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Universal Components of Random Nodal Sets

Damien Gayet[1,2], Jean-Yves Welschinger[3,4]

[1] Univ. Grenoble Alpes, Institut Fourier, 38000 Grenoble, France
[2] CNRS, IF, 38000 Grenoble, France. E-mail: damien.gayet@ujf-grenoble.fr
[3] Université de Lyon, CNRS UMR 5208, Lyon, France
[4] Université Lyon 1, Institut Camille Jordan, 69622 Villeurbanne Cedex, France.
  E-mail: welschinger@math.univ-lyon1.fr

Abstract: We give, as L grows to infinity, an explicit lower bound of order $L^{\frac{n}{m}}$ for the expected Betti numbers of the vanishing locus of a random linear combination of eigenvectors of P with eigenvalues below L. Here, P denotes an elliptic self-adjoint pseudo-differential operator of order $m > 0$, bounded from below and acting on the sections of a Riemannian line bundle over a smooth closed n-dimensional manifold M equipped with some Lebesgue measure. In fact, for every closed hypersurface Σ of $\mathbb{R}^n$, we prove that there exists a positive constant p_Σ depending only on Σ, such that for every large enough L and every $x \in M$, a component diffeomorphic to Σ appears with probability at least p_Σ in the vanishing locus of a random section and in the ball of radius $L^{-\frac{1}{m}}$ centered at x. These results apply in particular to Laplace–Beltrami and Dirichlet-to-Neumann operators.

Contents

Introduction

Let M be a smooth closed manifold of positive dimension n and E be a real line bundle over M. We equip M with a Lebesgue measure $|dy|$, that is a positive measure that can be locally expressed as the absolute value of some smooth volume form, and E with a Riemannian metric h_E. These induce an L^2-scalar product on the space $\Gamma(M, E)$ of smooth global sections of E, which reads

$$\forall (s, t) \in \Gamma(M, E)^2, \, \langle s, t \rangle = \int_M h_E \big(s(y), t(y) \big) |dy|. \tag{0.1}$$

Let $P : \Gamma(M, E) \to \Gamma(M, E)$ be a self-adjoint elliptic pseudo-differential operator of positive order m that is bounded from below. The spectrum of such an operator is thus real, discrete and bounded from below. Its eigenspaces are finite dimensional with smooth eigenfunctions, see [9]. We set, for every $L \in \mathbb{R}$, $\mathbb{U}_L = \bigoplus_{\lambda \leq L} \ker(P - \lambda Id)$. The dimension N_L of $\mathbb{U}_L$ satisfies Weyl's asymptotic law $\frac{1}{L^{\frac{n}{m}}} N_L \underset{L \to +\infty}{\to} \frac{1}{(2\pi)^n} Vol\{\xi \in T^*M, \, | \, \sigma_P(\xi) \leq 1\}$, where σ_P denotes the homogenized principal symbol of P, see [9] and Definition $A.8$ of [6]. The space $\mathbb{U}_L$ inherits by restriction the L^2-scalar product (0.1) and its associated Gaussian measure defined by the density

$$\forall s \in \mathbb{U}_L, \, d\mu(s) = \frac{1}{\sqrt{\pi}^{N_L}} \exp(-\|s\|^2) |ds|, \tag{0.2}$$

where $|ds|$ denotes the Lebesgue measure of $\mathbb{U}_L$ associated to its scalar product. The measure of the discriminant $\Delta_L = \{s \in \mathbb{U}_L, \, s \text{ does not vanish transversally}\}$ vanishes when L is large enough, see Lemma A.1 of [6]. Recall that a section $s \in \Gamma(M, E)$ is said to vanish transversally if and only if for every $x \in s^{-1}(0)$, $\nabla s_{|x}$ is onto, where ∇ denotes any connection on E.

Our purpose is to study the topology of the vanishing locus $s^{-1}(0) \subset M$ of a section $s \in \mathbb{U}_L$ taken at random. More precisely, for every closed hypersurface Σ of $\mathbb{R}^n$ not necessarily connected, and every $s \in \mathbb{U}_L \backslash \Delta_L$, we denote by $N_\Sigma(s)$ the maximal number of disjoint open subsets of M with the property that every such open subset U' contains a hypersurface Σ' such that $\Sigma' \subset s^{-1}(0)$ and (U', Σ') is diffeomorphic to $(\mathbb{R}^n, \Sigma)$ (compare [7]). We then consider

$$\mathbb{E}(N_\Sigma) = \int_{\mathbb{U}_L \backslash \Delta_L} N_\Sigma(s) d\mu(s) \tag{0.3}$$

the mathematical expectation of the function N_Σ. Note that when Σ is connected, the expected number of connected components diffeomorphic to Σ of the vanishing locus of a random section of $\mathbb{U}_L$ is bounded from below by $\mathbb{E}(N_\Sigma)$.

Theorem 0.1. *Let M be a smooth closed manifold of positive dimension n, equipped with a Lebesgue measure $|dy|$. Let E be a real line bundle over M equipped with a Riemannian metric h_E. Let $P : \Gamma(M, E) \to \Gamma(M, E)$ be an elliptic pseudo-differential operator of positive order m, which is self-adjoint and bounded from below. Let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected. Then, there exists a positive constant $c_\Sigma(P)$, such that*

$$\liminf_{L \to +\infty} \frac{1}{L^{\frac{n}{m}}} \mathbb{E}(N_\Sigma) \geq c_\Sigma(P).$$

The constant $c_\Sigma(P)$ is in fact explicit, given by (2.3).

Now, as in [7], we denote by $\mathcal{H}_n$ the space of diffeomorphism classes of closed connected hypersurfaces of $\mathbb{R}^n$. For every $[\Sigma] \in \mathcal{H}_n$ and every $i \in \{0, \ldots, n-1\}$, we denote by $b_i(\Sigma) = \dim H_i(\Sigma, \mathbb{R})$ the i-th Betti number of Σ with real coefficients. Likewise, for every $s \in \mathbb{U}_L \backslash \Delta_L$, $b_i(s^{-1}(0))$ denotes the i-th Betti number of $s^{-1}(0)$, and we set

$$\mathbb{E}(b_i) = \int_{\mathbb{U}_L \backslash \Delta_L} b_i(s^{-1}(0)) d\mu(s) \tag{0.4}$$

for its mathematical expectation.

Corollary 0.2. *Let M be a smooth closed manifold of positive dimension n equipped with a Lebesgue measure $|dy|$. Let E be a real line bundle over M equipped with a Riemannian metric h_E. Let $P : \Gamma(M, E) \to \Gamma(M, E)$ be an elliptic pseudo-differential operator of positive order m, which is self-adjoint and bounded from below. Then, for every $i \in \{0, \ldots, n-1\}$,*

$$\liminf_{L \to \infty} \frac{1}{L^{\frac{n}{m}}} \mathbb{E}(b_i) \geq \sum_{[\Sigma] \in \mathcal{H}_n} \sup_{\Sigma \in [\Sigma]} \big(c_\Sigma(P)\big) b_i(\Sigma),$$

where $c_\Sigma(P)$ is defined in Theorem 0.1.

Note that an upper bound for $\mathbb{E}(b_i)$ of the same order in L is given by Theorem 0.2 of [6].

Theorem 0.1 is in fact a consequence of Theorem 0.3, which is local and more precise. Let $Met_{|dy|}(M)$ be the space of Riemannian metrics of M whose associated Lebesgue measure equals $|dy|$. For every $g \in Met_{|dy|}(M)$, every $R > 0$ and every point $x \in M$, we set

$$Prob_\Sigma^x(R) = \mu \bigg\{ s \in \mathbb{U}_L \backslash \Delta_L \mid \big(s^{-1}(0) \cap B_g(x, RL^{-\frac{1}{m}})\big) \supset \Sigma_L$$

$$\text{with } \big(B_g(x, RL^{-\frac{1}{m}}), \Sigma_L\big) \text{ diffeomorphic to } (\mathbb{R}^n, \Sigma)\bigg\}, \tag{0.5}$$

where $B_g(x, RL^{-\frac{1}{m}})$ denotes the ball centered at x of radius $RL^{-\frac{1}{m}}$ for the metric g.

Theorem 0.3. *Under the hypotheses of Theorem 0.1, let $g \in Met_{|dy|}(M)$. Then, for every $x \in M$ and every $R > 0$, there exists $p_\Sigma^x(R) \geq 0$ for which*

$$\liminf_{L \to +\infty} Prob_\Sigma^x(R) \geq p_\Sigma^x(R).$$

Moreover, $p_\Sigma(R) = \inf_{x \in M} p_\Sigma^x(R)$ is positive as soon as R is large enough.

It is worthwhile to get quantitative versions of the lower estimates in Theorems 0.1 and 0.3, and in particular to compare Corollary 0.2 with the quantitative upper estimates for the expected Betti numbers obtained in [6]. In fact, the constant p_Σ defined in Theorem 0.3 as well as $c_\Sigma(P)$ defined in Theorem 0.1 turn out to be explicit, see (2.2) (see also (1.7) and (1.8)) and (2.3) (see also (2.4)). In the following Theorem 0.4, we give quantitative estimates of these constants in the case where Σ is the product of spheres $S^i \times S^{n-i-1}$, $i \in \{0, \ldots, n-1\}$. Indeed, these manifolds embed as hypersurfaces in $\mathbb{R}^n$ and

$$\forall i \in \{0, \ldots, n-1\}, \ b_i(S^i \times S^{n-i-1}) \geq 1. \tag{0.6}$$

These estimates only depend on the principal symbol of P, the volume of M and its dimension.

Theorem 0.4. *Under the hypotheses of Theorem 0.1, let $g \in Met_{|dy|}(M)$ and $c_{P,g}$, $d_{P,g} > 0$ be such that for every $\xi \in T^*M$, $d_{P,g}^{-1} \|\xi\| \leq \sigma_P(\xi)^{\frac{1}{m}} \leq c_{P,g}^{-1} \|\xi\|$, where σ_P denotes the homogenized principal symbol of P. Then, for every $i \in \{0, \ldots, n-1\}$ and every $R \geq \frac{48\sqrt{5}n}{c_{P,g}}$,*

$$c_{S^i \times S^{n-i-1}}(P) \geq \frac{e^{-(2\tau+1)^2}}{2^{n+1}\sqrt{\pi}\,Vol(B(0, 48\sqrt{5}n))} c_{P,g}^n Vol_{|dy|}(M) \quad and$$

$$p_{S^i \times S^{n-i-1}}(R) \geq \frac{1}{2\sqrt{\pi}} \exp\left(-(2\tau+1)^2\right),$$

where $\tau = 20 \frac{(n+6)^{11/2}}{\sqrt{\Gamma(\frac{n}{2}+1)}} \left(48n\frac{d_{P,g}}{c_{P,g}}\right)^{\frac{n+2}{2}} \exp\left(48\sqrt{5}n^{3/2}\frac{d_{P,g}}{c_{P,g}}\right)$.

Remark 0.5. Note that for any $g \in Met_{|dy|}(M)$, constants $c_{P,g}$ and $d_{P,g}$ satisfying the hypotheses of Theorem 0.4 do exist, since σ_P is smooth, homogeneous and M is compact.

In the case of Laplace–Beltrami operators, using (0.6) we obtain the following corollary.

