EXPONENTIAL RAREFACTION OF REAL CURVES WITH MANY COMPONENTS

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ABSTRACT

Given a positive real Hermitian holomorphic line bundle L over a smooth real projective manifold X, the space of real holomorphic sections of the bundle L^d inherits for every $d \in \mathbf{N}^*$ a L^2-scalar product which induces a Gaussian measure. When X is a curve or a surface, we estimate the volume of the cone of real sections whose vanishing locus contains many real components. In particular, the volume of the cone of maximal real sections decreases exponentially as d grows to infinity.

Introduction

Let (X, c_X) be a smooth real projective manifold of dimension n and $(L, c_L) \xrightarrow{\pi} (X, c_X)$ be a real ample holomorphic line bundle. In particular, c_X and c_L are antiholomorphic involutions on X and L respectively, such that $c_X \circ \pi = \pi \circ c_L$. Let h be a real Hermitian metric on (L, c_L) with positive curvature ω. It induces a Kähler structure on (X, c_X). For every nonnegative integer d, this metric induces a Hermitian metric h^d on L^d and then a L^2-Hermitian product on the complex vector space $H^0(X, L^d)$ of holomorphic sections of L^d. This product is defined by $(\sigma, \tau) \in H^0(X, L^d) \times H^0(X, L^d) \mapsto \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau) dx \in \mathbf{C}$, where $dx = \omega^n / \int_X \omega^n$ is the normalized volume induced by the Kähler form. Let $\mathbf{R}H^0(X, L^d)$ be the space of real sections $\{\sigma \in H^0(X, L^d) \mid c_L \circ \sigma = \sigma \circ c_X\}$ and $\Delta_k \subset H^0(X, L^d)$ (resp. $\mathbf{R}\Delta_k \subset \mathbf{R}H^0(X, L^d)$) be the discriminant locus (resp. its real part), that is the set of sections which do not vanish transversally. For every $\sigma \in H^0(X, L^d) \setminus \{0\}$, denote by $C_\sigma = \sigma^{-1}(0)$ the vanishing locus of σ and when σ is real, by $\mathbf{R}C_\sigma$ its real part. The divisor C_σ is smooth whenever $\sigma \in \mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d$. In this case, we denote by $b_0(\sigma) = b_0(\mathbf{R}C_\sigma)$ the number of connected components of $\mathbf{R}C_\sigma$.

0.1. *Real projective surfaces*

When X is two-dimensional, we know from Harnack-Klein's inequality [12], [14] that $b_0(\mathbf{R}C_\sigma) \leq g(C_\sigma) + 1$, where equality holds for the well-known maximal curves involved in Hilbert's sixteenth problem. Here, the genus $g(C_\sigma)$ of these smooth curves C_σ gets computed by the adjunction formula and equals $g(C_\sigma) = \frac{1}{2}(d^2L^2 - dc_1(X).L + 2)$, where $c_1(X)$ denotes the first Chern class of the surface X. It grows quadratically with respect to d. For every $d \in \mathbf{N}^*$ and $a \in \mathbf{Q}_+^*$, denote by $\mathcal{M}_d^a$ the open cone $\{\sigma \in \mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d \mid b_0(\mathbf{R}C_\sigma) \geq g(C_\sigma) + 1 - ad\}$. There always exists a constant $a \in \mathbf{Q}_+^*$ such that this set $\mathcal{M}_d^a$ is non-empty for d large enough, see Theorem 5. The main purpose of this article is to prove the following

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Theorem **1.** — *Let* (X, c_X) *be a smooth real projective surface and* (L, c_L) *be a real Hermitian holomorphic line bundle on* X *with positive curvature. Then, for every sequence* $d \in \mathbf{N}^* \mapsto a(d) \geq 1$ *of rationals, there exist constants* $C, D > 0$, *such that*

$$\forall d \in \mathbf{N}^*, \quad \mu(\mathcal{M}_d^{a(d)}) \leq Ca(d)^4 e^{-D\frac{d}{a(d)}},$$

where $\mu(\mathcal{M}_d^{a(d)})$ *denotes the Gaussian measure of* $\mathcal{M}_d^{a(d)}$.

The Gaussian measure μ on the Euclidian space $(\mathbf{R}H^0(X, L^d), \langle\ ,\ \rangle)$ is defined by the formula

$$\forall A \subset \mathbf{R}H^0(X, L^d), \quad \mu(A) = \frac{1}{\sqrt{\pi}^{N_d}} \int_A e^{-|x|^2} dx,$$

where dx denotes the Lebesgue measure associated to $\langle\ ,\ \rangle$ and N_d the dimension of $\mathbf{R}H^0(X, L^d)$. This Gaussian measure is a probability measure on $\mathbf{R}H^0(X, L^d)$ which is invariant under its isometry group.

Remark **1.** — Likewise, the scalar product $\langle\ ,\ \rangle$ induces a Fubini-Study volume form on the linear system $P(\mathbf{R}H^0(X, L^d))$. The volume of the projection $P(\mathcal{M}_d^a)$ for this form just coincides with the measure $\mu(\mathcal{M}_k^a)$ computed in Theorem 1. Note also that the choice of a scalar product on $\mathbf{R}H^0(X, L^d)$ is equivalent to the choice of a real embedding $\Phi_d : X \to \mathbf{C}P^{N_d-1}$ up to composition by a real isometry of $\mathbf{C}P^{N_d-1}$, see Section 1.2. Our choice of $\langle\ ,\ \rangle$ is such that this embedding is asymptotically an isometry, see [21].

When the sequence $a(d)$ is bounded, Theorem 1 implies that the measure of the set $\mathcal{M}_d^{a(d)}$ decreases exponentially with the degree d. This exponential rarefaction holds in particular for the maximal curves.

0.2. *Real curves*

When X is one-dimensional, we get the following result:

Theorem **2.** — *Let* X *be a closed smooth real curve and* L *be a real Hermitian holomorphic line bundle on* X *with positive curvature. Then, for every positive sequence* $(\epsilon(d))_{d\in\mathbf{N}}$ *of rational numbers, there exist constants* $C, D > 0$ *such that*

$$\forall d \in \mathbf{N}, \quad \mu\{\sigma \in \mathbf{R}H^0(X, L^d) \setminus \mathbf{R}\Delta_d,\ |\ \#(\sigma^{-1}(0) \cap \mathbf{R}X) \geq \sqrt{d}\epsilon(d)\}$$
$$\leq C\frac{\sqrt{d}}{\epsilon(d)} e^{-D\epsilon^2(d)},$$

where μ *denotes the Gaussian measure of the space* $\mathbf{R}H^0(X, L^d)$.

0.3. *Roots of polynomials*

When (X, c_X) is the projective space of dimension $n \geq 1$ and (L, h, c_L) is the degree one holomorphic line bundle equipped with its standard Fubini-Study metric, the vector space $\mathbf{R}H^0(X, L^d)$ gets isomorphic to the space $\mathbf{R}_d[X_1, \dots, X_n]$ of polynomials with n variables, real coefficients and degree at most d. The scalar product induced on $\mathbf{R}_d[X_1, \dots, X_n]$ by this isomorphism is the one turning the basis $\left(\sqrt{\binom{d+n}{j}} x_1^{j_1} \cdots x_n^{j_n}\right)_{0 \leq j_1 + \cdots + j_n \leq d}$ into an orthonormal one, where $\binom{d+n}{j} = \frac{(d+n)!}{n! j_1! \cdots j_n! (d - j_1 - \cdots - j_n)!}$. Thus, the induced measure μ on $\mathbf{R}_d[X_1, \dots, X_n]$ is the Gaussian measure associated to this basis. As a special case of Theorems 1 and 2, we get the following

Corollary **1.**

1. *For every positive sequence* $(\epsilon(d))_{d \in \mathbf{N}^*}$ *of rational numbers, there exist positive constants* C, D *such that the measure of the set of polynomials* $P \in \mathbf{R}_d[X]$ *which have at least* $\epsilon(d)\sqrt{d}$ *real roots is bounded from above by* $C\frac{\sqrt{d}}{\epsilon(d)} \exp(-D\epsilon^2(d))$.
2. *For every sequence* $d \in \mathbf{N}^* \mapsto a(d) \geq 1$ *of rationals, there exist positive constants* C, D *such that the measure of the set of polynomials* $P \in \mathbf{R}_d[X, Y]$ *having at least* $\frac{1}{2}d^2 - da(d)$ *connected components in their vanishing locus in* $\mathbf{R}^2$ *is bounded from above by* $Ca(d)^4 \exp(-D\frac{d}{a(d)})$.

0.4. *Strategy of the proof of Theorems 1 and 2*

Every curve $C_\sigma \subset X$, $\sigma \in \mathbf{R}H^0(X, L^d) \setminus \{0\}$, defines a current of integration which we renormalize by d, for its mass not to depend on $d \in \mathbf{N}^*$. In order to prove Theorems 1 and 2, we first obtain large deviation estimates for the random variable defined by this current, see Proposition 3. The expectation of this variable converges outside of $\mathbf{R}X$ to the curvature form ω of L when d grows to infinity. These results thus go along the same lines as the ones of Shiffman and Zelditch [17]. In particular, they make use of the asymptotic isometry theorem of Tian [21] (see also [3], [4] and [23]), as well as smoothness results [13] and behaviour close to the diagonal for the Bergman kernel [19], [2]. In order to deduce from these results informations on the random variable b_0, we use the theory of laminar currents introduced by Bedford, Lyubich and Smillie [1]. Indeed, we show as a corollary of a theorem of de Thélin [5] that every current in the closure of the ones arising from $\mathcal{M}_d^a$ (in particular every limit current of a sequence of real maximal curves) is weakly laminar outside of the real locus $\mathbf{R}X$, see Theorem 4. As a consequence, these currents remain in a compact set away from ω. At this point, our large deviation estimates provide the exponential decay.

0.5. *Description of the paper*

In the first paragraph, we bound from above the Markov moments needed for our large deviation estimates. In the second paragraph, we recall some elements of the theory of laminar currents in order to establish Theorem 4, that is laminarity outside of the real locus of currents in the closure of the union of the sets $\mathcal{M}^a_d$. We then get our large deviation estimates. We prove Theorems 1 and 2 in the third paragraph, dealing separately with the cases of bounded and unbounded sequences a. The last paragraph is devoted to some final discussion on the existence of real maximal curves on general real projective surfaces and on non-emptyness of sets $\mathcal{M}^a_d$, see Theorem 5, as well as on the expectation of the current of integration on real divisors.

1. Moments on real linear systems of divisors

In this paragraph, we estimate from above some Markov moments closely related to Poincaré-Lelong's formula, see Section 2.3. The upshot is Corollary 2 which, together with Poincaré-Lelong's formula, leads to the proof of our large deviation estimates given by Proposition 3. The latter provide the exponential decay in Theorems 1 and 2.

1.1. *Case of projective spaces*

Let $\mathcal{O}_{\mathbf{C}P^k}(1)$ be the degree one line bundle over $\mathbf{C}P^k$ and $\|.\|$ be its standard Hermitian metric with curvature the Fubini-Study Kähler form ω_{FS}. The sections $\sigma_i([z_0 : \cdots : z_k]) = \sqrt{k+1} z_i$, $i \in \{0, \ldots, k\}$, provide an orthonormal basis of $\mathbf{R}\mathrm{H}^0(\mathbf{C}P^k, \mathcal{O}_{\mathbf{C}P^k}(1))$. For every integer $m \geq 1$, set

$$\begin{array}{rcl} \mathrm{M}^m_{\mathbf{C}P^k} : \mathbf{C}P^k & \to & \mathbf{R} \\ z & \mapsto & \displaystyle\int_{\mathbf{R}\mathrm{H}^0(\mathbf{C}P^k, \mathcal{O}_{\mathbf{C}P^k}(1))} \left| \log\left(\frac{\|\sigma(z)\|^2}{k+1} \right) \right|^m d\mu(\sigma), \end{array}$$

where $d\mu$ denotes the Gaussian measure of the $(k+1)$-dimensional Euclidian space $\mathbf{R}\mathrm{H}^0(\mathbf{C}P^k, \mathcal{O}_{\mathbf{C}P^k}(1))$.

Proposition **1.** — *For every $m \geq 1$, the function $\mathrm{M}^m_{\mathbf{C}P^k}$ satisfies*

$$\forall z \in \mathbf{C}P^k \setminus \mathbf{R}P^k, \quad \mathrm{M}^m_{\mathbf{C}P^k}(z) \leq \frac{4m!}{1 - \|\tau(z)\|},$$

where τ denotes the section of $\mathcal{O}_{\mathbf{C}P^k}(2)$ defined by $\tau(z_0, \ldots, z_k) = z_0^2 + \cdots + z_k^2$.

