J. Eur. Math. Soc. 18, 733–772

DOI 10.4171/JEMS/601

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Betti numbers of random real hypersurfaces and determinants of random symmetric matrices

Received January 29, 2013 and in revised form February 18, 2014

Abstract. We asymptotically estimate from above the expected Betti numbers of random real hypersurfaces in smooth real projective manifolds. Our upper bounds grow as the square root of the degree of the hypersurfaces as the latter grows to infinity, with a coefficient involving the Kählerian volume of the real locus of the manifold as well as the expected determinant of random real symmetric matrices of given index. In particular, for large dimensions, these coefficients get exponentially small away from mid-dimensional Betti numbers. In order to get these results, we first establish the equidistribution of the critical points of a given Morse function restricted to the random real hypersurfaces.

Keywords. Real projective manifold, ample line bundle, random matrix, random polynomial

Contents

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Mathematics Subject Classification (2010): Primary 14P25; Secondary 32U40, 60F10, 60B20

1. Introduction

How many real roots does a random real polynomial have? This question was answered by M. Kac [20] in the 40's and, for a different measure, by E. Kostlan [21] and by M. Shub and S. Smale [36] in the 90's. In higher dimensions, this question may become: what is the topology of a random real hypersurface in a given smooth real projective manifold? The mean Euler characteristic of such random real hypersurfaces in $\mathbb{R}P^n$ has been computed by S. S. Podkorytov [33] and P. Bürgisser [5], while the mean total Betti number has been estimated from above by the present authors [16] (see also [14]). In the case of spherical harmonics in dimension two, rather precise estimates have been obtained by F. Nazarov and M. Sodin [31], partially extended to higher dimensions using the same approach by A. Lerario and E. Lundberg [24].

Our aim is to improve our previous results [16] by getting upper bounds for all individual Betti numbers of random real hypersurfaces. Let X be a smooth n-dimensional complex projective manifold defined over the reals and let $\mathbb{R}X$ be its real locus. Let L be a real ample line bundle over X. We equip L with a real Hermitian metric h of positive curvature ω, and X with a normalized volume form dx. These induce an L^2 inner product on all spaces of global holomorphic real sections $\mathbb{R}H^0(X, L^d)$ for all tensor powers L^d of L, $d > 0$ (see §2.1.2). The latter spaces then inherit Gaussian probability measures $\mu_{\mathbb{R}}$ (see (2.3)), with respect to which we are going to consider random sections; see [16, §3.1.1] for a discussion on this choice (previously considered in [21], [36], [33], [5], [22]) and on other possible ones (compare [35], [4], [24]).

For every generic section $\sigma \in \mathbb{R}H^0(X, L^d)$, the real locus $\mathbb{R}C_\sigma$ of its vanishing locus C_σ is a smooth hypersurface of $\mathbb{R}X$, if non-empty. For every $i \in \{0, \dots, n-1\}$, we denote by $m_i(\mathbb{R}C_\sigma)$ the infimum over all Morse functions f on $\mathbb{R}C_\sigma$ of the number of critical points of index i of f. This Morse number (compare [29, Definition 2.4]) bounds from above all ith Betti numbers of $\mathbb{R}C_\sigma$, for any coefficient ring, as follows from Morse theory, the upper bound being strict in general due to the Morse inequalities (see, e.g., [28]). We then denote by $E(m_i)$ the average value of this Morse number,

$$E(m_i) = \int_{\mathbb{R}H^0(X, L^d)} m_i(\mathbb{R}C_\sigma)\, d\mu_{\mathbb{R}}(\sigma). \tag{1.1}$$

Our aim is to prove the following upper bound for this expectation:

Theorem 1.1. *Let X be a smooth real projective manifold of dimension $n > 0$ and (L, h) be a real Hermitian line bundle of positive curvature over X. Then, for every $i \in \{0, \ldots, n-1\}$,*

$$\limsup_{d\to\infty} \frac{1}{\sqrt{d}^n} E(m_i) \leq \frac{\mathrm{Vol}_h(\mathbb{R}X)}{\sqrt{\pi}} e_{\mathbb{R}}(i, n-1-i).$$

Moreover, when $n = 1$, the $\limsup$ *is a limit and the inequality an equality, so that*

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}} E(m_0) = \frac{\mathrm{Length}_h(\mathbb{R}X)}{\sqrt{\pi}}. \tag{1.2}$$

In Theorem 1.1, $\mathrm{Vol}_h(\mathbb{R}X)$ denotes the Riemannian volume of $\mathbb{R}X$ for the Kähler metric induced by the curvature form ω of h. In contrast with $\mathrm{Vol}_h(X) = \int_X c_1(L)^n$, it depends on the germ of the metric h on $\mathbb{R}X$. Note that for $n = 1$, the Morse number $m_0(\mathbb{R}C_\sigma)$ just corresponds to the number of real zeros of the section σ. It turns out that Theorem 1.1, as well as Theorem 1.2 and Corollary 1.5, does not depend on the normalized volume form dx chosen on X to define the L^2 inner product on $\mathbb{R}H^0(X, L^d)$ (compare [35]).

The coefficient $e_{\mathbb{R}}(i, n-1-i)$ is itself part of a mathematical expectation, namely the integral of (the absolute value of) the determinant on symmetric matrices of signature $(i, n-1-i)$ (see §2.1.5). More precisely, the space $\mathrm{Sym}(n-1, \mathbb{R})$ of symmetric $(n-1) \times (n-1)$ matrices has a natural Gaussian measure that we also denote by $\mu_{\mathbb{R}}$. Let $\mathrm{Sym}(i, n-1-i, \mathbb{R})$ be the open subset of matrices of signature $(i, n-1-i)$. Then, for every $i \in \{0, \ldots, n-1\}$,

$$e_{\mathbb{R}}(i, n-1-i) = \int_{\mathrm{Sym}(i,n-1-i,\mathbb{R})} |\det A| \, d\mu_{\mathbb{R}}(A), \tag{1.3}$$

$$e_{\mathbb{R}}(n-1) = \int_{\mathrm{Sym}(n-1,\mathbb{R})} |\det A| \, d\mu_{\mathbb{R}}(A). \tag{1.4}$$

Here, by convention, $e_{\mathbb{R}}(0) = e_{\mathbb{R}}(0, 0) = 1$.

Note that when $X = \mathbb{C}P^1$, $L = \mathcal{O}_{\mathbb{C}P^1}(1)$ and h is the Fubini–Study metric, $\mathrm{Vol}_{\mathrm{FS}}(\mathbb{R}P^1) = \sqrt{\pi}$, so that the limit (1.2) in Theorem 1.1 recovers asymptotically the results of Kostlan and Shub–Smale, according to which a random degree d real polynomial in one variable has $\sqrt{d}$ roots for our choice of the probability measure. The initial result of M. Kac was instead expecting asymptotically $(2/\pi)\log d$ real roots, but for a different probability measure (see [16, §3.1.1]). When $X = \mathbb{C}P^2$, P. Sarnak and I. Wigman informed us in 2011 that they were also able to bound $E(b_0)$ from above by an $O(d)$ term as in Theorem 1.1. It has just been shown by A. Lerario and E. Lundberg [24] in projective spaces that when the Gaussian measure arises from the L^2 scalar product defined by integration over $\mathbb{R}P^n$ instead of $\mathbb{C}P^n$, the expected number of connected components of a random real hypersurface of degree d is bounded from below by ϵd^n for some positive ϵ. They follow the approach of F. Nazarov and M. Sodin [31]. Theorem 1.1 improves our previous results of [16], where the best upper bounds we could get were $O(\sqrt{d \log d}^n)$ in some cases.

Theorem 1.1 turns out to be a consequence of a more precise equidistribution result. Namely, when $n > 1$, we equip $\mathbb{R}X$ with a fixed Morse function $p : \mathbb{R}X \to \mathbb{R}$. Then, for every generic section $\sigma \in \mathbb{R}H^0(X, L^d)$, p restricts to a Morse function on $\mathbb{R}C_\sigma$ and we denote by $\mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})$ the set of critical points of index i of this restriction. We set

$$\nu_i(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n} \sum_{x \in \mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})} \delta_x, \tag{1.5}$$

the empirical measure on these critical points, where δ_x denotes the Dirac measure at x, and

$$E(\nu_i) = \int_{\mathbb{R}H^0(X,L^d)} \nu_i(\mathbb{R}C_\sigma)\, d\mu_{\mathbb{R}}(\sigma). \tag{1.6}$$

When $n = 1$, ν_0 denotes the empirical measure on $\mathbb{R}C_\sigma$. Then we get

Theorem 1.2. *Let X be a smooth real projective manifold of dimension $n > 0$ and L be a real ample holomorphic line bundle over X equipped with a real Hermitian metric of positive curvature ω. Let $p : \mathbb{R}X \to \mathbb{R}$ be a Morse function. Then, for every $i \in \{0, \dots, n-1\}$, the measure $E(\nu_i)$ weakly converges to $(1/\sqrt{\pi})e_{\mathbb{R}}(i, n-1-i)d\mathrm{vol}_h$ as $d \to \infty$ (see (1.3)).*

In Theorem 1.2, $d\mathrm{vol}_h$ denotes the Lebesgue measure of $\mathbb{R}X$ induced by its Riemannian metric, which is itself induced by the Kähler metric of X defined by ω.

We also establish such an equidistribution result for critical points of complex hypersurfaces, where the Morse function p on $\mathbb{R}X$ is replaced by a Lefschetz pencil on X and $\mathbb{R}H^0(X, L^d)$ by $H^0(X, L^d)$. We set, for every $d > 0$,

$$E(\nu) = \int_{H^0(X,L^d)} \nu(C_\sigma)\, d\mu_{\mathbb{C}}(\sigma), \tag{1.7}$$

where $\mu_{\mathbb{C}}$ is the Gaussian measure on $H^0(X, L^d)$ induced by the L^2 Hermitian product (see (2.1) and (2.2) below), C_σ is the vanishing locus of $\sigma \in H^0(X, L^d)$, and

$$\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in \mathrm{Crit}(p_{|C_\sigma})} \delta_x. \tag{1.8}$$

Theorem 1.3. *Let X be a smooth complex projective manifold of dimension $n > 0$ and let L be an ample holomorphic line bundle on X equipped with a Hermitian metric of positive curvature ω. Let $p : X \dashrightarrow \mathbb{C}P^1$ be a Lefschetz pencil. Then the measure $E(\nu)$ defined by (1.7) weakly converges to ω^n as $d \to \infty$.*

By "weak convergence" in Theorem 1.3, we mean that for every continuous function $\chi : X \to \mathbb{R}$, $\langle E(\nu), \chi \rangle$ converges to $\int_X \chi \omega^n$ as $d \to \infty$, where

$$\langle E(\nu), \chi \rangle = \frac{1}{d^n} \int_{H^0(X,L^d)\setminus\Delta_p^d} \Big(\sum_{x \in \mathrm{Crit}(p_{|C_\sigma})} \chi(x) \Big)\, d\mu_{\mathbb{C}}(\sigma)$$

(see §2.1.3 for the definition of the singular locus Δ_p^d). Note that Theorem 1.3 slightly improves [16, Theorem 3] and that similar equidistribution results can be found in [10], [11], [25], [16], or also [13], [2], [7], [32], where random functions are studied.

