $p : \tilde{X} \rightarrow \mathbb{C}P^1$  (resp.  $p|_{\tilde{C}_{\sigma_d}} : \tilde{C}_{\sigma_d} \rightarrow \mathbb{C}P^1$ ). Let  $F$  be a regular fiber of  $p$  transverse to  $C_{\sigma_d}$  and  $\tilde{F}$  be the corresponding fiber in  $\tilde{X}$ . By Proposition 1,

$$(-1)^{n-1} \# \text{Crit}(p|_{C_{\sigma_d}}) = \chi(\tilde{C}_{\sigma_d}) - 2\chi(\tilde{F} \cap \tilde{C}_{\sigma_d}).$$

From additivity of the Euler characteristic, we know that  $\chi(\tilde{C}_{\sigma_d}) = \chi(C_{\sigma_d}) + \chi(B \cap C_{\sigma_d})$ . The exceptional divisor of  $\tilde{C}_{\sigma_d}$  over  $B \cap C_{\sigma_d}$  is indeed a ruled surface over  $B \cap C_{\sigma_d}$  of Euler characteristic  $2\chi(B \cap C_{\sigma_d})$ . Likewise,  $\chi(\tilde{F} \cap \tilde{C}_{\sigma_d}) = \chi(F \cap C_{\sigma_d})$ , since the projection  $\tilde{F} \cap \tilde{C}_{\sigma_d} \rightarrow F \cap C_{\sigma_d}$  is a diffeomorphism. The result now follows from Lemma 2, which provides the equivalents

$$\chi(C_{\sigma_d}) \sim_{d \rightarrow \infty} (-1)^{n-1} \left( \int_X c_1(L)^n \right) d^n,$$

$$\begin{aligned} \chi(B \cap C_{\sigma_d}) &\sim_{d \rightarrow \infty} (-1)^{n-3} \left( \int_B c_1(L)|_B^{n-2} \right) d^{n-2}, \\ \chi(F \cap C_{\sigma_d}) &\sim_{d \rightarrow \infty} (-1)^{n-2} \left( \int_F c_1(L)|_F^{n-1} \right) d^{n-1}. \quad \square \end{aligned}$$

## 2. Random divisors and distribution of critical points

Let  $X$  be a smooth complex projective manifold equipped with a Lefschetz pencil. The restriction of this pencil to a generic smooth hypersurface  $C$  of  $X$  is a Lefschetz pencil of  $C$ . The aim of this section is to prove the equidistribution in average of critical points of such a restriction to a random hypersurface  $C$  of large degree, see Theorem 3. The estimations of the total Betti number of real hypersurfaces will be obtained in Section 3 as a consequence of a real analogue of this Theorem 3, see Theorem 6.

We first formulate this equidistribution Theorem 3, then introduce the main ingredients of the proof, namely Poincaré–Martinelli’s formula and Hörmander’s peak sections. Finally, we prove Theorem 3. Note that this section is independent of the remaining part of the paper, it does not involve any real geometry.

**2.1. Notations and result.** Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  with base locus  $B \subset X$ . Let  $L \rightarrow X$  be a holomorphic line bundle equipped with a Hermitian metric  $h$  of positive curvature  $\omega \in \Omega^{(1,1)}(X; \mathbb{R})$ . The latter is defined in the neighborhood of every point  $x \in X$  by the relation

$$\omega = \frac{1}{2i\pi} \partial\bar{\partial} \log h(e, e),$$

where  $e$  is a local non-vanishing holomorphic section of  $L$  defined in the neighborhood of  $x$ . The curvature form induces a Kähler metric on  $X$  and we denote by

$$dx = \frac{\omega^n}{\int_X \omega^n}$$

its associated normalized volume form. For every integer  $d > 0$ , we denote by  $h^d$  the induced Hermitian metric on the bundle  $L^d$  and by  $\langle \cdot, \cdot \rangle$  the induced  $L^2$ -Hermitian product on the space  $H^0(X; L^d)$  of global sections of  $L^d$ . This product is defined by the relation

$$(1) \quad (\sigma, \tau) \in H^0(X; L^d) \times H^0(X; L^d) \mapsto \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau) dx \in \mathbb{C}.$$

Denote by  $N_d$  the dimension of  $H^0(X; L^d)$  and by  $\mu$  its Gaussian measure, defined by the relation

$$\forall A \subset H^0(X; L^d), \quad \mu(A) = \frac{1}{\pi^{N_d}} \int_A e^{-\|\sigma\|^2} d\sigma,$$

where  $\|\sigma\|^2 = \langle \sigma, \sigma \rangle$  and  $d\sigma$  denotes the Lebesgue measure associated to  $\langle \cdot, \cdot \rangle$ . Denote by  $\Delta_d \subset H^0(X; L^d)$  the discriminant locus, that is, the set of sections of  $H^0(X; L^d)$  which do not vanish transversally. Likewise, denote by  $\tilde{\Delta}_d \in H^0(X; L^d)$  the union of  $\Delta_d$  with the set of sections  $\sigma \in H^0(X; L^d)$  such that either the restriction of  $p$  to  $C_\sigma$  is not Lefschetz, or

this vanishing locus  $C_\sigma$  meets the critical set  $\text{Crit}(p)$ . By Bertini’s theorem (see for example [10, Theorem 8.18]),  $\tilde{\Delta}_d$  is a hypersurface of  $H^0(X; L^d)$  as soon as  $d$  is large enough, which will be assumed throughout this article.

For every section  $\sigma \in H^0(X; L^d) \setminus \tilde{\Delta}_d$ , denote by  $\mathcal{R}_\sigma$  the set of critical points of the restriction  $p|_{C_\sigma}$  of  $p$  to  $C_\sigma$ , so that by Proposition 2, the cardinal  $\#\mathcal{R}_\sigma$  of this set is equivalent to  $(\int_X \omega^n)d^n$  as  $d$  grows to infinity. For every  $x \in X$ , we finally denote by  $\delta_x$  the Dirac measure  $\chi \in C^0(X, \mathbb{R}) \mapsto \chi(x) \in \mathbb{R}$ .

**Definition 3.** For every  $\sigma \in H^0(X; L^d) \setminus \tilde{\Delta}_d$ , the measure  $\nu_\sigma = \frac{1}{\#\mathcal{R}_\sigma} \sum_{x \in \mathcal{R}_\sigma} \delta_x$  is called the probability measure of  $X$  carried by the critical points of  $p|_{C_\sigma}$ .

Our goal in this section is to prove the following theorem which asymptotically computes the expected probability measure given by Definition 3.

**Theorem 3.** *Let  $X$  be a smooth complex projective manifold of dimension  $n$  greater than one equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  with critical locus  $\text{Crit}(p)$ . Let  $L \rightarrow X$  be a holomorphic line bundle equipped with a Hermitian metric  $h$  of positive curvature  $\omega$ . Then, for every function  $\chi : X \rightarrow \mathbb{R}$  of class  $C^2$  such that, when  $n > 2$ , the support of  $\partial\bar{\partial}\chi$  is disjoint from  $\text{Crit}(p)$ , we have*

$$\lim_{d \rightarrow \infty} E(\langle \nu_\sigma, \chi \rangle) = \int_X \chi dx,$$

where  $E(\langle \nu_\sigma, \chi \rangle) = \int_{H^0(X; L^d) \setminus \tilde{\Delta}_d} \langle \nu_\sigma, \chi \rangle d\mu(\sigma)$ .

Note that similar results on equidistribution of critical points of sections, have been obtained by M. R. Douglas, B. Shiffman and S. Zelditch [4, 5].

Note also that the equidistribution Theorem 3 as well as Theorem 6 are local in nature and does not depend that much on a Lefschetz pencil. Any local holomorphic Morse function could be used instead of a Lefschetz pencil, leading to the same proof and conclusions.

## 2.2. Poincaré–Martinelli’s formula and adapted atlas.

### 2.2.1. Adapted atlas and associated relative trivializations.

**Definition 4.** Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$ . An atlas  $\mathcal{U}$  of  $X$  is said to be *adapted* to  $p$  iff for every open set  $U \in \mathcal{U}$ , the restriction of  $p$  to  $U$  is conjugated to one of the following three models in the neighborhood of the origin in  $\mathbb{C}^n$ :

- (r)  $(z_1, \dots, z_n) \in \mathbb{C}^n \mapsto z_n \in \mathbb{C}$ ,
- (b)  $(z_1, \dots, z_n) \in \mathbb{C}^n \setminus \mathbb{C}^{n-2} \mapsto [z_{n-1} : z_n] \in \mathbb{C}P^1$ ,
- (c)  $(z_1, \dots, z_n) \in \mathbb{C}^n \mapsto z_1^2 + \dots + z_n^2 \in \mathbb{C}$ .

Every atlas of  $X$  becomes adapted in the sense of Definition 4 after refinement. Let  $x$  be a point in  $X$ . If  $x$  is a regular point of  $p$ , by the implicit function theorem it has a neighborhood biholomorphic to the model (r) of Definition 4. If  $x$  is a base point (resp. a critical point), it has by definition (resp. by the holomorphic Morse lemma) a neighborhood biholomorphic to the model (b) (resp. (c)) of Definition 4.

In the model (r), the vertical tangent bundle  $\ker(dp)$  is trivialized by the vector fields

$$(2) \quad \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-1}}$$

of  $\mathbb{C}^n$ . In the model (b), it is trivialized outside of the base locus by the vector fields

$$(3) \quad z_{n-1} \frac{\partial}{\partial z_{n-1}} + z_n \frac{\partial}{\partial z_n}, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-2}}$$

of  $\mathbb{C}^n$ . In the model (c), when  $n = 2$ , it is trivialized outside of the critical point by the vector field

$$(4) \quad z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1}$$

of  $\mathbb{C}^2$ .

**Definition 5.** Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  and an adapted atlas  $\mathcal{U}$ . A relative trivialization associated to  $\mathcal{U}$  is the data, for every open set  $U \in \mathcal{U}$ , of  $n - 1$  vector fields on  $U$  corresponding to the vector fields (2) in the model (r), to (3) in the model (b) and to (4) in the model (c) when  $n = 2$ .

Note that in the model (c) given by Definition 4 the vertical tangent bundle  $\ker(dp)$  restricted to  $\mathbb{C}^n \setminus \{0\}$  is isomorphic to the pullback of the cotangent bundle of  $\mathbb{C}P^{n-1}$  by the projection  $\pi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}P^{n-1}$ . Indeed, the fibers of this vertical tangent bundle are the kernels of the 1-form  $\alpha = \sum_{i=1}^n z_i dz_i$ , so that the restriction map induces an isomorphism  $(\mathbb{C}^n)^*/\langle \alpha \rangle \cong (\ker dp)^*$ . But the canonical identification between  $\mathbb{C}^n$  and  $(\mathbb{C}^n)^*$  gives an isomorphism between the bundle  $(\mathbb{C}^n)^*/\langle \alpha \rangle$  and  $\pi^*(TP^{n-1})$  over  $\mathbb{C}^n \setminus \{0\}$ . By duality, we get the isomorphism  $\ker(dp) \cong \pi^*T^*\mathbb{C}P^{n-1}$ . When  $n > 2$ , we no longer see trivializations of this bundle over  $\mathbb{C}^n \setminus \{0\}$  and thus restrict ourselves to  $n = 2$  for the model (c) in Definition 6.

**Definition 6.** Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  and with a holomorphic line bundle  $L \rightarrow X$ . An atlas  $\mathcal{U}$  is said to be adapted to  $(p, L)$  if it is adapted to  $p$  in the sense of Definition 4 and if for every open set  $U \in \mathcal{U}$ , the restriction of  $L$  to  $U$  is trivializable. A relative trivialization associated to  $\mathcal{U}$  is a relative trivialization in the sense of Definition 5 together with a trivialization  $e$  of  $L|_U$ , for every open set  $U$  of  $\mathcal{U}$ .