Remark **2.** — The holomorphic section τ in Proposition 1 is invariant under the action of the group $\mathrm{PO}_{k+1}(\mathbf{R})$ of real isometries of $\mathbf{C}P^k$. A slice of $\mathbf{C}P^k$ for this action is given by the interval $\mathrm{I} = \{z_r = [1 : ir : 0 : \cdots : 0], 0 \leq r \leq 1\}$, where the end $r = 0$ (resp. $r = 1$) corresponds to the orbit $\mathbf{R}P^k$ (resp. $\tau^{-1}(0)$) of this action.

Proof of Proposition 1. — Both members of the inequality are invariants under the action of $\mathrm{PO}_{k+1}(\mathbf{R})$, so that it is enough to prove it for z in the fundamental domain I. Let $\sigma_0, \ldots, \sigma_k$ be the orthonormal basis of $\mathbf{R}\mathrm{H}^0(\mathbf{C}\mathrm{P}^k, \mathcal{O}_{\mathbf{C}\mathrm{P}^k}(1))$ given by $\sigma_i([z_0 : \cdots : z_k]) = \sqrt{k+1}z_i$. This basis induces the isometry

$$a = (a_0, \ldots, a_k) \in \mathbf{R}^{k+1} \mapsto \sigma_a = a_0\sigma_0 + \cdots + a_k\sigma_k \in \mathbf{R}\mathrm{H}^0(\mathbf{C}\mathrm{P}^k, \mathcal{O}_{\mathbf{C}\mathrm{P}^k}(1)).$$

By definition, for every $a \in \mathbf{R}^{k+1}$ and $z \in \mathbf{C}\mathrm{P}^k$,

$$\frac{1}{k+1}\|\sigma_a(z)\|^2 = \frac{|a_0z_0 + \cdots + a_kz_k|^2}{|z|^2}.$$

We deduce that for every $0 < r \leq 1$ and $m \geq 1$,

$$\begin{aligned}
\mathrm{M}^m_{\mathbf{C}\mathrm{P}^k}(z_r) &= \int_{\mathbf{R}^{k+1}} \left|\log\left(\frac{\|\sigma_a(z_r)\|^2}{k+1}\right)\right|^m d\mu(a) \\
&= \int_{\mathbf{R}^2} \left|\log\left(\frac{|a_0 + ira_1|^2}{1+r^2}\right)\right|^m \frac{e^{-|a|^2}}{\pi} da_0 da_1 \\
&= \int_0^\infty \int_0^{2\pi} \left|\log(\rho^2) + \log\frac{|\cos\theta + ir\sin\theta|^2}{1+r^2}\right|^m \frac{e^{-\rho^2}}{\pi}\rho d\rho d\theta \\
&\leq 2\int_0^\infty \left||\log(\rho^2)| + \log\left(\frac{1+r^2}{r^2}\right)\right|^m e^{-\rho^2}\rho d\rho,
\end{aligned}$$

since for every $0 \leq r \leq 1$ and every $\theta \in [0, 2\pi]$, $r^2 \leq \cos^2\theta + r^2\sin^2\theta = |\cos\theta + ir\sin\theta|^2 \leq 1$. Hence,

$$\mathrm{M}^m_{\mathbf{C}\mathrm{P}^k}(z_r) \leq 2\int_0^1 \left(-\log\left(\frac{\rho}{\alpha}\right)^2\right)^m e^{-\rho^2}\rho d\rho + 2\int_1^\infty (\log(\alpha\rho)^2)^m e^{-\rho^2}\rho d\rho,$$

where

$$\alpha^2 = \frac{(1+r^2)}{r^2} = \frac{2}{1 - \|\tau(z_r)\|}.$$

We now compute these two integrals. First,

$$\begin{aligned}
2\int_0^1 \left(-\log\left(\frac{\rho}{\alpha}\right)^2\right)^m e^{-\rho^2}\rho d\rho &= 2\alpha^2\int_0^{1/\alpha} (-\log\rho^2)^m e^{-\alpha^2\rho^2}\rho d\rho \\
&\leq 2\alpha^2\int_0^1 (-\log\rho^2)^m \rho d\rho
\end{aligned}$$

$$= \alpha^2 \int_0^\infty t^m e^{-t} dt \quad \text{with} \quad t = -\log \rho^2$$
$$= \frac{2m!}{1 - \|\tau(z_r)\|}.$$

As for the second integral, we deduce from the estimate $\rho e^{-\rho^2} \leq e^{-1/\rho^2}/\rho^3$ valid for every $\rho \geq 1$, that

$$2\int_1^\infty (\log(\alpha\rho)^2)^m e^{-\rho^2} \rho d\rho \leq 2\int_1^\infty (\log(\alpha\rho)^2)^m \frac{e^{-1/\rho^2}}{\rho^3} d\rho.$$

With $t = 1/\rho$, the right hand side becomes $2\int_0^1 (-\log(\frac{t}{\alpha})^2)^m e^{-t^2} t dt \leq \frac{2m!}{1-\|\tau(z)\|}$. $\square$

1.2. *Asymptotic results in the general case*

Let X be a closed real Kähler manifold of dimension n and L be a real Hermitian line bundle over X with positive curvature ω. We denote by d_{L} the smallest integer such that L^d is very ample for every $d \geq d_{\mathrm{L}}$ and by $\Phi_d : \mathrm{X} \to \mathrm{P}(\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*)$ the associated embedding, where $x \in \mathrm{X}$ is mapped to the set of linear forms that vanish on the hyperplane $\{\sigma \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \mid \sigma(x) = 0\}$. The L^2-Hermitian product of $\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ induces a Fubini-Study metric on the complex projective space $\mathrm{P}(\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*)$ together with a Hermitian metric $\|.\|$ on the bundle $\mathcal{O}_{\mathrm{P}(\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)^*)}(1)$. We denote by h_{Φ_d} the pullback of $\|.\|$ on L^d under the canonical isomorphism $\mathrm{L}^d \to \Phi_d^* \mathcal{O}_{\mathrm{P}(\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)^*)}(1)$. Recall that the latter is induced by the isomorphism

$$(x, \alpha) \in \mathrm{L}^d \mapsto (\Phi_d(x), \alpha_x) \in \mathcal{O}_{\mathrm{P}(\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)^*)}(1),$$

where $\alpha_x : \sigma \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \mapsto \langle \sigma, \alpha \rangle_x \in \mathbf{C}$. In particular, the curvature form of h_{Φ_d} is $\Phi_d^* \omega_{\mathrm{FS}}$, where ω_{FS} denotes the Fubini-Study form of $\mathrm{P}(\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*)$. The quotient h^d / h_{Φ_d} of these metrics of L^d is given by the function $x \in \mathrm{X} \mapsto \sum_{i=1}^{\mathrm{N}_d} h^d(\sigma_i(x), \sigma_i(x))$, where $(\sigma_1, \ldots, \sigma_{\mathrm{N}_d})$ stands for any orthonormal basis of $\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$. Let $\|.\|_{\Phi_d}$ be the norm induced by h_{Φ_d}. For every $m \in \mathbf{N}^*$, we set

$$\begin{aligned} \mathrm{M}^m_{(\mathrm{X},\mathrm{L}^d)} : \mathrm{X} &\to \mathbf{R} \\ x &\mapsto \int_{\mathbf{R}\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)} \left|\log\left(\|\sigma(z)\|^2_{\Phi_d}\right)\right|^m d\mu(\sigma), \end{aligned}$$

so that $\mathrm{M}^m_{(\mathrm{X},\mathrm{L}^d)} = \mathrm{M}^m_{\mathrm{P}(\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)^*)} \circ \Phi_d$.

Proposition **2.** — *Let* X *be a closed real Kähler manifold of dimension n and* L *be a positive real Hermitian line bundle over* X. *For every sequence* $(\mathrm{K}_d)_{d\in\mathbf{N}^*}$ *of compact subsets of* $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$ *such that the sequence* $(d\,\mathrm{dist}(\mathrm{K}_d, \mathbf{R}\mathrm{X})^2)_{d\in\mathbf{N}^*}$ *grows to infinity, the sequence* $\|\|\tau\| \circ \Phi_d\|_{\mathrm{C}^0(\mathrm{K}_d)}$ *converges to zero as d grows to infinity, where* $\tau \in \mathbf{R}\mathrm{H}^0(\mathcal{O}_{\mathrm{P}(\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)^*)}(2))$ *is the section defined in Proposition 1. If* K *is a fixed compact, the same holds for any norm* $\mathrm{C}^q(\mathrm{K})$, $q \in \mathbf{N}$.

The L^2-Hermitian product on $\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ induces a scalar product on $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ and its dual $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*$. Let $\langle\ ,\ \rangle_{\mathbf{C}}$ be the extension of this scalar product to a complex bilinear product on $\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*$. The section τ of $\mathcal{O}(2)_{\mathrm{P}(\mathrm{H}^0(\mathrm{X},\mathrm{L}^d))^*}$ that appears in Propositions 1 and 2 is the one induced by $\sigma^* \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^* \mapsto \langle \sigma^*, \sigma^* \rangle_{\mathbf{C}} \in \mathbf{C}$.

Proof of Proposition 2. — Let $\mathrm{D}^* \in \mathrm{L}^*$ be the open unit disc bundle for the metric h and $dxdv$ the product measure on D^*, where $dx = \omega^n / \int_{\mathrm{X}} \omega^n$ is the measure on the base X of D^* and dv is the Lebesgue measure on the fibres. Let $\mathrm{L}^2(\mathrm{D}^*, \mathbf{C})$ be the space of complex functions of class L^2 on D^* for the measure $dxdv$ and $\mathcal{H}^2 \subset \mathrm{L}^2(\mathrm{D}^*, \mathbf{C})$ be the closed subspace of L^2 holomorphic functions on the interior of D^*. Every function $f \in \mathcal{H}^2$ has a unique expansion

$$f : (x, v) \in \mathrm{D}^* \mapsto \sum_{d=0}^{\infty} a_d(x) v^d \in \mathbf{C},$$

where for every $d \geq 0$, $a_d \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$. The series $\sum_{d=0}^{\infty} a_d(x) v^d$ converges uniformly on every compact subset of D^* as well as in L^2-norm. Let B be the Bergman kernel of D^*, defined by the relation:

$$\forall (y, w) \in \mathrm{D}^*, \forall f \in \mathcal{H}^2, \quad f(y, w) = \int_{\mathrm{D}^*} f(x, v) \overline{\mathrm{B}((x, v), (y, w))} dxdv,$$

where this function $\mathrm{B} : \mathrm{D}^* \times \mathrm{D}^* \to \mathbf{C}$ is holomorphic in the first variable and antiholomorphic in the second one. Kerzman [13] proved that the Bergman kernel can be extended smoothly up to the boundary outside of the diagonal of $\overline{\mathrm{D}^*} \times \overline{\mathrm{D}^*}$. Now, denote by $\mathrm{D}^*_{\mathrm{X} \setminus \mathbf{R}\mathrm{X}} = \{(x, v) \in \mathrm{D}^* \mid x \in \mathrm{X} \setminus \mathbf{R}\mathrm{X}\}$. The function

$$b : (x, v) \in \mathrm{D}^*_{\mathrm{X} \setminus \mathbf{R}\mathrm{X}} \mapsto \mathrm{B}((x, v), c_{\mathrm{L}^*}(x, v)) \in \mathbf{C}$$

is holomorphic on $\mathrm{D}^*_{\mathrm{X} \setminus \mathbf{R}\mathrm{X}}$ and can be extended smoothly on $\overline{\mathrm{D}^*_{\mathrm{X} \setminus \mathbf{R}\mathrm{X}}}$. For every $d \geq 0$ let $(\sigma_{i,d})_{1 \leq i \leq \mathrm{N}_d}$ be an orthonormal basis of $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ and

$$e_{i,d} = \sqrt{\frac{d+1}{\pi}} \widehat{\sigma_{i,d}} \in \mathcal{H}^2,$$

where $\widehat{\sigma_{i,d}} : (x, v) \in \mathrm{D}^* \mapsto \sigma_{i,d}(x) v^d \in \mathbf{C}$. The family $(e_{i,d})_{i,d}$ forms a Hilbertian basis of $\mathcal{H}^2$. Then,

$$\begin{aligned} \forall (x, v) \in \mathrm{D}^*_{\mathrm{X} \setminus \mathbf{R}\mathrm{X}}, \quad b(x, v) &= \mathrm{B}((x, v), c_{\mathrm{L}^*}(x, v)) \\ &= \sum_{d=0}^{\infty} \sum_{i=1}^{\mathrm{N}_d} e_{i,d}(x, v) \overline{e_{i,d}(c_{\mathrm{L}^*}(x, v))} \end{aligned}$$

$$= \sum_{d=0}^{\infty} \frac{d+1}{\pi} \sum_{i=1}^{N_d} \sigma_{i,d}(x) v^d \overline{\sigma_{i,d}(c_X(x)) c_{L^*}(v)^d}$$

$$= \sum_{d=0}^{\infty} \frac{d+1}{\pi} \sum_{i=1}^{N_d} \sigma_{i,d}^2(x) v^{2d}$$

since $\sigma_{i,d}$ is real, that is satisfies $\sigma_{i,d} \circ c_X = c_{L^d} \circ \sigma_{i,d}$. Note that $\tau \circ \Phi_d = \sum_{i=1}^{N_d} \sigma_{i,d}^2(x)$ and from Tian's asymptotic isometry theorem [21] (see also [4] and [23]) $\|.\|_{\Phi_d} \leq \frac{C}{d^n}\|.\|$. For a fixed compact K, the result thus just follows from Cauchy formula applied to b. In general, we may substitute L with L^d, K with K_d and deduce the result from Proposition2.1 of [19] (see also [2]) since the sequence $(d \sup_{x \in K_d} \mathrm{dist}(x, c_X(x))^2)_{d \in \mathbf{N}^*}$ grows to infinity as d grows to infinity. □

