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SMOOTH MODULI SPACES OF ASSOCIATIVE SUBMANIFOLDS

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Abstract

Let M^7 be a smooth manifold equipped with a G_2-structure ϕ, and Y^3 be a closed compact ϕ-associative submanifold. McLean [Deformations of calibrated submanifolds, *Comm. Anal. Geom.* **6** (1998), 705–747] proved that the moduli space $\mathcal{M}_{Y,\phi}$ of the ϕ-associative deformations of Y has vanishing virtual dimension. In this paper, we perturb ϕ into a G_2-structure ψ in order to ensure the smoothness of $\mathcal{M}_{Y,\psi}$ near Y. If Y is allowed to have a boundary moving in a fixed coassociative submanifold X, it was proved in Gayet and Witt [Deformations of associative submanifolds with boundary, *Adv. Math.* **226** (2011), 2351–2370] that the moduli space $\mathcal{M}_{Y,X}$ of the associative deformations of Y with boundary in X has finite virtual dimension. We show here that a generic perturbation of the boundary condition X into X' gives the smoothness of $\mathcal{M}_{Y,X'}$. In another direction, we use Bochner's technique to prove a vanishing theorem that forces $\mathcal{M}_Y$ or $\mathcal{M}_{Y,X}$ to be smooth near Y. For every case, some explicit families of examples will be given.

1. Introduction

In the Euclidean space $(\mathbb{R}^7, g_0)$ with its canonical coordinates $(x_i)_{i=1,\dots,7}$, consider the 3-form

$$\phi_0 = \mathrm{d}x_{123} + \mathrm{d}x_{145} + \mathrm{d}x_{167} + \mathrm{d}x_{246} - \mathrm{d}x_{257} - \mathrm{d}x_{347} - \mathrm{d}x_{356},$$

and G_2 the subgroup of SO(7) defined by $G_2 = \{g \in \mathrm{SO}(7), g^*\phi_0 = \phi_0\}$. If M is an oriented spin seven-dimensional Riemannian manifold, its structure group can be reduced to $G_2 \subset \mathrm{SO}(7)$. Given a set of trivialization charts for TM compatible with G_2, M inherits a non-degenerate 3-form ϕ and a metric g, which are the pullbacks of ϕ_0 and g_0 by these charts. We call the pair (ϕ, g) a *G_2-structure*. Moreover, TM inherits a vector product $\times$ defined by

$$\forall u, v, w \in TM, \quad \langle u \times v, w\rangle = g(u \times v, w) = \phi(u, v, w).$$

Note that in $\mathbb{R}^7$, the subspace $\mathbb{R}^3 \times \{0\}$ is stable under this vector product, which induces the classical vector product on $\mathbb{R}^3$. When ϕ is closed and coclosed for g, the structure is said to be *torsion-free*. In this situation, the holonomy of g is a subgroup of G_2, see [12].

A three-dimensional submanifold Y in (M, ϕ, g) is called *ϕ-associative*, or simply *associative* when there is no ambiguity, if its tangent bundle is stable under the vector product associated to ϕ. In other terms, ϕ restricted to Y is a volume form for Y. Likewise, a four-dimensional submanifold X is called *coassociative* if the fibres of its normal bundle are associative, or equivalently, $\phi_{|TX}$ vanishes.

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1.1. *Genericity*

Closed associative submanifolds

DEFINITION 1.1 Consider a smooth spin 7-manifold M and Y to be a smooth compact closed 3-submanifold. For every G_2-structure ϕ, define $\mathcal{M}_{Y,\phi}$ to be the set of smooth ϕ-associative submanifolds isotopic to Y.

It is known from [17] that the problem of associative deformations of a compact closed associative submanifold Y is related to an elliptic partial differential equation, namely a twisted Dirac operator, see Theorem 2.1. Hence, for a fixed G_2-structure ϕ, the moduli space $\mathcal{M}_{Y,\phi}$ has finite and vanishing virtual dimension. In general, the situation is obstructed. For instance, consider the torus $\mathbb{T}^3 \times \{t\}$ in the flat torus $(\mathbb{T}^7, \phi_0, g_0) = \mathbb{T}^3 \times \mathbb{T}^4$. This is an associative submanifold, and its moduli space $\mathcal{M}_{\mathbb{T}^3\times\{t\}}$ of associative deformations contains at least the four-dimensional $\mathbb{T}^4$. See also Proposition 4.6 for a more general situation in a product of a Calabi–Yau manifold with S^1.

A natural question is to find conditions which force the moduli space $\mathcal{M}_{Y,\phi}$ to be smooth at least near a ϕ-associative Y, or in other terms, which force the cokernel of the operator to vanish. One way to solve this is to perturb the G_2-structure and get generic smoothness. It turns out that in general we cannot do this in the realm of torsion-free structures, see Remark 2.4. On the other hand, G_2-structures with *closed* 3-form ϕ seem to be rich enough to work with, at least for the point of view of calibrated geometries, see [8]. Indeed, any closed G_2-structure ϕ defines a calibration, and when this form is closed, the calibrated submanifolds, here the associative ones, do minimize the volume in their homology class. As suggested to the author by D. Joyce, we will prove the following theorem.

THEOREM 1.2 *Let M be a manifold equipped with a closed G_2-structure ϕ, and Y be a smooth compact closed ϕ-associative submanifold. Then there is a neighbourhood V of Y, such that for every generic closed G_2-structure ψ close enough to ϕ, the subset of elements of $\mathcal{M}_{Y,\psi}$ lying in V is a finite set, possibly empty.*

A former result in this direction was proved by Abkulut and Salur [1], where the authors allow a certain freedom for the definition of associativity.

Associative submanifolds with boundary. In [7], the authors showed that the problem of associative deformations of an associative submanifold Y with boundary in a fixed coassociative submanifold X is an elliptic problem of finite index. Moreover, they proved that this virtual dimension equals the index of a natural Cauchy–Riemann operator related to the complex geometry of the boundary, see Theorem 3.1. As in the case of a closed associative, the situation can be obstructed. For instance, consider in $(\mathbb{T}^7, \phi_0, g_0)$ the T^2-family of associative submanifolds

$$Y_\lambda = \{(x_1, x_2, x_3, \lambda, \mu, 0, 0), 0 \le x_1 \le \tfrac{1}{2}, x_2, x_3 \in S^1\}, \quad (\lambda, \mu) \in T^2.$$

The two components of the boundary of Y_λ lie in the union X of the two coassociatives tori

$$X_i = \{(i/2, x_2, x_3, x_4, x_5, 0, 0), x_2, x_3, x_4, x_5 \in S^1\}, \quad i = 0, 1.$$

However, the index of this problem vanishes, see [7] or Theorem 3.1. For more general obstructed situations, see Theorem 4.12.

As in the case of a closed associative, we can perturb the closed G_2-structure ϕ of the manifold M into ψ to ensure the smoothness of the moduli space. Note that in this case, X has no reason to remain coassociative for the new structure. But it remains ψ*-free*, i.e. the tangent space of X does not contain any ψ-associative 3-plane, see [9] or [7, Section 5]. Indeed, ϕ-coassociativity implies ϕ-freedom, and for a submanifold to be ϕ-free is an open condition in the variable ϕ. For any G_2-structure ϕ, the problem of deformations of an associative submanifold with boundary in a fixed ϕ-free submanifold is still elliptic [7] and, in our present case, its index is the same as the index for the unperturbed situation.

DEFINITION 1.3 Consider a manifold equipped with a G_2-structure (ϕ, g) and Y a smooth compact associative submanifold with boundary in a ϕ-free submanifold X. We denote by $\mathcal{M}_{Y,X}$ the set of smooth associative submanifolds with boundary in X and isotopic to Y.

Instead of changing the G_2-structure, we can move the boundary condition, namely X. Still, if we demand that X remains coassociative, in general we cannot get smoothness. Indeed, it is known [17] that the moduli space of coassociative perturbations of X is smooth and has the dimension $b_2^+(X)$ of the space of harmonic self-dual 2-forms on X. In the former example of the flat torus, every coassociative deformation of X is a translation of the initial situation, hence the problem remains obstructed. Now, since any perturbation of a ϕ-free submanifold remains ϕ-free, we can fix ϕ and perturb X.

THEOREM 1.4 *Let Y be a smooth associative submanifold with boundary in a smooth coassociative submanifold X. If the virtual dimension of $\mathcal{M}_{Y,X}$ is non-negative, then for any sufficiently small generic smooth deformation X' of X, either $\mathcal{M}_{Y,X'}$ is locally empty, that means there is no associative manifold with boundary in X' close enough to Y, or there exists a small associative deformation Y' of Y such that the moduli space $\mathcal{M}_{Y',X'}$ is smooth near Y' and of dimension equal to the index computed for the unperturbed situation.*

1.2. *Metric conditions*

Concrete examples are often non-generic, so we would like too to get a condition that is not a perturbative one. For holomorphic curves in dimension 4, there are topological conditions on the degree of the normal bundle which imply the smoothness of the moduli space of complex deformations, see [10]. The main reason is that holomorphic curves intersect positively. In our case, there is no such phenomenon.

In [17, p. 30], McLean gives an example of an isolated associative submanifold. For this, he recalls that Bryant and Salamon constructed in [4] a metric of holonomy G_2 on the spin bundle $S^3 \times \mathbb{R}^4$ of the round 3-sphere. In this case, the base $Y = S^3 \times \{0\}$ is associative, the normal bundle of Y is the spin bundle of S^3, and the operator related to the associative deformations of Y is the Dirac operator on S^3. By the famous theorem of Lichnerowicz [16], there are no non-trivial harmonic spinors on S^3 for metric reasons (to be precise, because the Riemannian scalar curvature is positive), so the sphere is isolated as an associative submanifold.

Minimal submanifolds. Recall that in a manifold with a closed G_2-structure, associative submanifolds are minimal. In [19], Simons gives a metric condition for a minimal submanifold to be stable, i.e. isolated. For this, he introduces the following operator, a sort of partial Ricci operator.

DEFINITION 1.5 Let (M, g) be a Riemannian manifold and Y be a p-dimensional submanifold in M and ν be its normal bundle. Choose $\{e_1, \dots, e_p\}$ a local orthonormal frame field of TY, and define the 0-order operator $\mathcal{R} : \Gamma(Y, \nu) \to \Gamma(Y, \nu)$ with $\mathcal{R}s = \pi_\nu \sum_{i=1}^p R(e_i, s)e_i$, where R is the curvature tensor of g on M and π_ν the orthogonal projection to ν.

It turns out that the definition is independent of the chosen oriented orthonormal frame, and that $\mathcal{R}$ is symmetric. Simons defines another operator $\mathcal{A}$ related to the second fundamental form of Y.

DEFINITION 1.6 Let SY be the bundle over Y whose fibre at a point y is the space of symmetric endomorphisms of T_yY, and $A \in \mathrm{Hom}(\nu, SY)$ the second fundamental form defined by $A(s)(u) = -\nabla_u^\top s$, where $u \in TY$, $s \in \nu$ and $\nabla^\top$ is the projection to TY of the ambient Levi–Civita connection ∇, with $\nabla = \nabla^\top + \nabla^\perp$. Denote by $\mathcal{A}$ the operator $\mathcal{A} : \Gamma(Y, \nu) \longrightarrow \Gamma(Y, \nu)$, $\mathcal{A}s = A^t \circ A(s)$, where A^t is the transpose of A.

It is classical that $\mathcal{A}$ is a symmetric positive 0th order operator. Moreover, it vanishes if Y is totally geodesic. Using both operators and Bochner's technique, Simons gives a sufficient condition for a minimal submanifold to be stable.

THEOREM 1.7 [19] *Let Y be a minimal submanifold in M, and assume that $\mathcal{R} - \mathcal{A}$ is positive. Then Y cannot be deformed as a minimal submanifold.*

In particular, if Y is a compact closed associative submanifold satisfying the conditions of Theorem 1.7 in a manifold M with a closed G_2-structure, then it cannot be perturbed as an associative submanifold. Now, if Y is an associative submanifold with a boundary, we introduce another operator.

DEFINITION 1.8 In a manifold equipped with a G_2-structure, let Y be a smooth compact associative submanifold with boundary and ν be its normal bundle. Let L be a two-dimensional real subbundle of $\nu_{|\partial Y}$ invariant under the action of $n\times$, where n is the inward unit normal vector field along ∂Y. Choose $\{v, w = n \times v\}$ a local orthonormal frame for $T\partial Y$. We denote by $\mathcal{D}_L$ the operator $\mathcal{D}_L : \Gamma(\partial Y, L) \to \Gamma(\partial Y, L)$,

$$\mathcal{D}_L s = \pi_L(v \times \nabla_w^\perp s - w \times \nabla_v^\perp s),$$

where $\pi_L : \nu_{|\partial Y} \to L$ is the orthogonal projection to L and $\nabla^\perp$ the normal connection on ν induced by the Levi–Civita connection ∇ on M.

REMARK 1.9 Note that such subbundles always exist. First, it is easy to check that $\nu_{|\partial Y}$ is stable under the action of $n\times$. Secondly, $\nu_{|\partial Y}$ has real dimension $4 > 2$, so that it has a non-vanishing section e. Then, L generated by e and $n \times e$ satisfies the conditions of Definition 1.8.