In order to prove Theorem 1.2, we roughly follow the approach of [36]. We introduce the incidence variety $\Sigma_i = \{(\sigma, x) \in \mathbb{R}H^0(X, L^d) \times \mathbb{R}X \mid x \in \mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})\}$ and express $E(\nu_i)$ as the push-forward onto $\mathbb{R}X$ of the Gaussian measure $\mu_\mathbb{R}$ of $\mathbb{R}H^0(X, L^d)$ "pulled back" on Σ_i. This push-forward measure is then computed asymptotically thanks to the coarea formula and peak sections of Hörmander. The latter indeed make it possible to compute pointwise the measure in terms of the 2-jets of sections (see Section 2).

Now, what are the values of the expectations $e_\mathbb{R}(n)$, $n > 0$, and how do these distribute between the different $e_\mathbb{R}(i, n-i)$, $0 \le i \le n$? We devote the third section to this question and get:

Theorem 1.4. *When n is odd, $e_\mathbb{R}(n) = \frac{2\sqrt{2}}{\pi}\Gamma\left(\frac{n+2}{2}\right)$, while when $n = 2m$ is even,*

$$e_\mathbb{R}(n) = (-1)^m \frac{n!}{m!2^n} + (-1)^{m-1} \frac{4\sqrt{2}\,n!}{\sqrt{\pi}\,m!2^n} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(k+3/2)}{k!}. \tag{1.9}$$

Theorem 1.4 follows from Propositions 3.9 and 3.11 and Corollary 3.13. In both cases, $e_\mathbb{R}(n)$ is equivalent to $\frac{2\sqrt{2}}{\pi}\Gamma\left(\frac{n+2}{2}\right)$ as $n \to \infty$. The odd case in Theorem 1.4 was known (see [27, §26.5]), but we could only find the even case in terms of hypergeometric functions in the literature (see [8]). It turns out that $e_\mathbb{R}(n)$ is transcendental for odd n and algebraic in $\mathbb{Q}(\sqrt{2})$ for even n. We can now rewrite the bound deduced from Theorem 1.1 for the expected total Morse number $E(m_*) = \sum_{i=0}^{n-1} E(m_i)$ as follows (see Remark 2.14(3)).

Corollary 1.5. *Under the hypotheses of Theorem 1.1, for every even $n > 0$,*

$$\limsup_{d \to \infty} \frac{1}{\sqrt{d}^n} E(m_*) \le \frac{2\sqrt{2}}{\pi} \frac{\mathrm{Vol}_h(\mathbb{R}X)}{\mathrm{Vol}_{FS}(\mathbb{R}P^n)}.$$

For odd n, this inequality holds asymptotically in n. □

In particular, for every even-dimensional projective space, the right-hand side in Corollary 1.5 turns out not to depend on the dimension of the space. Finally, we get the following exponential decay away from the mid-dimensional Betti numbers:

Theorem 1.6. *For every $\alpha \in [0, 1/2[$, there exists $c_\alpha > 0$ such that for large values of $n \in \mathbb{N}^*$,*

$$\sum_{i=0}^{\lfloor \alpha n \rfloor} e_\mathbb{R}(i, n-i) \le \exp(-c_\alpha n^2).$$

This concentration near matrices having as many positive as negative eigenvalues actually follows from the large deviations estimates near the Wigner semi-circle law established in [3]. As a consequence of Theorem 1.6, for large values of n, the upper bound for the expected total Morse number of $\mathbb{R}C_\sigma$ given by the right-hand side of Theorem 1.1 distributes between the individual upper bounds for the different Morse numbers in such a way that it gets concentrated around the mid-dimensional ones and exponentially decreases away from them. The Betti numbers of C_σ themselves are actually concentrated at the mid-dimension ones, from the Lefschetz hyperplane theorem (see [16, Lemma 3]). Note that similar phenomena are observed in [22] and [23].

Let us finally mention the recent announcements [37] and [34], the numerical estimations carried out in [30] and our own papers [15] and [17]. In [15], we proved that for every closed possibly disconnected hypersurface Σ of $\mathbb{R}^n$, the expected number of appearances of the scheme $(\mathbb{R}^n, \Sigma)$ in the vanishing locus of a random real section of L^d grows faster than a positive constant times $\sqrt{d}^n$ as $d \to \infty$. In particular, we obtain a lower estimate for the expected Betti numbers similar to the upper ones of Theorem 1.1. In [17], we extended these results to random submanifolds of higher codimensions.

The second section of this paper is devoted to Theorems 1.1, 1.2 and 1.3. A key role is played by Hörmander peak sections (see §2.2). The third section is devoted to Theorems 1.4 and 1.6 and the study of determinants of random symmetric matrices, where we recall several classical results for the reader's convenience.

2. Expected Morse numbers of random real hypersurfaces

2.1. Notation

2.1.1. The polarized projective manifold (X, L, h). Let X be a smooth complex projective manifold of positive dimension n. When X is defined over $\mathbb{R}$, we denote by $c_X : X \to X$ the associated antiholomorphic involution, called the *real structure*, and by $\mathbb{R}X \subset X$ the *real locus* of the manifold, that is, the fixed point set of c_X. Likewise, let L be an ample holomorphic line bundle over X (sometimes called a *polarization*) equipped with a Hermitian metric h of positive curvature. We denote by ω the curvature form of h, so that for every local non-vanishing holomorphic section e of L defined over an open subset U of X,

$$\omega_{|U} = \frac{1}{2i\pi}\partial\bar{\partial}\log h(e, e).$$

We denote by $g = \omega(\cdot, J\cdot)$ the induced Kähler metric on X, where J denotes the complex structure of TX.

When X and L are defined over $\mathbb{R}$, we denote by c_L the associated real structure of L and assume that h is real, so that $\overline{c_L^* h} = h$. The restriction of g to $\mathbb{R}X$ is a Riemannian metric and we denote by $\mathrm{Vol}_h(\mathbb{R}X)$ the total volume of $\mathbb{R}X$ for the associated Lebesgue measure $d\mathrm{vol}_h$. Note that the volume of X is independent of the metric h and equals $\mathrm{Vol}(X) = \int_X \omega^n/n! = (1/n!)\int_X c_1(L)^n$. We denote by $dx = (1/\int_X \omega^n)\omega^n$ the normalized volume form of X, or any volume form on X with total volume one.

2.1.2. The Gaussian measures $\mu_{\mathbb{R}}$ *and* $\mu_{\mathbb{C}}$. For every $d > 0$, we denote by L^d the dth tensor power of L and by h^d the induced Hermitian metric on L^d. We denote by $H^0(X, L^d)$ the complex vector space of global holomorphic sections of L^d and by N_d the complex dimension of $H^0(X, L^d)$. In what follows, by dimension of a complex space we will always mean its complex dimension. We denote then by $\langle\cdot,\cdot\rangle$ the L^2 Hermitian product on this vector space, defined by the relation

$$\forall \sigma, \tau \in H^0(X, L^d), \qquad \langle \sigma, \tau \rangle = \int_X h^d(\sigma, \tau)\, dx. \tag{2.1}$$

The associated Gaussian measure is denoted by $\mu_{\mathbb{C}}$. It is defined, for every open subset U of $H^0(X, L^d)$, by

$$\mu_{\mathbb{C}}(U) = \frac{1}{\pi^{N_d}} \int_U e^{-\|\sigma\|^2} d\sigma, \tag{2.2}$$

where $d\sigma$ denotes the Lebesgue measure of $H^0(X, L^d)$. When L is defined over $\mathbb{R}$, we denote by $\mathbb{R}H^0(X, L^d)$ the real vector space of real sections of L^d, consisting of sections $\sigma \in H^0(X, L^d)$ satisfying $c_L \circ \sigma \circ c_X = \sigma$. Its real dimension equals N_d. The L^2 Hermitian product $\langle \cdot, \cdot \rangle$ given by (2.1) restricts to a scalar product on $\mathbb{R}H^0(X, L^d)$, which we also denote by $\langle \cdot, \cdot \rangle$. The associated Gaussian measure is denoted by $\mu_{\mathbb{R}}$ and defined for every open subset $\mathbb{R}U$ of $\mathbb{R}H^0(X, L^d)$ by

$$\mu_{\mathbb{R}}(\mathbb{R}U) = \frac{1}{\sqrt{\pi}^{N_d}} \int_{\mathbb{R}U} e^{-\|\sigma\|^2} d\sigma. \tag{2.3}$$

2.1.3. The discriminant loci Δ^d and Δ^d_p. For every $d > 0$, we denote by Δ^d (resp. $\mathbb{R}\Delta^d$) the discriminant hypersurface of $H^0(X, L^d)$ (resp. $\mathbb{R}H^0(X, L^d)$), that is, the set of sections $\sigma \in H^0(X, L^d)$ (resp. $\sigma \in \mathbb{R}H^0(X, L^d)$) which do not vanish transversely. For every $\sigma \in H^0(X, L^d) \setminus \{0\}$, we denote by C_σ (resp. $\mathbb{R}C_\sigma$) the vanishing locus of σ in X (resp. its real locus when $\sigma \in \mathbb{R}H^0(X, L^d)$). For every $\sigma \in H^0(X, L^d) \setminus \Delta^d$, C_σ is then a smooth hypersurface of X. When σ is real, $\mathbb{R}C_\sigma$ is of dimension $n - 1$ when non-empty and we denote, for $i \in \{0, \dots, n-1\}$, by $m_i(\mathbb{R}C_\sigma)$ the minimum number of critical points of index i of a Morse function on $\mathbb{R}C_\sigma$.