Proposition 2 and Proposition 1 imply the following

Corollary **2.** — *Let* X *be a real Kähler manifold and* L *be a real positive Hermitian line bundle over* X*. For every sequence* $(K_d)_{d\in\mathbf{N}}$ *of compact subsets of* $X \setminus \mathbf{R}X$ *such that the sequence* $(d\,\mathrm{dist}(K_d, \mathbf{R}X)^2)_{d\in\mathbf{N}^*}$ *grows to infinity, there exists a positive constant* c_K *such that as soon as* L^d *is very ample,*

$$\forall m \in \mathbf{N}^*, \quad \sup_{K_d} M^m_{(X,L^d)} \leq c_K m!$$

Remark **3.** — Actually, in Corollary 2, $\lim_{d\to\infty} \sup_{K_d} M^m_{(X,L^d)} \leq 4m!$. Also, Proposition 2 and even the exponential decay of the quantity $\sup_K \|\tau\| \circ \Phi_d$ are easy to establish in some cases, including the following ones.

Projective spaces. — When $X = \mathbf{C}P^n$, $\Phi_d : \mathbf{C}P^n \to \mathbf{C}P^{N_d-1}$ is equivariant with respect to the groups of real isometries $PO_{n+1}(\mathbf{R})$ and $PO_{N_d}(\mathbf{R})$. Since τ is invariant under these actions, $\tau \circ \Phi_d$ has to be a multiple of the section τ^d. Now $\|\tau\|_{\mathbf{R}P^n} \equiv 1$, so that $\tau \circ \Phi_d = \tau^d$ and $\sup_K \|\tau \circ \Phi_d\| = \sup_K \|\tau\|^d$. This was observed by Macdonald [15] in the case $X = \mathbf{C}^n$.

Ellipsoid quadrics. — Assume now that $X = \{[x_0 : \cdots : x_{n+1}] \in \mathbf{C}P^{n+1} \mid x_0^2 = x_1^2 + \cdots + x_{n+1}^2\}$ is the ellipsoid quadric and L the restriction of $\mathcal{O}_{\mathbf{C}P^{n+1}}(1)$ to X. Then, $\tau \circ \Phi_d$ is a multiple of the hyperplane section x_0^{2d}, since it is invariant under the group of isometries $O_{n+1}(\mathbf{R})$ acting on the coordinates $(x_1, \ldots, x_{n+1})$. At a real point $x = (x_0, \ldots, x_{n+1})$, $\|\tau\| \circ \Phi_d = 1$ and

$$\|x_0^2\| = \frac{|x_0^2|}{|x_0|^2 + \cdots + |x_{n+1}|^2}$$

$$= \frac{1}{2}\left(\frac{x_0^2}{x_0^2+\cdots+x_{n+1}^2} + \frac{x_1^2+\cdots+x_{n+1}^2}{x_0^2+\cdots+x_{n+1}^2}\right) = \frac{1}{2},$$

hence $\tau \circ \Phi_d = 2^d x_0^{2d}$ and $\sup_{\mathrm{K}} \|\sigma\| \circ \Phi_d = \sup_{\mathrm{K}}(2\|x_0^2\|)^d$.

The hyperboloid surface. — Finally, if $\mathrm{X} = (\mathbf{C}\mathrm{P}^1_1 \times \mathbf{C}\mathrm{P}^1_2; \mathrm{Conj} \times \mathrm{Conj})$ is the hyperboloid quadric surface and $\mathrm{L} = \mathcal{O}(a)_{\mathbf{C}\mathrm{P}^1_1} \otimes \mathcal{O}(b)_{\mathbf{C}\mathrm{P}^1_2}$ with $a, b > 0$, then $\tau \circ \Phi_d$ is $\mathrm{PO}_2(\mathbf{R}) \times \mathrm{PO}_2(\mathbf{R})$-invariant, hence a multiple of $(\tau_1^a \otimes \tau_2^b)^d$ where $\tau_i = \tau_{\mathbf{C}\mathrm{P}^1_i} \in \mathcal{O}(2)$ for $i = 1, 2$. Computed at a real point, this multiple is one, so that $\tau \circ \Phi_d = (\tau_1^a \otimes \tau_2^b)^d$ and $\sup_{\mathrm{K}} \|\sigma\| \circ \Phi_d = (\sup_{\mathrm{K}} \|\tau_1^a \otimes \tau_2^b\|)^d$.

2. Weakly laminar currents and large deviation estimates

This paragraph is devoted to two key ingredients of our proofs of Theorems 1 and 2. The first one is the large deviation estimates given by Proposition 3 which will produce the exponential decay. The second one, of independent interest, is the weak laminarity away from the real locus of limit currents of real maximal curves, see Theorem 4. To establish the latter, we first recollect some definitions and facts about weakly laminar currents, among which Theorem 3 of de Thélin which plays a crucial rôle.

2.1. *Weakly laminar currents*

Let (X, ω) be a smooth Kähler surface and $\mathcal{T}^{(1,1)}_{\mathrm{L}^2}$ be its space of closed positive currents of type $(1, 1)$ and mass $\mathrm{L}^2 = \int_{\mathrm{X}} \omega \wedge \omega$. Recall that by definition such a current is a continuous linear form on the space of smooth two-forms that vanishes on forms of type $(2, 0)$ and $(0, 2)$ as well as on the exact forms. Moreover, the mass $\langle \mathrm{T}, \omega \rangle$ equals L^2 and T is positive once evaluated on a positive $(1, 1)$-form. In particular, T is of measure type, that is continuous for the sup norm on two-forms. The space $\mathcal{T}^{(1,1)}_{\mathrm{L}^2}$, equipped with the weak* topology, is a compact and convex space. For every $d \in \mathbf{N}^*$ and every $\sigma \in \mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \{0\}$, denote by $\mathrm{Z}_\sigma \in \mathcal{T}^{(1,1)}_{\mathrm{L}^2}$ the current of integration

$$\mathrm{Z}_\sigma : \phi \in \Omega^2(\mathrm{X}) \mapsto \frac{1}{d}\int_{\mathrm{C}_\sigma} \phi \in \mathbf{R},$$

where $\mathrm{C}_\sigma = \sigma^{-1}(0)$. The following definition of weakly laminar currents was introduced in [1].

Definition **1.** — A positive current T of type $(1, 1)$ is called weakly laminar in the open set $\mathrm{U} \subset \mathrm{X}$ iff there exist a family of embedded discs $(\mathrm{D}_a)_{a \in \mathcal{A}}$ in U together with a measure da on $\mathcal{A}$, such that for every a and a' in $\mathcal{A}$, $D_a \cap D_{a'}$ is open in D_a and $D_{a'}$, and such that for every smooth two-form ϕ with support in U,

$$\langle T, \phi \rangle = \int_{a \in \mathcal{A}} \left(\int_{D_a} \phi \right) da.$$

Examples.

(1) The current of integration over a complex curve in a complex manifold is weakly laminar.
(2) Let $D = \{z \in \mathbf{C} \mid |z| < 1\}$ and $D_a = \{(z, f_a(z)) \in D^2\ ,\ z \in D\}$, $a \in \mathcal{A}$, be a measurable family of disjoint holomorphic graphs in the complex bidisk. Then, the integral over $\mathcal{A}$ of the current of integration over D_a is weakly laminar.
(3) Any current in a complex manifold locally modeled on the previous Example 2 is weakly laminar.
(4) The sum of the currents of integration over the three one-parameter families of complex disks with boundary on the Clifford torus $\partial D \times \partial D \subset \mathbf{C}P^2$ is a closed weakly laminar current, see [6] or [1]. Note that likewise, the preimage under the moment map of any tropical curve lying in the moment polytope of some toric surface, is the support of some closed weakly laminar current of this surface.
(5) The integral over $\rho \in \mathbf{R}_+^*$ of the current of integration over the disk of radius ρ is weakly laminar.

Some of these examples actually carry stronger laminarity properties than the one of Definition 1. Currents given by Example 3 are called uniformly laminar in [1] while the current in Example 4 is called laminar and is not uniformly laminar. For a current to be weakly laminar, the disks D_a need not be disjoint whereas for it to be laminar, they have to. Example 5 provides a weakly laminar current which is not laminar. A closed current of finite mass having this property, modelled on Demailly's Example 4, can be found in [9]. On the other hand, Kähler forms provide closed positive currents which are far from weakly laminar ones, as Lemma 1 shows.

Lemma **1.** — *Let ω be a Kähler form on a complex surface* X. *Then ω is nowhere weakly laminar.*

Proof. — Assume that there exists an open subset U of X and a measured family $(D_a)_{a \in \mathcal{A}}$ of embedded discs in U given by Definition 1 such that for every two-form ϕ with compact support in U,

$$\int_U \omega \wedge \phi = \int_{\mathcal{A}} \left(\int_{D_a} \phi \right) da.$$

For every two-form ψ defined and continuous on $\bigcup_{a\in\mathcal{A}} \mathrm{D}_a$, we denote by T_ψ the current

$$\phi \in \Omega^2_c(\mathrm{U}) \mapsto \mathrm{T}_\psi(\phi) = \int_{a\in\mathcal{A}} \left(\int_{\mathrm{D}_a} \left(\frac{\psi \wedge \phi}{\omega^2} \right) \omega \right) da.$$

Then $\mathrm{T}_\omega = \omega$, since for every $\phi \in \Omega^2_c(\mathrm{U})$,

$$\mathrm{T}_\omega(\phi) = \int_{\mathcal{A}} \left(\int_{\mathrm{D}_a} \left(\frac{\omega \wedge \phi}{\omega^2} \right) \omega \right) da = \int_{\mathrm{U}} \left(\frac{\omega \wedge \phi}{\omega^2} \right) \omega^2 = \int_{\mathrm{U}} \omega \wedge \phi.$$

Let ψ be the $(1, 1)$-form defined along $\bigcup_{a\in\mathcal{A}} \mathrm{D}_a$ in such a way that for every $a \in \mathcal{A}$ and $x \in \mathrm{D}_a$, $\mathrm{T}_x\mathrm{D}_a$ lies in the kernel of ψ_x and $\psi_x \wedge \omega = \omega^2$. Then $\mathrm{T}_\psi = \omega$, since

$$\forall \phi \in \Omega^2_c(\mathrm{U}), \quad \mathrm{T}_\psi(\phi) = \int_{\mathcal{A}} \left(\int_{\mathrm{D}_a} \left(\frac{\psi \wedge \phi}{\omega^2} \right) \omega \right) da = \int_{\mathcal{A}} \left(\int_{\mathrm{D}_a} \phi \right) da.$$

But ψ is integrable for both currents T_ψ and T_ω and $\mathrm{T}_\psi(\psi) = 0$ while $\mathrm{T}_\omega(\psi) = \omega^2$. This contradicts the equality $\mathrm{T}_\psi = \mathrm{T}_\omega$. □

The advantage for us to consider weakly laminar currents comes from the following Theorem 3 of H. de Thélin, see [5].