We will prove in Proposition 3.5 that $\mathcal{D}_L$ is independent of the chosen oriented frame, is of order 0 and is symmetric. Assume further that the boundary of Y lies in a coassociative submanifold X. It turns out that Y intersects X orthogonally, see Theorem 3.1. Denote by μ_X the two-dimensional orthogonal complement of n in the normal bundle of X over ∂Y, where n is the inward normal unit vector field in Y along ∂Y. Then we can state the following vanishing theorem.

THEOREM 1.10 *Let M be a manifold equipped with a torsion-free G_2-structure and Y be an associative submanifold with boundary in a coassociative submanifold X. If $\mathcal{D}_{\mu_X}$ and $\mathcal{R}-\mathcal{A}$ are positive, the moduli space $\mathcal{M}_{Y,X}$ is smooth near Y and of dimension given by the index in Theorem* 3.1.

Thanks to Theorem 1.10, we can find an explicit example, in the Bryant–Salamon manifold with G_2-holonomy, of a locally smooth one-dimensional moduli space of associative deformations with boundary in a coassociative submanifold, see Corollary 4.4. In Section 4, we explain other explicit examples, in particular for an ambient manifold which is the product of a Calabi–Yau manifold with S^1 or $\mathbb{R}$, see Theorem 4.12.

2. Closed associative submanifolds

2.1. *The operator D and the deformation problem*

We begin with the version of McLean's theorem proposed by Akbulut and Salur, and a proof of it.

THEOREM 2.1 [**2**, **17**] *Let M be a manifold equipped with a G_2-structure (ϕ, g), and Y be a closed compact associative submanifold with normal bundle ν. Then the Zariski tangent space at Y of $\mathcal{M}_Y$ can be identified with the kernel of the operator $D : \Gamma(Y,\nu) \to \Gamma(Y,\nu)$, where*

$$Ds = \sum_{i=1}^{3} e_i \times \nabla^{\perp}_{e_i} s + \sum_{k=1}^{4} (\nabla_s {*\phi})(\eta_k, \omega) \otimes \eta_k. \tag{1}$$

Here $(e_i)_{i=1,2,3}$ is any local orthonormal frame of the tangent space of Y with $e_3 = e_1 \times e_2$, $\omega = e_1 \wedge e_2 \wedge e_3$, $(\eta_k)_{k=1,2,3,4}$ is any local orthonormal frame of ν and $\nabla^{\perp}$ is the connection on ν induced by the Levi–Civita connection ∇ of (M, g).

Note that the second sum in the right-hand side of equation (1) is a 0th order operator that vanishes for a torsion-free G_2-structure, as proved in [**2**].

Proof. First, recall the existence on (M, ϕ, g) of an important object χ, the 3-form with values in TM and defined, for $u, v, w \in TM$ by $\chi(u, v, w) = -u \times (v \times w) - \langle u, v\rangle w + \langle u, w\rangle v$. It is easy to check [**2**] that $\chi(u, v, w)$ is orthogonal to the 3-plane $u \wedge v \wedge w$. Moreover, we will use the following useful formula [**8**]:

$$\forall u, v, w, \eta \in TM, \quad \langle \chi(u, v, w), \eta\rangle = {*\phi}(u, v, w, \eta),$$

where $*$ is the Hodge star associated to the metric g. So

$$\chi = \sum_k (\eta_k \lrcorner {*\phi}) \otimes \eta_k, \tag{2}$$

where $(\eta_k)_{k=1,2,\dots,7}$ is a local orthonormal frame of the tangent space of M. Further, if Y is a three-dimensional submanifold in (M, ϕ), then $\chi_{|TY} = 0$ if and only if Y is associative. As in [**17**], we use this characterization to study the moduli space of associative deformations of an associative Y. Let Y be any smooth closed associative submanifold in M. We parametrize its deformations by the sections of its normal bundle ν. Fix ω a non-vanishing global section of $\Lambda^3 TY$ writing locally $\omega = e_1 \wedge e_2 \wedge e_3$, with $(e_i)_{i=1,2,3}$ a local orthonormal frame of TY satisfying $e_3 = e_1 \times e_2$. For every smooth section $\sigma \in \Gamma(Y, \nu)$, define

$$F(\sigma) = \exp_\sigma^* \chi(\omega) \in \Gamma(Y, \nu_\sigma), \tag{3}$$

where ν_σ is the normal bundle of $\exp_\sigma(Y)$. Then $\exp_\sigma(Y)$ is associative if and only if $F(\sigma)$ vanishes. In order to compute the Zariski tangent space of $\mathcal{M}_Y$ at the vanishing section, consider a path of normal sections $(\sigma_t)_{t\in[0,1]} \in \Gamma(Y, \nu)$ and

$$s = \frac{\mathrm{d}\sigma_t}{\mathrm{d}t}_{|t=0} \in \Gamma(Y, \nu).$$

To differentiate F at $\sigma = 0$ in the direction of s, we use the Levi–Civita connection of (M, g). We have

$$\nabla_{\partial/\partial t} F(\sigma_t)_{|t=0} = \sum_k \mathcal{L}_s(\eta_k \lrcorner {*\phi})(\omega) \otimes \eta_k + (\eta_k \lrcorner {*\phi})(\omega) \otimes \nabla_s \eta_k, \tag{4}$$

where $\mathcal{L}_s$ is the Lie derivative in the direction s. Since Y is associative, $\omega \lrcorner {*\phi} = 0$ and the second term vanishes. Note that this implies that the result does not depend on the chosen connection. Thanks to classical Riemannian formulas, we compute the summand of the first term. For every k,

$$\mathcal{L}_s(\eta_k \lrcorner {*\phi}) = \eta_k \lrcorner\, \mathcal{L}_s({*\phi}) + [\eta_k, s] \lrcorner {*\phi},$$

and since $([\eta_k, s] \wedge \omega) \lrcorner {*\phi} = 0$, we get

$$\nabla_{\partial/\partial t} F(\sigma_t)_{|t=0} = \sum_k \mathcal{L}_s({*\phi})(\eta_k, \omega) \otimes \eta_k. \tag{5}$$

The Lie derivatives can be expressed in terms of the Levi–Civita connection, see, for instance, [11, Formula 3.3.26], so that

$$\begin{aligned}\mathcal{L}_s({*\phi})(\eta_k, \omega) &= (\nabla_s {*\phi})(\eta_k, \omega) + {*\phi}(\nabla_{\eta_k} s, \omega)\\ &\quad + {*\phi}(\eta_k, \nabla_{e_1} s, e_2, e_3) + {*\phi}(\eta_k, e_1, \nabla_{e_2} s, e_3) + {*\phi}(\eta_k, e_1, e_2, \nabla_{e_3} s).\end{aligned}$$

The second term of the right-hand side vanishes because $\omega \lrcorner {*\phi} = 0$ and the third one equals ${*\phi}(\eta_k, \nabla^\perp_{e_1} s, e_2, e_3) = -\langle \nabla^\perp_{e_1} s \times (e_2 \times e_3), \eta_k \rangle$. Using the relation $e_2 \times e_3 = e_1$ and adding up the two last similar terms, we obtain $\nabla_s F = \sum_i e_i \times \nabla^\perp_i s + \sum_k (\nabla_s {*\phi})(\eta_k, \omega) \otimes \eta_k$. Since $F(0)$ has values in ν, in fact we can assume that the η_ks form a local orthonormal frame of ν. $\square$

PROPOSITION 2.2 *Let Y be a smooth closed associative submanifold in a manifold M equipped with a G_2-structure. If the (co)kernel of the operator D given by (1) vanishes, then $\mathcal{M}_Y$ is smooth near Y and of vanishing dimension. In particular, Y is isolated among associative submanifolds isotopic to Y.*

Proof. Fix Y a smooth closed associative submanifold. For $kp > 3$, it makes sense to consider the Banach space $\mathcal{E} = W^{k,p}(Y, \nu)$ of sections with weak derivatives in L^p, up the kth one. Moreover, for

$(k-r)/3 > 1/p$, the inclusion $W^{k,p}(Y,\nu) \subset C^r(Y,\nu)$ holds and so $\sigma \in \mathcal{E}$ is C^1 if $k > 1 + 3/p$. In particular, one can define ν_σ the normal bundle to $\exp_\sigma(Y)$, and $\mathcal{F}$ the Banach bundle over $\mathcal{E}$ with fibre $\mathcal{F}_\sigma = W^{k-1,p}(Y,\nu_\sigma)$. It is clear that the operator F defined by (3) extends to a section $F_{k,p}$ of $\mathcal{F}$ over $\mathcal{E}$. The proof of Theorem 2.1 shows that $F_{k,p}$ is smooth and the derivative of F in the direction of a vector field $s \in T_0\mathcal{E} = W^{k,p}(Y,\nu)$ is computed by (1). Now, the operator $D : \Gamma(Y,\nu) \to \Gamma(Y,\nu)$ has symbol

$$\sigma(\xi) : s \mapsto \sum_i \xi_i s \times e_i = s \times \xi,$$

which is always invertible on ν as long as $\xi \in TY \setminus \{0\}$. This proves that D is elliptic. Note that $\sigma(\xi)^2 s = -|\xi|^2 s$, which is the symbol of the Laplacian. Hence, F is a Fredholm operator, and $\ker D$ and $\operatorname{coker} D$ have finite dimension. By the implicit function theorem for Banach bundles, if $\operatorname{coker} D = \{0\}$, then $F^{-1}(0)$ is a smooth Banach submanifold of $\mathcal{E}$ near the null section and of finite dimension equal to $\dim\ker D = \operatorname{index} D$, which vanishes since Y is odd-dimensional. Lastly, still thanks to the ellipticity of D, all elements of $\mathcal{M}_Y$ are smooth. □

2.2. *Varying the G_2-structure*

THEOREM 1.2 *Let M be a manifold equipped with a closed G_2-structure ϕ, and Y be a smooth compact closed ϕ-associative submanifold. Then there is a neighbourhood V of Y, such that for every generic closed G_2-structure ψ close enough to ϕ, the subset of elements of $\mathcal{M}_{Y,\psi}$ lying in V is a finite set, possibly empty.*

Proof. Consider Y a smooth closed associative submanifold in a manifold M equipped with a closed G_2-structure (ϕ, g). We modify the former map F defined in (3) in the following way. For every normal section $\sigma \in \Gamma(Y,\nu)$ and every G_2-structure ϕ', consider

$$F(\sigma,\phi') = \exp_\sigma^* \chi_{\phi'}(\omega) \in \Gamma(Y,\nu_\sigma). \tag{6}$$

Here the exponential map corresponds to the fixed metric g, whereas ν_σ, the normal vector bundle over $\exp_\sigma(Y)$, depends now on the metric associated to ϕ', as does $\chi_{\phi'}$. We will differentiate $F(0,\cdot)$ in the direction of $\mathcal{Z}^3(M)$, the subspace of smooth closed 3-forms on M. Recall that the set of 3-forms defining a G_2-structure is open in $\Omega^3(M)$, hence for every $\psi \in \mathcal{Z}^3(M)$ with small enough norm, $\phi + \psi$ still defines a closed G_2-structure. Let $(\phi_t)_{t\in[0,1]}$ be a smooth path of closed G_2-structures, with $\phi_0 = \phi$. In formula (2), the local orthonormal trivializations η_k of the tangent bundle TM are orthonormal for the metric g_t associated to ϕ_t, consequently we have to choose them as functions of t. On the other hand, we can keep ω constant. Hence, $F(0,\phi_t) = \sum_k (\eta_k(t)\wedge\omega) \lrcorner *_t\phi_t \otimes \eta_k(t)$, where $*_t$ denotes the Hodge star for g_t. Since $\omega \lrcorner *\phi = 0$, at $t = 0$ the two terms in the derivative containing $\nabla_{\partial/\partial t}\eta_k$ vanish, and we have

$$\nabla_{\partial/\partial t} F(0,\phi_t)_{|t=0} = \sum_k (\eta_k \wedge \omega) \lrcorner \frac{\partial}{\partial t}\Theta(\phi(t))_{|t=0} \otimes \eta_k.$$

The nonlinear function Θ is defined on the set of G_2-structures and has values in $\Omega^4(X)$, with

$$\Theta(\psi) = *_\psi \psi, \tag{7}$$

where the Hodge star $*_\psi$ is computed for the metric associated to the G_2-structure ψ. Proposition 10.3.5 in [12] shows that if ϕ is a G_2-structure, the derivative of Θ at ϕ satisfies

$$\forall \psi \in \mathcal{Z}^3(M), \quad d_\phi\Theta(\psi) = *\mathcal{P}(\psi), \tag{8}$$

where the Hodge star corresponds to g and

$$\mathcal{P} = \tfrac{4}{3}\pi_1 + \pi_7 - \pi_{27}. \tag{9}$$

Here π_1, π_7 and π_{27} are the orthogonal projections corresponding to the decomposition $\Lambda^3 T^*M = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ associated to the irreducible representations of G_2, see [6, Lemma 3.2] or [12, Proposition 10.1.4]. Hence, if $\psi = (\partial/\partial t)\phi(t)_{|t=0} \in \mathcal{Z}^3(M)$, we have

$$\nabla_\psi F = \sum_k (\eta_k \wedge \omega) \lrcorner *\mathcal{P}(\psi) \otimes \eta_k. \tag{10}$$

Lemma 2.3 *The operator $\nabla F : \mathcal{Z}^3(M) \to \Gamma(Y, \nu)$ defined by equation (10) is onto.*