**2.2.2. Poincaré–Martinelli’s formula.** Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  of critical locus  $\text{Crit}(p)$ . Let  $L \rightarrow X$  be an ample holomorphic line bundle equipped with a Hermitian metric  $h$  of positive curvature  $\omega \in \Omega^{(1,1)}(X; \mathbb{R})$ . Let  $\mathcal{U}$  be an atlas of  $X$  adapted to  $(p, L)$  and  $(v_1, \dots, v_{n-1}, e)$  be an associated relative trivialization given by Definition 6. Let  $U$  be an element of  $\mathcal{U}$ . For every section  $\sigma \in H^0(X; L^d)$ , we denote by  $f_{\sigma,U} : U \rightarrow \mathbb{C}$  the holomorphic function defined by the relation  $\sigma|_U = f_{\sigma,U} e|_U^d$ . When  $\sigma \notin \tilde{\Delta}_d$ , the set  $\mathcal{R}_\sigma \cap U$  coincides by definition with the transverse intersection of the hypersurfaces

$$\{f_{\sigma,U} = 0\}, \{\partial f_{\sigma,U}(v_1) = 0\}, \dots, \{\partial f_{\sigma,U}(v_{n-1}) = 0\}.$$

For every function  $\chi : X \rightarrow \mathbb{R}$  with compact support in  $U$ , Poincaré–Martinelli’s formula (see [8]) then reads

$$(5) \quad \langle v_\sigma, \chi \rangle = \left(\frac{i}{2\pi}\right)^n \frac{1}{\#\mathcal{R}_\sigma} \int_X \lambda_U \partial\bar{\partial}\chi \wedge (\partial\bar{\partial}\lambda_U)^{n-1},$$

where

$$\lambda_U = \log\left(d^2|f_{\sigma,U}|^2 + \sum_{i=1}^{n-1} |\partial f_{\sigma,U}(v_i)|^2\right).$$

Note that in this definition of  $\lambda_U$ , we have chosen for convenience to use  $df_{\sigma,U}$  instead of  $f_{\sigma,U}$  as the first function. Thus,  $d^2$  denotes the square of the degree and not some Hessian function. This formula of Poincaré–Martinelli computes the integral of  $\chi$  for the measure  $v_\sigma$  introduced in Definition 3; its left-hand side does not involve any trivialization of  $L$  over  $U$ , contrary to the right-hand side. It makes it possible to estimate the expectation of the random variable  $\langle v_\sigma, \chi \rangle$ . However, it appears to be useful for this purpose to choose an appropriate trivialization of  $L$  in the neighborhood of every point  $x \in X$ , whose norm reaches a local maximum at  $x$  where it equals one. We are going to make such a choice instead of the trivialization  $e$  defined on the whole  $U$  (see Proposition 3 and Section 2.3). For this purpose, we choose a family  $(g_x)_{x \in U}$  of germs of holomorphic functions such that  $g_x$  is defined in a neighborhood of  $x$  and  $\Re g_x(x) = -\frac{1}{2} \log h^d(e^d, e^d)|_x$ . This condition ensures that  $\exp(g_x)e^d$  is a holomorphic trivialization of norm one at  $x$ , so that  $h^d(\sigma, \sigma)|_x$  coincides with  $|f_{\sigma,x}|^2$ , where  $\sigma = f_{\sigma,x} \exp(g_x)e^d$  in the neighborhood of every point  $x \in U$ . Note that if

$$\Pi_d(z, x) = \sum_{i=1}^{N^d} \sigma_i(z) \overline{\sigma_i(x)}$$

denotes the Bergman kernel, which is the covariant function for random sections (see [1]), where  $z, x \in U$  and  $(\sigma_i)_{i=1, \dots, N^d}$  denotes an orthonormal basis of  $\mathbb{R}H^0(X; L^d)$ , and if  $B_d$  is the associated function defined by

$$\Pi_d(z, x) = B_d(z, x) e^d(z) \overline{e^d(x)},$$

then we may set

$$g_x(z) = \log\left(\frac{B_d(z, x)}{\|\Pi_d(x, x)\|} \|e^d(x)\|\right).$$

**Proposition 3.** *Let  $X$  be a smooth complex projective manifold of positive dimension  $n$  equipped with a Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$ . Let  $L$  be an ample holomorphic line bundle equipped with a Hermitian metric  $h$  of positive curvature  $\omega$ . Let  $U$  be an element of an atlas adapted to  $(p, L)$  and  $(v_1, \dots, v_{n-1}, e)$  be an associated relative trivialization. Then, for every function  $\chi : X \rightarrow \mathbb{R}$  of class  $C^2$  with support in  $U$ , disjoint from the critical set of  $p$  when  $n > 2$ , and for every  $\sigma \in H^0(X; L^d) \setminus \tilde{\Delta}_d$ , we have*

$$\langle v_\sigma, \chi \rangle = \frac{1}{\#\mathcal{R}_\sigma} d^n \int_X \chi \omega^n + \frac{1}{\#\mathcal{R}_\sigma} \sum_{k=0}^{n-1} \left(\frac{i}{2\pi}\right)^{n-k} d^k \int_X \partial\bar{\partial}\chi \wedge \omega^k \wedge \lambda_x (\partial\bar{\partial}\lambda_x)^{n-1-k},$$

where

$$\lambda_x = \log\left(d^2|f_{\sigma,x}|^2 + \sum_{i=1}^{n-1} |\partial f_{\sigma,x}(v_i) + f_{\sigma,x} \partial g_x(v_i)|^2\right).$$

Note that  $\lambda_x$  reads in the neighborhood of  $x$  as a function

$$z \mapsto \log \left( d^2 |f_{\sigma,x}|^2(z) + \sum_{i=1}^{n-1} |\partial f_{\sigma,x}(v_i)|_z + f_{\sigma,x}(z) \partial g_x(v_i)|_z \right)$$

and  $\partial\bar{\partial}\lambda_x$  in the formula given by Proposition 3 stands for its second derivative in the variable  $z$  computed at the point  $x$ . Note also that if  $\mathcal{U}$  is a locally finite atlas adapted to  $(p, L)$ , and if  $(\rho_U)_{U \in \mathcal{U}}$  is an associated partition of unity, then for every function  $\chi : X \rightarrow \mathbb{R}$  of class  $C^2$ , with support disjoint from the critical locus of  $p$  when  $n > 2$ , and for every open set  $U \in \mathcal{U}$ , the function  $\chi_U = \rho_U \chi$  satisfies the hypotheses of Proposition 3, while  $\chi = \sum_{U \in \mathcal{U}} \chi_U$  and  $\langle v_\sigma, \chi \rangle = \sum_{U \in \mathcal{U}} \langle v_\sigma, \chi_U \rangle$ .

*Proof.* Let  $\sigma \in H^0(X; L^d) \setminus \tilde{\Delta}_d$  and  $x \in U \setminus \text{Crit}(p)$ . By definition, we have  $f_{\sigma,U} = f_{\sigma,x} \exp(g_x)$  and, for every  $1 \leq i \leq n-1$ ,

$$\partial f_{\sigma,U}(v_i) = (\partial f_{\sigma,x}(v_i) + f_{\sigma,x} \partial g_x(v_i)) \exp(g_x).$$

As a consequence,  $\lambda_U = \Re g_x + \lambda_x$ , so that at the point  $x$ ,

$$\lambda_U(x) = -\log h^d(e^d, e^d)(x) + \lambda_x(x).$$

Since  $\partial\bar{\partial}\Re g_x$  vanishes, the equality  $\partial\bar{\partial}\lambda_U = \partial\bar{\partial}\lambda_x$  holds in a neighborhood of  $x$ . Hence, we can rewrite Poincaré–Martinelli's formula (5) as

$$\begin{aligned} \langle v_\sigma, \chi \rangle &= \frac{i^n}{(2\pi)^n \#\mathcal{R}_\sigma} \int_X \partial\bar{\partial}\chi(-\log h^d(e^d, e^d) + \lambda_{x_0}) (\partial\bar{\partial}\lambda_U)^{n-1} \\ &= \frac{i^{n-1}d}{(2\pi)^{n-1} \#\mathcal{R}_\sigma} \int_X \partial\bar{\partial}\chi \wedge \omega \wedge \lambda_U (\partial\bar{\partial}\lambda_U)^{n-2} \\ &\quad + \frac{i^n}{(2\pi)^n \#\mathcal{R}_\sigma} \int_X \partial\bar{\partial}\chi \wedge \lambda_{x_0} (\partial\bar{\partial}\lambda_x)^{n-1} \end{aligned}$$

The first part of the latter right-hand side follows from the relation

$$\partial\bar{\partial}(\lambda_U (\partial\bar{\partial}\lambda_U)^{n-2}) = (\partial\bar{\partial}\lambda_U)^{n-1},$$

the curvature equation  $\omega = \frac{i}{2\pi d} \partial\bar{\partial}(-\log h^d(e, e))$  and Stokes's theorem. The second part of this right-hand side comes from  $\partial\bar{\partial}\lambda_U = \partial\bar{\partial}\lambda_x$ . Applying this procedure  $(n-1)$  times, we deduce by induction and Stokes's theorem the relation

$$\langle v_\sigma, \chi \rangle = \frac{1}{\#\mathcal{R}_\sigma} d^n \int_X \chi \omega^n + \frac{1}{\#\mathcal{R}_\sigma} \sum_{k=0}^{n-1} \left( \frac{i}{2\pi} \right)^{n-k} d^k \int_X \partial\bar{\partial}\chi \wedge \omega^k \wedge \lambda_x (\partial\bar{\partial}\lambda_x)^{n-1-k}. \quad \square$$

**Corollary 1.** *Under the hypotheses of Proposition 3, we have*

$$\begin{aligned} E(\langle v_\sigma, \chi \rangle) &= \frac{1}{\#\mathcal{R}_\sigma} d^n \int_X \chi \omega^n + \frac{1}{\#\mathcal{R}_\sigma} \sum_{k=0}^{n-1} \left( \frac{i}{2\pi} \right)^{n-k} d^k \int_X \partial\bar{\partial}\chi \wedge \omega^k \\ &\quad \wedge \int_{H^0(X; L^d) \setminus \tilde{\Delta}_d} \lambda_x (\partial\bar{\partial}\lambda_x)^{n-1-k} d\mu(\sigma). \end{aligned}$$

*Proof.* The result follows by integration over  $H^0(X; L^d) \setminus \tilde{\Delta}_d$  of the relation given by Proposition 3.  $\square$

**2.3. Hörmander's peak sections.** Let  $L$  be a holomorphic line bundle over a smooth complex projective manifold, equipped with a Hermitian metric  $h$  of positive curvature  $\omega$ . Let  $x$  be a point of  $X$ . There exists, in the neighborhood of  $x$ , a holomorphic trivialization  $e$  of  $L$  such that the associated potential  $\phi = -\log h(e, e)$  reaches a local minimum at  $x$  with Hessian of type  $(1, 1)$ . The latter coincides, by definition, with  $\omega(\cdot, i\cdot)$ . The Hörmander  $L^2$ -estimates make it possible, for all  $d > 0$  and maybe after modifying a bit  $e^d$  in  $L^2$ -norm, to extend  $e^d$  to a global section  $\sigma$  of  $L^d$ . The latter is called peak section of Hörmander, see Definition 7. Moreover, in [25, Lemma 1.2], G. Tian showed that this procedure can be applied to produce global sections whose Taylor expansion at  $x$  can be controlled at every order, as long as  $d$  is large enough. We recall this result in the following lemma where we denote, for every  $r > 0$ , by  $B(x, r)$  the ball centered at  $x$  of radius  $r$  in  $X$ .

**Lemma 4** ([25, Lemma 1.2]). *Let  $(L, h)$  be a holomorphic Hermitian line bundle of positive curvature  $\omega$  over a smooth complex projective manifold  $X$ . Let  $x$  be any point in  $X$ ,  $(p_1, \dots, p_n) \in \mathbb{N}^n$  and  $p' > p_1 + \dots + p_n$ . There exists  $d_0 \in \mathbb{N}$  such that for every  $d > d_0$ , the bundle  $L^d$  has a global holomorphic section  $\sigma$  satisfying*

$$\int_X h^d(\sigma, \sigma) dx = 1 \quad \text{and} \quad \int_{X \setminus B(x, \frac{\log d}{\sqrt{d}})} h^d(\sigma, \sigma) dx = O\left(\frac{1}{d^{2p'}}\right).$$

Moreover, if  $z = (z_1, \dots, z_n)$  are local coordinates in the neighborhood of  $x$ , we can assume that in a neighborhood of  $x$ ,

$$\sigma(z) = \lambda(z_1^{p_1} \dots z_n^{p_n} + O(|z|^{2p'})) e^d \left(1 + O\left(\frac{1}{d^{2p'}}\right)\right),$$

where

$$\lambda^{-2} = \int_{B(x, \frac{\log d}{\sqrt{d}})} |z_1^{p_1} \dots z_n^{p_n}|^2 h^d(e^d, e^d) dx$$

and  $e$  is a trivialization of  $L$  in the neighborhood of  $x$  whose potential  $\phi = -\log h(e, e)$  reaches a local minimum at  $x$  with Hessian  $\pi\omega(\cdot, i\cdot)$ .

**Definition 7.** We call any section given by Lemma 4 with  $p_1 = \dots = p_n = 0$  and  $p' > 1$ , where  $d$  is large enough, a peak section of Hörmander of the ample line bundle  $L^d$  over the smooth complex projective manifold  $X$ .

Note that such a peak section  $\sigma_0$  given by Definition 7 has its norm concentrated in the neighborhood of the point  $x$  given by Lemma 4, so that it is close to the zero section outside of a  $\log d / \sqrt{d}$ -ball. Moreover, the derivatives and second derivatives of  $\sigma_0$  at  $x$  vanish, and

$$\|\sigma_0(x)\| \underset{d \rightarrow \infty}{\sim} \sqrt{\left(\int_X c_1(L)^n\right) d^n},$$

see [25, Lemma 2.1].

Note also that if the coordinates  $(z_1, \dots, z_n)$  in Lemma 4 are orthonormal at the point  $x$ , then two sections given by this lemma for different values of  $(p_1, \dots, p_n)$  are asymptotically orthogonal, see [25, Lemma 3.1].