Theorem **3** *(de Thélin). — Let* $(\mathrm{C}_n)_{n\in\mathbf{N}}$ *be a sequence of smooth holomorphic curves in the complex unit ball of dimension two, such that* $g(\mathrm{C}_n) = \mathrm{O}(\mathrm{A}(\mathrm{C}_n))$, *where* $g(\mathrm{C}_n)$ *stands for the genus of* C_n *and* $\mathrm{A}(\mathrm{C}_n)$ *for its area. If* $\frac{1}{\mathrm{A}(\mathrm{C}_n)}[\mathrm{C}_n]$ *converges to a current* T *as* n *grows to infinity, where* $[\mathrm{C}_n]$ *denotes the current of integration over* C_n, *then* T *is weakly laminar.*

Recall that the genus of C_n in Theorem 3 is by definition the genus of the closed surface obtained by capping a disk at every boundary component of C_n. The rough observation behind Theorem 3 is that a curve C_n of small genus in a conveniently chosen complex bidisk $\mathrm{D}_1 \times \mathrm{D}_2$ defines a branched cover of D_1 with few ramification points. One can then subdivide D_1 and remove from C_n pieces which are not graphs over elements of the subdivision. Taking the limit as n grows to infinity, ones gets from Montel's theorem a laminar current. Then, refining the subdivision so that the diameter of its elements converges to zero, one gets the result. Indeed, the mass of what has been removed converges to zero in the limiting process, see [5].

2.2. *Real maximal curves and laminarity*

For every open subset U of X, denote by $Lam(\mathrm{U}) \subset \mathcal{T}^{(1,1)}_{\mathrm{L}^2}$ the subset of closed positive currents of mass L^2 which are weakly laminar on U. For all $a \in \mathbf{Q}^*_+$, denote by $\mathcal{Z}^a$ the closure of the union $\bigcup_{d\in\mathbf{N}^*} \mathcal{Z}^a_d$ in $\mathcal{T}^{(1,1)}_{\mathrm{L}^2}$, where $\mathcal{Z}^a_d$ denotes the image of the set

$$\mathcal{M}^a_d = \{\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathbf{R}\Delta_d \mid b_0(\mathbf{R}\mathrm{C}_\sigma) \geq g(\mathrm{C}_\sigma) + 1 - ad\}$$

under the map $\sigma \mapsto Z_\sigma \in \mathcal{T}^{(1,1)}_{L^2}$.

Theorem **4.** — *Let* X *be a closed real projective surface and* L *be a positive real Hermitian line bundle on* X*. Then, for every* $a \in \mathbf{Q}^*_+$*, the inclusion* $\mathcal{Z}^a \subset Lam(X \setminus \mathbf{R}X)$ *holds.*

In particular, every limit of a sequence of real maximal curves is weakly laminar outside of the real locus of the manifold. The point in Theorem 4 is that curves in $\mathcal{M}^a_d$ have small genus outside of the real locus $\mathbf{R}X$, so that Theorem 4 quickly reduces to de Thélin's Theorem 3. For instance, smooth real maximal curves are just unions of two genus zero curves with boundary on $\mathbf{R}X$, exchanged by c_X (see for example Proposition 6.3 of [22]).

Proof. — Every current of integration is weakly laminar so that we just have to prove the result for $T \in \mathcal{Z}^a \setminus \bigcup_{d \in \mathbf{N}^*} \mathcal{Z}^a_d$. Let T be such a current and $(Z_{\sigma_d})_{d \in \mathbf{N}^*}$ be a sequence of currents of integration which converges to T as d grows to infinity. For every $d \in \mathbf{N}^*$, the genus of the complement $C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}$ satisfies $g(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) \leq ad$. Indeed, its Euler characteristic satisfies $\chi(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) = \chi(C_{\sigma_d}) - \chi(\mathbf{R}C_{\sigma_d}) = 2 - 2g(C_{\sigma_d})$. Every connected component of $\mathbf{R}C_{\sigma_d}$ creates two ends of $C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}$, so that the number $r(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d})$ of ends of the Riemann surface $C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}$ equals $2b_0(\mathbf{R}C_{\sigma_d})$. By definition, the genus $g(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d})$ of this Riemann surface is given by the formula $\chi(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) = 2b_0(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) - 2g(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) - r(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d})$, where $b_0(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d})$ denotes the number of connected components of $C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}$. It is well known that the latter is at most two, see for example Section 6 of [22], for such a component together with its image under the involution c_X compactifies as a closed irreducible component of C_{σ_d}. Moreover, by assumption, $r(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) = 2b_0(\mathbf{R}C_{\sigma_d}) \geq 2g(C_{\sigma_d}) + 2 - 2ad$. We deduce that $g(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) \leq 2 + (g(C_{\sigma_d}) - 1) + (ad - 1 - g(C_{\sigma_d})) \leq ad$. Let B be a closed complex ball in the complement $X \setminus \mathbf{R}X$, transversal to all curves C_{σ_d}. Once more, the genus of $C_{\sigma_d} \cap B$ is such that the formula $\chi(C_{\sigma_d} \cap B) = 2b_0(C_{\sigma_d} \cap B) - 2g(C_{\sigma_d} \cap B) - r(C_{\sigma_d} \cap B)$ holds, that is, it is the genus of the closed surface obtained by capping a disk at each boundary component of $C_{\sigma_d} \cap B$. Since the genus of an orientable surface can only increase under connected sums, we deduce that $g(C_{\sigma_d} \cap B) \leq g(C_{\sigma_d} \setminus \mathbf{R}C_{\sigma_d}) \leq ad$. Let $A(C_{\sigma_d} \cap B)$ be the area of the restriction of C_{σ_d} to B, so that $A(C_{\sigma_d} \cap B) \leq A(C_{\sigma_d}) = d \int_X \omega^2$. Without loss of generality, we can assume that $\frac{1}{d} A(C_{\sigma_d} \cap B)$ converges to $m_B \in [0, \int_X \omega^2]$. If $m_B = 0$, the restriction of T to B vanishes. Otherwise, $g(C_{\sigma_d} \cap B) = O(A(C_{\sigma_d} \cap B))$, where the area $A(C_{\sigma_d} \cap B)$ can be computed for the flat metric on the ball. Theorem 3 then implies that $1/m_B T_{|B}$ is weakly laminar. Hence the result. ☐

2.3. *Large deviation estimates*

The aim of this paragraph is to establish Proposition 3 which provides an estimate from above of the volume occupied by the sections $\sigma \in \mathbf{R}H^0(X, L^d)$ whose corresponding current of integration Z_σ is far from the Kähler form $\frac{1}{d}\Phi_d^* \omega_{FS}$. Recall indeed that from Poincaré-Lelong's formula, the current $1/(2i\pi d)\partial\bar{\partial}\log\|\sigma(x)\|^2_{\Phi_d}$ coincides with $\frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}} - \mathrm{Z}_\sigma$. Note also that this Kähler form $\frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}}$ turns out to coincide asymptotically with the average current of integration, see Proposition 5.

Proposition **3.** — *Let* X *be a real projective manifold of dimension* n *and* L *an ample real Hermitian line bundle over* X. *For every smooth* $(2n-2)$*-form* ϕ *with compact support in* $\mathrm{X}\setminus\mathbf{R}\mathrm{X}$, *every* $d \geq d_{\mathrm{L}}$ *and every* $\epsilon > 0$, *the measure of the set*

$$\left\{\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)\setminus \mathbf{R}\Delta_d \mid \frac{1}{d}\left|\int_{\mathrm{X}} \log\|\sigma(x)\|^2_{\Phi_d}\partial\bar{\partial}\phi\right| \geq \epsilon\right\}$$

is bounded from above by the quantity

$$\frac{2c_{\mathrm{K}_\phi}}{\mathrm{Vol}(\mathrm{K}_\phi)}\exp\left(\frac{-\epsilon d}{2\|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty}\mathrm{Vol}(\mathrm{K}_\phi)}\right),$$

where K_ϕ *can denote both the support of* ϕ *or the support of* $\partial\bar{\partial}\phi$, *while* c_{K_ϕ} *and* d_{L} *are defined in Section 1.2.*

Recall that X is equipped with the volume form $dx = \omega^n/\int_{\mathrm{X}}\omega^n$. The norm $\|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty}$ and the volume $\mathrm{Vol}(\mathrm{K}_\phi)$ are computed with respect to this volume form. Recall also that N_d denotes the dimension of $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$. Note that a similar result has been obtained by T.-C. Dinh and N. Sibony, see Corollary 7.4 of [7].

Proof. — We use Markov's trick. For every $\lambda > 0$,

$$\left|\int_{\mathrm{X}} \log\|\sigma\|^2_{\Phi_d}\partial\bar{\partial}\phi\right| \geq \epsilon d \Longleftrightarrow \exp\left(\lambda\left|\int_{\mathrm{X}} \log\|\sigma\|^2_{\Phi_d}\partial\bar{\partial}\phi\right|\right) \geq e^{\lambda\epsilon d},$$

where

$$\exp\left(\lambda\left|\int_{\mathrm{X}} \log\|\sigma\|^2_{\Phi_d}\partial\bar{\partial}\phi\right|\right) = \sum_{n=0}^{\infty}\frac{\lambda^m}{m!}\left|\int_{\mathrm{X}} \log\|\sigma\|^2_{\Phi_d}\partial\bar{\partial}\phi\right|^m.$$

From Hölder's inequality we get

$$\begin{aligned}
&\left|\int_{\mathrm{X}} \log\left(\|\sigma\|^2_{\Phi_d}\right)\partial\bar{\partial}\phi\right|^m \\
&\quad\leq \int_{\mathrm{X}}\left|\log\left(\|\sigma\|^2_{\Phi_d}\right)\right|^m dx\left(\int_{\mathrm{X}}\left|\partial\bar{\partial}\phi\right|^{\frac{m}{m-1}} dx\right)^{m-1} \\
&\quad\leq \frac{1}{\mathrm{Vol}(\mathrm{K}_\phi)}\left(\|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty}\mathrm{Vol}(\mathrm{K}_\phi)\right)^m\int_{\mathrm{X}}\left|\log\left(\|\sigma\|^2_{\Phi_d}\right)\right|^m dx.
\end{aligned}$$

As a consequence, for every $d \geq d_{\mathrm{L}}$, the measure $\mu_\epsilon^d(\phi)$ of our set satisfies

$$e^{\lambda\epsilon d}\mu_\epsilon^d(\phi) \leq \int_{\mathbf{R}\mathrm{H}^0(\mathrm{X},\mathrm{L}^d)} \exp\left(\lambda \left| \int_{\mathrm{X}} \log \|\sigma\|^2_{\Phi_d} \partial\bar{\partial}\phi \right| d\mu(\sigma)\right)$$

$$\leq \frac{1}{\mathrm{Vol}(\mathrm{K}_\phi)} \sum_{n=0}^{\infty} \frac{\lambda^m}{m!} \|\partial\bar{\partial}\phi\|^m_{\mathrm{L}^\infty} \mathrm{Vol}(\mathrm{K}_\phi)^m \int_{\mathrm{X}} \mathrm{M}^m_{(\mathrm{X},\mathrm{L}^d)} dx,$$

where $\mathrm{M}^m_{(\mathrm{X},\mathrm{L}^d)}$ is defined in Section 1.2. Thanks to Corollary 2, the latter right hand side is bounded from above by $\frac{c_{\mathrm{K}_\phi}}{\mathrm{Vol}(\mathrm{K}_\phi)} \sum_{n=0}^{\infty} (\lambda \|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty} \mathrm{Vol}(\mathrm{K}_\phi))^m$, that is

$$\frac{c_{\mathrm{K}_\phi}}{\mathrm{Vol}(\mathrm{K}_\phi)(1 - \lambda \|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty} \mathrm{Vol}(\mathrm{K}_\phi))}.$$

The result follows by choosing $\lambda = (2\|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty}\mathrm{Vol}(\mathrm{K}_\phi))^{-1}$. □

3. Proof of Theorems 1 and 2

3.1. *Proof of Theorem 2*

Lemma **2.** — *Let* X *be a closed real one-dimensional Kähler manifold. There exist nonnegative constants* $\eta_0, \mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4$ *and a family of smooth cutoff functions* $\chi_\eta : \mathrm{X} \to [0,1]$ *with support in* $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$, $0 < \eta \leq \eta_0$, *such that for every* $0 < \eta \leq \eta_0$,

1. $\mathrm{E}_1\eta \leq \mathrm{Vol}(\mathrm{supp}(\partial\bar{\partial}\chi_\eta))$
2. $\mathrm{Vol}(\mathrm{X} \setminus \chi_\eta^{-1}(1)) < \mathrm{E}_2\eta$
3. $\|\partial\bar{\partial}\chi_\eta\|_{\mathrm{L}^\infty} \leq \mathrm{E}_3/\eta^2$
4. $\mathrm{dist}(\mathrm{supp}(\chi_\eta), \mathbf{R}\mathrm{X}) \geq \mathrm{E}_4\eta$