Corollary 0.6. *Let (M, g) be a smooth closed n-dimensional Riemannian manifold and let Δ be its associated Laplace–Beltrami operator acting on functions. Then for every $i \in \{0, \ldots, n-1\}$,*

$$\liminf_{L \to +\infty} \frac{1}{\sqrt{L}^n} \mathbb{E}(b_i) \geq c_{S^i \times S^{n-i-1}}(\Delta) \geq \exp\left(-\exp(257n^{3/2})\right)Vol_g(M).$$

As a second example, Theorem 0.4 specializes to the case of the Dirichlet-to-Neumann operator on the boundary M of some $(n+1)$-dimensional compact Riemannian manifold (W, g).

Corollary 0.7. *Let (W, g) be a smooth compact Riemannian manifold of dimension $n+1$ with boundary M and let Λ_g be the associated Dirichlet-to-Neumann operator on M. Then, for every $i \in \{0, \ldots, n-1\}$,*

$$\liminf_{L \to +\infty} \frac{1}{L^n} \mathbb{E}(b_i) \geq c_{S^i \times S^{n-i-1}}(\Lambda_g) \geq \exp\left(-\exp(257n^{3/2})\right)Vol_g(M).$$

Note that the double exponential decay in Corollaries 0.6 and 0.7 has to be compared with the exponential decay observed in Proposition 0.4 of [6] and with the analogous double exponential decay already observed in Corollary 1.3 of [7].

Let us mention some related works. In [12], Nazarov and Sodin proved that the expected number of components of the vanishing locus of random eigenfunctions with eigenvalue L of the Laplace operator on the round 2-sphere is asymptotic to a constant times L. In the recent [13], they obtain similar results in a more general setting, in particular for all round spheres and flat tori (see also [15]). In [10], Lerario and Lundberg proved, for the Laplace operator on the round n-sphere, the existence of a positive constant c such that $\mathbb{E}(b_0) \geq c\sqrt{L}^n$ for large values of L. We got in [6] upper estimates for $\limsup_{L \to +\infty} L^{-\frac{n}{m}} \mathbb{E}(b_i)$ under the same hypotheses as Corollary 0.2, and previously obtained similar upper and lower estimates for the expected Betti numbers or N_Σ's of random real algebraic hypersurfaces of real projective manifolds (see [4,5,7,8]).

In [11], Letendre proved, under the hypotheses of Corollary 0.6, that the mean Euler characteristics (for odd n) is asymptotic to a constant times $\sqrt{L}^n$. Let us finally mention [14], where Sarnak and Wigman announce a convergence in probability for N_Σ in the case of Laplace–Beltrami operators (see also [1]).

In the first section, we introduce the space of Schwartz functions of $\mathbb{R}^n$ whose Fourier transforms have supports in the compact set $K_x = \{\xi \in T_x^* M \mid \sigma_P(\xi) \leq 1\}$, where $x \in M$ is given and $T_x^* M$ is identified with $\mathbb{R}^n$ via some isometry. This space appears to be asymptotically a local model for the space $\mathbb{U}_L$. Indeed, any function f in this space can be implemented in $\mathbb{U}_L$, in the sense that there exists a family of sections $(s_L \in \mathbb{U}_L)_{L \gg 1}$ whose restriction to a ball of radius of order $L^{-\frac{1}{m}}$ centered at x converges to f after rescaling, see Corollary 1.11. The vanishing locus of f then gets implemented as the vanishing locus of the sections s_L for L large enough. The second section is devoted to the proofs of Theorems 0.1 and 0.3, and of Corollary 0.2. For this purpose we follow the approach used in [7] (see also [5]), which was itself partially inspired by the works [3,12], see also [10]. We begin by estimating the expected local C^1-norm of elements of $\mathbb{U}_L$, see Proposition 2.1, and then compare it with the amount of transversality of s_L. We can then prove Theorem 0.3, see Sect. 2.2, and finally Theorem 0.1 and its Corollary 0.2, see §2.3. The last section is devoted to the explicit estimates and the proofs of Theorem 0.4 and Corollaries 0.6 and 0.7.

1. The Local Model and Its Implementation

In Sect. 1.1, we associate to any closed hypersurface Σ of $\mathbb{R}^n$ and any symmetric compact subset K of $\mathbb{R}^n$ with the origin in its interior, a Schwartz function f vanishing transversally along a hypersurface isotopic to Σ and whose Fourier transform has support in K. In Sect. 1.3, we implement the function f in the neighbourhood of every point x_0 in M, as the limit after rescaling of a sequence of sections of $\mathbb{U}_L$. Here, K is the pull-back of K_{x_0} under some measure-preserving isomorphism between $T_{x_0}^* M$ and $\mathbb{R}^n$. As a consequence, these sections of $\mathbb{U}_L$ vanish in a neighbourhood U_{x_0} of x_0 along a hypersurface Σ_L of M such that the pair (U_{x_0}, Σ_L) is diffeomorphic to $(\mathbb{R}^n, \Sigma)$. In Sect. 1.2, we quantify the transversality of the vanishing of the function f and thus of the associated sequence of sections, in order to prepare the estimates of the second section which involve perturbations.

1.1. The local model. Let K be a measurable subset of $\mathbb{R}^n$ and let χ_K be its characteristic function, so that $\chi_K(\xi) = 1$ if $\xi \in K$ and $\chi_K(\xi) = 0$ otherwise. This function χ_K induces the restriction $f \in L^2(\mathbb{R}^n) \mapsto \chi_K f \in L^2(\mathbb{R}^n)$. After conjugation by the Fourier transform $\mathcal{F}$ of $L^2(\mathbb{R}^n)$, defined for every $f \in L^2(\mathbb{R}^n)$ and every $\xi \in \mathbb{R}^n$ by $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} f(y) dy \in L^2(\mathbb{R}^n)$, we get the projector $\pi_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, defined for every $f \in L^2(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ by $\pi_K(f)(x) = \frac{1}{(2\pi)^n} \int_{\xi \in K} \int_{y \in \mathbb{R}^n} e^{i\langle x-y, \xi \rangle} f(y) d\xi dy$. Note that for $K = \mathbb{R}^n$, π_K is the identity map. Denote by $L_K^2(\mathbb{R}^n)$ the image of π_K. This is a Hilbert subspace of $L^2(\mathbb{R}^n)$, the kernel of the continuous operator $Id - \pi_K = \pi_{\mathbb{R}^n \setminus K}$. Denote by $C_0^\infty(K)$ the space of smooth functions on $\mathbb{R}^n$ whose support is included in K. Write $S(\mathbb{R}^n)$ for the space of Schwartz functions of $\mathbb{R}^n$ and set

$$S_K(\mathbb{R}^n) = \mathcal{F}^{-1}(C_0^\infty(K)). \tag{1.1}$$

Lemma 1.1. *Let K be a bounded measurable subset of $\mathbb{R}^n$. Then, $S_K(\mathbb{R}^n) \subset L^2_K(\mathbb{R}^n) \cap S(\mathbb{R}^n)$.*

Proof. Since K is bounded, $C_0^\infty(K) \subset S(\mathbb{R}^n)$ so that $S_K(\mathbb{R}^n) \subset \mathcal{F}^{-1}(S(\mathbb{R}^n)) = S(\mathbb{R}^n)$. Likewise, for every $f \in C_0^\infty(K)$, $\chi_K f = f$, so that by definition, $f \in L^2_K(\mathbb{R}^n)$. □

Lemma 1.2. *Let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected. Let K be a bounded measurable subset of $\mathbb{R}^n$, symmetric with respect to the origin and which contains the origin in its interior. Then, there exists a hypersurface $\widetilde{\Sigma}$ of $\mathbb{R}^n$, isotopic to Σ, and a function f_Σ in $S_K(\mathbb{R}^n)$ such that f_Σ vanishes transversally along $\widetilde{\Sigma}$.*

Recall that $\widetilde{\Sigma}$ is said to be isotopic to Σ if and only if there exists a continuous family $(\phi_t)_{t \in [0,1]}$ of diffeomorphisms of $\mathbb{R}^n$ such that $\phi_0 = Id$ and $\phi_1(\Sigma) = \widetilde{\Sigma}$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$ be a smooth compactly supported function of $\mathbb{R}^n$ which vanishes transversally along Σ and let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be an even function which equals 1 in a neighbourhood of the origin. For every $R > 0$, we set $\tilde{\chi}_R : \xi \in \mathbb{R}^n \mapsto \tilde{\chi}(\xi R^{-1}) \in \mathbb{R}$. Then $\mathcal{F}(f) \in S(\mathbb{R}^n)$ and $\tilde{\chi}_R \mathcal{F}(f)$ converges to $\mathcal{F}(f)$ in $S(\mathbb{R}^n)$ as R grows to infinity. Thus, $\mathcal{F}^{-1}(\tilde{\chi}_R \mathcal{F}(f))$ converges to f in $S(\mathbb{R}^n)$ as R grows to infinity, and $\mathcal{F}^{-1}(\tilde{\chi}_R \mathcal{F}(f))$ takes real values. We deduce that when R is large enough, the function $f_R = \mathcal{F}^{-1}(\tilde{\chi}_R \mathcal{F}(f))$ is real and vanishes transversally in a neighbourhood of Σ along a hypersurface isotopic to Σ. By construction, the support of $\mathcal{F}(f_R)$ is compact. By hypotheses, there exists thus $\rho > 0$ such that the function $\mathcal{F}_\rho(f_R) : \xi \in \mathbb{R}^n \mapsto \mathcal{F}(f_R)(\frac{\xi}{\rho}) \in \mathbb{R}$ has compact support in K. The function $f_\Sigma = \mathcal{F}^{-1}(\mathcal{F}_\rho(f_R))$ then belongs to $S_K(\mathbb{R}^n)$ and vanishes transversally along a hypersurface isotopic to Σ. □

1.2. Quantitative transversality. We now proceed as in [7] to introduce our needed quantitative transversality estimates.

Definition 1.3. Let W be a bounded open subset of $\mathbb{R}^n$ and $f \in S(\mathbb{R}^n)$. The pair (W, f) is said to be regular if and only if zero is a regular value of the restriction of f to W and the vanishing locus of f in W is compact.

Example 1.4. Let $f_\Sigma \in S_K(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ be a function given by Lemma 1.2. Then, there exists a tubular neighbourhood W of $\tilde{\Sigma} \subset f_\Sigma^{-1}(0)$ such that (W, f_Σ) is a regular pair in the sense of Definition 1.3.

Definition 1.5. For every regular pair (W, f) given by Definition 1.3, we denote by $\mathcal{T}_{(W, f)}$ the set of pairs $(\delta, \epsilon) \in (\mathbb{R}_+)^2$ such that

1. There exists a compact subset K_W of W such that $\inf_{W \setminus K_W} |f| > \delta$.
2. For every $z \in W$, if $|f(z)| \leq \delta$ then $\|d_{|z} f\| > \epsilon$, where $\|d_{|z} f\|^2 = \sum_{i=1}^n |\frac{\partial f}{\partial x_i}|^2(z)$.