**2.3.1. Evaluation of the two-jets of sections.** Again, let  $(L, h)$  be a holomorphic Hermitian line bundle of positive curvature  $\omega$  over a smooth  $n$ -dimensional complex projective manifold  $X$ . Let  $x$  be a point of  $X$  and  $d > 0$ . We denote by  $H_x$  the kernel of the evaluation map  $\sigma \in H^0(X; L^d) \mapsto \sigma(x) \in L_x^d$ , where  $L_x^d$  denotes the fiber of  $L^d$  over the point  $x$ . Likewise, we denote by  $H_{2x}$  the kernel of the map  $\sigma \in H_x \mapsto \nabla\sigma|_x \in T_x^*X \otimes L_x^d$ . This map does not depend on a chosen connection  $\nabla$  on  $L$ . Denote by  $H_{3x}$  the kernel of the map  $\sigma \in H_{2x} \mapsto \nabla^2\sigma|_x \in \text{Sym}^2(T_x^*X) \otimes L_x^d$ . We deduce from these the jet maps

$$\begin{aligned} \text{eval}_x &: \sigma \in H^0(X; L^d)/H_x \mapsto \sigma(x) \in L_x^d, \\ \text{eval}_{2x} &: \sigma \in H_x/H_{2x} \mapsto \nabla\sigma|_x \in T_x^*X \otimes L_x^d, \\ \text{eval}_{3x} &: \sigma \in H_{2x}/H_{3x} \mapsto \nabla^2\sigma|_x \in \text{Sym}^2(T_x^*X) \otimes L_x^d. \end{aligned}$$

When  $d$  is large enough, these maps are isomorphisms between finite dimensional normed vector spaces. We estimate the norm of these isomorphisms in the following proposition, closely following [25].

**Proposition 4.** *Let  $L$  be a holomorphic Hermitian line bundle of positive curvature over a smooth  $n$ -dimensional complex projective manifold  $X$ . Let  $x$  be a point of  $X$ . Then, the maps  $d^{-\frac{n}{2}} \text{eval}_x$ ,  $d^{-\frac{n+1}{2}} \text{eval}_{2x}$  and  $d^{-\frac{n+2}{2}} \text{eval}_{3x}$  as well as their inverses have norms and determinants bounded from above independently of  $d$  as long as  $d$  is large enough.*

Note that Proposition 4 provides an asymptotic result while the condition that  $d$  be large ensures that the three maps are invertible.

*Proof.* Let  $\sigma_0$  be a peak section of Hörmander given by Definition 7. By [25, Lemma 2.1],  $d^{-n}h^d(\sigma_0, \sigma_0)|_x$  converges to a positive constant as  $d$  grows to infinity. Let  $\sigma_0^{H_x}$  be the orthogonal projection of  $\sigma_0$  onto  $H_x$ . The Taylor expansion of  $\sigma_0^{H_x}$  does not contain any constant term, so that by [25, Lemma 3.1] (see also [19, Lemma 3.2]),

$$\left\langle \sigma_0, \frac{\sigma_0^{H_x}}{\|\sigma_0^{H_x}\|} \right\rangle = O\left(\frac{1}{d}\right),$$

where  $\|\sigma_0^{H_x}\|^2 = \langle \sigma_0^{H_x}, \sigma_0^{H_x} \rangle$  denotes the  $L^2$ -norm of  $\sigma_0^{H_x}$ . From the vanishing of the product  $\langle \sigma_0 - \sigma_0^{H_x}, \sigma_0^{H_x} \rangle$  we deduce that  $\|\sigma_0^{H_x}\|$  is  $O(\frac{1}{d})$ . It follows that the norm of  $\sigma_0 - \sigma_0^{H_x}$  equals  $1 + O(\frac{1}{d^2})$  and we set

$$\sigma_0^\perp = \frac{\sigma_0 - \sigma_0^{H_x}}{\|\sigma_0 - \sigma_0^{H_x}\|}.$$

As a consequence,  $d^{-n}h^d(\sigma_0^\perp, \sigma_0^\perp)|_x$  converges to a positive constant as  $d$  grows to infinity. Hence,  $d^{-\frac{n}{2}} \text{eval}_x$  as well as its inverse have norm and determinant bounded when  $d$  is large enough. The two remaining assertions of Proposition 4 follow along the same lines. For  $i \in \{1, \dots, n\}$ , let  $\sigma_i$  be a section given by Lemma 4, with  $p' = 2$ ,  $p_i = 1$  and  $p_j = 0$  if  $j \neq i, i \in \{1, \dots, n\}$ , and where the local coordinates  $(z_1, \dots, z_n)$  are orthonormal at the point  $x$ . By [25, Lemma 2.1], for  $i \in \{1, \dots, n\}$ ,  $d^{-(n+1)}h^d(\nabla\sigma_i, \nabla\sigma_i)|_x$  converges to a positive

constant as  $d$  grows to infinity. The sections  $\sigma_i, i \in \{1, \dots, n\}$ , belong by construction to  $H_x$  and we set as before

$$\sigma_i^\perp = \frac{\sigma_i - \sigma_i^{H_{2x}}}{\|\sigma_i - \sigma_i^{H_{2x}}\|},$$

where  $\sigma_i^{H_{2x}}$  denotes the orthogonal projection of  $\sigma_i$  onto  $H_{2x}$ . We deduce as before from [25, Lemma 3.1] that  $d^{-(n+1)}h^d(\nabla\sigma_i^\perp, \nabla\sigma_i^\perp)|_x$  converges to a positive constant when  $d$  grows to infinity and that  $h^d(\nabla\sigma_i^\perp, \nabla\sigma_j^\perp)|_x = 0$  if  $i \neq j$ . The norms of the sections  $\sigma_i^\perp, i \in \{1, \dots, n\}$ , all equal one but these sections are not a priori orthogonal. However, by [25, Lemma 3.1], the products  $\langle \nabla\sigma_i^\perp, \nabla\sigma_j^\perp \rangle$  are  $O(\frac{1}{d})$  if  $j \neq i$ , so that, asymptotically, the basis is orthonormal. We deduce that  $d^{-\frac{n+1}{2}}$   $\text{eval}_{2x}$  and its inverse have norms and determinants bounded as  $d$  is large enough. The last case follows along the same lines.  $\square$

### 2.4. Proof of Theorem 3.

**Proposition 5.** *Under the hypotheses and notations of Proposition 3, for all  $x \in U$  and  $k \in \{0, \dots, n - 1\}$ , the integral*

$$\frac{\|v_1\|^2}{d^k} \int_{H^0(X; L^d) \setminus \tilde{\Delta}_d} \|\lambda_x(\partial\bar{\partial}\lambda_x)^k\| d\mu(\sigma)$$

*is uniformly bounded by  $O((\log d)^2)$  on every compact subset of  $X \setminus \text{Crit}(p)$  when  $n > 2$  and on the whole  $X$  when  $n \leq 2$ .*

The norms  $\|\cdot\|$  appearing in the statement of Proposition 5 are induced by the Kähler metric of  $X$  on elements and  $2k$ -linear forms of  $T_x X$ , where  $x \in X$ . It follows from Definition 4 that  $\|v_1\|$  may vanish in the models (b) and (c). Before proving Proposition 5, for which we will spend the whole section, let us first deduce a proof of Theorem 3.

*Proof of Theorem 3.* Let  $\chi : X \rightarrow \mathbb{R}$  be a function of class  $C^2$  such that the support  $K$  of  $\partial\bar{\partial}\chi$  is disjoint from  $\text{Crit}(p)$  when  $n > 2$ . Choose a finite atlas  $\mathcal{U}$  adapted to  $(p, L)$  given by Definition 6, such that  $K$  is covered in  $X \setminus \text{Crit}(p)$  by elements of  $\mathcal{U}$  when  $n > 2$ . Let  $U$  be such an element of  $\mathcal{U}$  and  $(v_1, \dots, v_n, e)$  be an associated relative trivialization given by Definition 6. Without loss of generality, we can assume that  $\chi$  has support in  $U$ . By Proposition 3 and with the notations introduced there, we have, for  $d$  large enough,

$$\begin{aligned} E(\langle v_\sigma, \chi \rangle) &= \frac{1}{\#\mathcal{R}_\sigma} d^n \int_X \chi \omega^n + \frac{1}{\#\mathcal{R}_\sigma} \sum_{k=0}^{n-1} \left(\frac{i}{2\pi}\right)^{n-k} d^k \int_X \partial\bar{\partial}\chi \wedge \omega^k \\ &\quad \wedge \int_{H^0(X; L^d) \setminus \tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^{n-1-k} d\mu(\sigma). \end{aligned}$$

By Proposition 2,  $\#\mathcal{R}_\sigma$  is equivalent to  $(\int_X \omega^n)d^n$  as  $d$  grows to infinity, so that the first term converges to  $\int_X \chi dx$ . By Proposition 5, the last integral over  $H^0(X; L^d) \setminus \tilde{\Delta}_d$  is  $O(\frac{(\log d)^2}{d\|v_1\|^2})$ , since we integrate over the support of  $\partial\bar{\partial}\chi$  which is disjoint of  $\text{Crit}(p)$  when  $n > 2$ . The result now follows from the fact that the function  $\|v_1\|^{-2}$  is integrable over  $X$ .  $\square$

**2.4.1. Proof of Proposition 5 outside of the base and critical loci of  $p$ .** Recall that  $X$  is equipped with an atlas  $\mathcal{U}$  adapted to  $(p, L)$  and with an associated relative trivialization. The compact  $K$  given by Proposition 5 is covered by a finite number of elements of  $\mathcal{U}$ . Moreover, we can assume that these elements are all disjoint from the critical set  $\text{Crit}(p)$  when  $n > 2$ . Let  $U$  be such an element; it is either of type (r) given by Definition 4, or of type (b) or (c). Let us prove now Proposition 5 in the case when  $U$  is of type (r), and postpone the remaining cases to Section 2.4.2.

Let  $x$  be a point of  $K \cap U$ . For every  $\sigma \in H^0(X; L^d) \setminus \tilde{\Delta}_d$ , define  $h_0 = df_{\sigma,x}$  and for  $i \in \{1, \dots, n-1\}$ ,  $h_i = \partial f_{\sigma,x}(v_i) + df_{\sigma,x} \partial g_x(v_i)$ , so that

$$\partial h_i = \partial(\partial f_{\sigma,x}(v_i)) + d \partial g_x(v_i) \partial f_{\sigma,x} + df_{\sigma,x} \partial(\partial g_x(v_i)).$$

Recall that  $f_{\sigma,x}$  was introduced in Proposition 3 and defined by the relation

$$\sigma = f_{\sigma,x} \exp(g_x) e^d,$$

where the local section  $\exp(g_x) e^d$  has norm one at  $x$ . It is enough to bound by  $O((\log d)^2)$  the integral

$$\frac{1}{d^k} \int_{H^0(X; L^d) \setminus \tilde{\Delta}_d} \left\| \log \left( \sum_{i=0}^{n-1} |h_i|^2 \right) \left( \partial \bar{\partial} \log \left( \sum_{i=0}^{n-1} |h_i|^2 \right) \right)^k \right\| d\mu(\sigma),$$

since  $\|v_1\|^2$  is bounded from below and above by positive constants in the model (r). Recall that

$$\partial \bar{\partial} \left( \sum_{i=0}^{n-1} |h_i|^2 \right) = \frac{\sum_{i=0}^{n-1} \partial h_i \wedge \bar{\partial} h_i}{\sum_{i=0}^{n-1} |h_i|^2} + \frac{\sum_{i=0}^{n-1} h_i \bar{\partial} h_i \wedge \sum_{j=0}^{n-1} \bar{h}_j \partial h_j}{\left( \sum_{i=0}^{n-1} |h_i|^2 \right)^2}.$$

From this we deduce, for  $k \in \{1, \dots, n-1\}$ , the upper bound

$$\left\| \left( \partial \bar{\partial} \log \left( \sum_{i=0}^{n-1} |h_i|^2 \right) \right)^k \right\| \leq \frac{2^k}{\left( \sum_{i=0}^{n-1} |h_i|^2 \right)^k} \sum_{|I|=k, |J|=k} \|\partial h_I \wedge \partial h_J\|,$$

where  $I$  and  $J$  are ordered sets of  $k$  elements in  $\{1, \dots, n\}$  and  $\partial h_I = \partial h_{i_1} \wedge \dots \wedge \partial h_{i_k}$  if  $I = (i_1, \dots, i_k)$ . Our integral gets then bounded from above by

$$\frac{2^k}{d^k} \sum_{|I|=k, |J|=k} \int_{H^0(X; L^d) \setminus \tilde{\Delta}_d} \frac{|\log(\sum_{i=0}^{n-1} |h_i|^2)|}{\left( \sum_{i=0}^{n-1} |h_i|^2 \right)^k} \|\partial h_I \wedge \partial h_J\| d\mu(\sigma).$$

Denote by  $H_x^\perp$  the orthogonal complement of  $H_x$  in  $H^0(X; L^d)$ , see Section 2.3.1. Likewise, with a slight abuse of notation, denote by  $H_x/H_{2x}$  (resp.  $H_{2x}/H_{3x}$ ) the orthogonal complement of  $H_{2x}$  (resp.  $H_{3x}$ ) in  $H_x$  (resp.  $H_{2x}$ ). The space  $H^0(X; L^d)$  can then be written as a product

$$H^0(X; L^d) = H_x^\perp \times (H_x/H_{2x}) \times (H_{2x}/H_{3x}) \times H_{3x}$$

while its Gaussian measure  $\mu$  is a product measure. The terms in our integral only involve jets at the second order of sections and hence are constant on  $H_{3x}$ . Using Fubini's theorem, it becomes thus enough to bound the integral over the space  $H_x^\perp \times (H_x/H_{2x}) \times (H_{2x}/H_{3x})$ , whose dimension no longer depends on  $d$ .