Proof. — A neighborhood V of the real locus $\mathbf{R}\mathrm{X}$ is the union of a finite number of annuli isomorphic to $\mathrm{A} = \{z \in \mathbf{C} \mid 1 - \epsilon < |z| < 1 + \epsilon\}$. For every $\eta > 0$, choose χ_η such that $\chi_\eta(\mathrm{X} \setminus \mathrm{V}) = 1$ and the restriction of χ_η to A only depends on the modulus of $z \in \mathrm{A}$. That is, for every $z \in \mathrm{A}$, $\chi_\eta(z) = \rho_\eta(|z| - 1)$, where ρ_η is a function $]-\epsilon, \epsilon[\to [0,1]$. Let $\rho : \mathbf{R} \to [0,1]$ be an even function such that $\rho(x) = 1$ if $|x| \geq 1$ and $\rho(x) = 0$ if $|x| \leq 1/2$. For every $\eta > 0$, we set $\rho_\eta(x) = \rho(x/\eta)$. The family χ_η, $0 < \eta \leq \epsilon = \eta_0$ satisfies the required conditions. □

Proof of Theorem 2. — For every $d \in \mathbf{N}^*$, denote by

$$\mathcal{M}_d^{\epsilon(d)} = \{\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X},\mathrm{L}^d) \setminus \mathbf{R}\Delta_d \mid \#(\sigma^{-1}(0) \cap \mathbf{R}\mathrm{X}) \geq \sqrt{d}\epsilon(d)\}.$$

For every $\sigma \in \mathcal{M}_d^{\epsilon(d)}$, denote by $\mathrm{Z}_\sigma : \phi \in \mathrm{C}^0(\mathrm{X}) \mapsto \frac{1}{d}\int_{\mathrm{C}_\sigma} \phi \in \mathbf{R}$ the associated discrete measure, where $\mathrm{C}_\sigma = \sigma^{-1}(0)$. Let $(\chi_\eta)_{0<\eta\leq\eta_0}$ be a family of real cutoff functions given by

Lemma 2. By definition, for every $0 < \eta \leq \eta_0$ and every $\sigma \in \mathcal{M}_d^{\epsilon(d)}$,

$$\langle Z_\sigma, \chi_\eta \rangle = \frac{1}{d} \int_{C_\sigma} \chi_\eta \leq \int_X \omega - \frac{\epsilon(d)}{\sqrt{d}},$$

where ω denotes the curvature of L. Without loss of generality, we can assume that when d is large enough, $\epsilon(d)/\sqrt{d} \leq \eta_0$. We then set $\eta_d = \epsilon(d)/(2E_2\sqrt{d}\int_X \omega)$, where E_2 is given by Lemma 2. From Lemma 2, we deduce that for every $\sigma \in \mathcal{M}_d^{\epsilon(d)}$, $\langle \omega - Z_\sigma, \chi_{\eta_d} \rangle > \frac{\epsilon(d)}{2\sqrt{d}}$ and then from Poincaré-Lelong formula that

$$\frac{1}{d}\left| \int_X \log \|\sigma(x)\|^2_{\Phi_d} \partial\bar\partial \chi_{\eta_d} \right| \geq \frac{\pi\epsilon(d)}{\sqrt{d}} - 2\pi \left\| \frac{1}{d}\Phi^*\omega_{FS} - \omega \right\|_{L^\infty}.$$

We know from Tian's asymptotic isometry theorem [21] that $d\|\frac{1}{d}\Phi^*\omega_{FS} - \omega\|_{L^\infty}$ is bounded, so that for d large enough, the right hand side is greater than $\epsilon(d)/\sqrt{d}$. For every d large enough, denote by K_d the support of $\partial\bar\partial\chi_{\eta_d}$. Without loss of generality, we can assume that $\epsilon(d)$ grows to infinity as d grows to infinity. By Lemma 2, so does $d\,\mathrm{dist}(K_d, \mathbf{R}X)^2$. Proposition 3 and Lemma 2 then provide the result. □

3.2. *Proof of Theorem 1 when a is a bounded function*

Let $a \in \mathbf{Q}_+^*$. We have to prove the existence of two positive constants C and D such that $\mu(\mathcal{M}_d^a) \leq Ce^{-Dd}$. From Theorem 4 we know that the compact $\mathcal{Z}^a = \overline{\bigcup_{d\in\mathbf{N}^*} \mathcal{Z}_d^a}$ introduced in Section 2.1 is included in $Lam(X \setminus \mathbf{R}X)$, whereas from Lemma 1, $\omega \notin Lam(X \setminus \mathbf{R}X)$. As a consequence, there exists a finite set $(\phi_j)_{j\in\mathcal{J}}$ of two-forms with compact support in $X \setminus \mathbf{R}X$ such that $\forall T \in \mathcal{Z}^a, \exists j \in \mathcal{J},\ |\langle \omega - T, \phi_j \rangle| > 1$. Moreover, Poincaré-Lelong formula writes

$$\forall d \geq d_L,\ \forall \sigma \in \mathbf{R}H^0(X, L^d) \setminus \{0\}, \quad \frac{1}{2i\pi d}\partial\bar\partial \log \|\sigma\|^2_{\Phi_d} = \frac{1}{d}\phi_d^*\omega_{FS} - Z_\sigma,$$

where ω_{FS} denotes the Fubini-Study form of $P(H^0(X, L^d))^*$ defined in Section 1.1, and Z_σ the current of integration defined in Section 2.1. From Tian's asymptotic isometry theorem [21] (see also [3], [4] and [23]), $\frac{1}{d}\Phi_d^*\omega_{FS}$ converges to ω as d grows to ∞. Thus, there exists $d_1 \geq d_L$ such that

$$\forall d \geq d_1, \forall \sigma \in \mathcal{M}_d^a, \exists j \in \mathcal{J}, \quad \left| \left\langle \frac{1}{d}\partial\bar\partial \log \|\sigma\|^2_{\Phi_d}, \phi_j \right\rangle \right| > 2\pi.$$

From this relation and Proposition 3 we deduce

$$\mu(\mathcal{M}_d^a) \leq \sum_{j\in\mathcal{J}} \mu\left\{ \sigma \in \mathbf{R}H^0(X, L^d) \setminus \{0\} \mid \frac{1}{d}\left| \int_X \log \|\sigma\|_{\Phi_d} \partial\bar\partial\phi_j \right| > 2\pi \right\}$$

$$\leq \frac{2c_{\mathrm{K}}\#\mathcal{J}}{\inf \mathrm{Vol}(\mathrm{supp}(\phi_j))} \exp\left(-\frac{\pi d}{\max_{j\in\mathcal{J}} \|\partial\bar\partial\phi_j\|_{\mathrm{L}^\infty}\mathrm{Vol}(\mathrm{K})}\right),$$

where $\mathrm{K} \subset \mathrm{X} \setminus \mathbf{R}\mathrm{X}$ is a compact containing all supports of the ϕ_j's, $j \in \mathcal{J}$, and c_{K} is given by Corollary 2. Hence the result.

3.3. *Proof of Theorem 1, general case*

This paragraph is devoted to the proof of Theorem 1 in the general case. For this purpose, we first establish some preliminary consequences of the estimates given by Proposition 3.

3.3.1. *Preliminaries*

We first prove in Corollaries 3 and 4 that for generic sections $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$, the curve C_σ is uniformly distributed in $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$. More precisely, given a relatively compact open subset $\mathring{\mathrm{K}}$ of $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$, there exists a subset of $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ having exponentially decreasing size, such that for σ away from this subset, the portion of C_σ in K corresponds to the portion of K in X. We then prove in Lemma 4 a similar result for the genus of C_σ.

Corollary **3.** — *Under the hypotheses of Proposition 3, let* $\mathring{\mathrm{K}}$ *be a relatively compact open subset of* $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$. *Then, there exist constants* $\mathrm{C}_{\mathrm{K}}, \mathrm{D}_{\mathrm{K}}, \lambda_{\mathrm{K}} > 0$ *and, for every* $d \geq d_{\mathrm{L}}$, *a subset* $\mathcal{A}^d_{\mathrm{K}} \subset \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ *of measure bounded from above by* $\mathrm{C}_{\mathrm{K}} e^{-\mathrm{D}_{\mathrm{K}} d}$ *such that for every* $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathcal{A}^d_{\mathrm{K}}$, *the volume* $\mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{K})$ *of* $\mathrm{C}_\sigma \cap \mathrm{K}$ *satisfies* $\mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{K}) \geq \lambda_{\mathrm{K}} d$.

Proof. — Let $\mathring{\mathrm{K}}_1$ be a relatively compact open subset of $\mathring{\mathrm{K}}$ and $\chi : \mathrm{X} \to [0, 1]$ be a smooth cutoff function with support in K such that $\chi_{|\mathrm{K}_1} \equiv 1$. Applied to $\phi = \chi\omega^{n-1}$ and $\epsilon = \pi \int_{\mathrm{K}_1} \omega^n$, Proposition 3 provides us with constants C_{K} and D_{K} such that the set

$$\mathcal{A}^d_{\mathrm{K}} = \{0\} \cup \left\{\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \{0\} \mid \frac{1}{d}\left|\int_{\mathrm{X}} \log \|\sigma\|^2_{\Phi_d} \partial\bar\partial(\chi\omega^{n-1})\right| \right.$$
$$\left. \geq \pi \int_{\mathrm{K}_1} \omega^n \right\}$$

is of measure bounded from above by $\mathrm{C}_{\mathrm{K}} e^{-\mathrm{D}_{\mathrm{K}} d}$. From Poincaré-Lelong formula follows that for every $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathcal{A}^d_{\mathrm{K}}$,

$$\left|\frac{1}{d}\int_{\mathrm{C}_\sigma} \chi\omega^{n-1} - \int_{\mathrm{X}} \frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}} \wedge \chi\omega^{n-1}\right| \leq \frac{1}{2}\int_{\mathrm{K}_1} \omega^n.$$

Now, from Tian's asymptotic isometry theorem [21] (see also [3], [4] and [23]), $\frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}}$ converges to ω as d grows to ∞, so that for d large enough, $\int_{\mathrm{X}} \frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}} \wedge \chi\omega^{n-1} \geq \int_{\mathrm{K}_1} \omega^n$.

As a consequence,

$$\frac{(n-1)!}{d}\mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{K}) \geq \frac{1}{d}\int_{\mathrm{C}_\sigma} \chi\omega^{n-1} \geq \int_{\mathrm{X}} \frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}} \wedge \chi\omega^{n-1} - \frac{1}{2}\int_{\mathrm{K}_1} \omega^n.$$

The left hand side being bounded from below by a constant $(n-1)!\lambda_{\mathrm{K}}$, the result follows. $\square$

For every $n \in \mathbf{N}^*$ and $\rho > 0$, denote by $\mathrm{B}^{2n}(\rho) \subset \mathbf{C}^n$ the closed ball of radius ρ and volume $\pi^n\rho^{2n}/n!$. The standard Kähler form of $\mathbf{C}^n$ is denoted by ω_0.

Definition **2.** — Let (X, ω) be a Kähler manifold of dimension n. By abuse, we define a *ball of radius* ρ of X to be the image of a holomorphic embedding $\psi_\rho : \mathrm{B}^{2n}(\rho) \to \mathrm{X}$ whose differential at the origin is isometric and which everywhere satisfies the inequalities $1/2\omega_0 \leq \psi_\rho^*\omega \leq 2\omega_0$.

Corollary **4.** — *Under the hypotheses of Proposition 3, let* $\mathring{\mathrm{K}}$ *be a relatively compact open subset of* $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$. *Then, there exist constants* $\mathrm{D}_{\mathrm{K}}, \lambda_{\mathrm{K}}^1, \lambda_{\mathrm{K}}^2 > 0$ *such that for every ball* B *of radius* $\rho > 0$ *included in* K *and every* $d \geq d_{\mathrm{L}}$, *there exists a set* $\mathcal{A}_{\mathrm{B}}^d \subset \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ *of measure*

$$\mu(\mathcal{A}_{\mathrm{B}}^d) \leq \frac{2c_{\mathrm{K}}\int_{\mathrm{X}}\omega^n}{\rho^{2n}} e^{-\mathrm{D}_{\mathrm{K}} d\rho^2}$$

such that for every $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathcal{A}_{\mathrm{B}}^d$, *the volume of* $\mathrm{C}_\sigma \cap \mathrm{B}$ *satisfies* $\lambda_{\mathrm{K}}^1 d\rho^{2n} \leq \mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{B}) \leq \lambda_{\mathrm{K}}^2 d\rho^{2n}$.