The quantities and functions that are going to appear in the proof of our theorems are the following. Let K be a bounded measurable subset of $\mathbb{R}^n$. We set, for every positive R and every $j \in \{1, \ldots, n\}$,

$$\rho_K(R) = \frac{\sqrt{2}\lfloor \frac{n}{2} + 1 \rfloor}{\sqrt{2\pi}^n} \inf_{t \in \mathbb{R}_+^*} \left(\left(\frac{R+t}{t}\right)^{\frac{n}{2}} \sum_{i=0}^{\lfloor \frac{n}{2}+1 \rfloor} \frac{t^i}{i!} \left(\sum_{\substack{(j_1,\ldots,j_i) \\ \in \{1,\ldots,n\}^i}} \int_K \prod_{k=1}^{i} |\xi_{j_k}|^2 |d\xi| \right)^{\frac{1}{2}} \right) \quad (1.2)$$

$$\theta_K^j(R) = \frac{\sqrt{2}\lfloor \frac{n}{2} + 1 \rfloor}{\sqrt{2\pi}^n} \inf_{t \in \mathbb{R}_+^*} \left(\left(\frac{R+t}{t}\right)^{\frac{n}{2}} \sum_{i=0}^{\lfloor \frac{n}{2}+1 \rfloor} \frac{t^i}{i!} \left(\sum_{\substack{(j_1,\ldots,j_i) \\ \in \{1,\ldots,n\}^i}} \int_K |\xi_j|^2 \prod_{k=1}^{i} |\xi_{j_k}|^2 |d\xi| \right)^{\frac{1}{2}} \right).$$
$$(1.3)$$

Remark 1.6. Writing $\nu(K) = \int_K |d\xi|$ for the total measure of K and $d(K) = \sup_{\xi \in K} \|\xi\|$, we note that for every $(j_1, \ldots, j_i) \in \{1, \ldots, n\}^i$ and every $j \in \{1, \ldots, n\}$,

$$\int_K \prod_{k=1}^{i} |\xi_{j_k}|^2 |d\xi| \leq d(K)^{2i} \nu(K)$$

and $\int_K |\xi_j|^2 \prod_{k=1}^{i} |\xi_{j_k}|^2 \leq d(K)^{2(i+1)} \nu(K)$. It follows, after evaluation at $t = R$, that for every $j \in \{1, \ldots, n\}$,

$$\rho_K(R) \leq \frac{1}{\sqrt{\pi}^n} \sqrt{2\nu(K)} \lfloor \frac{n}{2} + 1 \rfloor \exp\left(R d(K) \sqrt{n} \right) \quad (1.4)$$

$$\theta_K^j(R) \leq \frac{1}{\sqrt{\pi}^n} \sqrt{2\nu(K)} \lfloor \frac{n}{2} + 1 \rfloor d(K) \exp\left(R d(K) \sqrt{n} \right). \quad (1.5)$$

For every regular pair (W, f) we set $R_W = \sup_{z \in W} \|z\|$ and for every bounded measurable subset K of $\mathbb{R}^n$ define

$$\tau_{(W,f)}^K = \|f\|_{L^2(\mathbb{R}^n)} \inf_{(\delta, \epsilon) \in \mathcal{T}_{(W,f)}} \left(\frac{1}{\delta} \rho_K(R_W) + \frac{n\sqrt{n}}{\epsilon} \sum_{i=1}^{n} \theta_K^j(R_W) \right) \quad (1.6)$$

and $p_{(W,f)}^K = \frac{1}{\sqrt{\pi}} \sup_{T \in [\tau_{(W,f)}^K, +\infty[} \left(1 - \frac{\tau_{(W,f)}^K}{T}\right) \int_T^{+\infty} e^{-t^2} dt.$ $\quad (1.7)$

Remark 1.7. Note that $p_{(W,f)}^K \geq \frac{1}{2\sqrt{\pi}} \exp\left(-(2\tau_{(W,f)}^K + 1)^2 \right).$

Now, let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected.

Definition 1.8. Let $\mathcal{I}_\Sigma^K$ be the set of regular pairs (W, f) given by Definition 1.3 such that $f \in S_K(\mathbb{R}^n)$ and such that the vanishing locus of f in W contains a hypersurface isotopic to Σ in $\mathbb{R}^n$. Likewise, for every $R > 0$, we set $\mathcal{I}_\Sigma^{K,R} = \{(W, f) \in \mathcal{I}_\Sigma^K \mid R_W \leq R\}.$

Finally, for every positive R we set

$$p_\Sigma^K(R) = \sup_{(W,f) \in \mathcal{I}_\Sigma^{K,R}} p_{(W,f)}^K. \quad (1.8)$$

Remark 1.9. It follows from Lemma 1.2 and Example 1.4 that when R is large enough and K satisfies the hypotheses of Lemma 1.2, $\mathcal{I}_\Sigma^{K,R}$ is not empty, and in particular $p_\Sigma^K(R) > 0$. Note moreover that if $K \subset K'$, then $p_\Sigma^K \leq p_\Sigma^{K'}$.

1.3. Implementation of the local model. In this paragraph, we prove that for every $x_0 \in M$ and every measure-preserving linear isomorphism A between $\mathbb{R}^n$ and $T_{x_0}^* M$, every function f in $S_{A^* K_{x_0}}(\mathbb{R}^n)$ can be implemented in $\mathbb{U}_L$ as a sequence of sections, see Proposition 1.10. Corollary 1.11 then estimates the amount of transversality of these sections along their vanishing locus, in terms of the one of f.

Proposition 1.10. *Under the hypotheses of Corollary* 0.2, *let* $x_0 \in M, \phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$ *be a measure-preserving chart and* $\tilde{\chi}_V \in C_c^\infty(V)$ *be an even function with support in V which equals 1 in a neighbourhood of 0. Then, for every* $f \in S_{(d_{|x_0}\phi_{x_0}^{-1})^* K_{x_0}}(\mathbb{R}^n)$, *there exists a family* $(s_L \in \mathbb{U}_L)_{L \in \mathbb{R}_+^*}$ *such that*

1. $\|s_L\|_{L^2(M)} \underset{L \to +\infty}{\to} \|f\|_{L^2(\mathbb{R}^n)}$
2. *the function* $z \in \mathbb{R}^n \mapsto L^{-\frac{n}{2m}} \tilde{\chi}_V(L^{-\frac{1}{m}}z)(s_L \circ \phi_{x_0}^{-1})(L^{-\frac{1}{m}}z) \in \mathbb{R}$ *converges to f in* $S(\mathbb{R}^n)$.

Note that the isomorphism $(d_{|x_0}\phi_{x_0})^{-1} : \mathbb{R}^n \to T_{x_0}M$ defines by the pull-back an isomorphism $((d_{|x_0}\phi_{x_0})^{-1})^* : T_{x_0}^* M \to \mathbb{R}^n$ that makes it possible to identify the compact

$$K_{x_0} = \{\xi \in T_{x_0}^* M \,|\, \sigma_P(\xi) \leq 1\} \tag{1.9}$$

with the compact $((d_{|x_0}\phi_{x_0})^{-1})^* K_{x_0}$ of $\mathbb{R}^n$. Moreover, the Riemannian metric h_E of E given in the hypotheses of Corollary 0.2 provides a trivialization of E in the neighbourhood U_{x_0} of x_0, choosing a smaller U_{x_0} if necessary, unique up to sign. This trivialization makes it possible to identify $\tilde{\chi}_V s_L \circ \phi_{x_0}^{-1}$ with a function from V to $\mathbb{R}$.

Proof. For every $L \in \mathbb{R}_+^*$, we set $\tilde{s}_L : x \in U_{x_0} \mapsto L^{\frac{n}{2m}} \tilde{\chi}_V(\phi_{x_0}(x)) f(L^{\frac{1}{m}}\phi_{x_0}(x)) \in E_{|x}$ that we extend by zero to a global section of E. We denote then by s_L the orthogonal projection of $\tilde{s}_L$ in $\mathbb{U}_L \subset L^2(M, E)$. This section reads

$$s_L = \langle e_L, \tilde{s}_L \rangle = \int_M h_E(e_L(x, y), \tilde{s}_L(y))|dy|,$$

where e_L denotes the Schwartz kernel of the orthogonal projection onto $\mathbb{U}_L$. Then, for every $z \in \mathbb{R}^n$, $L^{-\frac{1}{m}}z$ belongs to V when L is large enough and

$$L^{-\frac{n}{2m}} s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z) = L^{-\frac{n}{2m}} \int_M h_E\left(e_L(\phi_{x_0}^{-1}(L^{-\frac{1}{m}}z), y), \tilde{s}_L(y)\right)|dy|$$

$$= \int_{U_{x_0}} \tilde{\chi}_V(\phi_{x_0}(y)) e_L(\phi_{x_0}^{-1}(L^{-\frac{1}{m}}z), y) f(L^{\frac{1}{m}}\phi_{x_0}(y))(y)|dy|$$

$$= L^{-\frac{n}{m}} \int_{\mathbb{R}^n} \tilde{\chi}_V(L^{-\frac{1}{m}}h)(\phi_{x_0}^{-1})^* e_L(L^{-\frac{1}{m}}z, L^{-\frac{1}{m}}h) f(h)|dh|,$$

where we performed the substitution $h = L^{\frac{1}{m}}\phi_{x_0}(y)$, so that $|dh| = L^{\frac{n}{m}}|dy|$. From Theorem 4.4 of [9],

$$L^{-\frac{n}{m}}(\phi_{x_0}^{-1})^* e_L(L^{-\frac{1}{m}}z, L^{-\frac{1}{m}}h) \underset{L \to +\infty}{\to} \frac{1}{(2\pi)^n} \int_{K_{x_0}'} e^{i\langle z-h, \xi \rangle}|d\xi|,$$

where $K_{x_0}' = (d_{|x_0}\phi_{x_0}^{-1})^* K_{x_0}$. Moreover, there exists $\epsilon > 0$ such that this convergence holds in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for the semi-norms family defined by the supremum of the

derivatives of the functions on the bidisc $\bar{B}(\epsilon L^{\frac{1}{m}})^2$, where $\bar{B}(\epsilon L^{\frac{1}{m}})$ denotes the closed ball of $\mathbb{R}^n$ of radius $\epsilon L^{\frac{1}{m}}$, see [6]. As a consequence, after perhaps taking a smaller V so that V is contained in the ball of radius ϵ,

$$L^{-\frac{n}{m}} \tilde{\chi}_V(L^{-\frac{1}{m}}h)(\phi_{x_0}^{-1})^* e_L(L^{-\frac{1}{m}}z, L^{-\frac{1}{m}}h) f(h) \underset{L\to+\infty}{\to} \frac{1}{(2\pi)^n} \int_{K'_{x_0}} e^{i\langle z-h,\xi\rangle} f(h)|d\xi|$$

in this same sense, which implies convergence in the Schwartz space $S(\mathbb{R}^n)$ for each fixed z. After integration, it follows that

$$L^{-\frac{n}{2m}} s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z) \underset{L\to+\infty}{\to} \frac{1}{(2\pi)^n} \int_{K'_{x_0}} e^{i\langle z,\xi\rangle} \mathcal{F}(f)(\xi)|d\xi|$$

in $C^\infty(\mathbb{R}^n)$ for our family of semi-norms on $\bar{B}(\epsilon L^{\frac{1}{m}})$. Since $f \in S_{K'_{x_0}}(\mathbb{R}^n)$,

$$\frac{1}{(2\pi)^n} \int_{K'_{x_0}} e^{i\langle z,\xi\rangle} \mathcal{F}(f)(\xi)|d\xi| = f(z),$$

so that $z \mapsto L^{-\frac{n}{2m}} s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z)$ converges to f in $S(\mathbb{R}^n)$. This proves the second assertion.