Moreover, the subspace  $V^\perp \subset T_x^*X \otimes L_x^d$  of forms that vanish on the  $n - 1$  vectors  $v_1(x), \dots, v_{n-1}(x)$  given by the relative trivialization is one-dimensional. It induces an orthogonal decomposition  $T_x^*X \otimes L_x^d = V \oplus V^\perp$ , where  $V$  is of dimension  $n - 1$ . The inverse image of  $V^\perp$  in  $H_x/H_{2x}$  by the evaluation map  $\text{eval}_{2x}$  is the line  $D$  of  $H_x/H_{2x}$  containing the sections  $\sigma$  of  $H_x$  whose derivatives at  $x$  vanish against  $v_1(x), \dots, v_{n-1}(x)$ . We denote by  $\tilde{H}_x$  the orthogonal complement of  $D$  in  $H_x/H_{2x}$  and by  $\tilde{H}_{2x}$  the direct sum  $D \oplus (H_{2x}/H_{3x})$ .

We then write  $\sigma = (\sigma_0, \sigma_1, \sigma_2) \in H_x^\perp \times \tilde{H}_x \times \tilde{H}_{2x}$  and

$$|h_0(x)| = d \sqrt{h_d(\sigma_0, \sigma_0)}|_x = c_d \|\sigma_0\|,$$

where  $\|\sigma_0\| = \sqrt{\langle \sigma_0, \sigma_0 \rangle}$  and by Proposition 4,  $c_d d^{-\frac{n+2}{2}}$  remains bounded between positive constants as  $d$  grows to infinity. For  $i \in \{1, \dots, n - 1\}$ ,  $h_i$  linearly depends on  $\sigma_0$  and  $\sigma_1$ ; we write  $h_i(\sigma_0, \sigma_1)$  for this linear expression. The derivatives  $\partial h_I$  and  $\bar{\partial} h_J$  depend on  $\sigma_2$ . The expression  $\bar{\partial} h_I \wedge \partial h_J$  expands as a sum of  $3^k$  terms, some of which vanish since the forms  $\partial f_{\sigma,x}$  and  $\bar{\partial} f_{\sigma,x}$  can only appear once in the expression. Denote by  $h_{I\bar{J}}$  one of these  $3^k$  terms. It is a monomial of degree  $2k$  in  $\sigma_0, \sigma_1, \sigma_2$  and we denote by  $l_0$  the degree of  $\sigma_0$ , by  $l_1$  the degree of  $\sigma_1$  and by  $l_2 = 2k - l_0 - l_1$  the degree of  $\sigma_2$  in this monomial. Now, it is enough to bound from above by  $O((\log d)^2)$  the following integral, where  $I, J \subset \{1, \dots, n\}^k$  are given:

$$\begin{aligned} & \frac{1}{d^k} \int_{H_x^\perp \times \tilde{H}_x \times \tilde{H}_{2x}} \frac{|\log \sum_{i=0}^{n-1} |h_i(x)|^2|}{(\sum_{i=0}^{n-1} |h_i(x)|^2)^k} \|h_{I\bar{J}}(\sigma_0, \sigma_1, \sigma_2)\| d\mu(\sigma) \\ & \leq \frac{1}{c_d^{2k} d^k} \int_{H_x^\perp \times \tilde{H}_x \times \tilde{H}_{2x}} \frac{|\log(c_d^2 \|\sigma_0\|^2) + \log(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(\frac{\sigma_0}{\|\sigma_0\|}, \frac{\sigma_1}{\|\sigma_0\|})|^2)|}{(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(\frac{\sigma_0}{\|\sigma_0\|}, \frac{\sigma_1}{\|\sigma_0\|})|^2)^k \|\sigma_0\|^{2k}} \\ & \quad \times \|h_{I\bar{J}}(\frac{\sigma_0}{\|\sigma_0\|}, \sigma_1, \sigma_2)\| \|\sigma_0\|^{l_0} e^{-\|\sigma_0\|^2 - \|\sigma_1\|^2} d\sigma_0 d\sigma_1 d\mu(\sigma_2). \end{aligned}$$

We replace  $\frac{\sigma_0}{\|\sigma_0\|}$  by 1 in this integral, without loss of generality, since it remains bounded. Define  $\alpha_1 = \sigma_1/\|\sigma_0\|$ , so that  $d\alpha_1 = \frac{1}{\|\sigma_0\|^{2(n-1)}} d\sigma_1$ . We rewrite the integral as

$$\begin{aligned} & \frac{1}{c_d^{2k} d^k} \int_{H_x^\perp \times \tilde{H}_x \times \tilde{H}_{2x}} \frac{|\log(c_d^2 \|\sigma_0\|^2) + \log(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(1, \alpha_1)|^2)|}{(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(1, \alpha_1)|^2)^k} \\ & \quad \times \|h_{I\bar{J}}(1, \alpha_1, \sigma_2)\| \|\sigma_0\|^{2(n-1)-l_2} e^{-\|\sigma_0\|^2(1+\|\alpha_1\|^2)} d\sigma_0 d\alpha_1 d\mu(\sigma_2). \end{aligned}$$

Now set  $\beta_0 = \sigma_0 \sqrt{1 + \|\alpha_1\|^2}$  and  $\beta_1 = \alpha_1/\sqrt{d}$ , so that  $d\beta_0 d\beta_1 = \frac{1+\|\alpha_1\|^2}{d^{n-1}} d\sigma_0 d\alpha_1$ . The integral becomes

$$\begin{aligned} & \frac{1}{c_d^{2k} d^{1+\frac{l_0}{2}}} \\ & \times \int_{H_x^\perp \times \tilde{H}_x \times \tilde{H}_{2x}} \frac{|\log(c_d^2 \|\beta_0\|^2) - \log(1 + d \|\beta_1\|^2) + \log(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(1, \sqrt{d} \beta_1)|^2)|}{(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(1, \sqrt{d} \beta_1)|^2)^k (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_2}{2}}} \\ & \times \|h_{I\bar{J}}(1, \beta_1, \sigma_2)\| \|\beta_0\|^{2(n-1)-l_2} e^{-\|\beta_0\|^2} d\beta_0 d\beta_1 d\mu(\sigma_2). \end{aligned}$$

The only terms depending on  $\beta_0$  in this integral are  $\log\|\beta_0\|^2$ ,  $\|\beta_0\|^{2(n-1)-l_2}$  and  $e^{-\|\beta_0\|^2}$ . They can be extracted from it and integrated over  $H_x^\perp$  thanks to Fubini's theorem. The latter integral over  $H_x^\perp$  turns out to be bounded independently of  $d$ . As a consequence, it suffices to bound from above the following:

$$\frac{1}{c_d^{2k} d^{1+\frac{l_0}{2}}} \int_{\tilde{H}_x \times \tilde{H}_{2x}} \frac{|\log(c_d^2/d) - \log(\frac{1}{d} + \|\beta_1\|^2) + \log(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(1, \sqrt{d}\beta_1)|^2)|}{(1 + \sum_{i=1}^{n-1} \frac{1}{c_d^2} |h_i(1, \sqrt{d}\beta_1)|^2)^k (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_2}{2}}} \times \|h_{I\bar{J}}(1, \beta_1, \sigma_2)\| d\beta_1 d\mu(\sigma_2).$$

Note that by Proposition 4 and by definition of the functions  $h_i$ ,  $1 \leq i \leq n-1$ , the expressions  $\frac{1}{c_d} h_i(1, \sqrt{d}\beta_1)$  are affine in  $\beta_1$  with coefficients bounded independently of  $d$ . Indeed, only the first term  $\partial f_{\sigma,x}(v_i)$  of  $h_i = \partial f_{\sigma,x}(v_i) + d f_{\sigma,x} \partial g_x(v_i)$  depends on  $\beta_1$  and, by Proposition 4,

$$\|\partial f_{\sigma,x}\|_x^2 = \sqrt{h^d(\nabla\sigma_1, \nabla\sigma_1)}|_x$$

grows as  $\|\sigma_1\|$  times  $d^{\frac{n+1}{2}}$  while  $c_d$  grows as  $d^{\frac{n+2}{2}}$ . Let us denote by  $g_i(\beta_1)$  these expressions.

Likewise, the monomial  $h_{I\bar{J}}$  is the product of three monomials of degrees  $l_0$ ,  $l_1$  and  $l_2$  in  $\sigma_0$ ,  $\sigma_1$  and  $\sigma_2$  respectively. By Proposition 5, the coefficients of these monomials are  $O(d^{\frac{n+2}{2}l_0})$ ,  $O(d^{\frac{n+1}{2}l_1+l'_1})$  and  $O(d^{\frac{n+2}{2}l_2})$  respectively, where  $l'_1$  equals 0 if neither  $\partial f_{\sigma,x}$  nor  $\partial f_{\sigma,x}$  appears in the monomial  $h_{I\bar{J}}$ , equals 1 if one of these two forms appears and 2 if they both appear. As a consequence,  $\|h_{I\bar{J}}(1, \beta_1, \sigma_2)\|$  is bounded from above, up to a constant, by  $d^{(n+2)k+l'_1-l_1/2} \|\beta_1\|^{l_1} \|\sigma_2\|^{l_2}$ . Now, only the term in  $\|\sigma_2\|^{l_2}$  depends on  $\sigma_2$  in our integral. Again using Fubini's theorem, we may first integrate over  $\tilde{H}_{2x}$  equipped with the Gaussian measure  $d\mu(\sigma_2)$  to get an integral bounded independently of  $d$ . The upshot is that we just need to bound the integral

$$I = \frac{1}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \times \int_{\tilde{H}_x} \frac{|\log(c_d^2/d) - \log(\frac{1}{d} + \|\beta_1\|^2) + \log(1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2)|}{(1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2)^k (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_2}{2}}} \|\beta_1\|^{l_1} d\beta_1.$$

There is a compact subset  $Q$  of  $\tilde{H}_x$  independent of  $d$  and a constant  $C > 0$  independent of  $x$  and  $d$  such that

$$\forall \beta_1 \in Q, \quad 1 \leq 1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2 \leq C$$

and

$$\forall \beta_1 \in \tilde{H}_x \setminus Q, \quad 1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2 \geq \frac{1}{C} \|\beta_1\|^2,$$

since by Definition 4 the vector fields  $v_i$  remain uniformly linearly independent on the whole  $U$ .

Bounding from above the term  $\|\beta_1\|^{l_1}$  by  $(\frac{1}{d} + \|\beta_1\|^2)^{l_1/2}$ , we finally just have to estimate from above the integrals

$$I_1 = \frac{\log d}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \int_Q \frac{1}{(\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_1+l_2}{2}}} d\beta_1$$

and

$$I_2 = \frac{1}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \int_{\tilde{H}_x \setminus Q} \frac{\log d + \log(\|\beta_1\|^2) d \beta_1}{\|\beta_1\|^{2k} (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_1+l_2}{2}}},$$

since  $\log(1/d + \|\beta_1\|^2)$  over  $Q$  and  $\log(c_d/d)$  are  $O(\log d)$ . Note that  $l'_1 \leq \max(2, l_1)$ , so that the exponent  $1 + \frac{l_0+l_1}{2} - l'_1$  is never negative and vanishes if and only if  $l_0 = 0, l_1 = l'_1 = 2$  and thus  $l_2 = 2k - 2$ . There exists  $R > 0$  such that

$$I_1 \leq \frac{\text{Vol}(S^{2n-3}) \log d}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \int_{\frac{1}{d}}^R \frac{du}{u^{2-\frac{l_1+l_2}{2}}} du = O((\log d)^2),$$

where  $u = \frac{1}{d} + \|\beta_1\|^2$ . Likewise, there exists  $T > 0$  such that

$$\int_{\tilde{H}_x \setminus Q} \frac{\log d + \log \|\beta_1\|^2 d \beta_1}{\|\beta_1\|^{2k} (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_1+l_2}{2}}} \leq \text{Vol}(S^{2n-3}) \int_T^\infty \frac{\log d + \log u}{u^{2+l_0/2}} du.$$

This last integral is  $O(\log d)$ , which implies Proposition 5 when  $U$  is of type (r).