Proof. Due to the properties of χ, in this formula we can restrict our η_ks to a local orthonormal frame of ν for the metric g. Now, recall [6] that $\Lambda^3_7 = \{*(\phi \wedge \alpha), \alpha \in \Lambda^1 T^*M\}$. Consider $s \in \Gamma(Y, \nu)$, and α the dual 1-form of s. More precisely, $\alpha \in \Gamma(Y, T^*M)$ satisfies

$$\forall y \in Y,\ \forall v \in T_yM, \quad \alpha_y(v) = \langle s(y), v\rangle. \tag{11}$$

We choose ω such that $\phi(\omega) = 1$, which is always possible since Y is associative. Since $\mathcal{P}$ acts as the identity on Λ^3_7 and $*$ is an involution, it is straightforward to see that

$$\sum_l (\eta_l \wedge \omega) \lrcorner *\mathcal{P}(*(\phi \wedge \alpha)) \otimes \eta_l = s. \tag{12}$$

In order to prove the existence of $\psi \in \mathcal{Z}^3(M)$ such that $\nabla_\psi F = s$, we need to extend $*(\phi \wedge \alpha)$ outside Y as a closed form. For this, let $p \in Y$, U be an open set of M containing p and local co-ordinates $y_1, y_2, y_3, x_1, x_2, x_3, x_4$ on U, where the y_is are coordinates on Y and the x_is are transverse coordinates. Because Y is associative, the 3-form $\psi' = *(\phi \wedge \alpha) \in \Gamma(Y, \Lambda^3 T^*M)$ is of the form $\sum_{i=1}^4 \mathrm{d}x_i \wedge \beta_i$ over $Y \cap U$, where for all i, β_i is a 2-form. We extend arbitrarily the β_is as smooth 2-forms on U. Assume first that s has compact support in $U \cap Y$. Then so do the β_is on $U \cap Y$. Define

$$\psi' = \mathrm{d}\left(\chi_U \sum_i x_i \beta_i\right),$$

where χ_U is a cut-off function with support in U and equal to 1 in the neighbourhood of the support of s. Then ψ' is a global closed 3-form with $\psi'_{|Y} = \psi$ and hence satisfying $\nabla_{\psi'} F = s$. For a general section $s \in \Gamma(Y, \nu)$, a partition of unity allows us to find $\psi \in \mathcal{Z}^3(M)$ such that $\nabla_\psi F = s$. We conclude that ∇F is onto in the direction of $\mathcal{Z}^3(M)$. $\square$

We can now complete the proof of Theorem 1.2. If $\mathcal{Z}_D$ is the finite-dimensional subspace of $\mathcal{Z}^3(M)$ generated by the former closed 3-forms ψ associated to every $s \in \operatorname{coker} D$ given by Lemma 2.3, by the inverse mapping theorem, the set

$$\mathcal{M} = \{(\sigma, \psi) \in W^{k,p}(Y, \nu) \times \mathcal{Z}_D(M), F(\sigma, \psi) = 0\}$$

is a smooth manifold near $(0, \phi)$ if $k > 1 + 3/p$. By the Sard–Smale theorem applied to the projection $\pi : \mathcal{M} \to \mathcal{Z}_D$, for every generic $\psi \in \mathcal{Z}_D$ close enough to ϕ, the slice

$$\pi^{-1}(\psi) = \{\sigma \in W^{k,p}(Y, \nu), \exp_\sigma(Y) \text{ is } \psi\text{-associative}\}$$

is a smooth manifold or an empty set. As usual, the sections in $\pi^{-1}(\psi)$ are in fact smooth, hence the result. □

REMARK 2.4 By Joyce [12, Theorem 10.4.4], if ϕ is a torsion-free G_2-structure, the tangent space at ϕ of the set of torsion-free structures can be identified with $\mathcal{L} \oplus \mathcal{H}^3(M, \mathbb{R})$, where $\mathcal{L}$ is the subspace of the Lie derivatives of ϕ, i.e. $\mathcal{L} = \{\mathcal{L}_X\phi, X \in C^0(M, TM)\}$, and $\mathcal{H}^3(M, \mathbb{R})$ is the space of the real harmonic 3-forms on M. If $\psi = \mathcal{L}_X\phi \in \mathcal{L}$, Lemma 2.5 shows that the derivative of F along ψ equals $DX^\perp$, where $X^\perp \in \Gamma(Y, \nu)$ is the normal projection of X onto the normal bundle of Y. Hence, $\mathcal{L}$ is of no use for ∇F to be onto. But the dimension of coker D is not in general less than $b^3(M)$, and even when it is, $\mathcal{H}^3(M) \to \operatorname{coker} D$ might well be non-injective (see the end of the Section 4.4 for examples of every situation). This is the reason why we use the wider space of closed G_2-structures.

LEMMA 2.5 *Let M be a manifold equipped with a torsion-free G_2-structure ϕ, Y be a smooth compact closed ϕ-associative submanifold and X be a smooth vector field of TM in the neighbourhood of Y. Then*

$$\mathrm{d}F_{|(0,\phi)}(\mathcal{L}_X\phi) = DX^\perp,$$

where $\mathrm{d}F_{|(0,\phi)}(\mathcal{L}_X\phi)$ denotes the derivative of the section F given by (6) at $(0, \phi)$ in the direction $\mathcal{L}_X\phi$, D is the Dirac-like operator given by (1) and $X^\perp$ is the orthogonal projection of X onto the normal bundle ν over Y.

Proof. Denote by $(\Phi_X^t)_{t\in[0,\epsilon]}$ the flow generated by X near Y and $\phi_t = \Phi_X^{t*}\phi$ the pull-back of ϕ. Hence, the metric g_t associated to ϕ_t is $\Phi_X^{t*}g$, so that $\Theta(\phi_t) = \Phi_X^{t*}(\Theta(\phi))$, where Θ is defined by (7). Let $(\eta_k^t)_{k=1,\dots,4}$ be an orthonormal framing of the normal bundle of Y for the metric g_t, depending smoothly on t. Then

$$F(0, \phi_t) = \sum_k (\eta_k^t \wedge \omega) \lrcorner \Theta(\phi_t) \otimes \eta_k^t = \sum_k (\eta_k^t \wedge \omega) \lrcorner \Phi_X^{t*}(\Theta(\phi)) \otimes \eta_k^t,$$

which implies

$$\mathrm{d}F_{|(0,\phi)}(\mathcal{L}_X\phi) = \sum_k (\eta_k^0 \wedge \omega) \lrcorner \mathcal{L}_X(\Theta(\phi)) \otimes \eta_k^0$$

(there is no derivative of η_k^t because $\omega \lrcorner \Theta(\phi) = 0$). This is the right-hand side of (5) with X instead of s. The end of the proof of Theorem 2.1 shows that $\mathrm{d}F_{|(0,\phi)}(\mathcal{L}_X\phi) = DX^\perp$. □

In the following Proposition 2.6, we give a situation where we can find a way to isolate an associative after perturbing the G_2-structure.

PROPOSITION 2.6 *Let Y be a smooth closed ϕ-associative submanifold, such that* $\ker D$ *is generated by a non-vanishing normal vector field. Then there is a neighbourhood V of Y and a closed perturbation ψ of ϕ, such that the only element of $\mathcal{M}_{Y,\psi}$ lying in V is Y.*

Proof. Let $\xi_1 \in \ker D \setminus \{0\}$. Since the normal bundle ν of Y is trivial, we can find normal vector fields ξ_2, ξ_3 and ξ_4 such that the ξ_is form a global framing of ν. Let $(x_i)_{i=1,\dots,4}$ the coordinates near Y defined by exponentiating the ξ_is. The 3-form $\xi_1 \lrcorner *\phi$ writes $\sum_{i=2,3,4} \mathrm{d}x_i \wedge \beta_i$. The closed form

$$\psi = \mathrm{d}\left(x_1 \sum_{i=2,3,4} x_i \wedge \beta_i\right)$$

is defined near Y and vanishes on Y. If $\phi_\lambda = \phi + \lambda\psi$, denote by g_λ the associated metric and by D^λ the Dirac-like operator associated to ϕ_λ. We will prove that for λ small enough, the only solution to $D^\lambda s = 0$ is the null section, which will prove Proposition 2.6 by Proposition 2.2.

Derivative (5) together with equation (8) giving the derivative of the Hodge star imply that for every $s \in \Gamma(Y, \nu)$,

$$\begin{aligned} D^\lambda s &= \sum_k \eta_k \wedge \omega \lrcorner \left(\mathcal{L}_s\big(\Theta(\phi) + \lambda *\mathcal{P}(\psi) + O(\lambda^2)\big)\right) \otimes \eta_k \\ &= Ds + \lambda \sum_k \eta_k \wedge \omega \lrcorner (\mathcal{L}_s(*\mathcal{P}(\psi))) \otimes \eta_k + O(\lambda^2\, s), \end{aligned}$$

where $*$ is the Hodge star associated to ϕ and $\mathcal{P}$ is given by (9) (note that η_k is an orthonormal framing for every λ since $g_\lambda = g$ on Y). In particular, if $s \in \ker D^\lambda$,

$$Ds = O(\lambda s). \tag{13}$$

Near Y, we have $*\mathcal{P}(\psi) = x_1 \xi_1 \lrcorner *\phi + \sum_{i=2,3,4} x_i *\mathcal{P}(\mathrm{d}x_1 \wedge \beta_i) + O(x^2)$, so that on Y,

$$\mathcal{L}_s(*\mathcal{P}(\psi)) = s_1 \xi_1 \lrcorner *\phi + \sum_{i=2,3,4} s_i *\mathcal{P}(\mathrm{d}x_1 \wedge \beta_i),$$

where $s = \sum_{i=1,2,3,4} s_i \xi_i$. This implies

$$D^\lambda s = Ds + \lambda s_1 \xi_1 + O(\lambda (s_i)_{i=2,3,4}) + O(\lambda^2\, s). \tag{14}$$

Since D is a self-adjoint elliptic operator, there is a constant a (depending on λ and s) such that $s - a\xi_1 = O(Ds)$, see [**14**, Corollary 5.7] for instance. Assume now that $s \in \ker D^\lambda$. Then, estimate

(13) implies that $s - a\xi_1 = O(\lambda s)$, so that projecting onto the directions ξ_1 and ξ_i, $i = 2, 3, 4$, we get

$$s_1 - a = O(\lambda s) \tag{15}$$

and

$$s_i = O(\lambda s) \quad \text{for } i = 2, 3, 4. \tag{16}$$

The first estimate gives $\nabla s_1 = O(\lambda s)$, and the second one together with equation (14) implies

$$Ds = -\lambda s_1 \xi_1 + O(\lambda^2 s). \tag{17}$$

This gives

$$D^2 s = -\lambda \nabla s_1 \times \xi_1 + O(\lambda^2 s) = O(\lambda^2 s). \tag{18}$$

Since $\ker D^2 = \ker D$ and D^2 is elliptic, we have by Lawson and Michelsohn [**14**, Corollary 5.7] and relation (18) the estimate $s - a\xi_1 = O(D^2 s) = O(\lambda^2 s)$, so that

$$Ds = O(\lambda^2 s), \tag{19}$$

since $D\xi_1 = 0$. Now, from (17) and (19) we deduce $\lambda s_1 \xi_1 = O(\lambda^2 s)$. Since by (16), the norm of s is equivalent to the norm of s_1 when λ tends to zero, the last estimate is impossible for λ small enough and a non-zero $s \in \Gamma(Y, \nu)$, so that $s = 0$. □

This situation arises in particular in the Calabi–Yau extension, see Corollary 4.8.

2.3. *A vanishing theorem*

We turn now to the second way of getting the smoothness of the moduli space, namely Bochner's technique and Simons's theorem. We formulate the following theorem which can be deduced from Theorem 1.7, since any associative submanifold is minimal.

THEOREM 2.7 *Let Y be a smooth closed compact associative submanifold of a manifold M with a closed G_2-structure. If the spectrum of $\mathcal{R}_\nu = \mathcal{R} - \mathcal{A}$ is positive, then Y is isolated as an associative submanifold.*

For the reader's convenience, we give below a proof of this result in the case where the G_2-structure is torsion-free. We will compute D^2 to use Bochner's technique. For this, we introduce the normal equivalent of the invariant second derivative. More precisely, for any local vector fields v and w in $\Gamma(Y, TY)$, let $\nabla^{\perp 2}_{v,w}$ be the operator defined by $\nabla^{\perp 2}_{v,w} = \nabla^\perp_v \nabla^\perp_w - \nabla^\perp_{\nabla^\top_v w}$ acting on $\Gamma(Y, \nu)$. It is straightforward to see that it is tensorial in v and w. Moreover, define the equivalent of the connection Laplacian:

$$\nabla^{\perp *} \nabla^\perp = -\text{trace}\,(\nabla^{\perp 2}) = -\sum_i \nabla^{\perp 2}_{e_i, e_i},$$

where the e_is define a local orthonormal frame of TY.

THEOREM 2.8 *For Y an associative submanifold in a manifold with a torsion-free G_2-structure,* $D^2 = \nabla^{\perp *}\nabla^{\perp} + \mathcal{R}_\nu$.

We refer to the appendix for the proof of this theorem.