Definition 2.1. A *Lefschetz pencil* on X is a rational map $p : X \dashrightarrow \mathbb{C}P^1$ having only non-degenerate critical points and defined by two sections of a holomorphic line bundle with smooth and transverse vanishing loci.

We denote by B the base locus of a Lefschetz pencil p, given by Definion 2.1, that is, the codimension two submanifold of X where p is not defined. A Lefschetz pencil without base locus is called a *Lefschetz fibration*. Blowing up once the base locus of a Lefschetz pencil turns it into a Lefschetz fibration. When the dimension n of X equals one, the base locus is always empty and a Lefschetz fibration is nothing but a branched cover with simple ramifications.

When X is real (resp. complex) and $n > 1$, we equip its real locus with a Morse function $p : \mathbb{R}X \to \mathbb{R}$ (resp. with a Lefschetz pencil $p : X \dashrightarrow \mathbb{C}P^1$). We then denote, for every $d > 0$, by $\mathbb{R}\Delta^d_p$ (resp. Δ^d_p) the set of sections $\sigma \in \mathbb{R}H^0(X, L^d)$ (resp. $\sigma \in H^0(X, L^d)$) such that either $\sigma \in \mathbb{R}\Delta^d$ (resp. Δ^d), or $\mathbb{R}C_\sigma$ (resp. C_σ) intersects the critical locus of p, or the restriction of p to $\mathbb{R}C_\sigma$ (resp. C_σ) is not Morse (resp. not a Lefschetz pencil). For d large enough, this extended discriminant locus is of measure 0 for the measure $\mu_{\mathbb{R}}$ (resp. $\mu_{\mathbb{C}}$) (see Lemma 2.8).

2.1.4. The empirical measures ν and ν_i. For every $\sigma \in H^0(X, L^d) \setminus \Delta_p^d$, we denote by $\mathrm{Crit}(p_{|C_\sigma})$ the set of critical points of the restriction of p to C_σ and set

$$\nu(C_\sigma) = \frac{1}{d^n} \sum_{x \in \mathrm{Crit}(p_{|C_\sigma})} \delta_x,$$

where δ_x denotes the Dirac measure of X at the point x. When σ is real, we denote similarly, for every $i \in \{0, \dots, n-1\}$, by $\mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})$ the set of critical points of index i of $p_{|\mathbb{R}C_\sigma}$, and set

$$\nu_i(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}^n} \sum_{x \in \mathrm{Crit}_i(p_{|\mathbb{R}C_\sigma})} \delta_x. \tag{2.4}$$

When $n = 1$, we set likewise $\nu(C_\sigma) = \frac{1}{d} \sum_{x \in C_\sigma} \delta_x$, and when $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta^d$,

$$\nu_0(\mathbb{R}C_\sigma) = \frac{1}{\sqrt{d}} \sum_{x \in \mathbb{R}C_\sigma} \delta_x.$$

2.1.5. Random symmetric matrices. For every $n \in \mathbb{N}^*$, denote by $\mathrm{Sym}(n, \mathbb{R})$ (resp. by $\mathrm{Sym}(n, \mathbb{C})$) the real (resp. complex) vector space of real (resp. complex) symmetric $n \times n$ matrices. These vector spaces are of dimension $n(n+1)/2$ and we equip them with the basis B given by the vectors $\widetilde{E}_{ii} = \sqrt{2}\, E_{ii}$ and $\widetilde{E}_{ij} = E_{ij} + E_{ji}$, $1 \le i < j \le n$, where for every $1 \le k, l \le n$, E_{kl} denotes the elementary matrix whose (i, j) entry equals 1 if $(i, j) = (k, l)$, and 0 otherwise. We then equip $\mathrm{Sym}(n, \mathbb{R})$ (resp. $\mathrm{Sym}(n, \mathbb{C})$) with the scalar (resp. Hermitian) product turning B into an orthonormal basis and we denote by $\| \cdot \|$ the associated norm. We denote by $\mu_\mathbb{R}$ (resp. $\mu_\mathbb{C}$) the associated Gaussian probability measure, so that for every open subset U of $\mathrm{Sym}(n, \mathbb{R})$ (resp. V of $\mathrm{Sym}(n, \mathbb{C})$),

$$\mu_\mathbb{R}(U) = \frac{1}{\sqrt{\pi}^{n(n+1)/2}} \int_U e^{-\|A\|^2} dA \quad \text{and} \quad \mu_\mathbb{C}(V) = \frac{1}{\pi^{n(n+1)/2}} \int_V e^{-\|A\|^2} dA, \tag{2.5}$$

where dA denotes the Lebesgue measure. This classical measure turns $(\mathrm{Sym}(n, \mathbb{R}), \mu_\mathbb{R})$ into the Gaussian orthogonal ensemble $\mathrm{GOE}(n)$ (see [27] for instance).

For every $p, q \in \mathbb{N}$, we denote by $\mathrm{Sym}(p, q, \mathbb{R})$ the open subset of $\mathrm{Sym}(p+q, \mathbb{R})$ consisting of non-degenerate matrices of signature (p, q). We then define, for any integers p, q, n, $e_\mathbb{R}(n)$ and $e_\mathbb{R}(p, q)$ as in (1.3) and (1.4), and

$$e_\mathbb{C}(n) = E_\mathbb{C}(|\det|^2) = \int_{\mathrm{Sym}(n, \mathbb{C})} |\det A|^2 \, d\mu_\mathbb{C}(A). \tag{2.6}$$

We note that $\sum_{p+q=n,\, p,q \in \mathbb{N}} e_\mathbb{R}(p, q) = e_\mathbb{R}(n)$; by convention, $e_\mathbb{R}(0) = e_\mathbb{C}(0) = e_\mathbb{R}(0, 0) = 1$. Multiplication of symmetric matrices by $-\mathrm{Id}$ preserves the measure of $\mathrm{Sym}(n, \mathbb{R})$ as well as the absolute value of the determinant, so that $e_\mathbb{R}(p, q) = e_\mathbb{R}(q, p)$ for all $p, q \in \mathbb{N}$.

2.2. *Hörmander peak sections*

Let (L, h) be a real holomorphic Hermitian line bundle of positive curvature ω over a smooth projective manifold X. We recall in this section the construction of peak sections used by Hörmander [18] to solve the Levi problem. These sections also played a crucial role in the study of the Bergman kernel by Tian [39] or in the construction by Donaldson [9] of symplectic codimension two submanifolds in any closed symplectic manifold. The peak sections are holomorphic sections of L^d whose L^2-norms get localized in the $1/\sqrt{d}$-neighbourhood of some point $x \in X$. The jet at x of such a section can be controlled up to any order. It is then possible to get an asymptotically orthonormal family of sections for the Hermitian product (2.1), each section being associated to some monomial: see Lemma 2.2 and, in the real setting, Lemma 2.3.

With the notation of §2.1, let x be a point of X (resp. $\mathbb{R}X$). In a neighbourhood of x in X there exists a local holomorphic (resp. real holomorphic) trivialization e of L whose associated potential $\Phi = -\log h(e, e)$ vanishes at x, where it reaches a local minimum with Hessian of type $(1, 1)$. Let $(y_1, \dots, y_n)$ be holomorphic (resp. real holomorphic) coordinates in the neighbourhood of $x = (0, \dots, 0)$ in X such that $(\partial/\partial y_1, \dots, \partial/\partial y_n)$ is orthonormal at x for the Kähler metric g. In these coordinates, the Taylor expansion of Φ reads

$$\Phi(y) = -\frac{1}{2i}\partial\bar{\partial}\Phi(y, iy) + o(\|y\|^2) = \pi\|y\|^2 + o(\|y\|^2),$$

where the norm is induced by the Kähler metric g at x.

The L^2-estimates of Hörmander make it possible, for every $d > 0$, and after a small perturbation of e^d in the L^2-norm, to extend e^d to a global holomorphic (resp. real holomorphic) section of L^d. The latter is called a *Hörmander peak section*. G. Tian [39, Lemma 1.2] showed that this procedure can be controlled up to any order, as long as d is large enough. We recall this result in the lemma below, where for every $r > 0$, $B(x, r)$ denotes the ball centred at x and of radius r in X.

Lemma 2.2 (see [39, Lemma 1.2]). *Let (L, h) be a holomorphic Hermitian line bundle of positive curvature ω over a smooth complex projective manifold X. Let $x \in X$ and $(p_1, \dots, p_n) \in \mathbb{N}^n$ and $p' > p_1 + \cdots + p_n$. There exists $d_0 \in \mathbb{N}$ such that for every $d > d_0$, the bundle L^d has a global holomorphic section σ satisfying $\int_X h^d(\sigma, \sigma)\, dy = 1$ for the volume form $dy = (1/\int_X \omega^n)\omega^n$ and*

$$\int_{X \setminus B(x, (\log d)/\sqrt{d})} h^d(\sigma, \sigma)\, dy = O(1/d^{2p'}). \tag{2.7}$$

Moreover, if $(y_1, \dots, y_n)$ are local holomorphic coordinates in a neighbourhood of x, we can assume that in this neighbourhood,

$$\sigma(y_1, \dots, y_n) = \lambda(y_1^{p_1} \cdots y_n^{p_n} + O(|y|^{2p'}))e^d(1 + O(1/d^{2p'})), \tag{2.8}$$

where

$$\lambda^{-2} = \int_{B(x, (\log d)/\sqrt{d})} |y_1^{p_1} \cdots y_n^{p_n}|^2 h^d(e^d, e^d)\, dy$$

and e is a local trivialization of L whose potential $\Phi = -\log h(e, e)$ reaches a local minimum at x with Hessian $\pi\omega(\cdot, i\cdot)$.