If $\tilde{\chi}_U = \tilde{\chi}_V \circ \phi_{x_0}$, we deduce that $\|s_L \tilde{\chi}_U\|_{L^2(M)} \underset{L\to+\infty}{\to} \|f\|_{L^2(\mathbb{R}^n)}$. We still need to prove that $\|s_L(1 - \tilde{\chi}_U)\|_{L^2(M)} \underset{L\to+\infty}{\to} 0$. Since s_L is the orthogonal projection of $\tilde{s}_L$ onto $\mathbb{U}_L$,

$$\|s_L\|_{L^2(M)} \leq \|\tilde{s}_L\|_{L^2(M)} \underset{L\to+\infty}{\to} \|f\|_{L^2(\mathbb{R}^n)}.$$

The result follows. $\square$

Corollary 1.11. *Under the hypotheses of Theorem 0.3, let $x_0 \in M$ and $\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$ be a measure-preserving chart such that $A = d_{|x_0}\phi_{x_0}^{-1}$ is an isometry. Let $(W, f_\Sigma) \in \mathcal{I}_\Sigma^{A^* K_{x_0}}$ (see Definition 1.5) and $(\delta, \epsilon) \in \mathcal{T}_{(W, f_\Sigma)}$ (see Definition 1.8). Then, there exist $L_0 \in \mathbb{R}$ and $(s_L)_{L \geq L_0}$ such that for every $L \geq L_0$,*

1. *$s_L \in \mathbb{U}_L$ and $\|s_L\|_{L^2(M)} \underset{L\to+\infty}{\to} \|f_\Sigma\|_{L^2(\mathbb{R}^n)}$*
2. *The vanishing locus of s_L contains a hypersurface Σ_L included in the ball $B_g(x_0, R_W L^{-\frac{1}{m}})$ such that the pair $\left(B(x_0, R_W L^{-\frac{1}{m}}), \Sigma_L\right)$ is diffeomorphic to the pair $(\mathbb{R}^n, \Sigma)$.*
3. *There exist two neighbourhoods K_L and W_L of Σ_L such that K_L is compact, W_L is open, $\Sigma_L \subset K_L \subset W_L \subset B_g(x_0, R_W L^{-\frac{1}{m}})$, $\inf_{W_L \setminus K_L} |s_L| > \delta L^{\frac{n}{2m}}$ and for every $y \in W_L$,*

$$|s_L(y)| < \delta L^{\frac{n}{2m}} \Rightarrow \|d_{|y}(s_L \circ \phi_{x_0}^{-1})\| > \epsilon L^{\frac{n+2}{2m}}.$$

Proof. Let $L_0 \in \mathbb{R}$ and $(s_L)_{L \geq L_0}$ be a family given by Proposition 1.10 for $f = f_\Sigma$. Then, the first condition is satisfied and the family of functions $z \in B(0, R_W) \mapsto L^{-\frac{n}{2m}} s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z)$ converges to f_Σ in $C^\infty(B(0, R_W))$. Let K be the compact given by Definition 1.5, $K_L = \phi_{x_0}^{-1}(L^{-\frac{1}{m}}K)$ and $W_L = \phi_{x_0}^{-1}(L^{-\frac{1}{m}}W)$. The conditions 2 and 3 in the corollary follow from this convergence and from Definition 1.5. $\square$

2. Probability of the Local Presence of a Hypersurface

In this section, we follow the method of [7] partially inspired by [3,12] (see also [5,10]) in order to prove Theorem 0.3. If Σ is a smooth closed hypersurface of $\mathbb{R}^n$, $x \in M$ and $s_L \in \mathbb{U}_L$ is given by Proposition 1.10 and vanishes transversally along Σ_L in a small ball $B(x, RL^{-\frac{1}{m}})$, then we decompose any random section $s \in \mathbb{U}_L$ as $s = as_L + \sigma$, where $a \in \mathbb{R}$ is Gaussian and σ is taken at random in the orthogonal complement of $\mathbb{R}s_L$ in $\mathbb{U}_L$. In Sect. 2.1, we estimate the average of the values of σ and its derivatives on $B(x, RL^{-\frac{1}{m}})$, see Proposition 2.1. In Sect. 2.2, we prove that with a probability greater than a positive number $p_\Sigma^x(R)$ which is independent of L, s vanishes in the latter ball along a hypersurface isotopic to Σ_L. This follows from Proposition 2.1 and the quantitative estimates of the transversality of s_L given in Corollary 1.11.

2.1. Expected local C^1-norm of sections. Recall that for $x_0 \in M$,

$$K_{x_0} = \{\xi \in T_{x_0}^* M \,|\, \sigma_P(\xi) \leq 1\}. \tag{2.1}$$

Proposition 2.1. *Under the hypotheses of Theorem 0.3, let $x_0 \in M$ and $\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$ be a measure-preserving map such that $A = d_{|x_0}\phi_{x_0}^{-1}$ is an isometry. Then, for every positive R and every $j \in \{1, \ldots, n\}$,*

$$\limsup_{L \to +\infty} L^{-\frac{n}{2m}} \mathbb{E}\left(\|s\|_{L^\infty(B_g(x_0, RL^{-\frac{1}{m}}))}\right) \leq \rho_{A^*K_{x_0}}(R)$$

$$\text{and } \limsup_{L \to +\infty} L^{-\frac{n+2}{2m}} \mathbb{E}\left(\left\|\frac{\partial(s \circ \phi_{x_0}^{-1})}{\partial x_j}\right\|_{L^\infty(B_g(0, RL^{-\frac{1}{m}}))}\right) \leq \theta_{A^*K_{x_0}}^j(R),$$

where $\rho_{A^*K_{x_0}}$ and $\theta_{A^*K_{x_0}}^j$ are defined by (1.2) and (1.3).

Proof. Let $t \in \mathbb{R}_+^*$. When L is large enough, the ball $B(0, (R+t)L^{-\frac{1}{m}})$ of $\mathbb{R}^n$ is contained in V. From the Sobolev inequality (see §2.4 of [2]), we deduce that for every $s \in \mathbb{U}_L$, every $k > n/2$ and every $z \in B(0, RL^{-\frac{1}{m}})$,

$$|s \circ \phi_{x_0}^{-1}(z)| \leq \frac{2k}{Vol(B(0, tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{k} (tL^{-\frac{1}{m}})^i$$

$$\times \left(\frac{1}{i!} \int_{B(0,(R+t)L^{-\frac{1}{m}})} |D^i(s \circ \phi_{x_0}^{-1})|^2(x)|dx|\right)^{1/2},$$

where by definition, the norm of the i-th derivative $D^i(s \circ \phi_{x_0}^{-1})$ of $s \circ \phi_{x_0}^{-1}$ satisfies

$$i!|D^i(s \circ \phi_{x_0}^{-1})(x)|^2 = \sum_{\substack{(j_1,\ldots,j_i) \\ \in\{1,\ldots,n\}^i}} \left|\frac{\partial^i}{\partial x_{j_1} \cdots \partial x_{j_i}}(s \circ \phi_{x_0}^{-1})(x)\right|^2.$$

Note indeed that the metric h_E of the bundle E makes it possible to identify $s_{|U_{x_0}}$ with a real valued function well defined up to a sign. As a consequence, we deduce from the Cauchy–Schwarz inequality that

$$\mathbb{E}\left(\|s \circ \phi_{x_0}^{-1}\|_{L^\infty(B(0,RL^{-\frac{1}{m}}))}\right) \leq \frac{2k}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{k} \frac{1}{i!}(tL^{-\frac{1}{m}})^i$$
$$\left(\int_{B(0,(R+t)L^{-\frac{1}{m}})} i!\mathbb{E}(|D^i(s \circ \phi_{x_0}^{-1})|^2(x))|dx|\right)^{1/2}.$$

Given $(j_1, \ldots j_i) \in \{1, \ldots, n\}^i$ and $z \in B(0, (R+t)L^{-\frac{1}{m}})$, we can choose an ortho-normal basis $(s_1, \ldots, s_{N_L})$ of $\mathbb{U}_L$ such that $\frac{\partial^i}{\partial x_{j_1} \cdots \partial x_{j_i}}(s_l \circ \phi_{x_0}^{-1})(z) = 0$ for every $l > 1$. Since the spectral function reads $(x, y) \in M \times M \mapsto e_L(x, y) = \sum_{i=0}^{N_L} s_i(x)s_i^*(y)$, we deduce, using the decomposition of s in the basis $(s_1, \ldots, s_{N_L})$, that

$$\mathbb{E}\left(\left|\frac{\partial^i}{\partial x_{j_1} \cdots \partial x_{j_i}}(s \circ \phi_{x_0}^{-1})\right|^2(z)\right)$$
$$= \left(\int_{\mathbb{R}} a^2 e^{-a^2} \frac{da}{\sqrt{\pi}}\right) \frac{\partial^{2i}}{\partial x_{j_1} \cdots \partial x_{j_i} \partial y_{j_1} \cdots \partial y_{j_i}}(e_L \circ \phi_{x_0}^{-1})(z, z).$$

Choosing $k = \lfloor \frac{n}{2} + 1 \rfloor$ and noting that $\int_{\mathbb{R}} a^2 e^{-a^2} \frac{da}{\sqrt{\pi}} = \frac{1}{2}$, we deduce that for L large enough, $\mathbb{E}\left(\|s \circ \phi_{x_0}^{-1}\|_{L^\infty(B(0,RL^{-\frac{1}{m}}))}\right)$ is bounded from above by

$$\inf_{t \in \mathbb{R}_+^*} \frac{\sqrt{2}\lfloor \frac{n}{2} + 1 \rfloor}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{\lfloor \frac{n}{2}+1 \rfloor} \frac{1}{i!}(tL^{-\frac{1}{m}})^i$$
$$\times \left(\int_{B(0,(R+t)L^{-\frac{1}{m}})} \sum_{\substack{(j_1,\ldots,j_i) \\ \in \{1,\ldots,n\}^i}} \frac{\partial^{2i} e_L(x, x)}{\partial x_{j_1} \cdots \partial x_{j_i} \partial y_{j_1} \cdots \partial y_{j_i}} |dx|\right)^{1/2}.$$

Likewise, for every $j \in \{1, \ldots, n\}$, $\mathbb{E}\left(\|\frac{\partial(s \circ \phi_{x_0}^{-1})}{\partial z_j}\|_{L^\infty(B(0,RL^{-\frac{1}{m}}))}\right)$ is bounded from above by

$$\inf_{t \in \mathbb{R}_+^*} \frac{\sqrt{2}\lfloor \frac{n}{2} + 1 \rfloor}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{\lfloor \frac{n}{2}+1 \rfloor} \frac{1}{i!}(tL^{-\frac{1}{m}})^i$$
$$\times \left(\int_{B(0,(R+t)L^{-\frac{1}{m}})} \sum_{\substack{(j_1,\ldots,j_i) \\ \in \{1,\ldots,n\}^i}} \frac{\partial^{2i+2} e_L(x, x)}{\partial x_j \partial x_{j_1} \cdots \partial x_{j_i} \partial y_j y_{j_1} \cdots \partial y_{j_i}} |dx|\right)^{1/2}.$$