Proof of Theorem 2.7. Let us assume that we are given a fixed closed associative submanifold Y. Consider a section $s \in \Gamma(Y, \nu)$. By classical computations using normal coordinates and thanks to Theorem 2.8, we have

$$-\frac{1}{2}\Delta|s|^2 = \sum_i \langle \nabla_i^\perp s, \nabla_i^\perp s\rangle + \langle s, \nabla_i^\perp \nabla_i^\perp s\rangle = |\nabla^\perp s|^2 - \langle D^2 s, s\rangle + \langle \mathcal{R}_\nu s, s\rangle.$$

Since the Laplacian equals $-\mathrm{div}(\vec{\nabla})$, its integral over the closed Y vanishes. We get

$$0 = \int_Y |\nabla^\perp s|^2 - \langle D^2 s, s\rangle + \langle \mathcal{R}_\nu s, s\rangle \, \mathrm{d}y. \tag{20}$$

Assume that s belongs to $\ker D$. Under the hypothesis that $\mathcal{R}_\nu$ is positive, the last equation implies $s = 0$. Hence, $\dim \operatorname{coker} D = \dim \ker D = 0$, and by Proposition 2.2, $\mathcal{M}_Y$ is a smooth manifold near Y with vanishing dimension. In particular, Y is isolated. □

3. Associative submanifolds with boundary

In this section, we explain our results in the case of an associative submanifold with boundary in a coassociative submanifold. We first give below the principal results of [7]. For this, recall that in a manifold with a G_2-structure and an associated vector product $\times$, given $x \in M$ and n a unit vector in $T_x M$, the application

$$n\times : T_x M \to T_x M, \quad v \mapsto n \times v$$

defines a complex structure on $n^\perp$, the orthogonal complement of n. A 2-plane $L \subset n^\perp$ invariant under $n\times$ will be called an *$n\times$-complex line*.

THEOREM 3.1 [7] *Let M be a manifold equipped with a G_2-structure (ϕ, g) and Y be a smooth compact associative submanifold with boundary in a coassociative submanifold X. Let ν_X be the normal complement of $T\partial Y$ in $TX_{|\partial Y}$, and n be the inward unit normal vector to ∂Y in Y. Then*

1. *The bundle ν_X is a subbundle of $\nu_{|\partial Y}$ and is an $n\times$-complex line, as is the orthogonal complement μ_X of ν_X in $\nu_{|\partial Y}$.*
2. *Viewing $T\partial Y$, ν_X and μ_X as $n\times$-complex line bundles, we have $\mu_X^* \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$.*
3. *Further, the problem of the associative deformations of Y with boundary in X is elliptic and of index* $\operatorname{index}(Y, X) = \operatorname{index} \bar{\partial}_{\nu_X} = c_1(\nu_X) + 1 - g$, *where g is the genus of ∂Y.*

PROPOSITION 3.2 *Let M be a smooth manifold equipped with a G_2-structure (ϕ, g) and let Y be a smooth compact associative submanifold with boundary in a coassociative submanifold X. Consider the adapted version of the linearization of (1) for our boundary problem:*

$$D : \mathcal{E}_X = \{s \in \Gamma(Y, \nu), s_{|\partial Y} \in \nu_X\} \to \Gamma(Y, \nu).$$

If the cokernel of $D : \mathcal{E}_X \to \Gamma(Y, \nu)$ vanishes, then $\mathcal{M}_{Y,X}$ is smooth near Y and of dimension equal to $\operatorname{index}(Y, X)$.

Proof. For $2k > 3$ and $(k-r)/3 > \frac{1}{2}$, define the adapted Banach space $\mathcal{E}_X$ by

$$\mathcal{E}_X = \{\sigma \in W^{k,2}(Y, \nu), \forall y \in \partial Y, \sigma(y) \in \nu_{X,y}\},$$

and $\mathcal{F}$ the bundle over $\mathcal{E}_X$, where the fibre $\mathcal{F}_\sigma$ denotes $W^{k-1,2}(Y, \nu_\sigma)$. As before, ν_σ is the normal bundle to $\exp_\sigma(Y)$. Let us assume first that X is totally geodesic for the metric g. Then $\mathcal{E}_X$ parametrizes the submanifolds with boundary in X and close enough to Y. Define the analogue of the map (3) in the proof of Theorem 2.1 by $F : \mathcal{E}_X \to \mathcal{F}$, $F(\sigma) = \exp_\sigma^* \chi$. By the proof of Theorem 2.1, F is smooth and its derivative at the vanishing section is $D : \mathcal{E}_X \to \Gamma(Y, \nu)$. Further, by Booß-Bavnbek and Wojciechowski [**3**, Theorem 20.8], the operator $D : \mathcal{E}_X \to \Gamma(Y, \nu)$ is Fredholm and Theorem 3.1 gives its index. Now, if the cokernel of D vanishes, then the inverse mapping theorem shows that $\mathcal{M}_{Y,X}$ is smooth near Y and of the expected dimension equal to index (Y, X). Lastly, [**3**, Theorem 19.1] shows that in fact, the sections belonging to $\mathcal{M}_{Y,X}$ are smooth and so are the associated deformations of Y. In general, X is not totally geodesic and as explained in [**5**, **13**], $\exp_\sigma(\partial Y)$ has no reason to lie in X. For this, we change the metric near X, as in the mentioned works.

LEMMA 3.3 *There exists a tubular neighbourhood U of X and a metric $\hat{g}$ such that $\hat{g}(x) = g(x)$ for every $x \in X$, $\hat{g}$ equals g outside U and X is totally geodesic for $\hat{g}$.*

Proof. The exponential gives a diffeomorphism Φ between a tubular neighbourhood U of X in M and a neighbourhood V of the vanishing section in the normal vector bundle N_X of X. Moreover, it sends X to the vanishing section. Consider on V the metric $h = \pi^* g_{|TX} \oplus g_N$, where g_N is the natural flat metric on the fibres induced by the metric g, $g_{|TX}$ is the induced metric on X and $\pi : N_X \to X$ denotes the natural projection. Now $H = \Phi^* h$ is a metric on U, for which X is clearly totally geodesic. Take χ to be a cut-off function with support in U, equal to 1 in a neighbourhood of X. Then $\hat{g} = \chi H + (1-\chi) g$ satisfies all the conditions of the lemma. □

Consider $\hat{\nu}$ the normal bundle over Y for the new metric $\hat{g}$. For every section $\sigma \in \Gamma(Y, \hat{\nu})$, we use the adapted function $\hat{F}(\sigma) = \widehat{\exp}_\sigma{}^* \chi(\omega)$, where ω can be chosen as before and χ is the form associated to ϕ, but $\widehat{\exp}$ is the exponential map for the new metric $\hat{g}$. The proof of Theorem 2.1 shows that differentiating $\hat{F}$ in the direction of $s \in \Gamma(Y, \hat{\nu})$ gives the same result $\nabla_s \hat{F} = Ds \in \Gamma(Y, \nu)$, even if s does not belong to $\Gamma(Y, \nu)$. Now, given a bundle isomorphism between $\hat{\nu}$ and ν, it is straightforward to see that the kernel and the cokernel of $\hat{\nabla} F$ are isomorphic to the ones of D. The former conclusion in the totally geodesic case still holds. □

3.1. *Varying the coassociative submanifold*

In Section 2.2, we perturbed the G_2-structure in order for the moduli space $\mathcal{M}_Y$ to become smooth. When the associative submanifold has a boundary, we can repeat the same arguments. We can also move the boundary condition. As explained in Section 1, we will perturb generically X as a smooth ϕ-free submanifold, and no longer as a coassociative one.

THEOREM 1.4 *Let Y be a smooth associative submanifold with boundary in a smooth coassociative submanifold X. If the virtual dimension of $\mathcal{M}_{Y,X}$ is non-negative, then for any sufficiently small generic smooth deformation X' of X, there exists a small associative deformation Y' of Y such that $\mathcal{M}_{Y',X'}$ is smooth near Y' and of dimension equal to the index computed for the unperturbed situation.*

Proof. Recall [17] that if X is a coassociative submanifold, then its normal bundle N_X can be identified with the space of its self-dual two-forms $\Omega^2_+(X)$. For $\alpha \in \Omega^2_+(X)$, define $\sigma_\alpha \in \Gamma(\partial Y, N_X)$ the restriction to ∂Y of the associated normal vector field along X. By Theorem 3.1, $N_{X|\partial Y} = n\mathbb{R} \oplus \mu_X$, with n the inward unit normal vector to $T\partial Y$ in TY. Consider the subspace

$$\mathcal{C} = \{\alpha \in \Omega^2_+(X), \sigma_\alpha \in \Gamma(\partial Y, \mu_X)\}.$$

Note that infinitesimal deformations of X in these directions are normal to Y. This will be considered as the parameter space. For every $\alpha \in \mathcal{C}$, extend σ_α to $\Gamma(Y, \nu)$ in the following way. The associative Y is diffeomorphic to $Y_\epsilon = \partial Y \times [0, \epsilon]$ near ∂Y, where ∂Y holds for $\partial Y \times \{0\}$. This allows us to identify $\nu_{|Y_\epsilon}$ with $\nu_{|\partial Y} \times [0, \epsilon]$ and so this gives an extension of σ_α on Y_ϵ. Take ρ to be a cut-off function satisfying $\rho = 1$ in the neighbourhood of ∂Y and with support in Y_ϵ. Then $\hat{\sigma}_\alpha = \rho\sigma_\alpha \in \Gamma(Y, \nu)$ is a smooth normal vector field along Y such that $\hat{\sigma}_\alpha = \sigma_\alpha$ near ∂Y. Now, let $\mathcal{E}_\partial$ be the set

$$\mathcal{E}_\partial = \{(\alpha, s) \in \mathcal{C} \times \Gamma(Y, \nu), \forall y \in \partial Y, s(y) \in T_y X\}.$$

Here, we will assume that X is totally geodesic as in the first part of the proof of Proposition 3.2. If not, we change the metric by Lemma 3.3. Hence, if $(\alpha, s) \in \mathcal{E}_\partial$ and if we define $\phi_{\alpha,s} = \exp_{\hat{\sigma}_\alpha} \circ \exp_s$, then $Y_{\alpha,s} = \phi_{\alpha,s}(Y)$ is a smooth submanifold with boundary in $X_\alpha = \exp_{\sigma_\alpha}(X)$. Let $\mathcal{F}$ be the bundle over $\mathcal{E}_\partial$, where the fibre $\mathcal{F}_{\alpha,s}$ equals $\Gamma(Y, \nu_{\alpha,s})$ and $\nu_{\alpha,s}$ denotes the normal bundle of $Y_{\alpha,s}$. Define the section $F : \mathcal{E}_\partial \to \mathcal{F}$ by $F(\alpha, s) = \phi^*_{\alpha,s}\chi(\omega)$. Then $Y_{\alpha,s}$ is an associative submanifold if and only if $F(\alpha, s) = 0$. Now for every fixed $\alpha \in \mathcal{C}$, consider the restriction map

$$\begin{aligned} F_\alpha : \{s \in \Gamma(Y, \nu), s_{|\partial Y} \in TX\} &\to \Gamma(Y, \nu_{\alpha,s}), \\ s &\mapsto F(\alpha, s). \end{aligned}$$

Two tedious computations analogous to the proof of Theorem 2.1 and the proof of Theorem 3.1 in [7, Section 4] show that for every $\alpha \in \mathcal{C}$, the derivative of F_α is elliptic in the sense of [3, Definition 18.1]. Further, F_α is clearly a deformation of F_0, hence F_α is a Fredholm map of index computed in Theorem 3.1. For a genericity result, we need the classical theorem.

THEOREM 3.4 [18, Theorem 1.5.19] *Let $\mathcal{C}, \mathcal{E}$ and $\mathcal{F}$ be Banach spaces, $F : \mathcal{C} \times \mathcal{E} \to \mathcal{F}$ be a smooth map, such that for every $\alpha \in \mathcal{C}$, $F_\alpha = F(\alpha, \cdot)$ is a Fredholm map between $\mathcal{E}$ and $\mathcal{F}$. If $\mathrm{d}F : \mathcal{C} \times \mathcal{E} \to \mathcal{F}$ is onto at (α_0, x_0), then $F^{-1}(y_0)$ is locally a smooth manifold, where $y_0 = F(\alpha_0, x_0)$. Further, for every generic $\alpha \in \mathcal{C}$ close enough to α_0, the fibre $F_\alpha^{-1}(y_0)$ is a smooth manifold of finite dimension equal to the index of F_α.*

We compute the derivative of F at $(0, 0) \in \mathcal{E}_\partial$. One can easily check using the proof of Theorem 2.1 that this is equal to

$$\begin{aligned} \nabla_{(0,0)} F : \mathcal{E}_\partial &\to \Gamma(Y, \nu), \\ (\alpha, s) &\mapsto D(s + \hat{\sigma}_\alpha). \end{aligned}$$

This derivative is onto. Indeed, let s' be a section in $\Gamma(Y, \nu)$. Since Y has a boundary, our Dirac-like operator D is onto by Theorem 9.1 of the book [3], so there is a section $s \in \Gamma(Y, \nu)$ such that

$Ds = s'$. Now decompose $s_{|\partial Y}$ as $s_\nu + s_\mu$ with $s_\nu \in \Gamma(\partial Y, \nu_X)$ and $s_\mu \in \Gamma(\partial Y, \mu_X)$. Choosing the 2-form $\alpha \in \mathcal{C}$ such that $s_\mu = \sigma_\alpha$, we have $D((s - \hat{\sigma}_\alpha) + \hat{\sigma}_\alpha) = s'$ with $(\alpha, s - \hat{\sigma}_\alpha) \in \mathcal{E}_\partial$, hence the result. □

As in Theorem 1.2, we can restrict our smoothing deformations to a finite-dimensional space of dimension equal to dim coker D.