This lemma admits a real counterpart, which is obtained by averaging the peak sections with the real structure:

Lemma 2.3. *Let (L, h) be a real holomorphic Hermitian line bundle of positive curvature ω over a smooth real projective manifold X. Let $x \in \mathbb{R}X$ and $(p_1, \dots, p_n) \in \mathbb{N}^n$ and $p' > p_1 + \dots + p_n$. There exists $d_0 \in \mathbb{N}$ such that for every $d > d_0$, the bundle L^d has a global real holomorphic section σ satisfying $\int_X h^d(\sigma, \sigma)\, dy = 1$ for $dy = (1/\int_X \omega^n)\omega^n$ and*

$$\int_{X \setminus B(x, (\log d)/\sqrt{d})} h^d(\sigma, \sigma)\, dy = O(1/d^{2p'}).$$

Moreover, if $(y_1, \dots, y_n)$ are local real holomorphic coordinates in a neighbourhood of x in X, we can assume that in this neighbourhood,

$$\sigma(y_1, \dots, y_n) = \lambda(y_1^{p_1} \cdots y_n^{p_n} + O(|y|^{2p'}))e^d(1 + O(1/d^{2p'})),$$

where

$$\lambda^{-2} = \int_{B(x, (\log d)/\sqrt{d})} |y_1^{p_1} \cdots y_n^{p_n}|^2 h^d(e^d, e^d)\, dy$$

and e is a local real trivialization of L whose potential $\Phi = -\log h(e, e)$ reaches a local minimum at x with Hessian $\pi\omega(\cdot, i\cdot)$.

Definition 2.4. Let σ_0 be a section as in (2.8) with $p' = 3$ and $p_1 = \dots = p_n = 0$. Likewise, for every $j \in \{1, \dots, n\}$, let σ_j be a section as in (2.8) with $p' = 3$, $p_j = 1$ and $p_k = 0$ for $k \in \{1, \dots, n\} \setminus \{j\}$. Finally, for every $1 \leq k \leq l \leq n$, let σ_{kl} be a section as in (2.8) with $p' = 3$, $p_j = 0$ for every $j \in \{1, \dots, n\} \setminus \{k, l\}$ and $p_k = p_l = 1$ if $k \neq l$, while $p_k = 2$ otherwise.

By (2.7), all these sections have norms concentrated in a neighbourhood of x, and are close to 0 outside a ball of radius $(\log d)/\sqrt{d}$ (from the mean value inequality, see [19, Theorem 4.2.13] for instance). Likewise, by (2.8), the Taylor expansions of these sections satisfy

$$\sigma_0(y) = (\lambda_0 + O(\|y\|^6))e^d(y)(1 + O(d^{-6})), \tag{2.9}$$
$$\sigma_j(y) = (\lambda_1 y_j + O(\|y\|^6))e^d(y)(1 + O(d^{-6})), \quad \forall j \in \{1, \dots, n\}, \tag{2.10}$$

and for all $k, l \in \{1, \dots, n\}$, $k \neq l$,

$$\sigma_{kl}(y) = (\lambda_{(1,1)} y_k y_l + O(\|y\|^6))e^d(y)(1 + O(d^{-6})), \tag{2.11}$$
$$\sigma_{kk}(y) = (\lambda_{(2,0)} y_k^2 + O(\|y\|^6))e^d(y)(1 + O(d^{-6})). \tag{2.12}$$

The asymptotic values of the constants λ_0, λ_1, $\lambda_{(1,1)}$ and $\lambda_{(2,0)}$ are given by (compare [39, Lemma 2.1]):

Lemma 2.5. *Under the hypotheses of Lemma 2.2, let $\delta_L = \int_X c_1(L)^n$ be the degree of L. Then*

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}^n}\lambda_0 = \sqrt{\delta_L}, \qquad \lim_{d\to\infty} \frac{1}{\sqrt{d}^{n+1}}\lambda_1 = \sqrt{\pi}\sqrt{\delta_L},$$
$$\lim_{d\to\infty} \frac{1}{\sqrt{d}^{n+2}}\lambda_{(1,1)} = \pi\sqrt{\delta_L}, \qquad \lim_{d\to\infty} \frac{1}{\sqrt{d}^{n+2}}\lambda_{(2,0)} = \frac{\pi}{\sqrt{2}}\sqrt{\delta_L},$$

for the L^2 inner product induced by the volume form $dy = (1/\int_X \omega^n)\omega^n$.

These values differ from the ones given in [39, Lemma 2.1] by a constant π^n since our choice of the Kähler metric slightly differs from that in [39].

Proof. From Lemma 2.2, λ_0^{-2} is equivalent, as $d\to\infty$, to

$$\frac{1}{\int_X \omega^n}\int_{\mathbb{C}^n} e^{-d\pi\|y\|^2}\, d\,\mathrm{vol}(y) = \frac{1}{\int_X d^n\omega^n\pi^n}\int_{\mathbb{C}^n} e^{-\|z\|^2}\, d\,\mathrm{vol}(z) = \frac{1}{\int_X d^n\omega^n},$$

so that $\lambda_0 \sim_{d\to\infty} \sqrt{\int_X c_1(L)^n d^n}$. Likewise, λ_1^{-2} is equivalent to

$$\begin{aligned}\frac{1}{\int_X \omega^n}\int_{\mathbb{C}^n} |y_1|^2 e^{-d\pi\|y\|^2}\, d\,\mathrm{vol}(y) &= \frac{1}{d\pi\int_X d^n\omega^n\pi^n}\int_{\mathbb{C}^n} |z_1|^2 e^{-\|z\|^2}\, d\,\mathrm{vol}(z)\\ &= \frac{1}{d\pi\int_X d^n\omega^n},\end{aligned}$$

so that $\lambda_1 \sim_{d\to\infty} \sqrt{d\pi}\,\lambda_0$. In the same way we obtain $\lambda_{(1,1)} \sim_{d\to\infty} d\pi\lambda_0$, whereas $\lambda_{(2,0)}^{-1}$ is equivalent to

$$\begin{aligned}\frac{1}{\int_X \omega^n}\int_{\mathbb{C}^n} |y_1|^4 e^{-d\pi\|y\|^2}\, d\,\mathrm{vol}(y) &= \frac{1}{(d\pi)^2\int_X d^n\omega^n\pi^n}\int_{\mathbb{C}^n} |z_1|^4 e^{-\|z\|^2}\\ &= \frac{2}{(d\pi)^2\int_X d^n\omega^n}.\end{aligned}$$

Hence, $\lambda_{(2,0)} \sim_{d\to\infty} (d\pi/\sqrt{2})\lambda_0$. □

Let ∇^X be a torsion-free connection on TX and ∇^L be a connection (real, that is, such that for every section σ of L, $\nabla^L(c_L\circ\sigma\circ c_X) = c_L\circ(\nabla^L\sigma)\circ dc_X$) on L. The connections ∇^X and ∇^L induce a connection denoted by $\nabla^{X,L}$ on $T^*X\otimes L$. We then set

$$\nabla^2\sigma = \nabla^{X,L}(\nabla^L\sigma) \in \mathrm{End}(TX, T^*X\otimes L^d).$$

Now, the sections $(\sigma_i)_{0\le i\le n}$ and $(\sigma_{kl})_{1\le k\le l\le n}$ define a basis of a complement of the subspace of sections of $H^0(X, L^d)$ (resp. $\mathbb{R}H^0(X, L^d)$) whose 2-jets at x vanish, which is denoted by

$$H_{3x} = \{\sigma \in H^0(X, L^d) \mid \sigma(x) = 0,\ \nabla^L\sigma_{|x} = 0 \text{ and } \nabla^2\sigma_{|x} = 0\}, \tag{2.13}$$
$$(\text{resp. } \mathbb{R}H_{3x} = \{\sigma \in \mathbb{R}H^0(X, L^d) \mid \sigma(x) = 0,\ \nabla^L\sigma_{|x} = 0 \text{ and } \nabla^2\sigma_{|x} = 0\}). \tag{2.14}$$

This basis is not orthonormal and the subspace it spans is not orthogonal to H_{3x}. However, from [39, Lemma 3.1], it becomes closer and closer to being orthonormal as $d \to \infty$ (see Lemma 2.6), as long as the chosen volume form is $dx = (1/\int_X \omega^n)\omega^n$ (see Remark 2.7(1)).

Lemma 2.6 (see [39, Lemma 3.1]). *The sections $(\sigma_i)_{0\leq i\leq n}$ and $(\sigma_{kl})_{1\leq k\leq l\leq n}$ have L^2-norm equal to one and their pairwise scalar products are dominated by an $O(d^{-1})$ which does not depend on $x \in X$. Likewise, their scalar products with every unitary element of H_{3x} are dominated by an $O(d^{-3/2})$ which does not depend on $x \in X$.*

Remark 2.7. (1) If the L^2 scalar product is induced by a volume form different from $dx = (1/\int_X \omega^n)\omega^n$, say $dx = (f(x)/\int_X \omega^n)\omega^n$, then Lemma 2.6 remains unchanged except that the L^2-norms of the sections, instead of being one, would converge to $\sqrt{f(x)}$ from (2.7).

(2) When (X, ω, c_X) is the projective space $(\mathbb{C}P^n, \omega_{\mathrm{FS}}, \mathrm{conj})$ and (L, h, c_L) is $(\mathcal{O}(1), h_{\mathrm{FS}}, \mathrm{conj})$, then $H^0(X, L^d)$ is the space of homogeneous polynomials of degree d in the variables $(X_0, \dots, X_n)$. The real peak section at $x = [1 : 0 : \cdots : 0]$ is then $\sqrt{(d+n)!/d!}\, X_0^d$. Moreover, for every $(p_1, \dots, p_n) \in \mathbb{N}^n$ with $\sum_{i=1}^n p_i \leq d$, the section σ given by Lemma 2.2 is the monomial

$$\sigma = \sqrt{\frac{(d+n)!}{(d-(p_1+\cdots+p_n))!\,p_1!\cdots p_n!}}\, X_0^{d-(p_1+\cdots+p_n)} X_1^{p_1}\cdots X_n^{p_n}.$$

In particular, this family of monomials is orthonormal for the Fubini–Study Hermitian product. Now, for every $x \in \mathbb{C}P^n$ (resp. $x \in \mathbb{R}P^n$), the action of the unitary group $U_{n+1}(\mathbb{C})$ (resp. the orthogonal group $O_{n+1}(\mathbb{R})$) on sections of L^d provides the holomorphic (resp. real holomorphic) peak sections at every point of $\mathbb{C}P^n$ (resp. $\mathbb{R}P^n$).