**2.4.2. Proof of Proposition 5 along the base and critical loci of  $p$ .** The compact  $K$  given by Proposition 5 is covered by a finite number of elements of the atlas  $\mathcal{U}$  adapted to  $(p, L)$ . Moreover, we may assume that such elements are all disjoint from the critical locus  $\text{Crit}(p)$  when  $n > 2$ . Proposition 5 was proved in Section 2.4.1 for elements  $U$  of type (r) of  $\mathcal{U}$ . Let us assume now that  $U$  is such an element of type (b) given by Definition 4 and that  $x \in K \cap U$ . The case where  $U$  is of type (c) when  $n = 2$  just follows along the same lines. The main part of the proof is similar to the one given in Section 2.4.1 and we have to bound from above the integral

$$I = \frac{1}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \times \int_{\tilde{H}_x} \frac{|\log(c_d^2/d) - \log(\frac{1}{d} + \|\beta_1\|^2) + \log(1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2)|}{(1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2)^k (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_2}{2}}} \|\beta_1\|^{l_1} d\beta_1.$$

However, the norm  $\|v_1\|$  of the vector  $v_1$  given by Definition 5 converges now to 0 when  $x$  approaches the base locus of  $p$ . Denote by  $\tilde{H}_x''$  the hyperplane of  $\tilde{H}_x$  consisting of the sections whose 1-jet at  $x$  vanishes against  $v_1$ , that is, the sections whose image under  $\text{eval}_{2x}$  vanishes against  $v_1$ , see Section 2.3.1. Denote then by  $\tilde{H}_x'$  the line orthogonal to  $\tilde{H}_x''$  in  $\tilde{H}_x$  and by  $\beta_1 = (\beta_1', \beta_1'')$  the coordinates on  $\tilde{H}_x = \tilde{H}_x' \times \tilde{H}_x''$ . This time there exists a compact subset  $Q = Q' \times Q''$  of  $\tilde{H}_x$ , independent of  $d$  and of  $x \in K \cap U$ , as well as a constant  $C > 0$  such that

$$\forall \beta_1 \in Q, \quad 1 \leq 1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2 \leq C$$

and

$$\forall \beta_1 \in \tilde{H}_x \setminus Q, \quad 1 + \sum_{i=1}^{n-1} |g_i(\beta_1)|^2 \geq \frac{1}{C} (1 + \|v_1(x)\|^2 \|\beta_1'\|^2 + \|\beta_1''\|^2).$$

The integral  $I$  over the compact  $Q$  is bounded from above by  $O((\log d)^2)$ , see Section 2.4.1. Only the second integral differs. In order to estimate the latter, let us bound from above  $\|\beta_1\|^{l_1}$  by  $(\frac{1}{d} + \|\beta_1\|^2)^{l_1/2}$ . We have to bound the integral

$$\frac{1}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \int_{\tilde{H}_x \setminus Q} \frac{\log d + \log \|\beta_1\|^2 d\beta_1}{(1 + \|v_1\|^2 \|\beta'_1\|^2 + \|\beta''_1\|^2)^k (\frac{1}{d} + \|\beta_1\|^2)^{n-\frac{l_1+l_2}{2}}}.$$

When  $n - \frac{l_1+l_2}{2} > 1$ , let us bound from above this integral by

$$\frac{1}{d^{1+\frac{l_0+l_1}{2}-l'_1}} \int_{\tilde{H}_x'' \setminus Q''} \frac{d\beta''_1}{\|\beta''_1\|^{2k}} \int_{\tilde{H}_x \setminus Q'} \frac{\log d + \log \|\beta_1\|^2 d\beta'_1}{(\|\beta_1\|^2)^{n-\frac{l_1+l_2}{2}}}.$$

There exists  $R > 0$  such that

$$\begin{aligned} \frac{1}{\pi} \int_{\tilde{H}_x' \setminus Q'} \frac{\log d + \log \|\beta_1\|^2 d\beta'_1}{(\|\beta_1\|^2)^{n-\frac{l_1+l_2}{2}}} &\leq \int_R^\infty \frac{\log d + \log(\|\beta''_1\|^2 + u) du}{(\|\beta''_1\|^2 + u)^{n-\frac{l_1+l_2}{2}}} \\ &= \left[ \frac{\log d + \log(\|\beta''_1\|^2 + u)}{(\frac{l_1+l_2}{2} - n + 1)(\|\beta''_1\|^2 + u)^{n-\frac{l_1+l_2}{2}-1}} \right]_R^\infty \\ &\quad + \int_R^\infty \frac{du}{(n - \frac{l_1+l_2}{2} - 1)(\|\beta''_1\|^2 + u)^{n-\frac{l_1+l_2}{2}}} \\ &= \frac{\log d + \log(\|\beta''_1\|^2 + R)}{(n - \frac{l_1+l_2}{2} - 1)(\|\beta''_1\|^2 + R)^{n-\frac{l_1+l_2}{2}-1}} \\ &\quad + \frac{1}{(n - \frac{l_1+l_2}{2} - 1)^2 (\|\beta''_1\|^2 + R)^{n-\frac{l_1+l_2}{2}-1}}. \end{aligned}$$

Hence, our integral gets bounded from above, up to a constant, by the integral

$$\int_{\tilde{H}_x'' \setminus Q''} \frac{\log d + \log(\|\beta''_1\|^2 + R)}{\|\beta''_1\|^{2(n-1)+l_0}} d\beta''_1,$$

which is itself  $O(\log d)$  since  $\tilde{H}_x''$  is of dimension  $n - 2$ . When  $n - \frac{l_1+l_2}{2} = 1$ , which implies that  $l_0 = 0$  and  $k = n - 1$ , we observe that

$$\begin{aligned} (1 + \|v_1\|^2 \|\beta'_1\|^2 + \|\beta''_1\|^2)^k &= (1 + \|v_1\|^2 \|\beta'_1\|^2 + \|\beta''_1\|^2)(1 + \|v_1\|^2 \|\beta'_1\|^2 + \|\beta''_1\|^2)^{k-1} \\ &\geq \|v_1\|^2 (1 + \|\beta_1\|^2)(1 + \|\beta''_1\|^2)^{k-1} \geq \|v_1\|^2 \|\beta_1\|^2 \|\beta''_1\|^{2(k-1)} \end{aligned}$$

as long as  $\|v_1\| \leq 1$ , which can be assumed. Our integral gets then bounded by

$$\frac{1}{d^{1+\frac{l_0+l_1}{2}-l'_1} \|v_1\|^2} \int_{\tilde{H}_x'' \setminus Q''} \frac{d\beta''_1}{\|\beta''_1\|^{2(k-1)}} \int_{\tilde{H}_x \setminus Q'} \frac{\log d + \log \|\beta_1\|^2 d\beta'_1}{\|\beta_1\|^4}$$

which is similar to the previous one. The latter is then bounded from above by  $O(\frac{\log d}{\|v_1\|^2})$ , implying the result.

**Remark 1.** As was pointed out to us by the referee, an alternative approach to our calculations would be, say in dimension two, to express the measure  $\nu_\sigma$  as the wedge product of the currents  $\partial\bar{\partial} \log(d^2|f_{\sigma,U}|^2)$  and  $\partial\bar{\partial} \log|\partial f_{\sigma,U}(v)|^2$ . To compute the expectation of this product, we may view the first (resp. second) function as a function in a variable  $z$  (resp.  $w$ ) and then set  $z = w$ . After exchanging the integrals over  $X$  and  $\mathbb{R}H^0(X; L^d)$ , we may then get both derivatives out of the integral and are led to compute the expectation of the product  $\log(d^2|f_{\sigma,U}|^2) \log|\partial f_{\sigma,U}(v)|^2$ . This becomes similar to what has been done in [1, 21, 22] and uses known asymptotics of the Bergman kernel near the diagonal.

### 3. Total Betti numbers of random real hypersurfaces

#### 3.1. Statement of the results.

**3.1.1. Expectation of the total Betti number of real hypersurfaces.** Let  $(X, c_X)$  be a smooth real projective manifold of positive dimension  $n$ , meaning that  $X$  is a smooth  $n$ -dimensional complex projective manifold equipped with an antiholomorphic involution  $c_X$ . Let  $\pi : (L, c_L) \rightarrow (X, c_X)$  be a real holomorphic ample line bundle, so that the antiholomorphic involutions satisfy  $\pi \circ c_L = c_X \circ \pi$ . For every  $d > 0$ , we denote by  $L^d$  the  $d$ -th tensor power of  $L$ , by  $\mathbb{R}H^0(X; L^d)$  the space of global real holomorphic sections of  $L^d$ , which are the sections  $\sigma \in H^0(X; L^d)$  satisfying  $\sigma \circ c_X = c_L \circ \sigma$ , and by  $\mathbb{R}\Delta_d = \Delta_d \cap \mathbb{R}H^0(X; L^d)$  the real discriminant locus.

For every section  $\sigma \in \mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\Delta_d$ ,  $C_\sigma = \sigma^{-1}(0)$  is a smooth real hypersurface of  $X$ . By Smith–Thom’s inequality (see Theorem 2), the total Betti number

$$b_*(\mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) = \sum_{i=0}^{n-1} \dim H_i(\mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z})$$

of its real locus is bounded from above by the total Betti number

$$b_*(C_\sigma; \mathbb{Z}/2\mathbb{Z}) = \sum_{i=0}^{2n-2} \dim H_i(C_\sigma; \mathbb{Z}/2\mathbb{Z})$$

of its complex locus. We deduce from Lemma 3 that

$$b_*(C_\sigma; \mathbb{Z}/2\mathbb{Z}) \underset{d \rightarrow \infty}{\sim} \left( \int_X c_1(L)^n \right) d^n.$$

What is the expectation of this real total Betti number? If we are not able to answer this question, we will estimate this number from above, see Theorems 4 and 5. Note that in dimension one, such an upper bound can be deduced from our recent work [7].

Let us first precise the measure of probability considered on  $\mathbb{R}H^0(X; L^d)$ . We proceed as in Section 2. We equip the holomorphic line bundle  $L$  with a real Hermitian metric  $h$  of positive curvature  $\omega \in \Omega^{(1,1)}(X, \mathbb{R})$ , real meaning that  $c_L^*h = h$ . As in Section 2.1, we denote by

$$dx = \frac{1}{\int_X \omega^n} \omega^n$$

the associated volume form of  $X$ , by  $\langle \cdot, \cdot \rangle$  the induced  $L^2$ -scalar product on  $\mathbb{R}H^0(X; L^d)$ , see (1), and by  $\mu_{\mathbb{R}}$  the associated Gaussian measure defined by the relation

$$(6) \quad \forall A \subset \mathbb{R}H^0(X; L^d), \quad \mu_{\mathbb{R}}(A) = \frac{1}{(\sqrt{\pi})^{N_d}} \int_A e^{-\|\sigma\|^2} d\sigma.$$

For every  $d > 0$ , we denote by

$$E_{\mathbb{R}}(b_*(\mathbb{R}C_{\sigma}; \mathbb{Z}/2\mathbb{Z})) = \int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\Delta_d} b_*(\mathbb{R}C_{\sigma}; \mathbb{Z}/2\mathbb{Z}) d\mu_{\mathbb{R}}(\sigma)$$

the expected total Betti number of real hypersurfaces linearly equivalent to  $L^d$ .

**Theorem 4.** *Let  $(X, c_X)$  be a smooth real projective manifold of dimension  $n$  greater than one equipped with a Hermitian real line bundle  $(L, c_L)$  of positive curvature. Then, the expected total Betti number  $E_{\mathbb{R}}(b_*(\mathbb{R}C_{\sigma}; \mathbb{Z}/2\mathbb{Z}))$  is  $o(d^n)$  and even  $O(d(\log d)^2)$  if  $n = 2$ .*

Note that the exact value of the expectation  $E_{\mathbb{R}}(b_*(\mathbb{R}C_{\sigma}; \mathbb{Z}/2\mathbb{Z}))$  is only known when  $X = \mathbb{C}P^1$ , see [6, 13, 23]. While writing this article, Peter Sarnak informed us that he and Igor Wigman can bound this expectation, using the same Gaussian measure, by  $O(d)$  when  $X = \mathbb{C}P^2$ . They suspect that it is equivalent to a constant times  $d$  when  $d$  grows to infinity. Such a guess was already made a couple of years ago by Christophe Raffalli, based on computer experiments. P. Sarnak and I. Wigman use a formula of Kac and Rice which is purely real and their method thus differs from ours (compare [5, 13, 16]).

It could be that this expectation is in fact equivalent to  $d^{\frac{n}{2}}$  times a constant as soon as the real locus of the manifold  $(X, c_X)$  is non-empty and in particular that the bound given by Theorem 4 can be improved by  $O(d^{\frac{n}{2}})$ . When  $(X, c_X)$  is a product of smooth real projective curves for instance, we can improve  $o(d^n)$  given by Theorem 4 by  $O(d^{\frac{n}{2}}(\log d)^n)$ , being much closer to an  $O(d^{\frac{n}{2}})$  bound, see Theorem 5 below. This  $d^{\frac{n}{2}}$  can be understood as the volume of a  $1/\sqrt{d}$  neighborhood of the real locus  $\mathbb{R}X$  in  $X$  for the volume form induced by the curvature of  $L^d$ , where  $1/\sqrt{d}$  is a fundamental scale in Kähler geometry and Hörmander's theory of peak sections. A peak section centered at  $x$  can be symmetrized to provide a real section having two peaks near  $x$  and  $c_X(x)$ , see Section 3.2. This phenomenon plays an important rôle in the proof of Theorem 4 and seems to be intimately related to the value of the expectation  $E_{\mathbb{R}}(b_*(\mathbb{R}C_{\sigma}; \mathbb{Z}/2\mathbb{Z}))$ .