3.2. *A vanishing theorem*

Given Y an associative submanifold with boundary in a coassociative submanifold X, we turn now to metric conditions on Y that insure local smoothness of the moduli space $\mathcal{M}_{Y,X}$. Let ν be the normal bundle of Y and n is the inward normal vector to ∂Y in Y. Recall that if $L \subset \nu$ is an $n\times$-complex line bundle over ∂Y, the operator $\mathcal{D}_L : \Gamma(\partial Y, L) \to \Gamma(\partial Y, L)$ was defined in Section 1 by $\mathcal{D}_L s = \pi_L(v \times \nabla^\perp_w s - w \times \nabla^\perp_v s)$, where $\pi_L : \nu \to L$ is the orthogonal projection to L and $\{v, w = n \times v\}$ a local orthonormal frame for $T\partial Y$. We refer to the appendix for the proof of the following proposition.

PROPOSITION 3.5 *The operator $\mathcal{D}_L$ is of order* 0, *symmetric and its trace is* $2H$, *where H is the mean curvature of ∂Y in Y with respect to $-n$.*

Moreover, consider the operator (D, L) defined by $D : \{s \in \Gamma(Y, \nu), s_{|\partial Y} \in L\} \to \Gamma(Y, \nu)$. We will use the following lemma, whose proof can be found in the appendix.

LEMMA 3.6 *We have* coker $(D, L) = \ker(D, L^\perp)$, *where $L^\perp$ is the orthogonal complement of L in $\nu_{|\partial Y}$.*

We now prove the vanishing theorem stated previously.

THEOREM 1.10 *Let M be a manifold equipped with a torsion-free G_2-structure and Y be an associative submanifold with boundary in a coassociative submanifold X. If $\mathcal{D}_{\mu_X}$ and $\mathcal{R} - \mathcal{A}$ are positive, the moduli space $\mathcal{M}_{Y,X}$ is smooth near Y and of dimension given by the virtual one.*

Proof. To prove Theorem 1.10, it is enough by Proposition 3.2 to show that coker (D, ν_X), which equals $\ker(D, \mu_X)$ by Lemma 3.6, is trivial. So let $s \in \ker(D, \mu_X)$. Since Y has a boundary, we need to change the integration (20), because the divergence has to be considered:

$$\int_Y |\nabla^\perp s|^2 + \langle \mathcal{R}_\nu s, s\rangle \,\mathrm{d}y = \frac{1}{2}\int_Y \operatorname{div} \nabla |s|^2 \,\mathrm{d}y. \tag{21}$$

By Stokes, the last equals

$$-\frac{1}{2}\int_{\partial Y} \mathrm{d}|s|^2(n)\,\mathrm{d}\sigma = -\int_{\partial Y} \langle \nabla^\perp_n s, s\rangle \,\mathrm{d}\sigma,$$

where n is the inward unit normal vector of ∂Y. Choosing a local orthonormal frame $\{v, w = n \times v\}$ of $T\partial Y$, $0 = Ds = n \times \nabla^\perp_n s + v \times \nabla^\perp_v s + w \times \nabla^\perp_w s$ implies that

$$\nabla^\perp_n s = -w \times \nabla^\perp_v s + v \times \nabla^\perp_w s.$$

Here, we used the formula

$$\forall u, v, w \in TM, \quad \chi(u, v, w) = -u \times (v \times w) - \langle u, v\rangle w + \langle u, w\rangle v,$$

so that for orthogonal vectors $u, v, w \in TM$,

$$u \times (v \times w) = w \times (u \times v) \tag{22}$$

and

$$u \times (u \times w) = -\|u\|^2 w. \tag{23}$$

Hence,

$$-\int_{\partial Y} \langle \nabla_n^\perp s, s\rangle \, \mathrm{d}\sigma = \int_{\partial Y} \langle w \times \nabla_v^\perp s - v \times \nabla_w^\perp s, s\rangle \, \mathrm{d}\sigma = -\int_{\partial Y} \langle \mathcal{D}_{\mu_X} s, s\rangle \, \mathrm{d}\sigma.$$

Summing up, we obtain the equation

$$\int_Y |\nabla^\perp s|^2 \, \mathrm{d}y + \int_Y \langle \mathcal{R}_\nu s, s\rangle \, \mathrm{d}y + \int_{\partial Y} \langle \mathcal{D}_{\mu_X} s, s\rangle \, \mathrm{d}\sigma = 0. \tag{24}$$

If $\mathcal{D}_{\mu_X}$ and $\mathcal{R}_\nu$ are positive, s vanishes, hence the result. □

4. Examples

4.1. *Flatland*

In flat spaces, the curvature tensor R vanishes, and so $\mathcal{R}_\nu = -\mathcal{A} \leq 0$. Consequently, *a priori* Theorem 1.10 does not apply. Nevertheless, we have the following corollary.

COROLLARY 4.1 *Let M be a manifold equipped with a torsion-free G_2-structure whose metric is flat, and Y be a totally geodesic associative submanifold with boundary in a coassociative X. If $\mathcal{D}_{\mu_X}$ is positive, then $\mathcal{M}_{Y,X}$ is smooth near Y and of the expected dimension.*

Proof. The hypotheses on M and Y imply that $\mathcal{R}_\nu = 0$. Consider $s \in \operatorname{coker}(D, \nu_X) = \ker(D, \mu_X)$. Formula (24) shows that $\nabla^\perp s = 0$ and $s_{|\partial Y} = 0$. Using $\mathrm{d}|s|^2 = 2\langle \nabla^\perp s, s\rangle = 0$. This implies $s = 0$ and the result. □

When $M = \mathbb{R}^7$ with its canonical flat metric, we get the following very explicit example considered in [7]. Take a ball Y in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$ with real analytic boundary, and choose any normal real analytic vector field $e \in \Gamma(\partial Y, \nu)$. By Harvey and Lawson [8], there is a unique local coassociative X_e containing ∂Y such that its tangent bundle $T_y X_e$ contains $e(y)$ at every boundary point y.

COROLLARY 4.2 *Let us assume that Y is a strictly convex ball in $\mathbb{R}^3$. Then there exists a positive constant ϵ, such that for every normal vector field $e \in \Gamma(\partial Y, \nu)$ satisfying $\|de\|_{L^\infty} \leq \epsilon$, the moduli space $\mathcal{M}_{Y,X_e}$ is smooth near Y and one-dimensional.*

Proof. Since the fibre bundle ν_{X_e} is trivial and the genus of ∂Y is zero, the index equals here $c_1(\nu_X) + 1 - g = 1$. We want to show that $\mathcal{D}_{\mu_X}$ is positive. To see that, we choose local orthogonal principal directions v and $w = n \times v$ in $T\partial Y$. From Theorem 3.1, we know that $v \times e$ is a non-vanishing section of μ_X. Let us assume first that e is constant. We compute, using relation (22),

$$\begin{aligned}\mathcal{D}_{\mu_X}(v \times e) &= v \times (\nabla^{\perp\partial}_w v \times e) - w \times (\nabla^{\perp\partial}_v v \times e)\\ &= -k_v w \times (n \times e) = k_v v \times e,\end{aligned}$$

where k_v is the principal curvature in the direction of v. This shows that k_v is an eigenvalue of $\mathcal{D}_{\mu_X}$, and since we know that its trace is $2H$ by Proposition 3.5, we get that the other eigenvalue is k_w, the other principal curvature of ∂Y. These eigenvalues are positive if the boundary of Y is strictly convex and Corollary 4.1 gives the result. Now, if e is close enough to be a constant vector field, the eigenvalues of $\mathcal{D}_{\mu_X}$ remain positive, hence the general result. □

In fact, in the case where e is constant, we can give a better statement. Indeed, let $s \in \ker(D, \nu_X)$, and decompose $s_{|\partial Y}$ as $s = s_1 e + s_2 n \times e$. Of course, e is in the kernel of $\mathcal{D}_{\nu_X}$, and hence by Proposition 3.5, the second term is an eigenvector of $\mathcal{D}_{\nu_X}$ for the eigenvalue $2H$. So formula (24) applied to s gives $\int_Y |\nabla^\perp s|^2 + \int_{\partial Y} 2H|s_2|^2 = 0$. If $H > 0$, this implies immediately that $s_2 = 0$ and s_1 is constant, so s is proportional to e. This proves that $\dim\ker(D, \nu_X) = 1$ under the weaker condition that $H > 0$. Lastly, in fact we can even show that $\mathcal{M}_{Y,X_e} = \mathbb{R}$.

4.2. *The Bryant Salamon construction*

The spin bundle and its metric. As recalled briefly in Section 1, Bryant and Salamon [**4**] found on the total spin bundle $\mathcal{S} \simeq S^3 \times \mathbb{R}^4$ of the round sphere S^3 a complete metric with holonomy precisely equal to G_2. This metric is of the form

$$g = \alpha(r)\pi^* g_S + \beta(r) g_v,$$

where g_v is the flat metric on the fibre $\mathcal{S}_x \simeq \mathbb{R}^4$ induced by g_S, r is its associated norm, g_S the round metric on S^3 and $\pi : \mathcal{S} \to S^3$ the natural projection. For some particular smooth functions α and β, the authors proved that the holonomy of the metric is G_2. In this situation, the base S^3 is associative and the Dirac operator D is the classical one for the spin bundle $\mathcal{S}$.

COROLLARY 4.3 [**17**] *The associative S^3 is isolated as an associative submanifold.*

Proof. By the famous computation of Lichnerowicz [**16**], $D^2 = \nabla^*\nabla + s/4$, where s is the scalar curvature of (S^3, g_S) and ∇ is the induced connection on the spin bundle, which is in our case the connection $\nabla^\perp$. Identifying with the equation in Theorem 2.8, we get that $\mathcal{R}_\nu = s/4$. Since S is positive, so is $\mathcal{R}_\nu$, and Theorem 1.7 then implies the result. □

Example with boundary. Choose a point p on the base S^3, a ball $B_\rho \subset \mathcal{S}$ of radius ρ around p and define $Y_\rho = B_\rho \cap S^3$. Take a normal vector field $e \in \Gamma(\partial Y_\rho, \nu)$ at the boundary of the associative Y_ρ. Here $\nu_y = \mathcal{S}_y$ for $y \in \partial Y_\rho$. The round sphere is real algebraic as is its metric g_S, hence we can find for ρ small enough a local chart $\Phi : B_\rho \to \mathbb{R}^7$ such that $\Phi(Y_\rho) \subset \mathbb{R}^3 \times \{0\}$, and $\Phi_* g$ is a real analytic metric. Further, we choose B_ρ and e in such a way that $\Phi(\partial Y_\rho)$ and $\Phi_* e$ are real analytic. Now, a straightforward generalization of the arguments in [8] based on the Cartan–Kähler theory proves that e and ∂Y_ρ generate a semi-local coassociative submanifold X_e containing ∂Y_ρ.

COROLLARY 4.4 *For ρ small enough, $\mathcal{M}_{Y_\rho, X_e}$ is smooth near Y_ρ and one-dimensional.*

Proof. The genus of ∂Y_ρ vanishes and the subbundle ν_{X_e} is trivial, hence the index of the associative deformations problem equals one. We can assume that $\Phi_* g(0)$ is the standard metric of $\mathbb{R}^7$, hence $d_p\Phi(\mathcal{S}_p) = 0 \oplus \mathbb{R}^4$. Moreover, we choose Φ such that the Levi–Civita connection of $\Phi_* g$ vanishes at 0. When ρ tends to zero, $\Phi(\partial Y_\rho)$ is asymptotically close to be the round ball $\rho B^3 \subset \mathbb{R}^3$ for the metric g_0. Then we know from the proof of Corollary 4.2 that the eigenvalues of the operator $\mathcal{D}_{\mu_{X_e}}$ computed in the model situation (i.e. with the flat metric and connection) equal the principal curvatures, here the inverse of ρ. Hence, for ρ small enough, $\mathcal{D}_{\mu_{X_e}}$ and $\mathcal{R}_\nu = s/4$ are both positive. Theorem 1.10 then implies the result. $\square$