2.3. Example: the projective spaces

Let us sketch here the proof of Theorem 1.2 for the projective space, so that (X, ω, c_X) is $(\mathbb{C}P^n, \omega_{\mathrm{FS}}, \mathrm{conj})$ and (L, h_L, c_L) is $(\mathcal{O}(1), h_{\mathrm{FS}}, \mathrm{conj})$. In this case, $\mathbb{R}H^0(X, L^d)$ is the space $\mathbb{R}_d$ of real homogeneous polynomials of degree d in $n+1$ variables. Let $p : \mathbb{R}P^n \to \mathbb{R}$ be a Morse function, and for $i \in \{1, \dots, n\}$, $x_i = X_i/X_0$ be the affine coordinate on $U = \mathbb{R}P^n \setminus \{X_0 = 0\}$. For every $\sigma \in \mathbb{R}_d$, set

$$\nu(\mathbb{R}C_\sigma) = \sum_{x\in \mathrm{Crit}(p_{|\mathbb{R}C_\sigma})} \delta_x$$

(see (1.5)), where $\mathbb{R}C_\sigma = \sigma^{-1}(0) \cap \mathbb{R}P^n$. If $x_0 \in U$ is a critical point of $p_{|\mathbb{R}C_\sigma}$, but not of p, and if $\sigma_0 \in \mathbb{R}_d$ vanishes transversely at x_0, then for every $\sigma \in \mathbb{R}_d$ near σ_0 there is a unique critical point $x \in \mathbb{R}P^n$ of σ near x_0. We denote by $\mathrm{ev}_{(\sigma_0,x_0)}$ the local map which sends σ to x (see (2.21)). Now, let $\chi : \mathbb{R}P^n \to \mathbb{R}$ be a test function. Then

$$\langle E(\nu), \chi\rangle = \frac{1}{\sqrt{d}^n}\int_{\mathbb{R}_d\setminus \Delta_p^d} \sum_{x\in \mathrm{Crit}(p_{|\mathbb{R}C_\sigma})} \chi(x)\, d\mu_{\mathbb{R}}(\sigma),$$

where $\mu_{\mathbb{R}}$ is the measure on $\mathbb{R}_d$ given by (2.3) and where Δ_p^d is defined in §2.1.3. By the coarea formula (see [12] and (2.22)), this equals

$$\frac{1}{\sqrt{d}^n}\int_{\mathbb{R}P^n}\chi(x)\Big(\int_{\sigma\in\mathbb{R}_d\setminus\Delta_p^d,\, x\in\mathrm{Crit}(p_{|\mathbb{R}C_\sigma})}|\det d_{|\sigma}\,\mathrm{ev}^{\perp}_{(\sigma,x)}|^{-1}\,d\mu_{\mathbb{R}}(\sigma)\Big)\,d\,\mathrm{vol}_{\mathrm{FS}}(x), \tag{2.15}$$

where $d\,\mathrm{vol}_{\mathrm{FS}}$ denotes the Lebesgue measure of $\mathbb{R}P^n$ induced by the Fubini–Study metric, and $d_{|\sigma}\,\mathrm{ev}^{\perp}_{(\sigma,x)}$ is the restriction of the differential $d_{|\sigma}\,\mathrm{ev}_{(\sigma,x)}$ to the orthogonal complement of the fibre $\{\sigma\in\mathbb{R}_d\setminus\Delta_p^d \mid x\in\mathrm{Crit}(p_{|\mathbb{R}C_\sigma})\}$ in $\mathbb{R}_d$ (see the beginning of §2.4.2). Now, fix $x_0\in U$ which is not a critical point of p, and a smooth local framing field $(v_2,\dots,v_n)$ of the bundle $\ker dp$. Then x is a critical point of $p_{|\mathbb{R}C_\sigma}$ near x_0 if and only if (σ,x) belongs to the vanishing locus of the map

$$F:(\sigma,x)\in\mathbb{R}_d\times\mathbb{R}P^n\mapsto(s(x),d_{|x}s(v_2),\dots,d_{|x}s(v_n))$$

(see (2.26)), where $s=\sigma/(\sqrt{(d+n)!/d!}\,X_0^d)$ denotes the affine polynomial associated to σ on U (see Remark 2.7(2) for the coefficient of X_0^d). After differentiation we find that for every $(\sigma_0,x_0)\in F^{-1}(0)$,

$$d_{1|(\sigma_0,x_0)}F+d_{2|(\sigma_0,x_0)}F\circ d_{|\sigma_0}\,\mathrm{ev}_{(\sigma_0,x_0)}=0$$

(see (2.27)). Assume that $x_0=[1:0:\cdots:0]$ and expand $\dot\sigma\in\mathbb{R}_d$ in the polynomial basis given by Remark 2.7(2), so that

$$\dot s(x)=\dot a_0+\sqrt{d}\sum_{i=1}^n\dot a_i x_i+O(\|x\|^2).$$

We may assume that $d_{|x_0}p=dx_1$ and $v_i(x_0)=\partial/\partial x_i$ for every $i\in\{2,\dots,n\}$. Then $d_{1|(\sigma_0,x_0)}F(\dot\sigma)=(\dot a_0,\sqrt{d}\,\dot a_2,\dots,\sqrt{d}\,\dot a_n)$, so that $|\det d_{1|(\sigma_0,x_0)}F^{\perp}|=\sqrt{d}^{n-1}$. Now, we expand σ_0 as

$$s_0(x)=a_0+\sqrt{d}\sum_{i=1}^n a_i x_i+d\sum_{i=1}^n\frac{b_{ii}}{\sqrt{2}}x_i^2+d\sum_{1\le i<j\le n}b_{ij}x_ix_j+O(\|x\|^3)$$

(see (2.29)). Note that $F(\sigma_0,x_0)=0$ reads $a_0=a_2=\cdots=a_n=0$ (see (2.30)). Then

$$d_{2|(\sigma_0,x_0)}F=\Big(\sqrt{d}\,a_1dx_1,\,d\sqrt{2}\,b_{22}dx_2+d\sum_{1\le i<2}b_{i2}dx_i+d\sum_{2<j\le n}b_{2j}dx_j+O(\sqrt{d}),$$
$$\dots,d\sqrt{2}\,b_{nn}dx_n+d\sum_{1\le i<n}b_{in}dx_i+O(\sqrt{d})\Big)$$

(see (2.38)). We conclude that $|\det d_{2|(\sigma_0,x_0)}F|\sim_{d\to\infty}\sqrt{d}\,d^{n-1}|a_1|\,|\det B|$ (see (2.44)), where in the notation of §2.1.5, B denotes the real symmetric matrix $\sum_{2\le k\le l\le n}b_{kl}\widetilde{E}_{kl}$. Hence,

$$\int_{\sigma_0\in\mathbb{R}_d,\,F(\sigma_0,x_0)=0}|\det d_{|\sigma_0}\,\mathrm{ev}^{\perp}_{(\sigma_0,x_0)}|^{-1}\,d\mu(\sigma_0)\underset{d\to\infty}{\sim}\frac{\sqrt{d}^n}{\sqrt{\pi}}e_{\mathbb{R}}(n-1)$$

(see (1.4)), since $\int_{a_1\in\mathbb{R}} |a_1| e^{-a_1^2} da_1/\sqrt{\pi} = 1/\sqrt{\pi}$. By Remark 2.7(2), for every $x_0 \in \mathbb{R}P^n$ which is not a critical point of p, we can use the orthogonal group $O_{n+1}(\mathbb{R})$ to get an orthonormal basis of $\mathbb{R}_d$ adapted to x_0, so that the integral $\int_{\sigma_0\in\mathbb{R}_d,\, F(\sigma_0,x_0)=0} |\det d_{|\sigma_0} \operatorname{ev}^{\perp}_{(\sigma_0,x_0)}|^{-1} d\mu_{\mathbb{R}}(\sigma_0)$ is still equivalent to $(\sqrt{d}^n/\sqrt{\pi}) e_{\mathbb{R}}(n-1)$ as $d \to \infty$. If we consider the index of the critical points, then instead of integrating over the whole $\operatorname{Sym}(n-1,\mathbb{R})$, we integrate over the real symmetric matrices of given index (see Lemma 2.9), so that for every $i \in \{0,\dots,n-1\}$,

$$\int_{\sigma\in\mathbb{R}_d,\, x\in \operatorname{Crit}_i(p_{|\mathbb{R}C_\sigma})} |\det d_{|\sigma} \operatorname{ev}^{\perp}_{(\sigma,x)}|^{-1} d\mu_{\mathbb{R}}(\sigma) \underset{d\to\infty}{\sim} \frac{\sqrt{d}^n}{\sqrt{\pi}} e_{\mathbb{R}}(i, n-1-i)$$

(see (1.3)). The convergence of these integrals is uniform in x on every compact subset of the complement of the critical points, so that $E(\nu_i)$ converges weakly to $(1/\sqrt{\pi})e(i, n-1-i) d\,\mathrm{vol}_{FS}$ on this compact subset. A local analysis near the critical points is handled in §2.5.1.

2.4. *Incidence varieties and evaluation maps*

2.4.1. Incidence varieties. Using the notation of §2.1.3, let us denote by $\operatorname{Crit}(p)$ (resp. $\operatorname{Crit}_i(p)$) the finite set of critical points of p (resp. of index i) and by $\operatorname{Base}(p)$ its base locus (see Definition 2.1). Under the hypotheses of Theorem 1.3 (resp. Theorem 1.2) and following [36], we set

$$\Sigma = \{(\sigma, x) \in (H^0(X, L^d) \setminus \Delta_p^d) \times (X \setminus (\operatorname{Crit}(p) \cup \operatorname{Base}(p)) \mid x \in \operatorname{Crit}(p_{|C_\sigma})\}, \tag{2.16}$$

$$\text{(resp. } \Sigma_i = \{(\sigma, x) \in (\mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_p^d) \times (\mathbb{R}X \setminus \operatorname{Crit}(p)) \mid x \in \operatorname{Crit}_i(p_{|\mathbb{R}C_\sigma})\}), \tag{2.17}$$

and

$$\pi_1 : (\sigma, x) \in \Sigma \mapsto \sigma \in H^0(X, L^d) \text{ and } \pi_2 : (\sigma, x) \in \Sigma \mapsto x \in X \tag{2.18}$$

$$\text{(resp. } \pi_1 : (\sigma, x) \in \Sigma_i \mapsto \sigma \in \mathbb{R}H^0(X, L^d) \text{ and } \pi_2 : (\sigma, x) \in \Sigma_i \mapsto x \in \mathbb{R}X) \tag{2.19}$$

the associated projections on these incidence varieties.