**Theorem 5.** *Let  $(X, c_X)$  be the product of  $n > 1$  smooth real projective curves, equipped with a real Hermitian line bundle  $(L, c_L)$  of positive curvature. Then, the expected total Betti number  $E_{\mathbb{R}}(b_*(\mathbb{R}C_{\sigma}; \mathbb{Z}/2\mathbb{Z}))$  is  $O(d^{\frac{n}{2}}(\log d)^n)$ .*

Another very natural measure is the Fubini–Study measure arising from the  $L^2$ -scalar product obtained by integration over  $\mathbb{R}X$  only and not over the whole  $X$  as we do. With this measure, Fedor Nazarov and Mikhail Sodin have shown in [18] that in dimension two, the expected number of components of the vanishing locus of spherical harmonics of degree  $d$  is equivalent to  $bd^2$  as  $d$  grows to infinity, where  $b$  is some positive constant. Moreover, P. Sarnak and I. Wigman informed us that they can prove a similar asymptotics for polynomials in dimension two (in addition to their result mentioned above for the measure (6)).

The choice of the measure we made here and in our previous work [7] is the same as in [3] and [6] for example. This measure was actually motivated by Tian's asymptotically isometry theorem ([2, 25]). Indeed, a scalar product on  $\mathbb{R}H^0(X; L^d)$  is the same as a real embedding of  $X$  into  $\mathbb{C}P^{N_d-1}$ , modulo composition by an isometry of  $\mathbb{C}P^{N_d-1}$ . The scalar product turning this embedding into an isometry asymptotically (after rescaling) seemed to be a natural one to us.

Note finally that the initial result by M. Kac [13] used a third choice of inner product on  $\mathbb{R}H^0(X; L^d)$  in the case  $X = \mathbb{C}P^1$ , namely the one turning the canonical basis  $1, x, \dots, x^{N_d}$  into an orthonormal one. However, in general the space  $H^0(X; L^d)$  has no canonical basis so that this choice is hard to keep.

**3.1.2. Random real divisors and distribution of critical points.** Our proof of Theorem 4 is based on a real analogue of Theorem 3 that we formulate here, see Theorem 6. We use the notations introduced in Section 3.1.1 and equip  $X$  with a real Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$ , see Section 1. For every  $d > 0$ , denote by  $\mathbb{R}\tilde{\Delta}_d = \tilde{\Delta}_d \cap \mathbb{R}H^0(X; L^d)$  the union of  $\mathbb{R}\Delta_d$  with the set of sections  $\sigma \in \mathbb{R}H^0(X; L^d)$  such that either  $C_\sigma$  contains a critical point of  $p$ , or  $p|_{C_\sigma}$  is not Lefschetz, see Section 2.1. Denote, as in Section 2.1, by  $\mathcal{R}_\sigma$  the critical locus of the restriction  $p|_{C_\sigma}$ , where  $\sigma \in \mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d$ . Then, for every continuous function  $\chi : X \rightarrow \mathbb{R}$ , denote by

$$E_{\mathbb{R}}(\langle \nu_\sigma, \chi \rangle) = \frac{1}{\#\mathcal{R}_\sigma} \int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \left( \sum_{x \in \mathcal{R}_\sigma} \chi(x) \right) d\mu_{\mathbb{R}}(\sigma)$$

the expectation of the probability measure  $\nu_\sigma$  carried by the critical points of  $p|_{C_\sigma}$ , see Definition 3, computed with respect to the real Gaussian measure  $\mu_{\mathbb{R}}$  and evaluated against  $\chi$ .

**Theorem 6.** *Let  $(X, c_X)$  be a smooth real projective manifold of positive dimension  $n$  equipped with a real Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  of critical locus  $\text{Crit}(p)$ . Let  $(L, c_L) \rightarrow (X, c_X)$  be a real ample line bundle equipped with a real Hermitian metric of positive curvature. Let  $(\chi_d)_{d \in \mathbb{N}^*}$  be a sequence of elements of  $C^2(X, \mathbb{R})$  which converges to  $\chi$  in  $L^1(X, \mathbb{R})$  as  $d$  grows to infinity. Assume that*

$$\sup_{x \in \text{Supp}(\chi_d)} d(x, c_X(x)) > 2 \frac{\log d}{\sqrt{d}}.$$

*When  $n > 2$ , assume moreover that the distance between  $\text{Crit}(p)$  and the supports of  $\chi_d$ ,  $d > 0$ , is uniformly bounded from below by some positive constant. Then, the real expectation  $E_{\mathbb{R}}(\langle \nu_\sigma, \chi \rangle)$  converges to the integral  $\int_X \chi dx$  as  $d$  grows to infinity. More precisely,*

$$E_{\mathbb{R}}(\langle \nu_\sigma, \chi \rangle) = \int_X \chi dx + O\left(\frac{(\log d)^2}{d} \|\partial\bar{\partial}\chi_d\|_{L^1}\right) + O\left(\frac{1}{d} \|\chi_d\|_{L^1}\right).$$

In this theorem,  $\text{Supp}(\chi_d)$  denotes the support of  $\chi_d$  and  $d(x, c_X(x))$  the distance between the points  $x$  and  $c_X(x)$  for the Kähler metric induced by the curvature  $\omega$  of  $L$ . Note that we do not control the concentration of the measure  $\nu_\sigma$  inside the  $2 \log d / \sqrt{d}$ -tube around the real locus. Some concentration phenomena are studied in [15, 16].

**3.2. Real peak sections and evaluation of two-jets of sections.** Let  $(X, c_X)$  be a smooth real projective manifold of positive dimension  $n$  and  $(L, c_L) \rightarrow (X, c_X)$  be a real holomorphic ample line bundle equipped with a real Hermitian metric  $h$  of positive curvature.

**Definition 8.** A real peak section of the ample real holomorphic line bundle  $(L, c_L)$  over the real projective manifold  $(X, c_X)$  is a section which reads  $\frac{\sigma + c^*\sigma}{\|\sigma + c^*\sigma\|}$ , where  $\sigma$  is a peak section of Hörmander given by Definition 7 and  $c^*\sigma = c_{L^d} \circ \sigma \circ c_X$ .

Recall from Lemma 4 that the  $L^2$ -norm of a peak section concentrates in a  $\log d / \sqrt{d}$  neighborhood of a point  $x \in X$ . When  $x$  is real, the real peak section  $\frac{\sigma + c^*\sigma}{\|\sigma + c^*\sigma\|}$  looks like a section of Hörmander given by Definition 7. When the distance between  $x$  and  $c_X(x)$  is bigger than  $\log d / \sqrt{d}$ , more or less half of the  $L^2$ -norm of this real section concentrates in a neighborhood of  $x$  and another half in a neighborhood of  $c_X(x)$ . Such a real section has thus two peaks near  $x$  and  $c_X(x)$ . When  $d(x, c_X(x)) < 2 \log d / \sqrt{d}$ , these two peaks interfere, interpolating the extreme cases just discussed. We are now interested in the case  $d(x, c_X(x)) > 2 \log d / \sqrt{d}$ , where we can establish a real analogue of Proposition 4.

**Lemma 5.** *Let  $(L, c_L)$  be a real holomorphic Hermitian line bundle of positive curvature over a smooth real projective manifold  $(X, c_X)$  of positive dimension  $n$ . Let  $(x_d)_{d \in \mathbb{N}^*}$  be a sequence of points such that  $d(x_d, c_X(x_d)) > 2 \log d / \sqrt{d}$  and let  $(\sigma_d)_{d \in \mathbb{N}^*}$  be an associated sequence of sections given by Lemma 4 with  $p' = 2$ . Then, the Hermitian product  $\langle \sigma_d, c^*\sigma_d \rangle$  is  $O(\frac{1}{d^2})$ , so that the norm  $\|\sigma_d + c^*\sigma_d\|$  equals  $\sqrt{2}\|\sigma_d\|(1 + O(1/d^2))$ .*

*Proof.* By definition and then Cauchy–Schwarz’s inequality,

$$\begin{aligned} \langle \sigma_d, c^*\sigma_d \rangle &= \int_X h^d(\sigma_d, c^*\sigma_d) dx \\ &= \int_{B(x, \frac{\log d}{\sqrt{d}})} h^d(\sigma_d, c^*\sigma_d) dx + \int_{X \setminus B(x, \frac{\log d}{\sqrt{d}})} h^d(\sigma_d, c^*\sigma_d) dx \\ &\leq \left( \int_{B(x, \frac{\log d}{\sqrt{d}})} h^d(\sigma_d, \sigma_d) dx \right)^{1/2} \left( \int_{B(x, \frac{\log d}{\sqrt{d}})} h^d(c^*\sigma_d, c^*\sigma_d) dx \right)^{1/2} \\ &\quad + \left( \int_{X \setminus B(x, \frac{\log d}{\sqrt{d}})} h^d(\sigma_d, \sigma_d) dx \right)^{1/2} \left( \int_{X \setminus B(x, \frac{\log d}{\sqrt{d}})} h^d(c^*\sigma_d, c^*\sigma_d) dx \right)^{1/2} \end{aligned}$$

We deduce that

$$\begin{aligned} \langle \sigma_d, c^*\sigma_d \rangle &\leq \|\sigma_d\| \left[ \left( \int_{X \setminus B(x, \frac{\log d}{\sqrt{d}})} h^d(\sigma_d, \sigma_d) dx \right)^{1/2} \right. \\ &\quad \left. + \left( \int_{B(x, \frac{\log d}{\sqrt{d}})} h^d(c^*\sigma_d, c^*\sigma_d) dx \right)^{1/2} \right] \end{aligned}$$

By assumption, the balls  $B(x, \log d / \sqrt{d})$  and  $B(c_X(x), \log d / \sqrt{d})$  are disjoint, so that by Lemma 4, these two last terms are  $O(1/d^2)\|\sigma_d\|$ . Hence,

$$\langle \sigma_d + c^*\sigma_d, \sigma_d + c^*\sigma_d \rangle = 2\|\sigma_d\|^2 + 2\Re \langle \sigma_d, c^*\sigma_d \rangle = 2\|\sigma_d\|^2 + O(1/d^2)\|\sigma_d\|^2,$$

so that  $\|\sigma_d + c^*\sigma_d\| = \sqrt{2}\|\sigma_d\|(1 + O(1/d^2))$ . □

Set

$$\begin{aligned}\mathbb{R}H_x &= H_x \cap \mathbb{R}H^0(X; L^d), \\ \mathbb{R}H_{2x} &= H_{2x} \cap \mathbb{R}H^0(X; L^d), \\ \mathbb{R}H_{3x} &= H_{3x} \cap \mathbb{R}H^0(X; L^d),\end{aligned}$$

where  $H_x$ ,  $H_{2x}$  and  $H_{3x}$  have been introduced in Section 2.3.1. Likewise, with a slight abuse of notation, denote by  $\text{eval}_x$ ,  $\text{eval}_{2x}$  and  $\text{eval}_{3x}$  the restrictions of the evaluation maps to the spaces  $\mathbb{R}H^0(X; L^d)/\mathbb{R}H_x$ ,  $\mathbb{R}H_x/\mathbb{R}H_{2x}$  and  $\mathbb{R}H_{2x}/\mathbb{R}H_{3x}$  respectively, so that

$$\begin{aligned}\text{eval}_x : \sigma \in \mathbb{R}H^0(X; L^d)/\mathbb{R}H_x &\mapsto \sigma(x) \in L_x^d, \\ \text{eval}_{2x} : \sigma \in \mathbb{R}H_x/\mathbb{R}H_{2x} &\mapsto \nabla\sigma|_x \in T_x^*X \otimes L_x^d, \\ \text{eval}_{3x} : \sigma \in \mathbb{R}H_{2x}/\mathbb{R}H_{3x} &\mapsto \nabla^2\sigma|_x \in \text{Sym}^2(T_x^*X) \otimes L_x^d.\end{aligned}$$

The following proposition is a real analogue of Proposition 4.

**Proposition 6.** *Let  $(L, c_L)$  be a real holomorphic Hermitian line bundle of positive curvature over a smooth complex projective manifold  $X$  of positive dimension  $n$ . Let  $(x_d)_{d \in \mathbb{N}^*}$  be a sequence of points in  $X$  such that  $d(x_d, c_X(x_d)) > 2 \log d / \sqrt{d}$ . Then, the maps  $d^{-\frac{n}{2}} \text{eval}_{x_d}$ ,  $d^{-\frac{n+1}{2}} \text{eval}_{2x_d}$  and  $d^{-\frac{n+2}{2}} \text{eval}_{3x_d}$  as well as their inverses have bounded norms and determinants, as long as  $d$  is large enough.*

Note that the evaluation maps  $\text{eval}_{x_d}$ ,  $\text{eval}_{2x_d}$  and  $\text{eval}_{3x_d}$  of Proposition 6 are only  $\mathbb{R}$ -linear.