4.3. *The Joyce construction*

Recall briefly the construction of the compact smooth manifold with holonomy G_2 constructed by Joyce [12, Section 12.2] and used in [7] for an example of an associative with boundary. On the flat torus (T^7, g_0) equipped with the G_2 structure $\phi_0 = \mathrm{d}x_{123} + \mathrm{d}x_{145} + \mathrm{d}x_{167} + \mathrm{d}x_{246} - \mathrm{d}x_{257} - \mathrm{d}x_{347} - \mathrm{d}x_{356}$, let

$$\begin{aligned}
\alpha &: (x_1, \dots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7),\\
\beta &: (x_1, \dots, x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7),\\
\gamma &: (x_1, \dots, x_7) \mapsto (-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7),\\
\sigma_0 &: (x_1, \dots, x_7) \mapsto (x_1, \tfrac{1}{2} - x_2, \tfrac{1}{2} - x_3, x_4, x_5, -x_6, \tfrac{1}{2} - x_7),\\
\tau_0 &: (x_1, \dots, x_7) \mapsto (x_1, x_2, \tfrac{1}{2} - x_3, \tfrac{1}{2} - x_4, x_5, x_6, \tfrac{1}{2} - x_7)
\end{aligned}$$

be isometric involutions, where $\sigma_0^*\phi_0 = \phi_0$ and $\tau_0^*\phi_0 = -\phi_0$. If $\pi : T^7 \to T^7/\Gamma$ is the quotient of T^7 by Γ the group generated by α, β and γ, one can check that the image Y by π of $\{(x_1, \frac{1}{4}, \frac{1}{4}, x_4, x_5, 0, \frac{1}{4}), x_{1,4,5} \in T^3\}$ in T^7/Γ is a smooth closed associative submanifold Y belonging to the fixed point set of the well-defined involution $\sigma = \pi_*\sigma_0$. Likewise, the image Y_∂ by π of $\{(x_1, \frac{1}{4}, \frac{1}{4}, x_4, x_5, 0, \frac{1}{4}), x_{1,5} \in T^2, \frac{1}{4} \le x_4 \le \frac{3}{4}\}$ is a smooth associative submanifold with boundary. This boundary is the union of two 2-tori embedded in the two disjoint smooth coassociatives X_1 and X_2, where X_i is the image by π of $\{(x_1, x_2, \frac{1}{4}, a_i, x_5, x_6, \frac{1}{4}), x_{1,2,5,6} \in T^4\}$ with $a_1 = \frac{1}{4}$ and $a_2 = \frac{3}{4}$. The latter submanifolds are components of the fixed point set of $\tau = \pi_*\tau_0$. Joyce's method to construct a metric with holonomy precisely equal to G_2 on a resolution M of the singularities of T^7/Γ can be made σ- and τ-equivariantly, so that after the process Y, Y_∂, X_1 and X_2 remain associative and coassociative, respectively. Now, the bundles ν_{X_i}, $i = 1, 2$, are clearly trivial over the two components of ∂Y_∂, so that the index of the deformation problem vanishes. From Theorems 1.2 and 1.4, we get that for every generic closed perturbation ψ of the G_2-structure, Y disappears or is perturbed into an isolated closed ψ-associative torus. Likewise, for every generic small ϕ-free deformation $\tilde{X}_i$ of X_i, there is a perturbation $\tilde{Y}_\partial$ of Y_∂ such that $\mathcal{M}_{\tilde{Y}_\partial, \tilde{X}}$ is a singleton near $\tilde{Y}$ or is empty.

REMARK 4.5 We would like to know which alternative holds. Unfortunately, even if we are far from the singularities of T^7/Γ, we do not know how to improve [12, Proposition 11.8.1] in order to get a control of the η_js in C^2-norm. Said otherwise, the perturbation of the metric *a priori* has effects on the whole M, and can be big in C^2 norm. Hence, our methods do not allow us to understand the effects of the perturbation on the associative submanifolds.

4.4. *Extensions from the Calabi–Yau world*

The closed case. Let (N, J, Ω, ω) be a Calabi–Yau six-dimensional manifold, where J is an integrable complex structure, Ω a non-vanishing holomorphic 3-form and ω a Kähler form. Then $M = N \times S^1$ is a manifold with holonomy in $\mathrm{SU}(3) \subset G_2$. An associated torsion-free G_2-structure on M is given by $\phi = \omega \wedge \mathrm{d}t + \mathrm{Re}\,\Omega$. Recall that a closed special Lagrangian L in N is a three-dimensional submanifold satisfying both conditions $\omega_{|TL} = 0$ and $\mathrm{Im}\,\Omega_{|TL} = 0$. We know from [17] that $\mathcal{M}_L$ the moduli space of special Lagrangian deformations of L is smooth and of dimension $b^1(L)$. Now for every $t \in S^1$, the product $Y = L \times \{t\}$ of a special Lagrangian and a point is a ϕ-associative submanifold of M. The following is inspired by an analogous result on coassociative submanifolds of Leung [15, Proposition 5].

PROPOSITION 4.6 *Let $t \in S^1$. The moduli space $\mathcal{M}_{L\times\{t\}}$ of associative deformations of $L \times \{t\}$ is always smooth, and can be identified with the product $\mathcal{M}_L \times S^1$, hence of dimension $b^1(L) + 1$.*

Proof. Consider a closed associative submanifold Y in the same homology class as $L \times \{t\}$. On the one hand, Y has a bigger volume than its projection $\pi(Y)$ to $N \times \{t\}$ and equality holds only if Y lies in $N \times \{t'\}$ for a constant t'. On the other hand, $\pi(Y)$ is in the same homology class as L, hence has volume larger than that of L, since special Lagrangians minimize the volume in their homology class. But Y is associative, hence has the same volume as L. Consequently, all these volumes equal, and Y is of the form $L' \times \{t'\}$. It is now immediate that ϕ-associativity of Y implies that L' is special Lagrangian. □

For the sequel, we will need another.

Proof of Proposition 4.6. Recall that since L is Lagrangian, its normal bundle NL is simply JTL, and the normal bundle ν of $Y = L \times \{t\}$ is isomorphic to $JTL \times \mathbb{R}\partial_t$, where ∂_t is the dual vector field of $\mathrm{d}t$. In this situation, we do not use the expression for D^2 given in Theorem 2.8. Instead, we give another formula for it. If $s = J\sigma \oplus \tau\partial_t$ is a section of ν, with $\sigma \in \Gamma(L, TL)$ and $\tau \in \Gamma(L, \mathbb{R}) = \Omega^0(L)$, we call $\sigma^\vee \in \Omega^1(L, \mathbb{R})$ the 1-form dual to σ, and we use the same symbol for its inverse. Moreover, we use the classical notation $* : \Omega^k(L) \to \Omega^{3-k}(L)$ for the Hodge star. Lastly, we define

$$\begin{aligned} D^\vee : \Omega^1(L) \times \Omega^0(L) &\longrightarrow \Omega^1(L) \times \Omega^0(L), \\ (\alpha, \tau) &\mapsto ((-J\pi_L D(J\alpha^\vee, \tau))^\vee, \pi_t D(J\alpha^\vee, \tau)), \end{aligned}$$

where π_L (respectively, π_t) is the orthogonal projection $\nu = NL \oplus \mathbb{R}$ to the first (respectively, the second) component. This is just a way to use forms on L instead of normal ambient vector fields.

PROPOSITION 4.7 *For every* $(\alpha, \tau) \in \Omega^1(L) \times \Omega^0(L)$,

$$D^\vee(\alpha, \tau) = (-{*}\mathrm{d}\alpha - \mathrm{d}\tau, {*}\mathrm{d}{*}\alpha),$$
$$(D^\vee)^2(\alpha, \tau) = -\Delta(\alpha, \tau),$$

where $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ (*note that it is* $\mathrm{d}^*\mathrm{d}$ *on* τ).

We refer to the appendix for the proof of this proposition. We see that for an infinitesimal associative deformation of $L \times \{t\}$, then α and τ are harmonic over the compact L. In particular, τ is constant and α describes an infinitesimal special Lagrangian deformation of L (see [17]). In other words, the only way to displace Y is to perturb L as special Lagrangian in N or translate it along the S^1-direction. Lastly, $\dim \operatorname{coker} D = \dim \ker D = b^1(L) + 1$ and by an immediate refinement of Proposition 2.2 for cokernels with constant dimension, $\mathcal{M}_Y$ is smooth and of dimension $b^1(L) + 1$. □

Symmetry breaking. Although the moduli space is smooth, the deformation problem for $L \times \{\cdot\}$ is always obstructed. Theorem 1.2 proves that any closed generic perturbation of the G_2-structure ϕ will make the S^1-symmetry disappear as well as the $\mathcal{M}_L$-family of associative submanifolds. We give here a family of examples of this phenomenon.

COROLLARY 4.8 *Let* L *be a smooth closed special Lagrangian sphere in* N, $t_0 \in S^1$ *and* $Y = L \times \{t_0\}$ *in* $N \times S^1$ *equipped with the* G_2*-form* $\phi = \operatorname{Re}\Omega + \omega \wedge \mathrm{d}t$ *and* $f : S^1 \to \mathbb{R}$ *be a smooth function vanishing transversally at a finite number of points in* S^1. *Then, there is a closed perturbation* ψ *of* ϕ *such that the connected components of* $L \times f^{-1}(0)$ *are associative with respect to* ψ, *and are the only* ψ*-associatives near* $\{L \times \{t\} : t \in S^1\}$.

Proof. Define $\tilde{\psi} = -f(t){*}(\phi \wedge \mathrm{d}t) = -f(t)(\partial/\partial t) \lrcorner {*}\phi = f(t)\operatorname{Im}\Omega$ on $L \times S^1$, since ${*}\phi = \operatorname{Im}\Omega \wedge \mathrm{d}t + \omega^2/2$. We extend $\tilde{\psi}$ as a closed 3-form ψ following the proof of Lemma 2.3: since L is special Lagrangian, $\operatorname{Im}\Omega_{|L} \in \Gamma(L, \Lambda^3 T^*N)$ can locally be written as $\sum_{i=1}^3 \mathrm{d}x_i \wedge \beta_i$, where $(x_i)_{i=1,2,3}$ are local normal coordinates in N over L. If $(\chi_U)_U$ is a finite set of cut-off functions in a neighbourhood of L, then the closed 3-form $\psi = \mathrm{d}(f(t)\sum_{U,i} \chi_U x_i \beta_i)$ is well defined on $N \times S^1$ and satisfies $\psi = f(t)\operatorname{Im}\Omega + O(\operatorname{dist}(\cdot, L \times S^1))$. We choose as a closed perturbation the 3-form $\phi_\lambda = \phi + \lambda\psi$. Now take $t_0 \in S^1$, such that $f(t_0) = 0$. If we choose coordinates on S^1 such that $t_0 = 0$, then there exists $a \neq 0$ with $f(t) = at + O(t^2)$. Proposition 2.6 shows that for λ small enough, $L \times \{t_0\}$ is the only local ψ-associative. Now take t_0 such that $f(t_0) \neq 0$. The following lemma holds in a general situation.

LEMMA 4.9 *Let* Y *be a compact smooth associative submanifold of* M *equipped with a closed* G_2*-structure* ϕ, *such that near* Y, $\mathcal{M}_{Y,\phi}$ *is one-dimensional and* $\dim \ker D = 1$ *at every element of* $\mathcal{M}_{Y,\phi}$. *Let* $\xi \in \Gamma(Y, N_Y)$ *be a non-trivial normal vector field in* $\ker D$ *and* $\tilde{\psi}$ *be the* 3*-form* $\xi \lrcorner {*}\phi \in \Gamma(Y, \Lambda^3 T^*M)$. *If* ψ *is any closed extension of* ψ *in a neighbourhood of* Y *and* $\phi_\lambda = \phi + \lambda\psi$, *then for* $\lambda \neq 0$ *small enough the moduli space* $\mathcal{M}_{Y,\phi_\lambda}$ *near* Y *is empty.*

Proof. By definition of ϕ_λ and Lemma 2.3, the derivative of $F(\lambda, s) = \exp_s^* \chi_{\phi_\lambda}(\omega)$ is of index 1 and surjective at $(\lambda = 0, s = 0)$, so that the vanishing locus of F is locally smooth, of dimension 1 and contains $\mathcal{M}_{Y,\phi}$. These sets must be locally equal, hence the result. □

We come back to the situation described in Proposition 4.8. If t is such that $f(t) \neq 0$, Lemma 4.9 shows that $\mathcal{M}_{L\times\{t_0\},\phi_\lambda}$ is empty for λ small enough. $\square$

Coclosed deformations. If we prefer *coclosed* deformations of the G_2-structure, we get a more precise statement and a very short proof.

PROPOSITION 4.10 *Let L be a smooth closed special Lagrangian sphere in N, $Y = L \times \{1\} \subset N \times S^1$ and $f : S^1 \to \mathbb{R}$ be a smooth function vanishing at a finite number of points in S^1. For every $\lambda \in \mathbb{R}$, define $\phi_\lambda = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\lambda f(t)}\Omega) + \omega \wedge \mathrm{d}t$ a family of coclosed G_2-structures. Then, if $\lambda \neq 0$, $\mathcal{M}_{Y,\phi_\lambda} = f^{-1}(0)$ near $L \times S^1$.*

Note that in particular, the transversality condition for f is no more needed.

Proof. The proof is almost the same as the proof of Proposition 4.6. Take Y to be a ϕ_λ-associative submanifold of $N \times S^1$ in the same class of homology as $L \times \{1\}$. Since the metric associated to ϕ_λ is independent of λ, the arguments of Proposition 4.6 still hold, and Y writes $L' \times \{t'\}$ for some submanifold $L' \in N$ and $t \in S^1$. The latter L' must be a special Lagrangian for $\mathrm{e}^{\mathrm{i}\lambda f(t)}\Omega$ since Y is ϕ_λ-associative. Hence, $\mathrm{Im}(\mathrm{e}^{\mathrm{i}\lambda f(t)}\Omega)$ vanishes on TL'. But L' lies in the same class of homology as L, so $\int_L \mathrm{Im}(\mathrm{e}^{\mathrm{i}\lambda f(t)}\Omega)$ should vanish because Ω is closed. Now, this is in fact $\int_L \sin(\lambda f(t))\mathrm{Re}\,\Omega = \sin(\lambda f(t))\mathrm{Vol}(L)$ which is non-zero if $\lambda \neq 0$ is small enough (independently of t) and $f(t) \neq 0$. If $f(t) = 0$ and L' is close enough to L, then $L' = L$ since a special Lagrangian sphere is isolated. Note that ϕ_λ is coclosed because $*\phi_\lambda = \mathrm{Im}(\mathrm{e}^{\mathrm{i}\lambda f(t)}\Omega) \wedge \mathrm{d}t + \frac{1}{2}\omega^2$. $\square$

REMARK 4.11 If L is not a sphere, then the same proof shows that $\mathcal{M}_{Y,\phi_\lambda} = \mathcal{M}_L \times f^{-1}(0)$ for $\lambda \neq 0$ small enough. This remains an obstructed situation, in the G_2 point of view.