Now, the result is a consequence of the asymptotic estimate

$$\frac{\partial^{2i} e_L(x, x)}{\partial x_{j_1} \cdots \partial x_{j_i} \partial y_{j_1} \cdots \partial y_{j_i}} \underset{L \to +\infty}{\sim} \frac{1}{(2\pi)^n} L^{\frac{n+2i}{m}} \int_{K_0} |\xi_{j_1}|^2 \cdots |\xi_{j_i}|^2 |d\xi|,$$

see Theorem 2.3.6 of [6]. We used here that the balls $B_g(x_0, RL^{-\frac{1}{m}})$ and $\phi_{x_0}^{-1}(B(0, RL^{-\frac{1}{m}}))$ coincide to first order in L. $\square$

2.2. Proof of Theorem 0.3. Let $x_0 \in M$, $R > 0$ and $A \in Isom_g(\mathbb{R}^n, T_{x_0}M)$. Let $\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$ be a measure-preserving map such that $A = d_{|x_0}\phi_{x_0}^{-1}$. Let $(W, f_\Sigma) \in \mathcal{I}_\Sigma^{A^*K_{x_0}, R}$ and $(\delta, \epsilon) \in \mathcal{T}_{(W, f_\Sigma)}$. Let $(s_L)_{L \geq L_0}$ be a family given by Corollary 1.11 associated to f_Σ, where K_{x_0} is defined by (2.1). Denote by $s_L^\perp$ the hyperplane orthogonal to s_L in $\mathbb{U}_L$. Then,

$$\int_{s_L^\perp} \|s \circ \phi_{x_0}^{-1}\|_{L^\infty(B(0, R_W L^{-\frac{1}{m}}))} d\mu(s) \leq \int_{\mathbb{U}_L} \|s \circ \phi_{x_0}^{-1}\|_{L^\infty(B(0, R_W L^{-\frac{1}{m}}))} d\mu(s)$$

and for every $j \in \{1, \ldots, n\}$,

$$\int_{s_L^\perp} \left\|\frac{\partial}{\partial x_j}(s \circ \phi_{x_0}^{-1})\right\|_{L^\infty(B(0, R_W L^{-\frac{1}{m}}))} d\mu(s) \leq \int_{\mathbb{U}_L} \left\|\frac{\partial}{\partial x_j}(s \circ \phi_{x_0}^{-1})\right\|_{L^\infty(B(0, R_W L^{-\frac{1}{m}}))} d\mu(s),$$

compare the proof of Proposition 3.1 of [7]. From Proposition 2.1 and Markov's inequality we deduce that for every $T \in \mathbb{R}_+^*$,

$$\mu\left\{s \in s_L^\perp \mid \sup_{B_g(x_0, R_W L^{-\frac{1}{m}})} |s| \geq \frac{T\delta L^{\frac{n}{2m}}}{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}\right\} \leq \frac{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}{T\delta} \rho_{A^*K_{x_0}}(R_W) + o(1)$$

and for every $j \in \{1, \ldots, n\}$,

$$\mu\left\{s \in s_L^\perp \mid \sup_{B(0, R_W L^{-\frac{1}{m}})} |\frac{\partial}{\partial x_j}(s \circ \phi_{x_0}^{-1})| \geq \frac{T\epsilon L^{\frac{n+2}{2m}}}{\sqrt{n}\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}\right\}$$

$$\leq \frac{\sqrt{n}\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}{T\epsilon} \theta_{A^*K_{x_0}}^j(R_W) + o(1).$$

It follows that the measure of the set

$$\mathcal{E}_{s_L^\perp} = \left\{s \in s_L^\perp \mid \sup_{B_g(x_0, R_W L^{-\frac{1}{m}})} |s| < \frac{T\delta L^{\frac{n}{2m}}}{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}} \text{ and } \sup_{B(0, R_W L^{-\frac{1}{m}})} |d(s \circ \phi_{x_0}^{-1})| < \frac{T\epsilon L^{\frac{n+2}{2m}}}{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}\right\}$$

satisfies $\mu(\mathcal{E}_{s_L^\perp}) \geq 1 - \frac{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}{T}\left(\frac{1}{\delta}\rho_{A^*K_{x_0}}(R_W) + \frac{n\sqrt{n}}{\epsilon}\sum_{j=1}^n \theta_{A^*K_{x_0}}^j(R_W)\right) + o(1)$, where the $o(1)$ term can be chosen independently of x_0 since M is compact. Taking the supremum over the pairs $(\delta, \epsilon) \in \mathcal{T}_{(W, f_\Sigma)}$ and taking liminf, we deduce from (1.6) the estimate

$$\liminf_{L \to +\infty} \mu(\mathcal{E}_{s_L^\perp}) \geq 1 - \frac{\tau_{(W, f_\Sigma)}^{A^*K_{x_0}}}{T}.$$

Now, let $\mathcal{F}_T = \left\{a\frac{s_L}{\|s_L\|_{L^2(M)}} + \sigma \mid a > T \text{ and } \sigma \in \mathcal{E}_{s_L^\perp}\right\}$. From Lemma 3.6 of [7], every section $s \in \mathcal{F}_T$ vanishes transversally in $B_g(x_0, R_W L^{-\frac{1}{m}})$ along a hypersurface Σ_L such that $(B_g(x_0, R_W L^{-\frac{1}{m}}), \Sigma_L)$ is diffeomorphic to $(\mathbb{R}^n, \Sigma)$. Moreover, since μ is a product measure, $\liminf_{L \to +\infty} \mu(\mathcal{F}_T) \geq \left(\frac{1}{\sqrt{\pi}}\int_T^{+\infty} e^{-t^2} dt\right)\left(1 - \frac{\tau_{(W, f_\Sigma)}^{A^*K_{x_0}}}{T}\right)$. Taking the supremum over $T \in [\tau_{(W, f_\Sigma)}, +\infty[$, we deduce from (1.7) that

$$\liminf_{L \to +\infty} Prob_{x_0, \Sigma}(R_W) \geq \liminf_{L \to +\infty} \mu(\mathcal{F}_T) \geq p_{(W, f_\Sigma)}^{A^*K_{x_0}}.$$

Taking the supremum over all pairs $(W, f_\Sigma) \in \mathcal{I}_\Sigma^{A^* K_{x_0}, R}$, see (1.8), and then over every $A \in Isom_g(\mathbb{R}^n, T_{x_0} M)$, we obtain the first part of Theorem 0.3 by choosing

$$p_\Sigma^x(R) = \sup_{A \in Isom_g(\mathbb{R}^n, T_x M)} (p_\Sigma^{A^* K_x}(R)). \tag{2.2}$$

From Remark 0.5, there exists $c_{P,g} > 0$ such that for every $x \in M$ and $A \in Isom_g$ $(\mathbb{R}^n, T_x M)$, the ball $B(0, c_{P,g})$ is contained in $A^* K_x$. For every $R > 0$, we then have $\inf_M p_\Sigma^x(R) \geq p_\Sigma^{B(0, c_{P,g})}(R)$, where the right hand side is positive for R large enough, see Remark 1.9.

2.3. Proofs of Theorem 0.1 and Corollary 0.2.

Proof of Theorem 0.1. Let us denote by $\mathcal{R} = C^\infty(M, \mathbb{R}_+)$ the space of smooth positive functions on M. Let $g \in Met_{|dy|}(M)$, $\rho \in \mathcal{R}$, and $\tilde{g}$ be the normalized metric g/ρ^2. For every L large enough, let Λ_L be a subset of M such that the distance between any two distinct points of Λ_L is larger than $2L^{-\frac{1}{m}}$ in the metric $\tilde{g}$ and which is maximal with respect to this property. This means that Λ_L is not contained in any larger set having the same property. The $\tilde{g}$−balls centered at points of Λ_L and of radius $L^{-\frac{1}{m}}$ are then disjoint, whereas the ones of radius $2L^{-\frac{1}{m}}$ cover M. For every $s \in \mathbb{U}_L \backslash \Delta_L$ and every $x \in \Lambda_L$, we set $N_{x,\Sigma}(s) = 1$ if $B_{\tilde{g}}(x, L^{-\frac{1}{m}})$ contains a hypersurface $\tilde{\Sigma}$ such that $\tilde{\Sigma} \subset s^{-1}(0)$ and $(B_{\tilde{g}}(x, L^{-\frac{1}{m}}), \tilde{\Sigma})$ is diffeomorphic to $(\mathbb{R}^n, \Sigma)$, and $N_{x,\Sigma} = 0$ otherwise. Note that

$$\int_{\mathbb{U}_L \backslash \Delta_L} N_{x,\Sigma}(s) d\mu(s) \underset{L \to +\infty}{\sim} Prob_\Sigma^x(\rho(x)).$$

Thus,

$$\liminf_{L \to +\infty} \frac{1}{L^{\frac{n}{m}}} \mathbb{E}(N_\Sigma) \geq \liminf_{L \to +\infty} \frac{1}{L^{\frac{n}{m}}} \int_{\mathbb{U}_L \backslash \Delta_L} \Big(\sum_{x \in \Lambda_L} N_{x,\Sigma}(s) \Big) d\mu(s)$$

$$= \liminf_{L \to +\infty} \frac{1}{L^{\frac{n}{m}}} \sum_{x \in \Lambda_L} Prob_{x,\Sigma}(\rho(x))$$

$$\geq \frac{1}{2^n} \liminf_{L \to +\infty} \sum_{x \in \Lambda_L} Vol(B_{\tilde{g}}(x, 2L^{-\frac{1}{m}})) \rho^n(x) \Big(\frac{p_\Sigma^x(\rho(x))}{Vol_{eucl} B(0, \rho(x))} \Big)$$

by Theorem 0.3. Now, set

$$\mathcal{M}_\Sigma = \Big\{ \psi \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}_{\geq 0}), \ \forall (x, R) \in M \times \mathbb{R}_+, \ \psi(x, R) \leq \frac{p_\Sigma^x(R)}{Vol_{eucl} B(0, R)} \Big\}.$$

Then, for every $\psi \in \mathcal{M}_\Sigma$,

$$\liminf_{L \to +\infty} \frac{1}{L^{\frac{n}{m}}} \mathbb{E}(N_\Sigma) \geq \frac{1}{2^n} \liminf_{L \to +\infty} \int_M \sum_{x \in \Lambda_L} \mathbb{1}_{B_{\tilde{g}}(x, 2L^{-\frac{1}{m}})}(y) \rho^n(x) \psi(x, \rho(x)) |dvol_{\tilde{g}}(y)|$$

$$\geq \frac{1}{2^n} \liminf_{L \to +\infty} \int_M \max_{x \in \Lambda_L \cap B_{\tilde{g}}(y, 2L^{-\frac{1}{m}})} \Big(\rho^n \psi(x, \rho(x)) \Big) |dvol_{\tilde{g}}(y)|$$

$$= \frac{1}{2^n} \int_M \psi(y, \rho(y)) \rho^n(y) |dvol_{\tilde{g}}(y)| = \frac{1}{2^n} \int_M \psi \circ \rho |dy|.$$

We deduce Theorem 0.1 by defining $c_\Sigma(P)$ to be the supremum

$$c_\Sigma(P) = \frac{1}{2^n} \sup_{(g,\rho,\psi)\in Met_{|dy|}(M)\times\mathcal{R}\times\mathcal{M}_\Sigma} \int_M \psi(y,\rho(y))|dy|. \qquad (2.3)$$

Choosing $\psi = \frac{\inf p_\Sigma^x}{Vol_{eucl}B(0,\cdot)} \in \mathcal{M}_\Sigma$ and $\rho = R$ we obtain

$$c_\Sigma(P) \geq \frac{1}{2^n} Vol_{|dy|}(M)\frac{\inf_M p_\Sigma^x(R)}{Vol_{eucl}B(0,R)} \qquad (2.4)$$

which is positive for R large enough by Theorem 0.3. □

Remark 2.2. It might be that (2.3) can be rewritten as

$$c_\Sigma(P) = \frac{1}{2^n} \sup_{g\in Met_{|dy|}(M)} \int_M \sup_{R>0}\left(\frac{p_\Sigma^x(R)}{Vol_{eucl}(B(0,R))}\right)|dx|,$$

but (2.3) or actually (2.4) suffices for the purpose of this paper.