Recall that the constant c_{K} is given by Corollary 2, while d_{L} is defined in Sect. 1.2.

Proof. — Let $\chi : \mathbf{C}^n \to [0, 1]$ be a smooth cutoff function with support in the unit ball and such that $\chi^{-1}(1)$ contains the ball of radius $\sqrt{2/\pi}$. For any $\rho > 0$, let $\chi_\rho : x \in \mathbf{C}^n \mapsto \chi(x/\rho)$ be the associated cutoff function with support in the ball of radius ρ. Let $\psi : \mathrm{B} \to \mathrm{B}^{2n}(\rho) \subset \mathbf{C}^n$ be a biholomorphism given by Definition 2, and $\mathrm{B}_1 = (\chi_\rho \circ \psi)^{-1}(1)$. Let $\phi = (\chi_\rho \circ \psi)\omega^{n-1}$, $\mathrm{K}_\phi = \mathrm{supp}(\phi)$, and for every $d \geq d_{\mathrm{L}}$,

$$\mathcal{A}_{\mathrm{B}}^d = \{0\} \cup \left\{\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \{0\} \mid \frac{1}{d}\left|\int_{\mathrm{X}} \log \|\sigma\|_{\Phi_d}^2 \partial\bar{\partial}\phi\right| \geq \pi\int_{\mathrm{B}_1}\omega^n\right\}.$$

From Proposition 3 follows that

$$\mu(\mathcal{A}_{\mathrm{B}}^d) \leq \frac{2c_{\mathrm{K}_\phi}}{\mathrm{Vol}(\mathrm{K}_\phi)} \exp\left(\frac{-\pi d\int_{\mathrm{B}_1}\omega^n}{2\|\partial\bar{\partial}\phi\|_{\mathrm{L}^\infty}\mathrm{Vol}(\mathrm{K}_\phi)}\right),$$

where

$$\Big(\int_X \omega^n\Big)\mathrm{Vol}(K_\phi) = \int_{K_\phi} \omega^n \geq \frac{1}{2^n}\int_{\chi_\rho^{-1}(1)} \omega_0^n \geq \rho^{2n}.$$

The metric on B is bounded from above and below by the flat metric, see Definition 2. The quotient $\int_{B_1}\omega^n/\mathrm{Vol}(K_\phi)$ is thus bounded from below by a positive constant, since $\int_{\chi_\rho^{-1}(1)}\omega_0^n/\int_{\mathrm{supp}(\chi_\rho)}\omega_0^n$ does not depend on ρ. Likewise, $\|\partial\bar\partial\phi\|_{L^\infty}$ is bounded from above by a multiple of $\sup_{B^{2n}(\rho)}|\partial\bar\partial\chi_\rho\wedge\omega_0^{n-1}/\omega_0^n|$. The latter being of the order of $1/\rho^2$, we deduce the existence of a positive constant D_K such that

$$\mu(\mathcal{A}_B^d) \leq 2\Big(\int_X \omega^n\Big)\frac{c_K}{\rho^{2n}}\exp(-D_K\rho^2 d).$$

But for every $\sigma \in \mathbf{R}H^0(X, L^d)\setminus\mathcal{A}_B^d$, we have

$$\left|\frac{1}{d}\int_{C_\sigma}(\chi_\rho\circ\psi)\omega^{n-1} - \int_X \frac{1}{d}\Phi_d^*\omega_{FS}\wedge(\chi_\rho\circ\psi)\omega^{n-1}\right| \leq \frac{1}{2}\int_{B_1}\omega^n.$$

The term $\frac{1}{d}\int_X\Phi_d^*\omega_{FS}\wedge(\chi_\rho\circ\psi)\omega^{n-1}$ is greater than

$$\int_{B_1}\omega^n + \int_{B\setminus B_1}(\chi_\rho\circ\psi)\omega^{n-1} - \left\|\Big(\frac{1}{d}\Phi_d^*\omega_{FS} - \omega\Big)\wedge\omega^{n-1}\right\|_{L^\infty}\mathrm{Vol}(B).$$

From Tian's asymptotic isometry theorem [21], $\frac{1}{d}\Phi_d^*\omega_{FS}$ converges to ω as d grows to infinity. Together with Definition 2, this implies that for d large enough,

$$\frac{1}{d}\int_X \Phi_d^*\omega_{FS}\wedge(\chi_\rho\circ\psi)\omega^{n-1} \geq \int_{B_1}\omega^n$$

and

$$\frac{(n-1)!}{d\rho^{2n}}A(C_\sigma\cap B) \geq \frac{1}{d\rho^{2n}}\int_{C_\sigma}(\chi_\rho\circ\psi)\omega^{n-1} \geq \frac{1}{2\rho^{2n}}\int_{B_1}\omega^n.$$

The right hand side being bounded from below by a positive constant, we deduce the lower bound for $A(C_\sigma\cap B)$. Likewise, we deduce that $(n-1)!/(d\rho^{2n})A(C_\sigma\cap B) \leq 3/(2\rho^{2n})\int_B\omega^n$. The right hand side being bounded from above by a positive constant, we deduce the upper bound for $A(C_\sigma\cap B)$ replacing B_1 by B in the proof. □

Lemma **3.** — *For every compact subset* K *of a n-dimensional Kähler manifold, there exist constants* r_K *and* $n_K > 0$ *such that for every* $\rho > 0$ *small enough,* K *can be covered by* r_K/ρ^{2n} *balls of radius* ρ*, in such a way that every point of* K *belongs to at most* n_K *balls.*

Proof. — The lattice $\mathbf{Z}^{2n}$ acts by translations on $\mathbf{C}^n$. The orbit of the ball $B^{2n}(\sqrt{n})$ under this action covers $\mathbf{C}^n$ in such a way that every point belongs to a finite number of balls. The images of this covering under homothetic transformations provides for every $\rho > 0$ a covering of $\mathbf{C}^n$ by balls of radius ρ such that every point belongs to a number of balls bounded independently of ρ. Let (X, ω) be a Kähler manifold. For every point $x \in K$, choose a holomorphic embedding $\phi_x : B' \to X$, where B' is a ball in $\mathbf{C}^n$ independent of x, $\phi_x(0) = x$ and ϕ_x is everywhere contracting. Let $B \subset B'$ be the ball of half radius. We extract a finite subcovering $\phi_1(B), \dots, \phi_k(B)$ from the covering $(\phi_x(B))_{x\in K}$ of K. For every $j \in \{1, \dots, k\}$ and every $p \in B$, there exists an affine expanding map $D^j_p : \mathbf{C}^n \to \mathbf{C}^n$ that fixes p and such that $\phi_j \circ D^j_p$ is an isometry at p. Then, there exists $\rho_0 > 0$ such that for every $0 < \rho \le \rho_0$ and $1 \le j \le k$, the restriction to B of the covering of $\mathbf{C}^n$ by balls of radius ρ satisfies the following: for every ball $B_p(\rho)$ of this covering centered at $p \in B$, we have $D^j_p(B_p(\rho)) \subset B'$, and $\phi_j \circ D^j_p(B_p(\rho))$ is a ball of radius ρ of (X, ω). Since D^j_p is expanding, $D^j_p(B_p(\rho))$ contains $B_p(\rho)$, so that the union of these balls of radius ρ covers K. Moreover, the norm of D^j_p is uniformly bounded on B. Thus, there exists a constant $h > 1$ such that $D^j_p(B_p(\rho)) \subset B_p(h\rho)$ for every p and j. From this and the construction of our coverings of $\mathbf{C}^n$, we deduce the existence of a constant $n_K > 0$ independent of p such that for every point $x \in K$ and every covering of K by balls of radius ρ obtained in this way, x belongs to at most n_K balls of the covering. Finally, the existence of r_K follows from the construction of the covering of $\mathbf{C}^n$ we used. □

Lemma **4.** — *Under the hypotheses of Theorem 1, let* $\mathring{K}$ *be a relatively compact open subset of* $X \setminus \mathbf{R}X$ *equipped with a covering by balls given by Lemma 3. Let* n_K *be given by Lemma 3 and* $\lambda_K > 0$, $\mathcal{A}^d_K \subset \mathbf{R}H^0(X, L^d)$ *be given by Corollary 3. Then for every* $d \ge d_L$ *and* $\sigma \in \mathcal{M}^{a(d)}_d \setminus \mathcal{A}^d_K$, *there is a ball* B *of the covering such that the genus* $g(C_\sigma \cap B)$ *of* $C_\sigma \cap B$ *satisfies* $g(C_\sigma \cap B) \le \frac{n_K}{\lambda_K} a(d) A(C_\sigma \cap B)$, *where* $A(C_\sigma \cap B)$ *denotes the area of* $C_\sigma \cap B$.

Recall that the integer d_L was defined in Section 1.2.

Proof. — By definition, the genus $g(C_\sigma \cap B)$ is such that the Euler characteristic $\chi(C_\sigma \cap B)$ of this curve be given by the formula

$$\chi(C_\sigma \cap B) = 2b_0(C_\sigma \cap B) - 2g(C_\sigma \cap B) - r(C_\sigma \cap B)$$

where $b_0(C_\sigma \cap B)$ (resp. $r(C_\sigma \cap B)$) denotes the number of the connected components of $C_\sigma \cap B$ (resp. of $\partial(C_\sigma \cap B)$). In particular, for every $\sigma \in \mathcal{M}^{a(d)}_d$, $g(C_\sigma \cap B) \le g(C_\sigma \setminus \mathbf{R}C_\sigma) \le a(d)d$, as established in the proof of Theorem 4. Denote by $(B_i)_{i\in\mathcal{I}}$ the covering of K. Since $\sigma \notin \mathcal{A}^d_K$, Corollary 3 implies that $\sum_{i\in\mathcal{I}} A(C_\sigma \cap B_i) \ge A(C_\sigma \cap K) \ge \lambda_K d$. Now, from Lemma 3 we conclude that

$$\sum_{i\in\mathcal{I}} g(C_\sigma \cap B_i) \le n_K g(C_\sigma \setminus \mathbf{R}C_\sigma) \le n_K a(d) d.$$

Hence the result. □

3.3.2. *Proof of Theorem 1*

From Section 3.2, we can assume that the sequence $a(d)$ grows to infinity. For every $d \in \mathbf{N}^*$, we set $\rho_d = a(d)^{-1/2}$. Let $\overset{\circ}{\mathrm{K}}$ be a relatively compact open subset of $\mathrm{X} \setminus \mathbf{R}\mathrm{X}$. For every d large enough, we cover K by balls of radius ρ_d as given by Lemma 3. This cover contains at most $r_{\mathrm{K}}/\rho_d^4 = r_{\mathrm{K}} a(d)^2$ balls. From Corollary 4, there is a subset $\mathcal{B}^d$ of $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ satisfying

$$\mu(\mathcal{B}^d) \leq 2 \int_{\mathrm{X}} \omega^2 c_{\mathrm{K}} r_{\mathrm{K}} a(d)^4 \exp\left(-\mathrm{D}_{\mathrm{K}} \frac{d}{a(d)}\right),$$

and such that for every $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathcal{B}^d$ and every ball B of the cover,

$$\lambda^1_{\mathrm{K}} \frac{d}{a(d)^2} \leq \mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{B}) \leq \lambda^2_{\mathrm{K}} \frac{d}{a(d)^2}.$$

Let $\mathcal{A}^d_{\mathrm{K}} \subset \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ be the set given by Corollary 3. By Lemma 4, for every $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathcal{A}^d_{\mathrm{K}}$, there is a ball B_σ of our cover such that

$$g(\mathrm{C}_\sigma \cap \mathrm{B}_\sigma) \leq \frac{n_{\mathrm{K}}}{\lambda_{\mathrm{K}}} a(d) \mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{B}).$$

Without loss of generality, we can assume that $\mathrm{B}_\sigma = \mathrm{B}^4(\rho_d) \subset \mathbf{C}^2$ and that the area of C_σ is computed with respect to the standard metric ω_0 of $\mathbf{C}^2$. Denote by $\widetilde{\mathrm{C}}_\sigma$ the image of $\mathrm{C}_\sigma \cap \mathrm{B}$ under the homothetic transformation with coefficient $1/\rho_d$, so that $\widetilde{\mathrm{C}}_\sigma \subset \mathrm{B}^4(1)$ and $g(\widetilde{\mathrm{C}}_\sigma) \leq \frac{n_{\mathrm{K}}}{\lambda_{\mathrm{K}}} \mathrm{A}(\widetilde{\mathrm{C}}_\sigma)$.