With boundary. Recall that if Σ is a complex surface of N and $t \in S^1$, then $X = \Sigma \times \{t\}$ is a co-associative submanifold of M. Consider the problem of associative deformations of $Y = L \times \{t\}$ with boundary in X.

THEOREM 4.12 *Let $t \in S^1$ and L be a special Lagrangian submanifold in a six-dimensional Calabi–Yau N, such that L has boundary in a complex surface Σ. Let $Y = L \times \{t\}$ in $N \times S^1$ and $X = \Sigma \times \{t\}$.*

(1) *The moduli space $\mathcal{M}_{Y,X}$ of associative deformations of $L \times \{t\}$ with boundary in the co-associative $\Sigma \times \{t\}$ can be identified with the moduli space of special Lagrangian deformations of L with boundary in the fixed Σ.*
(2) *If the Ricci curvature of L is positive and if the boundary of L has positive mean curvature in L, then $\mathcal{M}_{Y,X}$ is locally smooth and has dimension g, where g is the genus of ∂L.*

Although the moduli space is smooth, its dimension exceeds by 1 the index of the deformation problem, see the beginning of the proof of the second assertion. As a consequence, Theorem 1.4 shows that generic perturbations of the boundary condition will decrement by one the dimension of the initial moduli space.

Note, moreover, that the deformation theory in [5] concerns *minimal Lagrangian* submanifolds with boundary in Σ, a wider class than that of *special Lagrangian* submanifolds of fixed phase.

Proof of Theorem 4.12(1). First, if M is equipped with a closed G_2-structure ϕ, note that an associative submanifold Y with boundary in a coassociative X minimizes the volume in the relative homology class $[Y] \in H_3(M, X, \mathbb{Z})$. Indeed, let Z be any 3-cycle with boundary in X, such that $[Z] = [Y]$. There is a 4-chain S with boundary in X and T a 3-chain in X, such that $Z - Y = \partial S + T$. Since ϕ is a calibration,

$$\text{Volume}(Z) \geq \int_Z \phi = \int_Y \phi + \int_{\partial S} \phi + \int_T \phi = \int_Y \phi = \text{Volume}(Y),$$

by Stokes and the fact that ϕ vanishes on any coassociative submanifold. By the same arguments as in the closed case, this proves the identity of the two moduli spaces. □

Proof of Theorem 4.12(2). Consider a special Lagrangian L with boundary ∂L in a complex surface Σ. If $Y = L \times \{t\}$ and $X = \Sigma \times \{t\}$, it is clear that the orthogonal complement ν_X of $T\partial Y$ in TX is equal as a real bundle to $JT\partial L \oplus \{0\}$, and μ_X is the trivial $n\times$-complex line bundle generated by ∂_t, where n is the inward unit normal vector field of ∂Y in Y. We begin by computing the index of the boundary problem. This is very easy, since μ_X is trivial, and by Theorem 3.1, we have $\nu_X \cong T\partial L^*$ as $n\times$-bundles. Hence, the index equals $-c_1(T\partial L) + 1 - g = -(2 - 2g) + 1 - g = g - 1$, where g is the genus of ∂L. Now let $\psi = s + \tau(\partial/\partial t)$ belonging to coker $(D, \nu_X) = \ker(D, \mu_X)$, where s is a section of NL and $\tau \in \Gamma(L, \mathbb{R})$. Let $\alpha = -Js^\vee$. By Proposition 4.7, α is a harmonic 1-form, and τ is harmonic (note that Y is not closed, so τ may be not constant). By classical results for harmonic 1-forms, we have

$$\tfrac{1}{2}\Delta|\psi|^2 = \tfrac{1}{2}\Delta(|\alpha|^2 + |\tau|^2) = |\nabla_L \alpha|^2 + |\mathrm{d}\tau|^2 + \tfrac{1}{2}\text{Ric}\,(\alpha, \alpha).$$

Integrating on $L \times \{t\}$, we obtain the equivalent of formula (24):

$$-\int_{\partial Y} \langle \mathcal{D}_{\mu_X}\psi, \psi\rangle \,\mathrm{d}\sigma = \int_Y |\nabla_L \alpha|^2 + |\mathrm{d}\tau|^2 + \frac{1}{2}\text{Ric}\,(\alpha, \alpha)\,\mathrm{d}y.$$

Lastly, let us compute the eigenvalues of $\mathcal{D}_{\mu_X}$. The constant vector $\partial/\partial t$ over ∂Y lies clearly in the kernel of $\mathcal{D}_{\mu_X}$. By Proposition 3.5, the other eigenvalue of $\mathcal{D}_{\mu_X}$ is $2H$, with eigenspace generated by $n \times (\partial/\partial t)$. Over ∂Y, s lies in $JTL \cap \mu_X$, hence is proportional to $n \times (\partial/\partial t)$. Consequently, $\mathcal{D}_{\mu_X}\psi = 2Hs$ and

$$-\int_{\partial Y} 2H|s|^2 \,\mathrm{d}\sigma = \int_Y |\nabla_L \alpha|^2 + |\mathrm{d}\tau|^2 + \frac{1}{2}\text{Ric}\,(\alpha, \alpha)\,\mathrm{d}y.$$

This equation, the positivity of the Ricci curvature and the positivity of H show that α vanishes and τ is constant. So we see that dim coker $(D, \nu_X) = 1$, and by the constant rank theorem, $\mathcal{M}_{Y,X}$ is locally smooth and of dimension $\dim\ker(D, \nu_X) = g$. □

Theorem 4.12 shows an equivalent result for deformations of special Lagrangian submanifold with metric conditions and boundary in a complex surface. Certainly, a direct proof would be shorter. But it seems to us that our proof has didactic virtues in our context of associative deformations.

A family of examples where $b^3(M) <$ dim coker D. Let N be a projective Calabi–Yau 3-fold equipped with an ample holomorphic line bundle L, and N_d be the dimension of $\mathbb{P}H^0(N, L^d)$. Take d to be big enough, so that $N_d(N_d - 1)/2 > b^3(N \times S^1)$ and choose C a generic complex curve defined by the intersection of the vanishing locus of two sections of L^d. Then, its moduli space of complex deformations is of dimension $N_d(N_d - 1)/2$, so that the dimension of the kernel of the Dirac operator associated to the associative $C \times S^1$ is bigger than $b^3(N \times S^1)$.

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Appendix

We will need the following trivial lemma.

LEMMA A.1 *Let ∇ be the Levi–Civita connection on M and R its curvature tensor. For any vector fields w, z, u and v on M, we have*

$$\nabla(u\times v)=\nabla u\times v+u\times\nabla v,$$
$$R(w,z)(u\times v)=R(w,z)u\times v+u\times R(w,z)v.$$

If Y is an associative submanifold of M with normal bundle ν, $u\in\Gamma(Y,TY)$, $v\in\Gamma(Y,TY)$ and $\eta\in\Gamma(Y,\nu)$, then

$$\nabla^{\top}(u\times v)=\nabla^{\top}u\times v+u\times\nabla^{\top}v,$$
$$\nabla^{\perp}(u\times\eta)=\nabla^{\top}u\times v+u\times\nabla^{\perp}v,$$

where $\nabla^{\top}=\nabla-\nabla^{\perp}$ is the orthogonal projection of ∇ to TY.

Proof. Let $x_1,\dots,x_7$ be normal coordinates on M near x, and $e_i=\partial/\partial x_i$ be their derivatives, orthonormal at x. We have

$$u\times v=\sum_i\langle u\times v,e_i\rangle e_i=\sum_i\phi(u,v,e_i)e_i,$$

so that at x, where $\nabla_{e_j}e_i=0$,

$$\begin{aligned}\nabla(u\times v)&=\sum_i(\nabla\phi(u,v,e_i)+\phi(\nabla u,v,e_i)+\phi(u,\nabla v,e_i)+\phi(u,v,\nabla e_i))e_i\\&=\sum_i(\phi(\nabla u,v,e_i)+\phi(u,\nabla v,e_i))e_i=\nabla u\times v+u\times\nabla v,\end{aligned}$$

because $\nabla\phi=0$. Now if u and v are in TY, then we get the result after noting that $(\nabla u\times v)^{\top}=\nabla^{\top}u\times v$, because TY is invariant under $\times$. The last relation is implied by $TY\times\nu\subset\nu$ and $\nu\times\nu\subset TY$. The curvature relation is easily derived from the definition $R(w,z)=\nabla_w\nabla_z-\nabla_z\nabla_w-\nabla_{[w,z]}$ and the differentiation of the vector product. □

A.1. *Proof of Lemma* 3.6

In this paragraph, we will assume that the ambient manifold M has a torsion-free G_2-structure (ϕ,g). Consider Y an associative submanifold and ν its normal bundle in (M,g). We begin with the classical lemma.

LEMMA A.2 *For a torsion-free structure, the operator D defined in* (1) *is formally self-adjoint, i.e. for s and $s'\in\Gamma(Y,\nu)$,*

$$\int_Y\langle Ds,s'\rangle-\langle s,Ds'\rangle\,\mathrm{d}y=-\int_{\partial Y}\langle n\times s,s'\rangle\,\mathrm{d}\sigma,\tag{A.1}$$

where $\mathrm{d}\sigma$ is the volume induced by the restriction of g on the boundary, and n is the inward unit normal vector of ∂Y.

Proof. The proof of this lemma is *mutatis mutandis*, the one for the classical Dirac operator, see [3, Proposition 3.4], for example. For the reader's convenience, we give a proof of this.

$$\begin{aligned}\langle Ds, s'\rangle &= \left\langle \sum_i e_i \times \nabla_i^{\perp} s, s'\right\rangle = -\sum_i \langle \nabla_i^{\perp} s, e_i \times s'\rangle \\ &= -\sum_i d_{e_i}\langle s, e_i \times s'\rangle + \langle s, \nabla_i^{\perp}(e_i \times s')\rangle \\ &= -\sum_i d_{e_i}\langle s, e_i \times s'\rangle + \langle s, \nabla_i^{\top} e_i \times s' + e_i \times \nabla_i^{\perp} s'\rangle.\end{aligned}$$

By a classical trick, define the vector field $X \in \Gamma(Y, TY)$ by $\langle X, w\rangle = -\langle s, w \times s'\rangle\ \forall w \in TY$. Note that the product on the LHS is on TY, and the one on the RHS is on ν. Now

$$-\sum_i d_{e_i}\langle s, e_i \times s'\rangle = \sum_i d_{e_i}\langle X, e_i\rangle = \sum_i \langle \nabla_i^{\top} X, e_i\rangle + \langle X, \nabla_i^{\top} e_i\rangle = \sum_i \operatorname{div} X - \langle s, \nabla_i^{\top} e_i \times s'\rangle.$$

By Stokes, we get

$$\int_Y \langle Ds, s'\rangle\,\mathrm{d}y = \int_{\partial Y}\langle X, -n\rangle\,\mathrm{d}\sigma + \int_Y \langle s, Ds'\rangle\,\mathrm{d}y = \int_{\partial Y}\langle s, n \times s'\rangle\,\mathrm{d}\sigma + \int_Y \langle s, Ds'\rangle\,\mathrm{d}y,$$

which is what we wanted. □

Now, consider L a subbundle of $\nu_{|\partial Y}$ of real rank equal to 2 and invariant under the action of $n\times$. Let $s' \in \Gamma(Y, \nu)$ lying in coker (D, L). This means that for every $s \in \Gamma(Y, \nu)$ with $s_{|\partial Y} \in L$, we have $\int_Y \langle Ds, s'\rangle\,\mathrm{d}y = 0$. By the former result, we see that this is equivalent to

$$\int_Y \langle s, Ds'\rangle + \int_{\partial Y}\langle n \times s, s'\rangle = 0.$$

This clearly implies that $Ds' = 0$, and $s'_{|\partial Y} \perp L$, because L is invariant under the action of $n\times$. So $s' \in \ker(D, L^{\perp})$. The reverse inclusion holds too by similar reasons.