*Proof.* The proof is analogous to the one of Lemma 5. Let  $\sigma_0 = \frac{\sigma + c^*\sigma}{\|\sigma + c^*\sigma\|}$  be a real peak section given by Definition 8, where  $\sigma$  is a peak section given by  $x_d$  and Definition 7. From Lemma 5, we know that  $\|\sigma + c^*\sigma\|$  equals  $\sqrt{2} + O(1/d^2)$ . Moreover,

$$2 \frac{\log d}{\sqrt{d}} < d(x_d, c_X(x_d))$$

and by Lemma 4, the  $L^2$ -norm of  $c^*\sigma$  in a neighborhood of  $x_d$  is  $O(1/d^2)$ . From the mean inequality, we deduce the bound (see, e.g., [11, Theorem 4.2.13])  $h^d(c^*\sigma, c^*\sigma)|_{x_d} = O(1/d^2)$ , so that, by [25, Lemma 2.1],  $d^{-n}h^d(\sigma_0, \sigma_0)|_{x_d}$  converges to a non-negative constant as  $d$  grows to infinity. Denote by  $\sigma_0^{\mathbb{R}H_x}$  the orthogonal projection of  $\sigma_0$  onto  $\mathbb{R}H_x$ . We proceed as in the proof of Proposition 4 to get

$$\left\langle \sigma, \frac{\sigma_0^{\mathbb{R}H_x}}{\|\sigma_0^{\mathbb{R}H_x}\|} \right\rangle = O\left(\frac{1}{d}\right)$$

and likewise

$$\left\langle c^*\sigma, \frac{\sigma_0^{\mathbb{R}H_x}}{\|\sigma_0^{\mathbb{R}H_x}\|} \right\rangle = O\left(\frac{1}{d}\right),$$

since  $c$  is an isometry for the  $L^2$ -Hermitian product. Hence,

$$\left\langle \sigma_0, \frac{\sigma_0^{\mathbb{R}H_x}}{\|\sigma_0^{\mathbb{R}H_x}\|} \right\rangle = O\left(\frac{1}{d}\right).$$

Writing

$$\sigma_0^\perp = \frac{\sigma_0 - \sigma_0^{\mathbb{R}H_x}}{\|\sigma_0 - \sigma_0^{\mathbb{R}H_x}\|},$$

we deduce as in the proof of Proposition 4 that  $d^{-n}h^d(\sigma_0^\perp, \sigma_0^\perp)|_{x_d}$  converges to a positive constant as  $d$  grows to infinity. Replacing  $\sigma$  by  $i\sigma$ , we define

$$\tilde{\sigma}_0 = \frac{i\sigma_0 + c^*(i\sigma_0)}{\|i\sigma_0 + c^*(i\sigma_0)\|}$$

and check in the same way that  $d^{-n}h^d(\tilde{\sigma}_0^\perp, \tilde{\sigma}_0^\perp)|_{x_d}$  converges to a positive constant as  $d$  grows to infinity.

But the quotient  $\frac{\tilde{\sigma}_0(x_d)}{\sigma_0(x_d)}$  and thus  $\frac{\tilde{\sigma}_0^\perp(x_d)}{\sigma_0^\perp(x_d)}$  converge to  $i$  as  $d$  grows to infinity. Likewise,

$$\langle \sigma_0, \tilde{\sigma}_0 \rangle = \frac{\langle \sigma + c^*\sigma, i\sigma - ic^*\sigma \rangle}{\|\sigma + c^*\sigma\| \|i\sigma - ic^*\sigma\|} = \frac{2\Re(i\langle \sigma, c^*\sigma \rangle)}{\|\sigma + c^*\sigma\| \|i\sigma - ic^*\sigma\|} = O\left(\frac{1}{d}\right),$$

so that  $\langle \sigma_0^\perp, \tilde{\sigma}_0^\perp \rangle = O\left(\frac{1}{d}\right)$  and  $d^{-\frac{n}{2}} \text{eval}_x$  as well as its inverse have bounded norms and determinants when  $d$  grows to infinity. The remaining cases are obtained by similar modifications of the proof of Corollary 1.  $\square$

### 3.3. Proof of the main results.

**3.3.1. Proof of Theorem 6.** The proof goes along the same lines as the one of Theorem 3. We begin with the following analogue of Corollary 1.

**Corollary 2.** *Under the hypotheses of Proposition 3, we assume moreover that the manifold  $X$  and Hermitian bundle  $L$  are real. Then,*

$$E_{\mathbb{R}}(\langle v_\sigma, \chi \rangle) = \frac{1}{\#\mathcal{R}_\sigma} d^n \int_X \chi \omega^n + \frac{1}{\#\mathcal{R}_\sigma} \sum_{k=0}^{n-1} \left(\frac{i}{2\pi}\right)^{n-k} d^k \int_X \partial\bar{\partial}\chi \wedge \omega^k \\ \wedge \int_{\mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \lambda_x (\partial\bar{\partial}\lambda_x)^{n-1-k} d\mu(\sigma),$$

where

$$\lambda_x = \log\left(d^2 |f_{\sigma,x}|^2 + \sum_{i=1}^{n-1} |\partial f_{\sigma,x}(v_i) + f_{\sigma,x} \partial g_x(v_i)|^2\right).$$

*Proof.* The result follows after integration over  $\mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d$  of the relation given by Proposition 3.  $\square$

*Proof of Theorem 6.* We apply Corollary 2 to the functions  $\chi_d$ ,  $d > 0$ , and use the notations of this corollary. The first term in the right-hand side of the formula given by this corollary is

$$\int_X \chi_d dx + O\left(\frac{1}{d} \|\chi_d\|_{L^1}\right)$$

as follows from Proposition 2. It is thus enough to prove that for  $k \in \{0, \dots, n - 1\}$  the integral

$$\int_{\mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^k d\mu(\sigma)$$

is uniformly bounded on the support of  $\partial\bar{\partial}\chi_d$  by  $O(d^k(\log d)^2/\|v_1\|^2)$  since  $1/\|v_1\|^2$  is integrable over  $X$ . The space  $\mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d$  is equipped with its Gaussian measure, but the integrand only depends on the two-jets of sections  $\sigma \in \mathbb{R}H^0(X;L^d)$  at the point  $x \in X \setminus (\mathbb{R}X \cup \text{Crit}(p))$ . Since the Gaussian measure is a product measure, writing  $\mathbb{R}H^0(X;L^d)$  as the product of the space  $\mathbb{R}H_{3x}$  introduced in Section 3.2 with its orthogonal complement  $\mathbb{R}H_{3x}^\perp$ , we deduce that it is enough to prove this uniform bound for the integral

$$\int_{\mathbb{R}H_{3x}^\perp \setminus \mathbb{R}\tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^k d\mu(\sigma).$$

By Proposition 6, the evaluation maps  $\text{eval}_x$ ,  $\text{eval}_{2x}$  and  $\text{eval}_{3x}$  of jets up to order two at  $x$  provide an isomorphism between  $\mathbb{R}H_{3x}^\perp$  and  $L_x^d \oplus (T_x^*X \otimes L_x^d) \oplus (\text{Sym}^2(T_x^*X) \otimes L_x^d)$ . In particular, this implies that the space  $\mathbb{R}H_{3x}^\perp$  is also a complement of the space  $H_{3x}$  introduced in Section 2.3.1. By Proposition 4, these evaluation maps factor through an isomorphism  $I_d^{-1}$  between  $\mathbb{R}H_{3x}^\perp$  and the orthogonal complement  $H_{3x}^\perp$  of  $H_{3x}$  in  $H^0(X;L^d)$ . By Proposition 4 and Proposition 6, the Jacobian of  $I_d$  is bounded independently of  $d$ , while the Gaussian measure  $\mu_{\mathbb{R}}$  is bounded from above on  $\mathbb{R}H_{3x}^\perp$  by the measure  $(I_d)_*\mu$  up to a positive dilation of the norm, independent of  $d$ . After a change of variables given by this isomorphism  $I_d$  and after the one given by the dilation, it suffices to prove that the integral

$$\int_{H^0(X;L^d) \setminus \tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^k d\mu(\sigma)$$

is uniformly bounded on the support of  $\chi_d$  by  $O(d^k(\log d)^2/\|v_1\|^2)$ . The latter follows from Proposition 5. □

**3.3.2. Proof of Theorem 4.** Equip  $(X, c_X)$  with a real Lefschetz pencil  $p : X \dashrightarrow \mathbb{C}P^1$  and denote by  $F$  a regular fiber of  $p$ . For every section  $\sigma \in \mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d$ , the restriction of  $p$  to  $\mathbb{R}C_\sigma$  satisfies the hypotheses of Lemma 1, so that

$$b_*(\mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) \leq b_*(\mathbb{R}F \cap \mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) + \#\text{Crit}(p|_{\mathbb{R}C_\sigma}).$$

By Smith–Thom’s inequality, see Theorem 2,

$$b_*(\mathbb{R}F \cap \mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) \leq b_*(F \cap C_\sigma; \mathbb{Z}/2\mathbb{Z}),$$

while from Lemma 3 applied to  $L|_F$  we deduce that  $b_*(F \cap C_\sigma; \mathbb{Z}/2\mathbb{Z})$  is  $O(d^{n-1})$ . As a consequence, we have to prove that the expectation of  $\#\text{Crit}(p|_{\mathbb{R}C_\sigma})$  is  $o(d^n)$  and even  $O(d(\log d)^2)$  when  $n = 2$ .

Let us identify a neighborhood  $V$  of  $\mathbb{R}X$  in  $X$  with the cotangent bundle of  $\mathbb{R}X$ . We can assume that  $V \setminus \mathbb{R}X$  does not contain any critical point of  $p$ . Let  $\chi : X \rightarrow [0, 1]$  be a function of class  $C^2$  satisfying  $\chi = 1$  outside of a compact subset of  $V$  and  $\chi = 0$  in a neighborhood of  $\mathbb{R}X$ . For every  $d > 0$ , let  $\chi_d : X \rightarrow [0, 1]$  be the function which equals one outside of  $V$  and whose restriction to  $V$  can be written in local coordinates as

$$(q, p) \in V \simeq T^*\mathbb{R}X \mapsto \chi\left(q, \frac{\sqrt{d}}{\log d}p\right) \in [0, 1],$$

where  $q$  is the coordinate along  $\mathbb{R}X$  and  $p$  the coordinate along the fibers of  $T^*\mathbb{R}X$ . This sequence  $(\chi_d)_{d>0}$  converges to the constant function 1 in  $L^1(X, \mathbb{R})$  as  $d$  grows to infinity, while the norm  $\|\partial\bar{\partial}\chi_d\|_{L^1(X, \mathbb{R})}$  is  $O((\log d/\sqrt{d})^{n-2})$ . Moreover, for every  $x \in \text{Supp}(\chi_d)$ , we have  $d(x, c_X(x)) > 2 \log d/\sqrt{d}$ , so that when  $n = 2$ , Theorem 6 applies. From Proposition 2, we thus deduce that

$$\int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \left( \sum_{x \in \mathcal{R}_\sigma} \chi_d(x) \right) d\mu_{\mathbb{R}}(\sigma) = \#\mathcal{R}_\sigma + O(d(\log d)^2).$$

Moreover, for every  $\sigma \in \mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d$  and every  $d > 0$ , we have

$$\#\text{Crit}(p|_{\mathbb{R}C_\sigma}) \leq \#\mathcal{R}_\sigma - \sum_{x \in \mathcal{R}_\sigma} \chi_d(x).$$

After integration we get

$$\int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \#\text{Crit}(p|_{\mathbb{R}C_\sigma}) d\mu_{\mathbb{R}}(\sigma) = O(d(\log d)^2),$$

and hence the result for  $n = 2$ .

When  $n > 2$ , we apply Theorem 6 to the function  $\chi$  and deduce that

$$\int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \left( \sum_{x \in \mathcal{R}_\sigma} \chi(x) \right) d\mu_{\mathbb{R}}(\sigma) = \left( \int_X \chi dx \right) \#\mathcal{R}_\sigma + O(d^{n-1}(\log d)^2).$$

The expectation of the number of real critical points satisfies now the bound

$$\int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \#\text{Crit}(p|_{\mathbb{R}C_\sigma}) d\mu_{\mathbb{R}}(\sigma) \leq \left( 1 - \int_X \chi dx \right) \#\mathcal{R}_\sigma + O(d^{n-1}(\log d)^2).$$

Changing the function  $\chi$  if necessary, the difference  $(1 - \int_X \chi dx)$  can be made as small as we want. We thus deduce from Proposition 2 that

$$d^{-n} \int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \#\text{Crit}(p|_{\mathbb{R}C_\sigma}) d\mu_{\mathbb{R}}(\sigma)$$

converges to zero as  $d$  grows to infinity, which completes the proof of Theorem 4.

**3.3.3. Proof of Theorem 5.** We will prove Theorem 5 by induction on the dimension  $n$  of  $(X, c_X)$ . When  $n = 2$ , Theorem 5 is a consequence of Theorem 4. Let us assume now that  $n > 2$  and that  $(X, c_X)$  is the product of a real curve  $(\Sigma, c_\Sigma)$  by a product of curves  $(F, c_F)$  of dimension  $n - 1$ . We denote by  $p : (X, c_X) \rightarrow (\Sigma, c_\Sigma)$  the projection onto the first factor. Again, for every section  $\sigma \in \mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d$ , the restriction of  $p$  to  $\mathbb{R}C_\sigma$  satisfies the hypotheses of Lemma 1, so that

$$b_*(\mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) \leq 4b_*(\mathbb{R}F \cap \mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) + \#\text{Crit}(p|_{\mathbb{R}C_\sigma}).$$

Let us bound from above each term of the right-hand side by  $O(d^{\frac{n}{2}}(\log d)^n)$ .