Let $\mathcal{T}^{(1,1)}_{\pi^2}(\mathrm{B}(1))$ be the space of positive closed currents of bidegree $(1, 1)$ on the unit ball $\mathrm{B}^4(1)$ with mass π^2, and $\widetilde{\mathrm{Z}}_\sigma \in \mathcal{T}^{(1,1)}_{\pi^2}(\mathrm{B}(1))$ the current of integration

$$\widetilde{\mathrm{Z}}_\sigma : \phi \in \Omega^{(1,1)}_c(\mathrm{B}^4(1)) \mapsto \widetilde{\mathrm{Z}}_\sigma(\phi) = \frac{\pi^2}{\mathrm{A}(\widetilde{\mathrm{C}}_\sigma)} \int_{\widetilde{\mathrm{C}}_\sigma} \phi.$$

We set

$$\mathcal{Z}^a = \overline{\bigcup_{d \geq d_{\mathrm{L}}} \{\widetilde{\mathrm{Z}}_\sigma, \sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \mathcal{A}^d_{\mathrm{K}}\}} \subset \mathcal{T}^{(1,1)}_{\pi^2}(\mathrm{B}(1)).$$

By Theorem 3, $\mathcal{Z}^a$ is contained in the space of weakly laminar currents of the unit ball $\mathrm{B}^4(1)$. In particular, from Lemma 1 we know that $\omega_0 \notin \mathcal{Z}^a$. Since $\overline{\mathrm{B}^4(1)}$ is compact, $\mathcal{Z}^a$ is compact and there exists a finite number of two-forms $(\widetilde{\phi}_j)_{j \in \mathcal{J}}$ with compact support in $\mathrm{B}^4(1)$ such that

$$\forall \lambda \in \left[\frac{\lambda^1_{\mathrm{K}}}{\pi^2}, \frac{\lambda^2_{\mathrm{K}}}{\pi^2}\right], \forall \mathrm{T} \in \mathcal{Z}^a, \exists j \in \mathcal{J}, \quad |\langle \lambda \mathrm{T} - \omega_0, \widetilde{\phi}_j \rangle| > 1.$$

Applying this inequality to $\mathrm{T} = \widetilde{\mathrm{Z}}_\sigma$ and $\lambda = a(d)^2 \mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{B})/\pi^2 d$, we get

$$\left| \frac{a(d)^2 \mathrm{A}(\mathrm{C}_\sigma \cap \mathrm{B})}{d\mathrm{A}(\tilde{\mathrm{C}}_\sigma)} \int_{\tilde{\mathrm{C}}_\sigma} \widetilde{\phi}_j - \int_{\mathrm{B}(1)} \omega_0 \wedge \widetilde{\phi}_j \right| > 1.$$

Denote by ϕ_j the pullback of $\widetilde{\phi}_j$ under the homothetic transformation of coefficient $1/\rho_d$, so that the support of ϕ_j lies in $\mathrm{B}^4(\rho_d)$. We get

$$\left| \frac{1}{d} \int_{\mathrm{C}_\sigma} \phi_j - \int_{\mathrm{B}^4(\rho_d)} \omega_0 \wedge \phi_j \right| > 1/a(d),$$

as long as $\sigma \notin \mathcal{B}^d$. Finally, since by Definition 2 $a(d) \int_{\mathrm{B}_\sigma} (\omega - \omega_0) \wedge \phi_j$ converges to zero, we deduce for d large enough the relation

$$\forall \sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus (\mathcal{A}^d_{\mathrm{K}} \cup \mathcal{B}^d),\ \exists j \in \mathcal{J},$$
$$\left| \frac{1}{d} \int_{\mathrm{C}_\sigma} \phi_j - \int_{\mathrm{X}} \omega \wedge \phi_j \right| \geq 1/a(d).$$

Likewise, from Tian's asymptotic isometry theorem [21], $a(d) \int_{\mathrm{X}} (\omega - \frac{1}{d} \Phi_d^* \omega_{\mathrm{FS}}) \wedge \phi_j$ converges to zero as d grows to ∞. Applying Proposition 3 to every ball of our cover and every ϕ_j, $j \in \mathcal{J}$, with support in this ball, we finally obtain the existence of positive constants C, D, such that $\mu(\mathcal{M}_d^{a(d)}) \leq \mathrm{C}a(d)^4 \exp(-\mathrm{D}\frac{d}{a(d)})$.

4. Final remarks

4.1. *Average current of integration*

For every $k \geq 1$, denote by

$$\begin{aligned} \mathrm{E}_{\mathbf{C}\mathrm{P}^k} : \mathbf{C}\mathrm{P}^k &\to \mathbf{R} \\ z &\mapsto \int_{\mathbf{R}\mathrm{H}^0(\mathbf{C}\mathrm{P}^k, \mathcal{O}_{\mathbf{C}\mathrm{P}^k}(1))} \log \|\sigma(z)\|^2 d\mu(\sigma) \end{aligned}$$

the expectation of the random variable $\sigma \mapsto \log \|\sigma\|^2$.

Proposition **4.** — *For every $k \geq 1$ and $z \in \mathbf{C}\mathrm{P}^k \setminus \mathbf{R}\mathrm{P}^k$,*

$$\mathrm{E}_{\mathbf{C}\mathrm{P}^k}(z) = \log\left(\frac{k+1}{4}\right) + \int_0^\infty e^{-\rho} \log \rho \, d\rho + \log(1 + \sqrt{1 - \|\tau\|^2(z)}),$$

where τ is the section introduced in Proposition 1.

This result is very close to Lemma 2.5 of [15].

Proof. — As in the proof of Proposition 1 and using the notations of Remark 2, we get for every $0 < r \leq 1$:

$$\begin{aligned}
\mathrm{E}_{\mathbf{C}\mathrm{P}^k}(z_r) &= \int_{\mathbf{R}^2} \log\left((k+1)\frac{|a_0 + ira_1|^2}{1+r^2}\right)\frac{e^{-|a|^2}}{\pi} da_0 da_1 \\
&= \log\left(\frac{k+1}{1+r^2}\right) + \int_0^\infty e^{-\rho}\log\rho\, d\rho + \frac{1}{2\pi}\int_0^{2\pi} \log|\cos\theta \\
&\quad + ir\sin\theta|^2 d\theta \\
&= \log\left(\frac{k+1}{4(1+r^2)}\right) + \int_0^\infty e^{-\rho}\log\rho\, d\rho \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{2i\theta}(1+r) + 1 - r|^2 d\theta.
\end{aligned}$$

From Jensen formula, as soon as $r > 0$,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log|e^{2i\theta}(1+r) + 1 - r|^2 d\theta &= \log|1-r|^2 + \log\left|\frac{1+r}{1-r}\right|^2 \\
&= \log(1+r)^2.
\end{aligned}$$

From this we deduce

$$\begin{aligned}
\mathrm{E}_{\mathbf{C}\mathrm{P}^k}(z_r) &= \log\left(\frac{k+1}{4}\right) + \int_0^\infty e^{-\rho}\log\rho\, d\rho + \log\left(\frac{(1+r)^2}{1+r^2}\right) \\
&= \log\left(\frac{k+1}{4}\right) + \int_0^\infty e^{-\rho}\log\rho\, d\rho + \log\left(1 + \sqrt{1 - \|\tau\|^2(z)}\right),
\end{aligned}$$

since $\|\tau\|(z_r) = |\tau(z_r)|/|z_r|^2 = (1-r^2)/(1+r^2)$. The result follows from the invariance of $\mathrm{E}_{\mathbf{C}\mathrm{P}^k}$ and $\|\tau\|$ under the action of $\mathrm{PO}_{k+1}(\mathbf{R})$, see Remark 2. $\square$

Corollary **5.** — *For every $k \geq 1$ and every real line* D *in* $\mathbf{C}\mathrm{P}^k$, *the restriction of the current* $\frac{1}{2i\pi}\partial\bar\partial \mathrm{E}_{\mathbf{C}\mathrm{P}^k}$ *to* $\mathrm{D} \setminus \mathbf{R}\mathrm{D}$ *coincides with the Fubini-Study form, while its restriction to the quadric* $\{\tau = 0\}$ *vanishes.*

Proof. — Proposition 4 implies that the restriction of $\mathrm{E}_{\mathbf{C}\mathrm{P}^k}$ to the quadric $\{\tau = 0\}$ is constant, so that the current $\partial\bar\partial \mathrm{E}_{\mathbf{C}\mathrm{P}^k}$ vanishes on this quadric. In the same way, Proposition 4 implies that the restriction of $\frac{1}{2i\pi}\partial\bar\partial \mathrm{E}_{\mathbf{C}\mathrm{P}^k}$ to D does not depend on k. Thus, we may assume $k = 1$ and $\mathrm{D} = \mathbf{C}\mathrm{P}^1$. Now, every $\sigma \in \mathbf{R}\mathrm{H}^0(\mathbf{C}\mathrm{P}^1, \mathcal{O}_{\mathbf{C}\mathrm{P}^1}(1))$ does not vanish on $\mathbf{C}\mathrm{P}^1 \setminus \mathbf{R}\mathrm{P}^1$, so that by definition

$$\begin{aligned}\forall z \in \mathbf{C}\mathrm{P}^1 \setminus \mathbf{R}\mathrm{P}^1, \quad & \frac{1}{2i\pi}\partial\bar{\partial}\mathrm{E}_{\mathbf{C}\mathrm{P}^k}(z) \\ & = \int_{\mathbf{R}\mathrm{H}^0(\mathbf{C}\mathrm{P}^1, \mathcal{O}_{\mathbf{C}\mathrm{P}^1}(1))} \frac{1}{2i\pi}\partial\bar{\partial}\log\|\sigma(z)\|^2 d\mu(\sigma) \\ & = \int_{\mathbf{R}\mathrm{H}^0(\mathbf{C}\mathrm{P}^1, \mathcal{O}_{\mathbf{C}\mathrm{P}^1}(1))} \omega_{\mathrm{FS}}(z) d\mu(\sigma) \\ & = \omega_{\mathrm{FS}}(z). \qquad \square\end{aligned}$$

Now, let L be a real Hermitian line bundle with positive curvature on a smooth real Kähler manifold X of dimension $n \geq 1$. For every $d \in \mathbf{N}^*$ and every $(2n-2)$-form $\phi \in \Omega^{2n-2}(\mathrm{X})$, we denote by

$$\mathrm{Z}^\phi : \sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \{0\} \mapsto \mathrm{Z}^\phi_\sigma = \frac{1}{d}\int_{\mathrm{C}_\sigma} \phi \in \mathbf{R}$$

the associated random variable, where the space $\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)$ is equipped with the L^2 Gaussian probability measure μ. We write

$$\mathrm{E}_d(\mathrm{Z}^\phi) = \int_{\mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)} \mathrm{Z}^\phi_\sigma d\mu(\sigma)$$

for the expectation of this random variable, and $\mathrm{E}_d(\mathrm{Z}) : \phi \in \Omega^2(\mathrm{X}) \mapsto \mathrm{E}_d(\mathrm{Z}^\phi) \in \mathbf{R}$ for the associated closed positive $(1,1)$-current.

Proposition **5.** — *Let* L *be a real Hermitian line bundle with positive curvature on a smooth closed real Kähler manifold* X. *Then, for every* $d \geq d_{\mathrm{L}}$,

$$\mathrm{E}_d(\mathrm{Z}) = \frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}} - \frac{1}{2i\pi d}\Phi_d^*\partial\bar{\partial}\mathrm{E}_{\mathrm{P}(\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*)}.$$

Moreover, the restriction of this current to the complement of the real locus converges to ω *as* d *grows to infinity.*

Recall that the embedding $\Phi_d : \mathrm{X} \to \mathrm{P}(\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*)$, $d \geq d_{\mathrm{L}}$, and the Fubini-Study form ω_{FS} of the projective space $\mathrm{P}(\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d)^*)$ were introduced in Section 1.2.