A.2. *Proof of Proposition* 3.5

Proof. Let Y be a smooth compact associative with boundary, and L be a subbundle of $\nu_{|\partial Y}$ invariant under the action of $n\times$. It is straightforward to check that $\mathcal{D}_L$ defined in Definition 1.8 does not depend on the chosen orthonormal frame $\{v, w = n \times v\}$. For every $\psi \in \Gamma(\partial Y, L)$ and f a function,

$$\begin{aligned}\mathcal{D}_L(f\psi) &= \pi_L(v \times \nabla_w(f\psi) - w \times \nabla_v(f\psi)) \\ &= f\mathcal{D}_L\psi + (d_w f)\pi_L(v \times \psi) - (d_v f)\pi_L(w \times \psi) = f\mathcal{D}_L\psi,\end{aligned}$$

because $w \times L$ and $v \times L$ are orthogonal to L. Now, decompose the connexion $\nabla^{\top}$ on TY as $\nabla^{\top} = \nabla^{\top\partial} + \nabla^{\perp\partial}$ into its two projections along $T\partial Y$ and along the normal (in TY) n-direction. For the computations, choose v and $w = n \times v$ the two orthogonal characteristic directions on $T\partial Y$, i.e. $\nabla_v^{\top\partial} n = -k_v v$ and $\nabla_w^{\top\partial} n = -k_w w$, where k_v and k_w are the two principal curvatures. We have $\nabla_v^{\perp\partial} v = k_v n$ and $\langle \nabla_w^{\perp\partial} v, n\rangle = 0$, and the same, *mutatis mutandis*, for w. Then, for ψ and $\phi \in \Gamma(\partial Y, L)$, using the fact that $T\partial Y \times L$ is orthogonal to L and Lemma A.1,

$$\begin{aligned}\langle \mathcal{D}_L\psi, \phi\rangle &= \langle \nabla_w^{\perp}(v \times \psi) - (\nabla_w^{\perp\partial} v) \times \psi - \nabla_v^{\perp}(w \times \psi) + (\nabla_v^{\perp\partial} w) \times \psi, \phi\rangle \\ &= \langle \nabla_w^{\perp}(v \times \psi) - \nabla_v^{\perp}(w \times \psi), \phi\rangle = -\langle v \times \psi, \nabla_w^{\perp}\phi\rangle + \langle w \times \psi, \nabla_v^{\perp}\phi\rangle \\ &= \langle \psi, v \times \nabla_w^{\perp}\phi - w \times \nabla_v^{\perp}\phi\rangle = \langle \psi, \mathcal{D}_L\phi\rangle.\end{aligned}$$

To prove that the trace of $\mathcal{D}_L$ is $2H$, let $e \in L$ be a local unit section of L. We have $n \times e \in L$ too, and using again Lemma A.1 and relation (22),

$$\begin{aligned}\langle \mathcal{D}_L(n \times e), n \times e\rangle &= \langle v \times ((\nabla_w^{\top\partial} n) \times e) + v \times (n \times \nabla_w^{\perp} e), n \times e\rangle \\ &\quad - \langle w \times ((\nabla_v^{\top\partial} n) \times e) - w \times (n \times \nabla_v^{\perp} e), n \times e\rangle \\ &= \langle v \times (-k_w w \times e) - w \times (-k_v v \times e), n \times e\rangle \\ &\quad + \langle v \times (n \times \nabla_w^{\perp} e) - w \times (n \times \nabla_v^{\perp} e), n \times e\rangle \\ &= \langle k_w n \times e + k_v n \times e, n \times e\rangle + \langle w \times \nabla_w^{\perp} e + v \times \nabla_v^{\perp} e, n \times e\rangle.\end{aligned}$$

Using again relations (22) and (23) and the fact that $n\times$ is an isometry on the orthogonal complement of n, we get

$$\begin{aligned}\langle \mathcal{D}_L(n \times e), n \times e\rangle &= k_w + k_v - \langle n \times (w \times \nabla_w^{\perp} e + v \times \nabla_v^{\perp} e), e\rangle \\ &= 2H - \langle v \times \nabla_w^{\perp} e - w \times \nabla_v^{\perp} e, e\rangle \\ &= 2H - \langle \mathcal{D}_L e, e\rangle.\end{aligned}$$

This shows that trace $\mathcal{D}_L = 2H$. □

A.3. *Computation of* D^2

Proof of Theorem 2.8. We compute D^2 at a point $x \in Y$. For this, we choose normal coordinates on Y and $e_i \in \Gamma(Y, TY)$ their associated derivatives, orthonormal at x. To be explicit, $\nabla^{\top} e_i = 0$ at x. Let $\psi \in \Gamma(Y, \nu)$.

$$\begin{aligned}D^2\psi &= \sum_{i,j} e_i \times \nabla_i^{\perp}(e_j \times \nabla_j^{\perp}\psi) \\ &= \sum_{i,j} e_i \times (e_j \times \nabla_i^{\perp}\nabla_j^{\perp}\psi) + \sum_{i,j} e_i \times (\nabla_i^{\top} e_j \times \nabla_j^{\perp}\psi).\end{aligned}$$

The second sum of the right-hand side vanishes, so that using relations (22) and (23) for the first sum we get

$$\begin{aligned}D^2\psi &= -\sum_i \nabla_i^{\perp}\nabla_i^{\perp}\psi - \sum_{i\neq j}(e_i \times e_j) \times \nabla_i^{\perp}\nabla_j^{\perp}\psi \\ &= \nabla^{\perp *}\nabla^{\perp}\psi - \sum_{i<j}(e_i \times e_j) \times (\nabla_i^{\perp}\nabla_j^{\perp} - \nabla_j^{\perp}\nabla_i^{\perp})\psi \\ &= \nabla^{\perp *}\nabla^{\perp}\psi - \sum_{i<j}(e_i \times e_j) \times R^{\perp}(e_i, e_j)\psi.\end{aligned}$$

Since $(e_i \times e_j) \times R^{\perp}(e_i, e_j)$ is symmetric in i, j, this is equal to

$$\nabla^{\perp *}\nabla^{\perp}\psi - \frac{1}{2}\sum_{i,j}(e_i \times e_j) \times R^{\perp}(e_i, e_j)\psi.$$

The main tool for what follows is the Ricci equation. Let u, v be sections of $\Gamma(Y, TY)$ and ϕ, ψ be elements of $\Gamma(Y, \nu)$.

$$\langle R^{\perp}(u, v)\psi, \phi\rangle = \langle R(u, v)\psi, \phi\rangle + \langle (A_\psi A_\phi - A_\phi A_\psi)u, v\rangle,$$

where $A_\phi(u) = A(\phi)(u) = -\nabla_u^\top \phi$. Choosing $\eta_1, \dots, \eta_4$ to be an orthonormal basis of ν at the point x, we get

$$\begin{aligned}
-\frac{1}{2}\sum_{i,j}(e_i \times e_j) \times R^\perp(e_i, e_j)\psi &= -\frac{1}{2}\sum_{i,j,k}\langle (e_i \times e_j) \times R^\perp(e_i, e_j)\psi, \eta_k\rangle \eta_k \\
&= \frac{1}{2}\sum_{i,j,k}\langle R^\perp(e_i, e_j)\psi, (e_i \times e_j) \times \eta_k\rangle \eta_k \\
&= -\frac{1}{2}\pi_\nu \sum_{i,j}(e_i \times e_j) \times R(e_i, e_j)\psi \\
&\quad + \frac{1}{2}\sum_{i,j,k}\langle (A_\psi A_{(e_i\times e_j)\times\eta_k} - A_{(e_i\times e_j)\times\eta_k}A_\psi)e_i, e_j\rangle \eta_k.
\end{aligned}$$

Using the classical Bianchi relation $R(e_i, e_j)\psi = -R(\psi, e_i)e_j - R(e_j, \psi)e_i$, the first part of the sum $-\frac{1}{2}\pi_\nu \sum_{i,j}(e_i \times e_j) \times R(e_i, e_j)\psi$ is equal to

$$\begin{aligned}
I &= -2\pi_\nu(e_1 \times R(e_2, \psi)e_3 + e_2 \times R(e_3, \psi)e_1 + e_3 \times R(e_1, \psi)e_2) \\
&= -2\pi_\nu(e_1 \times R(e_2, \psi)(e_1 \times e_2) + e_2 \times R(e_3, \psi)(e_2 \times e_3) + e_3 \times R(e_1, \psi)(e_3 \times e_1)) \\
&= -2\pi_\nu(e_1 \times (R(e_2, \psi)e_1 \times e_2 + e_1 \times R(e_2, \psi)e_2) + e_2 \times (R(e_3, \psi)e_2 \times e_3 + e_2 \times R(e_3, \psi)e_1) \\
&\quad + e_3 \times (R(e_1, \psi)e_3 \times e_1 + e_3 \times R(e_1, \psi)e_2)) \\
&= -I + 2\pi_\nu \sum_i R(e_i, \psi)e_i,
\end{aligned}$$

which gives $I = \pi_\nu \sum_i R(e_i, \psi)e_i$. The Weingarten endomorphisms are symmetric, so that the second part of the sum is

$$\frac{1}{2}\sum_{i,j,k}\langle A_{(e_i\times e_j)\times\eta_k}e_i, A_\psi e_j\rangle \eta_k - \frac{1}{2}\sum_{i,j,k}\langle A_\psi e_i, A_{(e_i\times e_j)\times\eta_k}e_j\rangle \eta_k.$$

It is easy to see that the second sum is the opposite of the first one. We compute

$$A_{(e_i\times e_j)\times\eta_k}e_i = -(\nabla_i^\perp e_i \times e_j) \times \eta_k - (e_i \times \nabla_i^\perp e_j) \times \eta_k + (e_i \times e_j) \times A_{\eta_k}e_i.$$

But we know that an associative submanifold is minimal, so that $\sum_i \nabla_i^\perp e_i = 0$. Moreover, differentiating the relation $e_3 = \pm e_1 \times e_2$, one easily checks that $\sum_i e_i \times \nabla_j^\perp e_i = 0$. Summing up, the only remaining term is

$$\sum_{i,j,k}\langle (e_i \times e_j) \times A_{\eta_k}e_i, A_\psi e_j\rangle \eta_k.$$

We now use the classical formula for vectors u, v and w in TY:

$$(v \times w) \times u = \langle u, v\rangle w - \langle u, w\rangle v,$$

hence

$$(e_i \times e_j) \times A_{\eta_k}e_i = \langle A_{\eta_k}e_i, e_i\rangle e_j - \langle A_{\eta_k}e_i, e_j\rangle e_i.$$

One more simplification comes from $\sum_i \langle A_{\eta_k}e_i, e_i\rangle = 0$ for all k because Y is minimal, so our sum is now equal to

$$-\sum_{i,j,k}\langle A_{\eta_k}e_i, e_j\rangle\langle e_i, A_\psi e_j\rangle \eta_k = -\mathcal{A}\psi. \qquad \square$$

A.4. *Computation of D in the Calabi–Yau extension*

Proof of Proposition 4.7. We will use the simple formula $\nabla^{\perp} Js = J\nabla^{\top} s$ for all sections $s \in \Gamma(L, NL)$. For $(s, \tau) \in \Gamma(L, NL) \times \Gamma(L, \mathbb{R})$, and e_i local orthonormal frame on L,

$$\begin{aligned} D(s,\tau) &= \sum_{i,j} \langle e_i \times \nabla_i^{\perp} s, Je_j\rangle Je_j + \sum_i \langle e_i \times \nabla_i^{\perp} s, \partial_t\rangle \partial_t + \sum_i \partial_i \tau \, e_i \times \partial_t \\ &= J \sum_{i,j} \phi(e_i, \nabla_i^{\perp} s, Je_j) e_j + \sum_i \phi(e_i, \nabla_i^{\perp} s, \partial_t)\partial_t + J \sum_{i,j} \partial_i \tau \, \langle e_i \times \partial_t, Je_j\rangle e_j, \end{aligned}$$

where we used that $e_i \times \partial_t \perp \partial_t$.

$$\begin{aligned} &= J \sum_{i,j} \operatorname{Re}\Omega(e_i, \nabla_i^{\perp} s, Je_j) e_j + \sum_i \omega(e_i, \nabla_i^{\perp} s)\partial_t + J \sum_{i,j} \partial_i \tau \, \phi(e_i, \partial_t, Je_j) e_j \\ &= J \sum_{i,j} \operatorname{Re}\Omega(e_i, J\nabla_i^{\top} \sigma, Je_j) e_j + \sum_i \omega(e_i, J\nabla_i^{\top} \sigma)\partial_t + J \sum_{i,j} \partial_i \tau \, \omega(Je_j, e_i) e_j, \end{aligned}$$

where $\sigma = -Js \in \Gamma(L, TL)$.

$$\begin{aligned} &= -J \sum_{i,j} \operatorname{Re}\Omega(e_i, \nabla_i^{\top} \sigma, e_j) e_j + \sum_i \langle e_i, \nabla_i^{\top} \sigma\rangle \partial_t - J \sum_{i,j} \partial_i \tau \langle e_j, e_i\rangle e_j \\ &= -J \sum_{i,j} \operatorname{Vol}(e_i, \nabla_i^{\top} \sigma, e_j) e_j + \sum_i \langle e_i, \nabla_i^{\top} \sigma\rangle \partial_t - J \sum_i \partial_i \tau e_i, \end{aligned}$$

since $\operatorname{Re}\Omega$ is the volume form on TL. It is easy to find that this is equivalent to

$$D(s,\tau) = -J(*\mathrm{d}\sigma^{\vee})^{\vee} + (*\mathrm{d} * \sigma^{\vee})\partial_t - J(\mathrm{d}\tau)^{\vee},$$

and so $D^{\vee}(\sigma^{\vee}, \tau) = (-*\mathrm{d}\sigma^{\vee} - \mathrm{d}\tau, *\mathrm{d} * \sigma^{\vee})$. Now, since $\mathrm{d}^* = (-1)^{3p+1} * \mathrm{d} *$ on the p-forms, one easily checks the formula for D^2. $\square$