Lemma 2.8. *Under the hypotheses of Theorem 1.3 (resp. Theorem 1.2), let $d > 0$ be such that L^d is very ample. Then Σ (resp. Σ_i) is a codimension n submanifold of $H^0(X, L^d) \times X$ (resp. $\mathbb{R}H^0(X, L^d) \times \mathbb{R}X$). Moreover, the critical locus of π_1 coincides with Δ_p^d (resp. $\mathbb{R}\Delta_p^d$)). In particular, Δ_p^d and $\mathbb{R}\Delta_p^d$ are of measure zero in $H^0(X, L^d)$ and $\mathbb{R}H^0(X, L^d)$ respectively.*

Proof. Let us prove the complex case. Let d be such that L^d is very ample and

$$F : (\sigma, y) \in H^0(X, L^d) \times (X \setminus (\operatorname{Crit}(p) \cup \operatorname{Base}(p))) \mapsto (\sigma(y), \nabla^L \sigma_y) \in L^d_y \times (K^*_y \otimes L^d_y),$$

where $K_y = \ker dp_{|y} \subset T_y X$. Then Σ defined by (2.16) coincides with the vanishing locus of F. Let $(\sigma, y) \in \Sigma$. Then

$$\begin{aligned} \nabla_{|(\sigma,y)} F : H^0(X, L^d) \times T_y X &\to L^d_y \times (K^*_y \otimes L^d_y), \\ (\dot\sigma, \dot y) &\mapsto (\dot\sigma(y) + \nabla^L_{\dot y}\sigma, \nabla^L \dot\sigma + \nabla^{X,L}_{\dot y} \nabla^L \sigma). \end{aligned}$$

Since L^d is very ample, $\nabla_{|(\sigma,y)}F$ is onto and even its restriction to $H^0(X, L^d) \times \{0\}$ is. From the implicit function theorem it follows that Σ is a codimension n submanifold of $H^0(X, L^d) \times X$. Moreover, for every $(\sigma, y) \in \Sigma$,

$$T_{(\sigma,y)}\Sigma = \left\{ (\dot\sigma, \dot y) \in H^0(X, L^d) \times T_y X \;\middle|\; \begin{array}{l} \dot\sigma(y) + \nabla^L_{\dot y}\sigma = 0, \\ \nabla^L \dot\sigma + \nabla^{X,L}_{\dot y} \nabla^L \sigma = 0 \end{array} \right\}.$$

The differential of the first projection π_1 reads

$$d\pi_{1|(\sigma,y)} : T_{(\sigma,y)}\Sigma \to H^0(X, L^d), \qquad (\dot\sigma, \dot y) \mapsto \dot\sigma.$$

It is then onto if and only if for every $\dot\sigma \in H^0(X, L^d)$ there exists $\dot y \in T_y X$ such that $\nabla_{\dot y}\sigma = -\dot\sigma(y)$ and $\nabla^{X,L}_{\dot y}\nabla^L\sigma = -\nabla^L\dot\sigma$. The first condition is satisfied if and only if σ vanishes transversely, thus if σ does not belong to the discriminant hypersurface $\Delta^d \subset H^0(X, L^d)$. The second condition is satisfied if and only if the restriction of the bilinear form $\nabla^2\sigma$ to $K_y \times K_y$ is non-degenerate. From Lemma 2.9, this holds whenever σ does not belong to Δ^d_p. The last part of the lemma now follows from Sard's theorem. The proof in the real case goes along the same lines. $\square$

Lemma 2.9. *Under the hypotheses of Theorem 1.2, let $(\sigma, x) \in \Sigma_i$. Let $\phi_x : \mathbb{R}L_x \to \mathbb{R}$ be an isomorphism such that $\phi_x \circ \nabla^L_{|x}\sigma = -dp_x$. Then $\phi_x \circ \nabla^2\sigma_{|K_x} = \nabla^2(p_{|\mathbb{R}C_\sigma})_{|x}$, so that the quadratic form $\phi_x \circ \nabla^2\sigma_{|K_x}$ is non-degenerate of index i.*

Proof. Let v, w be vector fields on $\mathbb{R}C_\sigma$ in a neighbourhood of x. By definition, $0 = \nabla^L_v(\nabla^L_w\sigma) = \nabla^2_{v,w}\sigma + \nabla^L_{\nabla^X_v w}\sigma$, so that

$$\phi_x \circ \nabla^2_{v,w}\sigma = -\phi_x \circ \nabla^L_{\nabla^X_v w}\sigma = dp_{|x}(\nabla^X_v w).$$

Applying the same equality to the function p, we get

$$d_v(d_w p) = \nabla^X(dp)(v, w) + dp(\nabla^X_v w) = dp(\nabla^X_v w)$$

by hypothesis on ∇^X, so that $\phi_x \circ \nabla^2_{v,w}\sigma = d_v(d_w p)$. Finally, applying this equality to the restriction $p_{|\mathbb{R}C_\sigma}$, we get

$$d_v(d_w p)_{|x} = \nabla^2_{v,w} p_{|\mathbb{R}C_\sigma|x} + dp_{|\mathbb{R}C_\sigma}(\nabla^{C_\sigma}_v w)_{|x} = \nabla^2_{v,w}(p_{|\mathbb{R}C_\sigma})_{|x},$$

where ∇^{C_σ} denotes any connection on $T\mathbb{R}C_\sigma$. Hence the result. $\square$

2.4.2. Evaluation maps. For every $(\sigma_0, x_0) \in \Sigma$ (resp. $(\sigma_0, x_0) \in \Sigma_i$), there exists a neighbourhood U (resp. $\mathbb{R}U$) of σ_0 in $H^0(X, L^d)$ (resp. $\mathbb{R}H^0(X, L^d)$) and a neighbourhood V (resp. $\mathbb{R}V$) of x_0 in X (resp. $\mathbb{R}X$) such that for every $\sigma \in U$ (resp. $\sigma \in \mathbb{R}U$), the function $p_{|C_\sigma}$ (resp. $p_{|\mathbb{R}C_\sigma}$) has a unique critical point (resp. critical point of index i) in V (resp. $\mathbb{R}V$). We deduce from this an evaluation map at the critical point

$$\mathrm{ev}_{(\sigma_0,x_0)} : \sigma \in U \mapsto x \in \mathrm{Crit}(p_{|C_\sigma}) \cap V \tag{2.20}$$

$$(\text{resp. } \mathrm{ev}_{(\sigma_0,x_0)} : \sigma \in \mathbb{R}U \mapsto x \in \mathrm{Crit}(p_{|\mathbb{R}C_\sigma}) \cap \mathbb{R}V), \tag{2.21}$$

so that $\Sigma \cap (U \times V)$ (resp. $\Sigma_i \cap (\mathbb{R}U \times \mathbb{R}V)$) is the graph of $\mathrm{ev}_{(\sigma_0,x_0)}$. This evaluation map is constant on $\pi_1(\pi_2^{-1}(x_0)) \cap U$, where π_1 and π_2 are defined by (2.18), so that its differential $d_{|\sigma_0} \operatorname{ev}_{(\sigma_0,x_0)}$ vanishes on $T_{\sigma_0}\pi_1(\pi_2^{-1}(x_0)) \simeq \pi_1(\pi_2^{-1}(x_0))$. When $n = 1$, we agree that

$$\pi_1(\pi_2^{-1}(x)) = \{\sigma \in H^0(X, L^d) \mid \sigma(x) = 0\}.$$

We denote by $d_{|\sigma_0} \operatorname{ev}^{\perp}_{(\sigma_0,x_0)}$ the restriction of $d_{|\sigma_0} \operatorname{ev}_{(\sigma_0,x_0)}$ to the orthogonal complement of $\pi_1(\pi_2^{-1}(x_0))$ in $H^0(X, L^d)$ (resp. $\mathbb{R}H^0(X, L^d)$).

Proposition 2.10. *Under the hypotheses of Theorem 1.3 (resp. Theorem 1.2),*

$$E(\nu) = \frac{1}{d^n}(\pi_2)_*(\pi_1^* d\mu_{\mathbb{C}}) \quad (\textit{resp. } E(\nu_i) = \frac{1}{\sqrt{d}^n}(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}})),$$

where $E(\nu)$ (resp. $E(\nu_i)$) is defined by (1.7) (resp. (1.6)). Moreover, at every point x of $X \setminus (\operatorname{Crit}(p) \cup \operatorname{Base}(p))$ (resp. $\mathbb{R}X \setminus \operatorname{Crit}(p)$),

$$(\pi_2)_*(\pi_1^* d\mu_{\mathbb{C}}) = \frac{1}{\pi^n}\left(\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} d\mu_{\mathbb{C}}(\sigma)\right)\frac{\omega^n}{n!}$$

$$(\textit{resp. } (\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}}) = \frac{1}{\sqrt{\pi}^n}\left(\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1} d\mu_{\mathbb{R}}(\sigma)\right) d\operatorname{vol}_h).$$

Note that π_1 is a map between manifolds of the same dimension, while $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ are absolute values of volume forms, so that the pull-backs $\pi_1^* d\mu_{\mathbb{R}}$ and $\pi_1^* d\mu_{\mathbb{C}}$ are well defined.