Proof of Corollary 0.2. For every $i \in \{0,\ldots,n-1\}$ and every large enough $L > 0$,

$$\mathbb{E}(b_i) = \int_{\mathbb{U}_L\setminus\Delta_L} b_i(s^{-1}(0))d\mu(s) \geq \int_{\mathbb{U}_L\setminus\Delta_L}\left(\sum_{[\Sigma]\in\mathcal{H}_n} N_\Sigma(s)b_i(\Sigma)\right)d\mu(s)$$

$$\geq \sum_{[\Sigma]\in\mathcal{H}_n} b_i(\Sigma)\mathbb{E}(N_\Sigma).$$

The result is a consequence of Theorem 0.1 after passing to the liminf in the latter bound. □

3. Explicit Estimates

The goal of this section is to obtain explicit lower bounds for the constants $c_\Sigma(P)$ and $\inf_{x\in M} p_\Sigma^x(R)$ appearing in Theorems 0.1 and 0.3, when Σ is diffeomorphic to the product of spheres $S^{i+1} \times S^{n-i-1}$ (whose i-th Betti number is at least one). In the first paragraph, we approximate quantitatively the product of a polynomial function and a Gaussian one by a function whose Fourier transform has compact support. We then apply this result to a particular degree four polynomial vanishing along a product of spheres to finally get Theorem 0.4, Corollary 0.6 and Corollary 0.7.

3.1. Key estimates for the approximation. Let $\tilde{\chi}_c : \mathbb{R}^n \to [0,1]$ be a smooth function with support in the ball of radius $c > 0$, such that $\tilde{\chi}_c = 1$ on the ball of radius $c/2$. For every $Q \in \mathbb{R}[X_1,\ldots,X_n]$ and every $\eta > 0$, we set $q : x \in \mathbb{R}^n \mapsto q(x) = Qe^{-\frac{\|x\|^2}{2}} \in \mathbb{R}$ and

$$q_\eta^c : x \in \mathbb{R}^n \mapsto q_\eta^c(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\tilde{\chi}_c(\eta\xi)\mathcal{F}(q(x))(\xi)e^{i\langle x,\xi\rangle}|d\xi|. \qquad (3.1)$$

Note that $q_\eta^c \in S_{B(0,c/\eta)}(\mathbb{R}^n)$, see (1.1).

Proposition 3.1. *Let $Q = \sum_{I \in \mathbb{N}^n} a_I x^I \in \mathbb{R}[X_1, \ldots, X_n]$ and $c, \eta > 0$. Then,*

1. $\|q_\eta^c - q\|_{L^\infty(\mathbb{R}^n)} \le \sqrt{\lfloor n/2 + 1 \rfloor} \left(\frac{c}{2\eta}\right)^{\frac{n-2}{2}} e^{-\frac{1}{4}\left(\frac{c}{2\eta}\right)^2} \left(\sum_{I \in \mathbb{N}^n} |a_I| \sqrt{I!}\right).$

2. $\forall k \in \{1, \ldots, n\}, \left\|\frac{\partial q_\eta^c}{\partial x_k} - \frac{\partial q}{\partial x_k}\right\|_{L^\infty(\mathbb{R}^n)} \le \sqrt{\lfloor n/2 + 3 \rfloor} \left(\frac{c}{2\eta}\right)^{\frac{n}{2}} e^{-\frac{1}{4}\left(\frac{c}{2\eta}\right)^2} \left(\sum_{I \in \mathbb{N}^n} |a_I| \sqrt{I!}\right).$

3. $\|q_\eta^c - q\|_{L^2(\mathbb{R}^n)}^2 \le \sqrt{2\pi}^n N(Q) \left(\sum_{I \in \mathbb{N}^n} a_I^2 I!\right) e^{-\frac{1}{2}\left(\frac{c}{2\eta}\right)^2}$, *where $N(Q)$ denotes the number of monomials of Q.*

Proof. For every $x \in \mathbb{R}^n$, we have

$$|q_\eta^c(x) - q(x)| \le \frac{1}{(2\pi)^n} \int_{\|\xi\| \ge \frac{c}{2\eta}} |\mathcal{F}(Q e^{-\frac{\|x\|^2}{2}})|(\xi)|d\xi|$$

$$\le \frac{1}{(2\pi)^n} \sum_{I \in \mathbb{N}^n} |a_I| \int_{\|\xi\| \ge \frac{c}{2\eta}} |\mathcal{F}(x_I e^{-\frac{\|x\|^2}{2}})|(\xi)|d\xi|.$$

Moreover,

$$\mathcal{F}(x_I e^{-\frac{\|x\|^2}{2}}) = i^{|I|} \frac{\partial}{\partial \xi_I}\left(\mathcal{F}(e^{-\frac{\|x\|^2}{2}})\right) = \sqrt{2\pi}^n i^{|I|}(-1)^{|I|} \prod_{j=1}^n \left(H_{i_j}(\xi_j) e^{-\frac{\xi_j^2}{2}}\right), \quad (3.2)$$

where we have set $I = (i_1, \ldots, i_n)$ and H_j the j-th Hermite polynomial. We deduce from Cauchy–Schwarz inequality that

$$|q_\eta^c(x) - q(x)| \le \frac{1}{\sqrt{2\pi}^n} \sum_{I \in \mathbb{N}^n} |a_I| \left(\prod_{j=1}^n \int_{\mathbb{R}} H_{i_j}^2(\xi_j) e^{-\frac{\xi_j^2}{2}} d\xi_j\right)^{1/2} \left(\int_{\|\xi\| \ge \frac{c}{2\eta}} e^{-\frac{\|\xi\|^2}{2}} d\xi\right)^{1/2}$$

$$\le \frac{1}{(2\pi)^{n/4}} \left(\sum_{I \in \mathbb{N}^n} |a_I| \sqrt{I!}\right) \sqrt{Vol(S^{n-1})} \left(\int_{\frac{c}{2\eta}}^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} dr\right)^{1/2},$$

since

$$\int_{\mathbb{R}} H_k^2(\xi) e^{-\frac{\xi^2}{2}} d\xi = k! \sqrt{2\pi}. \tag{3.3}$$

Likewise, after integration by parts we obtain

$$\int_{\frac{c}{2\eta}}^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} dr = [-r^{n-2} e^{-\frac{r^2}{2}}]_{\frac{c}{2\eta}}^{+\infty} + (n-2) \int_{\frac{c}{2\eta}}^{+\infty} r^{n-3} e^{-\frac{r^2}{2}} dr$$

$$\le \left(\frac{c}{2\eta}\right)^{n-2} e^{-\frac{1}{2}\left(\frac{c}{2\eta}\right)^2} + (n-2)\left(\frac{c}{2\eta}\right)^{n-4} e^{-\frac{1}{2}\left(\frac{c}{2\eta}\right)^2} + \cdots$$

From the latter we deduce, when $|\frac{c}{2\eta}| \ge 1$,

$$\int_{\frac{c}{2\eta}}^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} dr = \lfloor \frac{n}{2} + 1 \rfloor \left(\frac{c}{2\eta}\right)^{n-2} e^{-\frac{1}{2}\left(\frac{c}{2\eta}\right)^2} (n-2)(n-4) \cdots$$

Recall that $Vol(S^{n-1}) = \frac{\sqrt{2\pi}^n}{(n-2)(n-4)\cdots 2}$ if n is even and $\frac{\sqrt{2}\sqrt{2\pi}^n}{\sqrt{\pi}(n-2)(n-4)\cdots 3\times 1}$ if n is odd. We thus finally get $\|q_\eta^c - q\|_{L^\infty} \leq \sqrt{\lfloor\frac{n}{2} + 1\rfloor}\left(\frac{c}{2\eta}\right)^{\frac{n-2}{2}} e^{-\frac{1}{4}\left(\frac{c}{2\eta}\right)^2} \left(\sum_{I\in\mathbb{N}^n} |a_I|\sqrt{I!}\right)$. Likewise, for every $k \in \{1, \dots, n\}$,

$$\left\|\frac{\partial q_\eta^c}{\partial x_k} - \frac{\partial q}{\partial x_k}\right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} \int_{\|\xi\|\geq\frac{c}{2\eta}} |\xi_k||\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})(\xi)|d\xi|$$

$$\leq \frac{1}{\sqrt{2\pi}^n} \sum_{I\in\mathbb{N}^n} |a_I|\left(\prod_{j=1}^n \int_{\mathbb{R}} H_{i_j}^2(\xi)e^{-\frac{\xi_j^2}{2}}dx_j\right)^{\frac{1}{2}}$$

$$\times\left(\int_{\|\xi\|\geq\frac{c}{2\eta}} \|\xi\|^2 e^{-\frac{\|\xi\|^2}{2}}d\xi\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{(2\pi)^{n/4}} \sum_{I\in\mathbb{N}^n} |a_I|\sqrt{I!}Vol(S^{n-1})^{\frac{1}{2}}\left(\int_{\frac{c}{2\eta}}^{+\infty} r^{n+1}e^{-r^2/2}dr\right)^{\frac{1}{2}}$$

$$\leq \sqrt{\lfloor\frac{n}{2} + 3\rfloor}\left(\frac{c}{2\eta}\right)^{n/2} e^{-\frac{1}{4}\left(\frac{c}{2\eta}\right)^2}\left(\sum_{I\in\mathbb{N}^n} |a_I|\sqrt{I!}\right) \text{ from (3.3).}$$

Lastly,

$$\|q_\eta^c - q\|_{L^2(\mathbb{R}^n)}^2 \leq \left\|\mathcal{F}^{-1}\left(\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})(1 - \tilde{\chi}_c(\eta\xi))\right)\right\|_{L^2(\mathbb{R}^n)}^2$$

$$\leq \frac{1}{(2\pi)^n} \int_{\xi\geq\frac{c}{2\eta}} |\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})|^2|d\xi|$$

$$= \int_{\xi\geq\frac{c}{2\eta}} \left|\sum_{I\in\mathbb{N}^n} i^{|I|}a_I \prod_{j=1}^n H_{i_j}(\xi_j)e^{-\frac{\xi_j^2}{2}}\right|^2|d\xi|$$

$$\leq N(Q)e^{-\frac{1}{2}\left(\frac{c}{2\eta}\right)^2} \sum_{I\in\mathbb{N}^n} a_I^2 \prod_{j=1}^n \int_{\mathbb{R}} H_{i_j}^2(\xi_j)e^{-\frac{\xi_j^2}{2}}d\xi_j$$

$$\leq \sqrt{2\pi}^n N(Q)\left(\sum_{I\in\mathbb{N}^n} a_I^2 I!\right)e^{-\frac{1}{2}\left(\frac{c}{2\eta}\right)^2}.$$

Here, we used Plancherel's equality for the second inequality, (3.2) for the equality, and Cauchy–Schwarz and (3.3) for the last two inequalities respectively. □

3.2. The product of spheres.

For every $n > 0$ and every $i \in \{0, \dots, n - 1\}$, let $Q_i : (x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1} \mapsto (\|x\|^2 - 2)^2 + \|y\|^2 - 1 \in \mathbb{R}$. We recall that this polynomial vanishes in the ball of radius $\sqrt{5}$ along a hypersurface diffeomorphic to the product of spheres $S^i \times S^{n-i-1}$, see §2.3.2 of [7]. Let

$$q_i : (x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1} \mapsto Q_i(x, y)e^{-\frac{1}{2}(\|x\|^2 + \|y\|^2)} \in \mathbb{R}.$$

This function belongs to the Schwartz space and has the same vanishing locus as Q_i. Let us quantify the transversality of this vanishing. We set $W = \{(x, y) \in \mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}, \|x\|^2 + \|y\|^2 \leq 5\}$.