Proof. — Poincaré-Lelong's formula provides for every $\sigma \in \mathbf{R}\mathrm{H}^0(\mathrm{X}, \mathrm{L}^d) \setminus \{0\}$ the relation

$$\frac{1}{2i\pi d}\partial\bar{\partial}\log\|\sigma\|^2_{\Phi_d} = \frac{1}{d}\Phi_d^*\omega_{\mathrm{FS}} - \mathrm{Z}_\sigma.$$

The first part of Proposition 5 is obtained by integration of this relation over $\mathbf{R}H^0(X, L^d)$. Tian's asymptotic isometry theorem [21] implies that $\frac{1}{d}\Phi_d^*\omega_{FS}$ converges to the curvature ω of L. Proposition 2 combined with Proposition 4 imply that $\frac{1}{2i\pi d}\Phi_d^*\partial\bar\partial E_{P(H^0(X,L^d)^*)}$ converges to zero faster than every polynomial function in d, and even exponentially fast in the cases covered by Remark 3 (compare with [15]). Hence the result. □

Note that when the chosen probability space is the whole complex space $H^0(X, L^d)$, the expectation $\int_{H^0(X,L^d)} \log\|\sigma(z)\|^2 d\mu(\sigma)$ is a function of $z \in \mathbf{C}P^k$ invariant under the whole $PU_{k+1}(\mathbf{C})$, thus is constant. Hence, $E(Z^\phi_{\mathbf{C}}) = \frac{1}{d}\Phi_d^*\omega_{FS}$, see [17]. Moreover, Shiffman and Zelditch proved in [18] that the law of $Z^\phi_{\mathbf{C}}$ converges to a normal law as d grows to infinity, a result that was already obtained in dimension one in [20]. It would be here of interest to understand in more details the convergence of the law of Z^ϕ.

4.2. *Existence of real maximal curves*

A smooth real algebraic curve C of genus $g(C)$ is said to be maximal when the number of components of its real locus coincides with $g(C) + 1$, the maximum allowed by Harnack-Klein's inequality [12], [14]. Our Theorem 1 implies, in particular, that if L is a real ample Hermitian line bundle over a real projective surface X, the measure of the set of real maximal curves linearly equivalent to L^d decreases exponentially as d increases. When $X = \mathbf{C}P^2$, Harnack [12] proved that such maximal curves exist in any degree. These curves were later involved in Hilbert's sixteenth problem and quite studied since that time. Nevertheless, such curves do not always exist. For instance, if X is the product of two non maximal real curves, then for every ample real line bundle L over X and every $d \in \mathbf{N}^*$, the linear system $\mathbf{R}H^0(X, L^d)$ contains no maximal curve.

However, every real closed symplectic manifold (X, ω, c_X) of dimension four with rational form ω supports, when d is large enough, symplectic real surfaces Poincaré dual to $d\omega$ and whose real locus contains at least ϵd components, where ϵ depends on the manifold (X, ω, c_X), see [11]. Applying Harnack's method to these curves, we can prove that there even always exist symplectic real surfaces whose real locus contains at least $\epsilon' d^2$ connected components. For real projective surfaces, this method gives more, as was pointed out to us by V. Kharlamov, see Theorem 5. For every ample real line bundle L on a real projective surface X and every $d \in \mathbf{N}^*$, denote by

$$m(L^d) = \sup_{\sigma\in \mathbf{R}H^0(X,L^d)\setminus \mathbf{R}\Delta_d} b_0(\mathbf{R}C_\sigma)$$

the maximal number of connected components that a smooth real divisor linearly equivalent to L^d may contain. Likewise, denote by $g(L^d) = \frac{1}{2}(d^2L^2 - dc_1(X)L + 2)$ the genus of such a smooth divisor, so that Harnack-Klein's inequality provides $m(L^d) \leq g(L^d) + 1$.

Theorem **5.** — *Let* C *be a smooth ample real divisor with non-empty real part on a smooth real projective surface* X*, and* L *be the associated line bundle. Then, as soon as* $d \in \mathbf{N}^*$ *is large enough,* $m(\mathrm{L}^d) \geq g(\mathrm{L}^d) - 2dg(\mathrm{L}) + \mathrm{O}(1)$.

In particular, this Theorem 5 implies that there always exists a constant $a \in \mathbf{Q}^*_+$ such that the set $\mathcal{M}^a_d$ introduced in Section 0.1 is non-empty for d sufficiently large. From Theorem 1, its volume decays exponentially as d grows to infinity.

Proof. — Let σ be a real holomorphic section of L vanishing transversally along C. From Riemann-Roch's theorem, denoting by $\mathrm{L}|_\mathrm{C}$ the restriction of L to C, we know that there exists $d_0 \in \mathbf{N}$ such that for every $d \geq d_0$, $\dim \mathrm{H}^0(\mathrm{C}; \mathrm{L}|^d_\mathrm{C}) = \deg(\mathrm{L}|^d_\mathrm{C}) + 1 - g(\mathrm{L})$, since $\mathrm{H}^1(\mathrm{C}; \mathrm{L}|^d_\mathrm{C}) \cong \mathrm{H}^0(\mathrm{C}; \mathrm{K}_\mathrm{C} \otimes \mathrm{L}|^{-d}_\mathrm{C}) = 0$. Let then $d \geq d_0$ and $\underline{x}_d$ be a set of $\deg(\mathrm{L}|^d_\mathrm{C}) - g(\mathrm{L})$ distinct points of C. There exists a non-zero section $\mu_{\underline{x}_d}$ of $\mathrm{L}|^d_\mathrm{C}$ vanishing along $\underline{x}_d$. Moreover, from the short exact sequence of sheaves $0 \to \mathcal{O}_\mathrm{X}(\mathrm{L}^{d-1}) \to \mathcal{O}_\mathrm{X}(\mathrm{L}^d) \to \mathcal{O}_\mathrm{C}(\mathrm{L}|^d_\mathrm{C}) \to 0$ we deduce the exact sequence $\mathrm{H}^0(\mathrm{X}; \mathrm{L}^d) \to \mathrm{H}^0(\mathrm{C}; \mathrm{L}|^d_\mathrm{C}) \to \mathrm{H}^1(\mathrm{X}; \mathrm{L}^{d-1})$, while from ampleness of L there exists $d_1 \geq d_0$ such that for $d \geq d_1$, $\mathrm{H}^1(\mathrm{X}; \mathrm{L}^{d-1}) = 0$. As a consequence, for every $d \geq d_1$, the section $\mu_{\underline{x}_d}$ extends to a real section of L over X which is still denoted by $\mu_{\underline{x}_d}$. We now apply Harnack's method [12]. Let us choose a connected component of $\mathbf{R}\mathrm{C}$ which we identify with $\mathbf{R} \cup \{\infty\}$. Let then $\sigma_{d_1} = \mu_{\underline{x}_{d_1}}$ and for $d > d_1$, $\sigma_d = \sigma \otimes \sigma_{d-1} + \epsilon_d \mu_{\underline{x}_d}$. We choose the sets of points $\underline{x}_d \subset \mathbf{R} \subset \mathbf{R}\mathrm{C}$, $d \geq d_1$, such that $\mu_{\underline{x}_d}$ does not vanish on the intervals $[\inf \underline{x}_{d-1}, \sup \underline{x}_{d-1}]$ and $[\inf \underline{x}_{d+1}, \sup \underline{x}_{d+1}]$. Such a choice is always possible by induction. Indeed, we may start with $g(\mathrm{L}) + 1$ sets of distinct points $\underline{x}^1_{d_1}, \ldots, \underline{x}^{g(\mathrm{L})+1}_{d_1}$ with disjoint corresponding intervals $[\inf \underline{x}^i_{d_1}, \sup \underline{x}^i_{d_1}]$, $1 \leq i \leq g(\mathrm{L}) + 1$. Then, we choose $g(\mathrm{L})(g(\mathrm{L}) + 1) + 1$ sets of distinct points $\underline{x}^1_{d_1+1}, \ldots, \underline{x}^{g(\mathrm{L})(g(\mathrm{L})+1)+1}_{d_1+1}$ with disjoint corresponding intervals away from the union $\bigcup_{i=1}^{g(\mathrm{L})+1} ([\inf \underline{x}^i_{d_1}, \sup \underline{x}^i_{d_1}] \cup \{\mu_{\underline{x}^i_{d_1}} = 0\})$. For every $j \in \{1, \ldots, g(\mathrm{L})(g(\mathrm{L}) + 1) + 1\}$, there exists $i_j \in \{1, \ldots, g(\mathrm{L}) + 1\}$ such that $\mu_{\underline{x}^j_{d_1+1}}$ does not vanish on $[\inf \underline{x}^{i_j}_{d_1}, \sup \underline{x}^{i_j}_{d_1}]$, since $\mu_{\underline{x}^j_{d_1+1}}$ has only $g(\mathrm{L})$ vanishing points different from $\underline{x}^j_{d_1+1}$. Then, there exists $i \in \{1, \ldots, g(\mathrm{L}) + 1\}$ such that the fiber $f^{-1}(i)$ of the map $f : j \in \{1, \ldots, g(\mathrm{L})(g(\mathrm{L}) + 1) + 1\} \mapsto i_j \in \{1, \ldots, g(\mathrm{L}) + 1\}$ contains at least $g(\mathrm{L}) + 1$ elements. We choose $\underline{x}^i_{d_1}$ and continue our induction for $d = d_1 + 1$ with $g(\mathrm{L}) + 1$ elements chosen in $f^{-1}(i)$. Now, choosing real numbers ϵ_d small enough with adequate signs, we can ensure that the vanishing locus C_{d_1+2} of σ_{d_1+2} has $\deg(\mathrm{L}|^{d_1+1}_\mathrm{C}) - g(\mathrm{L}) - 1$ real components near $[\inf \underline{x}_{d_1+1}, \sup \underline{x}_{d_1+1}] \subset \mathbf{R}\mathrm{C}$, see Figure 1.

By induction, for every $d > d_1 + 1$, C_d has $\deg(\mathrm{L}|^j_\mathrm{C}) - g(\mathrm{L}) - 1$ real components near $[\inf \underline{x}_j, \sup \underline{x}_j] \subset \mathbf{R}\mathrm{C}$, $d_1 + 1 \leq j < d$. We deduce that

$$\begin{aligned}\forall d > d_1 + 1, \quad b_0(\mathbf{R}\mathrm{C}_d) &\geq \sum_{j=1}^{d-1} (j\mathrm{L}^2 - g(\mathrm{L}) - 1) + \mathrm{O}(1) \\ &\geq \frac{1}{2} d(d-1)\mathrm{L}^2 - d(g(\mathrm{L}) + 1) + \mathrm{O}(1)\end{aligned}$$

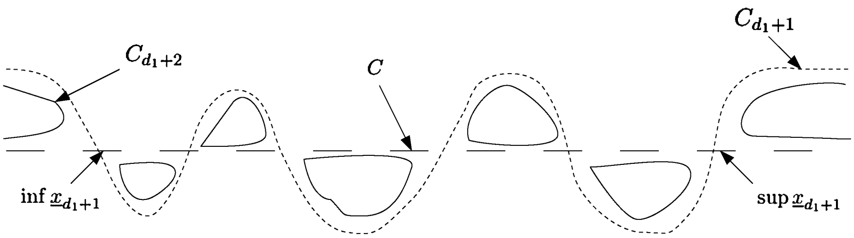

FIG. 1. — Harnack's method

$$\begin{aligned}&\geq \frac{1}{2}d^2\mathrm{L}^2 - \frac{d}{2}(\mathrm{L}^2 + \mathrm{L}^2 - c_1(\mathrm{X})\mathrm{L} + 4) + \mathrm{O}(1)\\ &\geq \frac{1}{2}(d^2\mathrm{L}^2 - dc_1(\mathrm{X})\mathrm{L}) - d(\mathrm{L}^2 - c_1(\mathrm{X})\mathrm{L} + 2)\\ &\quad + \mathrm{O}(1)\end{aligned}$$

and the result. □

Note that the constant O(1) in Theorem 5 is actually quite explicit. Is the linear term in this Theorem 5 optimal? That is, what is the best lower bound for the quantity $\limsup_{d\to\infty} \frac{1}{d}(m(\mathrm{L}^d) - \frac{1}{2}d^2\mathrm{L}^2)$? The same questions holds within the realm of four-dimensional real symplectic manifolds. Recall that the real symplectic surfaces built in [11] are obtained via Donaldson's method [8], so that their current of integration converges to ω as d grows to infinity. Theorem 4 provides an obstruction to get real maximal curves using this method (Donaldson's quantitative transversality gives another one, as observed in [11]). This phenomenon was in fact the starting point of our work.

This work raises several questions. It is known [10] that the expectation of the number of real roots of a real polynomial in one variable is $\sqrt{n}$. What is the expected value of $b_0(\mathbf{R}\mathrm{C}_\sigma)$ in dimension two? How to improve Theorem 1 to get decays till this expectation, as in Theorem 2? What happens for values below this expectation? Note that for spherical harmonics on the two-dimensional sphere, such kinds of results have been obtained in [16]. More generally, what is the asymptotic law of the random variable b_0? What happens in higher dimensions?

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