Proof. Let $\chi : X \to \mathbb{R}$ be a continuous function. By definition,

$$\begin{aligned}\langle E(\nu), \chi\rangle &= \frac{1}{d^n}\int_{H^0(X,L^d)\setminus\Delta_p^d}\Big(\sum_{x\in\operatorname{Crit}(p_{|C_\sigma})}\chi(x)\Big) d\mu_{\mathbb{C}}(\sigma)\\ &= \frac{1}{d^n}\int_{\Sigma}(\pi_2^*\chi)(\pi_1^* d\mu_{\mathbb{C}}) = \frac{1}{d^n}\int_X \chi\, (\pi_2)_*(\pi_1^* d\mu_{\mathbb{C}}).\end{aligned}$$

But from the coarea formula (see [12, Theorem 3.2.3] or [36, Theorem 1]), for every $x \in X \setminus (\operatorname{Crit}(p) \cup \operatorname{Base}(p))$,

$$(\pi_2)_*(\pi_1^* d\mu_{\mathbb{C}})_{|x} = \frac{1}{\pi^{N_d}}\left(\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} e^{-\|\sigma\|^2}\, d\sigma\right)\frac{\omega^n}{n!},$$

since the Jacobian of $d_{|\sigma}\operatorname{ev}^{\perp}_{(\sigma,x)}$, which is $\mathbb{C}$-linear and computed with respect to the volume forms $d\sigma$ at the source and $\omega^n/n!$ at the target, equals $|\det d_{|\sigma}\operatorname{ev}^{\perp}_{(\sigma,x)}|^2$. Hence

$$(\pi_2)_*(\pi_1^* d\mu_{\mathbb{C}})_{|x} = \frac{1}{\pi^n}\left(\int_{\pi_1(\pi_2^{-1}(x))} |\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-2} d\mu_{\mathbb{C}}(\sigma)\right)\frac{\omega^n}{n!}_{|x}.$$

Under the hypotheses of Theorem 1.2, if χ now denotes a continuous function χ : $\mathbb{R}X \to \mathbb{R}$, we obtain likewise

$$\langle E(\nu_i), \chi\rangle = \frac{1}{\sqrt{d}^n}\int_{\Sigma_i}(\pi_2^*\chi)(\pi_1^* d\mu_{\mathbb{R}}) = \frac{1}{\sqrt{d}^n}\int_{\mathbb{R}X}\chi(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}}).$$

The coarea formula implies now for every $x \in \mathbb{R}X \setminus \mathrm{Crit}(p)$ the relation

$$(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}})_{|x} = \frac{1}{\sqrt{\pi}^n}\left(\int_{\pi_1(\pi_2^{-1}(x))}|\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|^{-1}\, d\mu_{\mathbb{R}}(\sigma)\right) d\,\mathrm{vol}_{h|x}\,. \tag{2.22}$$

□

Remark 2.11. The pointwise expression for $(\pi_2)_*(\pi_1^* d\mu_{\mathbb{C}})$ (resp. $(\pi_2)_*(\pi_1^* d\mu_{\mathbb{R}})$) is invariant under dilation of the L^2 inner product $\langle\cdot,\cdot\rangle$ on $H^0(X, L^d)$ (resp. on $\mathbb{R}H^0(X, L^d)$). Indeed, for every $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{R}$), (σ_0, x_0) in Σ (resp. in Σ_i) and σ in a neighbourhood of σ_0, $\mathrm{ev}_{(\sigma_0,x_0)}(\sigma) = \mathrm{ev}_{(\lambda\sigma_0,x_0)}(\lambda\sigma)$ so that $d_{|\sigma_0}\mathrm{ev}_{(\sigma_0,x_0)} = \lambda d_{|\lambda\sigma_0}\mathrm{ev}_{(\lambda\sigma_0,x_0)}$. We deduce that

$$\det d_{|\sigma_0}\mathrm{ev}^{\perp}_{(\sigma_0,x_0)} = \lambda^n \det d_{|\lambda\sigma_0}\mathrm{ev}^{\perp}_{(\lambda\sigma_0,x_0)}$$

if both determinants are computed in the orthonormal basis for the same inner product $\langle\cdot,\cdot\rangle$ at the source; but $\det d_{|\sigma_0}\mathrm{ev}^{\perp}_{(\sigma_0,x_0)}$ becomes equal to $\det d_{|\lambda\sigma_0}\mathrm{ev}^{\perp}_{(\lambda\sigma_0,x_0)}$ when the latter is computed in an orthonormal basis for the inner product dilated by λ^2. Since under such a dilation the associated Gaussian measures are just push-forwards of each other by the corresponding homothety, the invariance follows.

Combined with Remark 2.7(1), Remark 2.11 explains why Theorems 1.1, 1.2 and 1.3 do not depend on the choice of a normalised volume form dx on X to define the L^2-product (2.1) on $H^0(X, L^d)$ and $\mathbb{R}H^0(X, L^d)$.

2.4.3. Computation of the Jacobian. We are going to compute the Jacobian $|\det d_{|\sigma} ev^{\perp}_{(\sigma,x)}|$ appearing in Proposition 2.10. For every $x \in X \setminus (\mathrm{Crit}(p) \cup \mathrm{Base}(p))$ (resp. $x \in \mathbb{R}X \setminus \mathrm{Crit}(p)$), we denote by

$$K_x = \ker d_x p \subset T_x X$$
$$(\text{resp. } \mathbb{R}K_x = \ker d_x p \subset T_x\mathbb{R}X)$$

the kernel of $d_x p$ and set

$$H_x = \{\sigma \in H^0(X, L^d) \mid \sigma(x) = 0\} \tag{2.23}$$
$$(\text{resp. } \mathbb{R}H_x = \{\sigma \in \mathbb{R}H^0(X, L^d) \mid \sigma(x) = 0\}). \tag{2.24}$$

We now assume that the torsion-free connection ∇^X preserves the distribution K on $X \setminus (\mathrm{Crit}(p) \cup \mathrm{Base}(p))$. This means that for every local vector field v of X taking values in K, $\nabla^X v$ also has values in K. For every $(\sigma, x) \in \Sigma$, we set

$$\lambda'_{(\sigma,x)} = \frac{\nabla^L\sigma}{\dot\sigma}\dot\sigma \in \mathrm{End}(T_xX/K_x, H^0(X, L^d)/H_x),$$

where $\dot\sigma$ denotes any non-trivial element of $H^0(X, L^d)/H_x$. We consider $\nabla^2\sigma$ as a bilinear form on K_x, that is, $\nabla^2\sigma \in \mathrm{End}(K_x, K_x^* \otimes L_x^d)$. Hence,

$$\det(\nabla^2\sigma) \in \mathrm{End}(\textstyle\bigwedge^{n-1} K_x, \bigwedge^{n-1} K_x^* \otimes L_x^{d(n-1)}).$$

Define also the bilinear form

$$\nabla^L : (v, \dot\sigma) \in K_x \times H_x/\pi_1(\pi_2^{-1}(x)) \mapsto \nabla_v^L \dot\sigma \in L_x^d$$

which we consider as an element of $\mathrm{End}(K_x, (H_x/\pi_1(\pi_2^{-1}(x)))^* \otimes L_x^d)$ and denote abusively by ∇^L. It follows that

$$\det(\nabla^L) \in \mathrm{End}\big(\textstyle\bigwedge^{n-1} K_x, \bigwedge^{n-1}(H_x/\pi_1(\pi_2^{-1}(x)))^* \otimes L_x^{d(n-1)}\big)$$

and we set

$$\lambda''_{(\sigma,x)} = \frac{\det(\nabla^2\sigma)}{\det(\nabla^L)} \in \mathrm{End}\big(\textstyle\bigwedge^{n-1} K_x, \bigwedge^{n-1}(H_x/\pi_1(\pi_2^{-1}(x)))\big).$$

Finally, we set

$$\lambda_{(\sigma,x)} = \lambda'_{(\sigma,x)} \wedge \lambda''_{(\sigma,x)} \in \mathrm{End}\big(\textstyle\bigwedge^n T_x X, \bigwedge^n(H^0(X, L^d)/\pi_1(\pi_2^{-1}(x)))\big) \tag{2.25}$$

when $n > 1$ and $\lambda_{(\sigma,x)} = \lambda'_{(\sigma,x)}$ when $n = 1$.

In the real case, ∇^X denotes a torsion-free connection on $T\mathbb{R}X_{|\mathbb{R}X\setminus \mathrm{Crit}(p)}$ which preserves the distribution $\mathbb{R}K$, while ∇^L is real. For every $(\sigma, x) \in \Sigma_i$, $\lambda'_{(\sigma,x)}$ then belongs to $\mathrm{End}(T_x\mathbb{R}X/\mathbb{R}K_x, \mathbb{R}H^0(X, L^d)/\mathbb{R}H_x)$ and $\nabla^2\sigma$ to $\mathrm{End}(\mathbb{R}K_x, \mathbb{R}K_x^* \otimes \mathbb{R}L_x^d)$, so that

$$\det(\nabla^2) \in \mathrm{End}(\textstyle\bigwedge^{n-1} \mathbb{R}K_x, \bigwedge^{n-1} \mathbb{R}K_x^* \otimes \mathbb{R}L_x^{d(n-1)}).$$

The bilinear form

$$(v, \dot\sigma) \in \mathbb{R}K_x \times \mathbb{R}H_x/\pi_1(\pi_2^{-1}(x)) \mapsto \nabla_v^L \dot\sigma \in \mathbb{R}L_x^d$$

is considered as an element of $\mathrm{End}(\mathbb{R}K_x, (\mathbb{R}H_x/\pi_1(\pi_2^{-1}(x)))^* \otimes \mathbb{R}L_x^d)$, so that

$$\det(\nabla^L) \in \mathrm{End}\big(\textstyle\bigwedge^{n-1} \mathbb{R}K_x, \bigwedge^{n-1}(\mathbb{R}H_x/\pi_1(\pi_2^{-1}(x)))^* \otimes \mathbb{R}L_x^{d(n-1)}\big).$$

Finally,

$$\lambda''_{(\sigma,x)} \in \mathrm{End}\big(\textstyle\bigwedge^{n-1} \mathbb{R}K_x, \bigwedge^{n-1}(\mathbb{R}H_x/\pi_1(\pi_2^{-1}(x)))\big),$$
$$\text{while } \lambda_{(\sigma,x)} \in \mathrm{End}\big(\textstyle\bigwedge^n T_x\mathbb{R}X, \bigwedge^n(\mathbb{R}H^0(X, L^d)/\pi_1(\pi_2^{-1}(x)))\big)$$

when $n > 1$, and $\lambda_{(\sigma,x)} = \lambda'_{(\sigma,x)}$ when $n = 1$.

Proposition 2.12. *Under the hypotheses of Theorem 1.3 (resp. Theorem 1.2), let* $(\sigma_0, x_0) \in \Sigma$ *(resp.* $(\sigma_0, x_0) \in \Sigma_i$*). Then*

$$\det(d_{|\sigma_0}\mathrm{ev}^{\perp}_{(\sigma_0,x_0)})^{-1} = (-1)^n \lambda_{(\sigma_0,x_0)},$$

where $\lambda_{(\sigma_0,x_0)}$ *is given by* (2.25).