Every connected component  $R$  of  $\mathbb{R}X$  has a neighborhood in  $X$  biholomorphic to a product of annuli in  $\mathbb{C}$  and thus satisfies the conditions of Definition 6. Indeed, every complex

annulus has a non-vanishing holomorphic vector field. By product, every component of  $\mathbb{R}F$  has a neighborhood in  $F$  trivialized by  $n - 1$  vector fields  $v_1, \dots, v_{n-1}$ . We deduce a trivialization of  $\ker dp = TF$  in the neighborhood of  $R$ . This open neighborhood can be completed into an atlas adapted to  $p$  with open sets disjoint from  $\mathbb{R}X$ . For every  $d > 0$ , set  $\theta_d = 1 - \chi_d$ , where  $(\chi_d)_{d>0}$  is the sequence of functions introduced in Section 3.3.2. Corollary 2 applies to  $\theta_d$  whose  $L^1$ -norm is  $O((\log d/\sqrt{d})^n)$  whereas  $\|\partial\bar{\partial}\theta_d\|_{L^1(X,\mathbb{R})}$  is  $O((\log d/\sqrt{d})^{n-2})$ , so that the conclusions of Theorem 6 also hold for  $\theta_d$ . As a consequence,

$$\int_{\mathbb{R}H^0(X;L^d)\setminus\mathbb{R}\tilde{\Delta}_d} \left( \sum_{x\in\mathcal{R}_\sigma} \theta_d(x) \right) d\mu_{\mathbb{R}}(\sigma) = O(d^{\frac{n}{2}}(\log d)^n).$$

Since for every  $\sigma \in \mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d$  and every  $d > 0$ ,

$$\#\text{Crit}(p|_{\mathbb{R}C_\sigma}) \leq \sum_{x\in\mathcal{R}_\sigma} \theta_d(x),$$

we deduce that  $\#\text{Crit}(p|_{\mathbb{R}C_\sigma})$  is  $O(d^{\frac{n}{2}}(\log d)^n)$ .

It remains to prove that the same holds for the integral

$$\int_{\mathbb{R}H^0(X;L^d)\setminus\mathbb{R}\tilde{\Delta}_d} b_*(\mathbb{R}F \cap \mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) d\mu_{\mathbb{R}}(\sigma).$$

If the space of integration was the space of real sections of the restriction  $L|_F$ , this would follow from Theorem 5 in dimension  $n - 1$ . We will thus reduce the space of integration to this one. Let us write  $(F, c_F) = (\Sigma_2, c_{\Sigma_2}) \times (Y, c_Y)$  where  $(\Sigma_2, c_{\Sigma_2})$  is a smooth real curve and  $(Y, c_Y)$  a  $(n - 2)$ -dimensional product of curves. From Lemma 1, for every  $\sigma \in \mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d$ , we get

$$b_*(\mathbb{R}F \cap \mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) \leq 4b_*(\mathbb{R}Y \cap \mathbb{R}C_\sigma; \mathbb{Z}/2\mathbb{Z}) + \#\text{Crit}(p_2|_{\mathbb{R}F \cap \mathbb{R}C_\sigma}),$$

where  $p_2 : F \rightarrow \Sigma_2$  is the projection onto the first factor. Denote, with a slight abuse of notation, by  $\theta_d$  the restriction of  $\theta_d$  to  $F$ . We have

$$\begin{aligned} \int_{\mathbb{R}H^0(X;L^d)\setminus\mathbb{R}\tilde{\Delta}_d} \#\text{Crit}(p_2|_{\mathbb{R}F \cap \mathbb{R}C_\sigma}) d\mu_{\mathbb{R}}(\sigma) \\ \leq \int_{\mathbb{R}H^0(X;L^d)\setminus\mathbb{R}\tilde{\Delta}_d} \left( \sum_{x\in\text{Crit}(p|_{F \cap C_\sigma})} \theta_d(x) \right) d\mu_{\mathbb{R}}(\sigma). \end{aligned}$$

After integration over  $\mathbb{R}H^0(X;L^d) \setminus \mathbb{R}\tilde{\Delta}_d$  of the relation given by Proposition 3 applied to  $F$  and  $v_1, \dots, v_{n-2}$ , we deduce that

$$\begin{aligned} \int_{\mathbb{R}H^0(X;L^d)\setminus\mathbb{R}\tilde{\Delta}_d} \left( \sum_{x\in\text{Crit}(p|_{F \cap C_\sigma})} \theta_d(x) \right) d\mu_{\mathbb{R}}(\sigma) \leq d^{n-1} \int_F \theta_d \omega^{n-1} \\ + \sum_{k=0}^{n-2} \left( \frac{i}{2\pi} \right)^{n-k-1} d^k \int_F \partial\bar{\partial}\theta_d \wedge \omega^k \wedge \int_{\mathbb{R}H^0(X;L^d)\setminus\mathbb{R}\tilde{\Delta}_d} \lambda_x (\partial\bar{\partial}\lambda_x)^{n-2-k} d\mu_{\mathbb{R}}(\sigma), \end{aligned}$$

where

$$\lambda_x = \log \left( d^2 |f_{\sigma,x}|^2 + \sum_{i=1}^{n-2} |\partial f_{\sigma,x}(v_i) + f_{\sigma,x} \partial g_x(v_i)|^2 \right),$$

compare with Corollary 1. We then proceed as in the proof of Theorem 4, noting that the latter integral only depends on the two-jets of sections  $\sigma \in \mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d$ . Let us thus decompose  $\mathbb{R}H^0(X; L^d)$  as the product of the subspace  $\mathbb{R}\hat{H}_{3x}$  of sections whose two-jet at  $x \in F$  of their restriction to  $F$  vanishes, with its orthogonal complement  $\mathbb{R}\hat{H}_{3x}^\perp$ . We get

$$\int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^{n-2-k} d\mu_{\mathbb{R}}(\sigma) = \int_{\mathbb{R}\hat{H}_{3x}^\perp \setminus \tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^{n-2-k} d\mu_{\mathbb{R}}(\sigma)$$

for every  $k$ . Likewise, the space of sections  $\mathbb{R}H^0(F; L^d|_F)$  of the restriction  $L|_F$  decomposes into the product  $\mathbb{R}H_{3x} \times \mathbb{R}H_{3x}^\perp$ . From Proposition 6, the map

$$\sigma \in \mathbb{R}\hat{H}_{3x}^\perp \mapsto \sigma_F = \frac{1}{\sqrt{d}} \text{pr}^\perp(\sigma|_F) \in \mathbb{R}H_{3x}^\perp,$$

which is a composition of the restriction map to  $F$ , a contraction by  $\sqrt{d}$  and the projection  $\text{pr}^\perp$  onto  $\mathbb{R}H_{3x}^\perp$ , has bounded norm and determinant. The same holds for its inverse. This map in fact asymptotically coincides with the isomorphism

$$\mathbb{R}\hat{H}_{3x}^\perp \rightarrow L_x^d \oplus T_x^*F \otimes L_x^d \oplus \text{Sym}^2(T_x^*F) \otimes L_x^d$$

given by the evaluation maps, composed with the inverse of the isomorphism

$$\mathbb{R}H_{3x}^\perp \rightarrow L_x^d \oplus T_x^*F \otimes L_x^d \oplus \text{Sym}^2(T_x^*F) \otimes L_x^d$$

given by the evaluation maps, see Proposition 5. From [25, Lemma 3.1], the restricted section  $\sigma|_F$  is indeed asymptotically orthogonal to  $\mathbb{R}H_{3x}$ . Hence, there is a constant  $C > 0$  such that

$$(7) \quad \left\| \int_{\mathbb{R}\hat{H}_{3x}^\perp \setminus \tilde{\Delta}_d} \lambda_x(\partial\bar{\partial}\lambda_x)^{n-2-k} d\mu_{\mathbb{R}}(\sigma) \right\| \leq C \left\| \int_{\mathbb{R}H_{3x}^\perp \setminus \tilde{\Delta}_d} (\log d + \lambda_x)(\partial\bar{\partial}\lambda_x)^{n-2-k} d\mu_{\mathbb{R}}(\sigma_F) \right\|,$$

since  $f_{\sqrt{d}\sigma_F, x} = \sqrt{d} f_{\sigma_F, x}$ , so that asymptotically,  $\lambda_x(\sigma_F) = \log d + \lambda_x(\sigma)|_F$ . Proceeding as in the proof of Theorem 4, the right-hand side of (7) is  $O(d^{n-2-k}(\log d)^3)$ . We deduce that

$$\begin{aligned} \int_{\mathbb{R}H^0(X; L^d) \setminus \mathbb{R}\tilde{\Delta}_d} \#\text{Crit}(p_2|_{\mathbb{R}F \cap \mathbb{R}C_\sigma}) &= O(d^{n-1} \|\theta_d\|_{L^1(F, \mathbb{R})}) \\ &\quad + O(d^{n-2}(\log d)^3 \|\partial\bar{\partial}\theta_d\|_{L^1(F, \mathbb{R})}) \\ &= O(d^{\frac{n-1}{2}} (\log d)^n). \end{aligned}$$

Hence, the result follows by recurrence over the dimension  $n$ .

**3.4. Final remarks.** Several technical issues prevent us from improving Theorem 4 with an  $O(d^{\frac{n}{2}}(\log d)^n)$  bound in general or an  $O(d^{\frac{n}{2}})$  bound.

1. First of all, Theorem 6, which is central in the proof of Theorem 4, contains an  $O(\frac{1}{d} \|\chi\|_{L^1})$  term. It comes from the fact that the number  $\#\mathcal{R}_\sigma$  of critical points of our

Lefschetz pencil does not coincide with the leading term  $(\int_X c_1(L)^n)d^n$  given by Poincaré–Martinelli’s formula, but is rather a polynomial of degree  $n$  given by Proposition 2 having  $(\int_X c_1(L)^n)d^n$  as leading term. It would be of interest to identify every monomial of the latter with a term of Poincaré–Martinelli’s formula or of any analytic formula. Anyway, because of this  $O(\frac{1}{d}\|\chi\|_{L^1})$  term in Theorem 6, we cannot use the function  $\chi$  with support disjoint from  $\mathbb{R}X$  which we used in the proof of Theorem 4. We rather have to use the function  $\theta$  with support in a neighborhood of  $\mathbb{R}X$  and which equals 1 on  $\mathbb{R}X$  which we used in the proof of Theorem 5.

2. The use of Poincaré–Martinelli’s formula with local trivializations  $(v_1, \dots, v_{n-1})$  forces us to choose an atlas  $\mathcal{U}$  on  $X$  and an associated partition of unity  $(\rho_U)_{U \in \mathcal{U}}$ . As a consequence, even if a function  $\theta$  on  $X$  equals one in a neighborhood of  $\mathbb{R}X$ , so that  $\partial\bar{\partial}\theta$  has support disjoint from  $\mathbb{R}X$ , this is not true for the functions  $\rho_U\theta$ , so that Theorem 6 or Corollary 2 cannot be used. Recall that on  $\mathbb{R}X$ , or near  $\mathbb{R}X$ , at a smaller scale than  $\log d/\sqrt{d}$ , the two peaks of a real peak section interfere so that results of Proposition 6 on evaluation maps no longer hold. This forces us to have a neighborhood of each connected component of  $\mathbb{R}X$  on which the vector fields  $v_1, \dots, v_n$  are globally defined, as it is the case for products of curves for instance.

3. Our Lefschetz pencils do not produce Morse functions from  $\mathbb{R}X$  to  $\mathbb{R}$ , but rather from  $\mathbb{R}X$  to  $\mathbb{R}P^1$ . As a consequence, the total Betti number of  $\mathbb{R}X$  is not bounded from above by the number of critical points of this pencil, one has to take into account also the total Betti number of a fiber, see Lemma 1, and thus prove the result by induction on the dimension. Such a fiber of the pencil becomes then submitted to the same constraints as  $X$ .

4. The weak convergence of the measure given by Theorems 3 and 6 was only proved at bounded distance of the critical set of the Lefschetz pencil in dimensions  $n > 2$ . It is actually not hard to prove it on the critical set for  $n = 3$  but seems less clear to us for  $n > 3$ . This is another obstacle since the pencils have real critical points in general, which have to be approached by the supports of our test functions  $\theta$ .

5. The  $\log d/\sqrt{d}$ -scale which we use throughout the paper comes from Lemma 4 taken out from [25]. It ensures that outside of the ball of radius  $\log d/\sqrt{d}$ , the  $L^2$ -norm of peak sections is  $O(\frac{1}{d^{2p'}})$ . This  $\log d/\sqrt{d}$  might be improved by a  $1/\sqrt{d}$  instead if a weaker upper bound for this  $L^2$ -norm, such as  $O(1)$ , suffices. However, even with such a  $1/\sqrt{d}$ -scale, we would still have some  $\log d$  term in our Theorem 4, because some  $\log d$  term shows up in our estimates of integrals arising from Poincaré–Martinelli’s formula, at the end of Section 2.4.1.

Note finally that what is the exact value of the expectation in Theorem 4 remains a mystery, as well as what happens below this expectation. Is there any exponential rarefaction below this expectation similar to the one observed in [7]? Is indeed the expectation a constant times  $d^{\frac{n}{2}}$  as soon as  $\mathbb{R}X$  is non-empty? It is also of interest to investigate the question with a different Gaussian measure, namely the one arising from the  $L^2$ -scalar product obtained by integration over  $\mathbb{R}X$  instead of  $X$ , see the end of Section 3.1. As pointed out by the referee, a new question shows up: what is the topology of the vanishing locus of sections which are extremals for the ratio between these two “complex” and “real”  $L^2$ -norms.

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Eingegangen 15. Juli 2011