Lemma 3.2. *For every* $\delta \leq 1/2$, $\left(\delta e^{-5/2}, \frac{e^{-5/2}}{2}(2-\delta)\right) \in \mathcal{T}_{(W,q_i)}$, *where* $\mathcal{T}_{(W,q_i)}$ *is defined in Definition* 1.5.

Proof. Let $(x, y) \in \mathbb{R}^{i-1} \times \mathbb{R}^{n-i-1}$ be such that $\|x\|^2 + \|y\|^2 \leq 5$, and $\delta \leq \frac{1}{2}$. If $|q_i(x, y)| < \delta e^{-5/2}$, then $|Q_i(x, y)| < \delta$, so that $1 - \delta < (\|x\|^2 - 2)^2 + \|y\|^2 < 1 + \delta$. This implies that $\frac{1}{2} < 2 - \sqrt{1 + \delta} < \|x\|^2$ and that either $\frac{1}{2} < \|x\|^2 - 2$ or $\frac{1}{4} < \|y\|^2$ since $\delta \leq \frac{1}{2}$. Moreover, for every $j \in \{1, \ldots, i+1\}$, $\left|\frac{\partial q_i}{\partial x_j}\right| \geq \left|\frac{\partial Q_i}{\partial x_j}\right| e^{-5/2} - |x_j|\delta e^{-5/2}$ which is greater or equal to $\geq 4|x_j|\left|\|x\|^2 - 2\right|e^{-5/2} - |x_j|\delta e^{-5/2} \geq |x_j|e^{-5/2}(4\left|\|x\|^2 - 2\right| - \delta)$ and for every $k \in \{1, \ldots, n-i-1\}$, $\left|\frac{\partial q_i}{\partial y_k}\right| \geq \left|\frac{\partial Q_i}{\partial y_k}\right| e^{-5/2} - |y_k|\delta e^{-5/2} \geq |y_k|e^{-5/2}(2 - \delta)$.

Summing up, we deduce $|d_{(x,y)}q_i|^2 \geq \|x\|^2 e^{-5}(4\left|\|x\|^2 - 2\right| - \delta)^2 + \|y\|^2 e^{-5}(2 - \delta)^2$ which is greater or equal to $\frac{e^{-5}}{2}(4\left|\|x\|^2 - 2\right| - \delta)^2 + \|y\|^2 e^{-5}(2 - \delta)^2 \geq \frac{e^{-5}}{4}(2 - \delta)^2$. Since on the boundary of the ball W, we have either $\|x\|^2 \geq 7/2$ or $\|y\|^2 \geq 3/2$, the values of the function q_i are greater than $\frac{1}{2}e^{-5/2}$ and we get the result. $\square$

We now estimate the L^2-norm of q_i.

Lemma 3.3. *For every* $i \in \{0, \ldots, n-1\}$, $\|q_i\|_{L^2(\mathbb{R}^n)} \leq \sqrt{\frac{3}{2}}\pi^{n/4}(n+6)^2$.

Proof. We have

$$\|q_i\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}} (\|x\|^4 - 4\|x\|^2 + 3 + \|y\|^2)^2 e^{-\|x\|^2 - \|y\|^2} dx dy$$

$$\leq \sqrt{\pi}^{n-i-1} \int_{\mathbb{R}^{i+1}} (\|x\|^8 + 16\|x\|^4) e^{-\|x\|^2} dx$$

$$+ \sqrt{\pi}^{i+1} \int_{\mathbb{R}^{n-i-1}} (\|y\|^4 + 6\|y\|^2 + 9) e^{-\|y\|^2} dy$$

$$+ 2\left(\int_{\mathbb{R}^{i+1}} \|x\|^4 e^{-\|x\|^2} dx\right)\left(\int_{\mathbb{R}^{n-i-1}} (\|y\|^2 + 3) e^{-\|y\|^2} dy\right).$$

Now,

$$\int_{\mathbb{R}^{i+1}} (\|x\|^8 + 16\|x\|^4) e^{-\|x\|^2} dx = \frac{1}{2} Vol(S^i) \int_0^{+\infty} (t^4 + 16t^2) t^{\frac{i-1}{2}} e^{-t} dt$$

$$= \frac{1}{2} Vol(S^i)\left(\Gamma\left(\frac{i+9}{2}\right) + 16\Gamma\left(\frac{i+5}{2}\right)\right) \leq \frac{17}{2} Vol(S^i)\Gamma\left(\frac{i+9}{2}\right)$$

and

$$\int_{\mathbb{R}^{n-i-1}} (\|y\|^4 + 6\|y\|^2 + 9) e^{-\|y\|^2} dy$$

$$= \frac{1}{2} Vol(S^{n-i-2}) \int_0^{+\infty} (t^2 + 6t + 9) t^{\frac{1}{2}(n-i-3)} e^{-t} dt$$

$$= \frac{1}{2} Vol(S^{n-i-2})\left(\Gamma\left(\frac{n-i+3}{2}\right) + 6\Gamma\left(\frac{n-i+1}{2}\right) + 9\Gamma\left(\frac{n-i-1}{2}\right)\right)$$

$$\leq \frac{25}{2} Vol(S^{n-i-2})\Gamma\left(\frac{n-i+3}{2}\right).$$

Likewise $\int_{\mathbb{R}^{i+1}} \|x\|^4 e^{-\|x\|^2} dx = \frac{1}{2} Vol(S^i) \int_0^{+\infty} t^{\frac{i+3}{2}} e^{-t} dt = \frac{1}{2} Vol(S^i) \Gamma(\frac{i+5}{2})$, and

$$
\begin{aligned}
\int_{\mathbb{R}^{n-i-1}} (\|y\|^2 + 3) e^{-\|y\|^2} dy &= \frac{1}{2} Vol(S^{n-i-2}) \int_0^{+\infty} (t+3) t^{\frac{n-i-3}{2}} e^{-t} dt \\
&= \frac{1}{2} Vol(S^{n-i-2}) \left(\Gamma(\frac{n-i+1}{2}) + 3\Gamma(\frac{n-i-1}{2}) \right) \\
&\le \frac{7}{2} Vol(S^{n-i-2}) \Gamma(\frac{n-i+1}{2}).
\end{aligned}
$$

Finally, since $Vol(S^i) = \frac{2\pi^{\frac{i+1}{2}}}{\Gamma(\frac{i+1}{2})}$ and $Vol(S^{n-i-2}) = \frac{2\pi^{\frac{n-i-1}{2}}}{\Gamma(\frac{n-i-1}{2})}$, we get

$$
\begin{aligned}
\|q_i\|^2_{L^2(\mathbb{R}^n)} &\le \sqrt{\pi}^n \left(17 \frac{\Gamma(\frac{i+9}{2})}{\Gamma(\frac{i+1}{2})} + 25 \frac{\Gamma(\frac{n-i+3}{2})}{\Gamma(\frac{n-i-1}{2})} + 14 \frac{\Gamma(\frac{n-i+1}{2})\Gamma(\frac{i+5}{2})}{\Gamma(\frac{n-i-1}{2})\Gamma(\frac{i+1}{2})} \right) \\
&\le \sqrt{\pi}^n \left(\frac{17}{16}(i+7)^4 + \frac{25}{4}(n-i+1)^2 + \frac{7}{4}(n-i-1)(i+3)^2 \right) \\
&\le \frac{3}{2}\sqrt{\pi}^n (n+6)^4.
\end{aligned}
$$

The last inequality follows from $n+6 \ge 7$, which implies that $\frac{25}{4}(n-i+1)^2 \le \frac{25}{4\times49}(n+6)^4$ and $\frac{7}{4}(n-i-1)(i+3)^2 \le \frac{1}{4}(n+6)^4$. $\square$

We now approximate q_i by a function whose Fourier transform has compact support. For every $i \in \{0, \dots, n-1\}$ and $c > 0$, we set

$$
q_{i,c,\eta} : x \in \mathbb{R} \mapsto q_{i,c,\eta}(x) = q^c_{i,\eta}(\eta x) = \frac{1}{\eta^n} \int_{\mathbb{R}^n} \tilde{\chi}_c(\xi) \mathcal{F}(Q_i e^{-\frac{\|x\|^2}{2}})(\frac{\xi}{\eta}) e^{i\langle x,\xi \rangle} |d\xi| \in \mathbb{R},
\tag{3.4}
$$

see (3.1). By construction, $q_{i,c,\eta}$ belongs to the Schwartz space of $\mathbb{R}^n$ and its Fourier transform has support in the ball of radius c, so that with the notations introduced in Sect. 1.1, $q_{i,c,\eta} \in S_{B(0,c)}(\mathbb{R}^n)$.

Corollary 3.4. *For every $i \in \{0, \dots, n-1\}$, every $c > 0$ and every $\eta \le \frac{c}{48n}$, $q_{i,c,\eta}$ vanishes in the ball $W_\eta = \{x \in \mathbb{R}^n, \|x\|^2 \le 5/\eta^2\}$ along a hypersurface diffeomorphic to $S^i \times S^{n-i-1}$. Moreover, $\left(\frac{e^{-5/2}}{4}, \frac{\eta}{\sqrt{2}} e^{-5/2} \right) \in \mathcal{T}_{(W_\eta, q_{i,c,\eta})}$ and $\|q_{i,c,\eta}\|_{L^2(\mathbb{R}^n)} \le \frac{3}{2\eta^{n/2}} \pi^{n/4} (n+6)^2$.*

Proof. We have $Q_i(x,y) = \sum_{k=1}^{i+1} x_k^4 + 2\sum_{1 \le j < k \le n} x_j^2 x_k^2 - 4\sum_{k=1}^{i+1} x_k^2 + \sum_{k=1}^{n-i-1} y_k^2 + 3$ so that, with the notations of Proposition 3.1,

$$
\sum_{I \in \mathbb{N}^n} |a_I| \sqrt{I!} = (i+1)\sqrt{4!} + 4\binom{i+1}{2} + 4\sqrt{2}(i+1) + (n-i-1)\sqrt{2} + 3
$$

$$
\le 5n + 2n^2 + 8n + 3 \le 18n^2
$$

and $\sum_{I \in \mathbb{N}^n} a_I^2 I! = (i+1)4! + 16\binom{i+1}{2} + 32(i+1) + 2(n-i-1) + 9$ so that $\sum_{I \in \mathbb{N}^n} a_I^2 I! \le 24n + 8n^2 + 34n + 9 \le 75n^2$, whereas $N(Q_i) = (i+1) + \binom{i+1}{2} + (i+1) + (n-i-1) + 1$

so that $N(Q_i) \leq 2n + 1 + \frac{n(n-1)}{2} \leq 3n^2$. Noting that $\sqrt{\lfloor \frac{n}{2} + 1 \rfloor} \leq \sqrt{\lfloor \frac{n}{2} + 3 \rfloor} \leq 2\sqrt{n}$, that $\left(\frac{c}{2\eta}\right)^{\frac{n-2}{2}} \leq \left(\frac{c}{2\eta}\right)^{\frac{n}{2}}$ as soon as $\frac{c}{2\eta} \geq 1$, and that $\frac{5}{2} \ln n + \frac{n}{2} \ln\left(\frac{c}{2\eta}\right) \leq 3n\left(\frac{c}{2\eta}\right)$ under the same hypothesis, we deduce from Proposition 3.1 that when $\eta \leq \frac{c}{48n}$,

$$\|q_{i,c,\eta}(x) - q_i(\eta x)\|_{L^\infty(\mathbb{R}^n)} \leq 36 e^{-\frac{1}{8}\left(\frac{c}{2\eta}\right)^2} \leq 36 e^{-72n^2}$$

and for every $k \in \{1, \ldots, n\}$, $\left\| \frac{\partial q_{i,c,\eta}}{\partial x_k}(x) - \eta \frac{\partial q_i}{\partial x_k}(\eta x) \right\|_{L^\infty(\mathbb{R}^n)} \leq 36\eta e^{-72n^2}$. For every $x \in \mathbb{R}^n$ such that $\|x\|^2 \leq 5/\eta^2$ and every $\eta \leq \frac{c}{48n}$, it follows from Lemma 3.2, after choosing $\delta = 1/2$, that

$$q_{i,c,\eta}(x) \leq \frac{e^{-5/2}}{4} \Rightarrow q_i(\eta x) \leq \frac{e^{-5/2}}{2} \Rightarrow |d_{|\eta x}q_i| > 3\frac{e^{-5/2}}{4} \Rightarrow |d_{|x}q_{i,c,\eta}| > \eta\frac{e^{-5/2}}{\sqrt{2}},$$

since $|d_{|x}q_{i,c,\eta}| \geq \eta|d_{|\eta x}q_i| - |d_{|x}q_{i,c,\eta} - \eta d_{|\eta x}q_i| > \eta\frac{3e^{-5/2}}{4} - \sqrt{\sum_{k=1}^{n} \left| \frac{\partial q_{i,c,\eta}}{\partial x_k}(x) - \eta \frac{\partial q_i}{\partial x_k}(\eta x) \right|^2}$ which is greater or equal to $\eta\left(\frac{3e^{-5/2}}{4} - 36\sqrt{n}e^{-72n^2}\right) > \eta\frac{e^{-5/2}}{\sqrt{2}}$. From Lemma 3.6 of [7], $q_{i,c,\eta}$ vanishes in the ball W_η along a hypersurface diffeomorphic to $S^i \times S^{n-i-1}$ and by definition, $\left(\frac{e^{-5/2}}{4}, \eta\frac{e^{-5/2}}{\sqrt{2}}\right) \in \mathcal{T}_{(W_\eta, q_{i,c,\eta})}$ if $\eta \leq \frac{c}{48n}$.

Lastly, we estimate the L^2-norm of $q_{i,c,\eta}$. By Proposition 3.1 and the bounds given above, $\|q_{i,\eta}^c - q_i\|_{L^2(\mathbb{R}^n)}^2 \leq \sqrt{2\pi}^n 225n^4 e^{-288n^2}$. Therefore by Lemma 3.3,

$$\|q_{i,c,\eta}\|_{L^2(\mathbb{R}^n)} = \frac{1}{\eta^{n/2}} \|q_{i,\eta}^c\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\eta^{n/2}} \left(\|q_i\|_{L^2(\mathbb{R}^n)} + \|q_{i,\eta}^c - q_i\|_{L^2(\mathbb{R}^n)}\right)$$

$$\leq \frac{1}{\eta^{n/2}} \left(\sqrt{\frac{3}{2}}\pi^{n/4}(n+6)^2 + (\sqrt{2\pi}^n 225n^4 e^{-288n^2})^{1/2}\right)$$

$$\leq \frac{3}{2\eta^{n/2}} \pi^{n/4}(n+6)^2.$$

$\square$

3.3. Proofs of Theorem 0.4, Corollary 0.6 and Corollary 0.7.

Proof of Theorem 0.4. Let us choose $c = c_{P,g}$ and $\eta = \frac{c_{P,g}}{48n}$. It follows from Corollary 3.4 that for $R \geq \frac{48\sqrt{5}n}{c_{P,g}}$ we have $(W_\eta, q_{i,c,\eta}) \in \mathcal{I}_{S^i \times S^{n-1-i}}^{B(0,c_{P,g}),R}$, see Definition 1.8 for the definition of $\mathcal{I}_{\Sigma}^{K,R}$ and (3.4) for the definition of $q_{i,c,\eta}$. This implies that $(W_\eta, q_{i,c,\eta}) \in \mathcal{I}_{S^i \times S^{n-1-i}}^{A^*K_x, R}$ for any $x \in M$ and any $A \in Isom_g(\mathbb{R}^n, T_x M)$, since $B(0, c_{P,g}) \subset A^*K_x \subset B(0, d_{P,g})$. From Remark 1.7, we get that for every $x \in M$ and every $R \geq \frac{48\sqrt{5}n}{c_{P,g}}$,

$$p_{S^i \times S^{n-i-1}}^x(R) \geq \frac{1}{2\sqrt{\pi}} \exp(-(2\tau + 1)^2).$$

From (1.4), (1.5), (1.6) and Corollary 3.4 with $\eta = \frac{c_{P,g}}{48n}$, using that $\nu(A^*K_x) \leq Vol(B(0, d_{P,g}))$, we deduce

$$\tau \leq \frac{3}{2}\pi^{n/4}(n+6)^2\left(\frac{48n}{c_{P,g}}\right)^{n/2}\left(\frac{4}{e^{-5/2}}\frac{1}{\sqrt{\pi}^n}\sqrt{2Vol(B(0,d_{P,g}))}\right.$$

$$\times \left\lfloor\frac{n}{2}+1\right\rfloor \exp\left(48\sqrt{5}n\sqrt{n}\frac{d_{P,g}}{c_{P,g}}\right)$$

$$\left. +\frac{48n\sqrt{2}}{e^{-5/2}c_{P,g}}n\sqrt{n}\frac{n}{\sqrt{\pi}^n}\sqrt{2Vol(B(0,d_{P,g}))}\left\lfloor\frac{n}{2}+1\right\rfloor d_{P,g}\exp\left(48\sqrt{5}n\sqrt{n}\frac{d_{P,g}}{c_{P,g}}\right)\right)$$

$$\leq \frac{3}{4\pi^{n/4}}(n+6)^3(48n)^{n/2}\sqrt{2Vol(B(0,1))}\left(\frac{d_{P,g}}{c_{P,g}}\right)^{n/2}\exp\left(48\sqrt{5}n^{3/2}\frac{d_{P,g}}{c_{P,g}}\right)$$

$$\left(4e^{5/2}+\sqrt{2}e^{5/2}n^{5/2}(48n)\frac{d_{P,g}}{c_{P,g}}\right)$$

$$\leq 20\frac{(n+6)^{11/2}}{\sqrt{\Gamma\left(\frac{n}{2}+1\right)}}\left(48n\frac{d_{P,g}}{c_{P,g}}\right)^{\frac{n+2}{2}}\exp\left(48\sqrt{5}n^{3/2}\frac{d_{P,g}}{c_{P,g}}\right).$$

The estimate for $c_{[S^i\times S^{n-i-1}]}$ follows from the above estimate with $R = 48\sqrt{5}\frac{n}{c_{P,g}}$, see (2.4). $\square$

Proof of Corollary 0.6. If P is the Laplace–Beltrami operator associated to a metric g on M, then we choose as the Lebesgue measure $|dy|$ on M the measure $|dvol_g|$ associated to g, so that $g \in Met_{|dy|}(M)$ and the principal symbol of P equals $\xi \in T^*M \mapsto \|\xi\|^2 \in \mathbb{R}$. Theorem 0.4 then applies with $m = 2$ and $c_{P,g} = d_{P,g} = 1$ and we deduce, using $\Gamma(\frac{n}{2}+1) \geq 1/2$, that

$$\tau \leq 20\frac{(7n)^{11/2}}{\sqrt{\Gamma\left(\frac{n}{2}+1\right)}}(48n)^{\frac{n+2}{2}}\exp(108n^{3/2})$$

$$\leq \exp\left(\ln(20\sqrt{2})+\frac{11}{2}\ln 7+\frac{n+2}{2}\ln 48+\frac{13}{2}\ln n+\frac{n}{2}\ln n+108n^{3/2}\right)$$

$$\leq \exp\left(18+\frac{17}{2}(n-1)+\frac{n}{2}(2\sqrt{n}-1)+108n^{3/2}\right) \leq \exp(127n^{3/2}).$$

Theorem 0.4 then provides, for every $i \in \{0,\ldots,n-1\}$, that

$$\left(Vol_g(M)\right)^{-1}c_{[S^i\times S^{n-i-1}]}(P) \geq \exp\left(-(2\tau+1)^2-(n+1)\ln 2-\frac{1}{2}\ln\pi\right.$$

$$-n\ln(48\sqrt{5}n)-\ln(\pi^{n/2})+\ln(\Gamma(n/2+1)))$$

$$\geq \exp\left(-(2\tau+1)^2-3/2-6\ln n-n\ln n\right)$$

$$\geq \exp\left(-\exp(256n^{3/2})-\exp\left(\ln(17/2)+\ln n+\ln(\ln n)\right)\right)$$

$$\geq \exp(-\exp(257n^{3/2})).$$

$\square$

Remark 3.5. Under the assumptions of Corollary 0.6, we get likewise for $R \geq 48\sqrt{5}n$,

$$\inf_{x\in M}\left(p^x_{S^i\times S^{n-i-1}}(R)\right) \geq \frac{1}{2\sqrt{\pi}}\exp(-\exp(256n^{3/2})) \geq \exp(-\exp(257n^{3/2})).$$

Proof of Corollary 0.7. If P denotes the Dirichlet-to-Neumann operator on M, then the principal symbol of P equals $\xi \in T^*M \mapsto \|\xi\| \in \mathbb{R}$. Theorem 0.4 then applies with $m = 1$ and $c_{P,g} = d_{P,g} = 1$. Thus, the proof is the same as the one of Corollary 0.6. ☐

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