Proof. Consider neighbourhoods U and V of σ_0 and x_0 respectively such that the evaluation map $\mathrm{ev}_{(\sigma_0,x_0)} : U \to V$ is well defined (see (2.20)). Under the hypotheses of Theorem 1.3, $\Sigma \cap (U \times V)$ is the vanishing locus of the map

$$F : (\sigma, y) \in U \times V \mapsto (\sigma(y), \nabla^L \sigma_{|y}) \in L_y^d \times (K_y^* \otimes L_y^d), \tag{2.26}$$

where we recall that $K_y = \ker dp_{|y}$. It follows that for every $\sigma \in U$, $F(\sigma, \mathrm{ev}_{(\sigma_0,x_0)}(\sigma)) = 0$. By hypothesis ∇^X restricts to a connection on the subbundle K^*. Hence, $\nabla^{X,L}$ restricts to a connection on $K^* \otimes L^d$, denoted by $D_2^{X,L}$. The latter makes it possible to differentiate F with respect to the second variable. After differentiation we deduce that

$$d_1 F_{|(\sigma_0,x_0)} + D_2^{X,L} F_{|(\sigma_0,x_0)} \circ d_{|\sigma_0} \mathrm{ev}_{(\sigma_0,x_0)} = 0, \tag{2.27}$$

where $d_1 F$ and $D_2^{X,L}$ denote the partial derivatives of F with respect to the first and second variables respectively. Hence the relation

$$d_{|\sigma_0} ev_{(\sigma_0,x_0)} = -(D_2^{X,L} F)^{-1}_{|(\sigma_0,x_0)} \circ d_1 F_{|(\sigma_0,x_0)}.$$

But the matrix of $D_2^{X,L} F \in \mathrm{End}(K_{x_0}^\perp \oplus K_{x_0}, L_{x_0}^d \oplus (K_{x_0}^* \otimes L_{x_0}^d))$ at the point (σ_0, x_0) is trigonal of the form $\begin{pmatrix} \nabla^L\sigma_0 & 0 \\ * & \nabla^2\sigma_0 \end{pmatrix}$, so that

$$\det D_2^{X,L} F_{|(\sigma_0,x_0)} = \nabla^L \sigma_0 \wedge \det(\nabla^2 \sigma_0).$$

Likewise, let $\dot\sigma_0$ be a Bergman section at x_0, that is, a unitary vector in the orthogonal complement of H_{x_0} (see (2.23)) in $H^0(X, L^d)$. The restriction of

$$d_1 F \in \mathrm{End}\big(\langle \dot\sigma_0 \rangle \oplus H_{x_0}/\pi_1(\pi_2^{-1}(x_0)), L_{x_0}^d \oplus (K_{x_0}^* \otimes L_{x_0}^d)\big)$$

at the point (σ_0, x_0) to the orthogonal complement of $\pi_1(\pi_2^{-1}(x_0))$ in $H^0(X, L^d)$ has matrix $\begin{pmatrix} \dot\sigma_0(x_0) & 0 \\ * & \nabla^L \end{pmatrix}$, so that

$$\det d_1 F_{|(\sigma_0,x_0)} = \dot\sigma_0(x_0) \det(\nabla^L).$$

Taking the quotient, we deduce the result under the hypotheses of Theorem 1.3. The proof goes along the same lines under the hypotheses of Theorem 1.2. □

2.5. Proofs of Theorems 1.1–1.3

The proofs of Theorems 1.2 and 1.3 are mostly based on Proposition 2.13 below which computes asymptotically the Jacobian given by Proposition 2.12. This yields the push-forward measure given by Proposition 2.10. This asymptotic computation is carried out using the peak sections of Hörmander introduced in §2.2.

Proposition 2.13. *Under the hypotheses of Theorem 1.3,*

$$\lim_{d\to\infty} \frac{1}{d^n} \int_{\pi_1(\pi_2^{-1}(x))} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}}\, d\mu_{\mathbb{C}}(\sigma) = \pi^n e_{\mathbb{C}}(n-1)\, d\,\mathrm{vol}_h, \tag{2.28}$$

whatever normalized volume form dx on X is chosen to define $d\mu_{\mathbb{C}}$ (see §2.1.2), and where the convergence is dominated by a function in $L^1(X, d\,\mathrm{vol}_h)$.

Likewise, under the hypotheses of Theorem 1.2,

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\lambda_{(\sigma,x)}|\, d\mu_{\mathbb{R}}(\sigma) = \sqrt{\pi}^{n-1} e_{\mathbb{R}}(i, n-1-i) d\,\mathrm{vol}_h,$$

whatever normalized volume form dx on X is chosen, where the convergence is dominated by a function in $L^1(\mathbb{R}X, d\,\mathrm{vol}_h)$.

In Proposition 2.13, the dominating function in $L^1(X, d\,\mathrm{vol}_h)$ (resp. $L^1(\mathbb{R}X, d\,\mathrm{vol}_h)$) has a pole of order at most $2n-2$ (resp. $n-1$) along $\mathrm{Crit}(p)$ and at most 2 along $\mathrm{Base}(p)$, whereas it is continuous everywhere else. Here, a function f is said to have a *pole of order at most k along a submanifold Y* if $r^k|f|$ is bounded near Y, where r denotes the distance function to Y.

2.5.1. Proof of Proposition 2.13 in the complex case. We begin by computing the various derivatives of the random section $\sigma \in H^0(X, L^d)$ involved in the integral (2.28). We are then able to estimate the integrand in (2.28) and to conclude the proof, the domination of the integrand by an L^1 function being a consequence of the last paragraph devoted to the study of the integral near the critical and base points.

Let $x \in X \setminus (\mathrm{Crit}(p) \cup \mathrm{Base}(p))$ (resp. $x \in \mathbb{R}X \setminus \mathrm{Crit}(p)$) and $(x_1, \dots, x_n)$ be local holomorphic (resp. real holomorphic) coordinates in the neighbourhood of $x = (0, \dots, 0)$ such that $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ is orthonormal at x and $(\partial/\partial x_2, \dots, \partial/\partial x_n)$ spans $K_x = \ker dp_{|x}$.

The derivatives of σ. Every element $\sigma \in H^0(X, L^d)$ (resp. $\sigma \in \mathbb{R}H^0(X, L^d)$) can be written, in the notation (2.9)–(2.12),

$$\sigma = \sum_{j=0}^{n} a_j \sigma_j + \sum_{1\le k\le l\le n} b_{kl}\sigma_{kl} + \tau, \tag{2.29}$$

where $a_j, b_{kl} \in \mathbb{C}$ (resp. $a_j, b_{kl} \in \mathbb{R}$) and $\tau \in H_{3x}$ (resp. $\tau \in \mathbb{R}H_{3x}$)—see (2.13) (resp. (2.14)). In the previous equality,

$$\sigma \in \pi_1(\pi_2^{-1}(x)) \;\Leftrightarrow\; \forall j \in \{0, \dots, n\} \setminus \{1\},\ a_j = 0, \tag{2.30}$$

and we assume that this condition holds true. Moreover, from Lemmas 2.2 and 2.5,

$$\sigma_0 = \lambda_0 e^d(x)(1 + O(d^{-6})), \tag{2.31}$$

$$\nabla^L \sigma_{j|x} = \sqrt{\pi d}\, \lambda_0 e^d(x)(1 + O(d^{-6}))dx_j, \tag{2.32}$$

$$\nabla^2 \sigma_{jj|x} = \frac{\pi d \lambda_0}{\sqrt{2}} e^d(x)(1 + O(d^{-6}))(2dx_j \otimes dx_j), \tag{2.33}$$

for all $j \in \{1, \dots, n\}$, while for $1 \le k < l \le n$,

$$\nabla^L \sigma_{kl|x} = 0, \tag{2.34}$$

$$\nabla^2 \sigma_{kl|x} = \pi d \lambda_0 e^d(x)(1 + O(d^{-6}))(dx_k \otimes dx_l + dx_l \otimes dx_k). \tag{2.35}$$

These equations do not depend on the chosen connections ∇^L, ∇^X. It follows from (2.32) that

$$\nabla^2 \sigma_{1|K_x} = \sqrt{\pi d}\, \lambda_0 e^d(x)(1 + O(d^{-6}))\nabla^X(dx_1) \tag{2.36}$$

since by hypothesis, the restriction of dx_1 to K_x vanishes. Likewise,

$$\frac{1}{\pi d \lambda_0} \frac{\nabla^2 \sigma_{|K_x}}{e^d(x)} = \sum_{j=2}^n \frac{b_{jj}}{\pi d \lambda_0} \frac{\nabla^2 \sigma_{jj|K_x}}{e^d(x)} + \sum_{2 \le k < l \le n} \frac{b_{kl}}{\pi d \lambda_0} \frac{\nabla^2 \sigma_{kl|K_x}}{e^d(x)} + \frac{a_1}{\pi d \lambda_0} \frac{\nabla^2 \sigma_{1|K_x}}{e^d(x)}, \tag{2.37}$$

so that this restriction reads

$$\sum_{j=2}^n \sqrt{2}\, b_{jj} dx_j \otimes dx_j + \sum_{2 \le k < l \le n} b_{kl}(dx_k \otimes dx_l + dx_l \otimes dx_k) + a_1 r(d,x) + s(d,x,B), \tag{2.38}$$

where, using the notation of §2.1.5, B denotes the matrix $\sum_{2 \le k \le l \le n} b_{kl} \widetilde{E}_{kl}$, while $\|r(d,x)\| = O(d^{-1/2}\|\nabla^X dx_1\|)$ and $\|s(d,x,B)\| = O(d^{-6}\|B\|)$. In particular, the norm $\|a_1 r(d,x) + s(d,x,B)\|$ is dominated by $c(|a_1|\|\nabla^X dx_1\| + \|B\|)$ for some positive constant c.

The estimation of the integrand in (2.28). Let us first assume that $dx = (1/\int_X \omega^n)\omega^n$, so that from Lemma 2.6, the sections $(\sigma_j)_{0 \le j \le n}$ and $(\sigma_{kl})_{0 \le k \le l \le n}$ are asymptotically orthonormal. We deduce from (2.32) and (2.38) that pointwise on $X \setminus (\mathrm{Crit}(p) \cup \mathrm{Base}(p))$,

$$\frac{1}{\sqrt{\pi d}^{\,n-1}} \left| \frac{\det(\nabla^2 \sigma_{|K_x})}{\det(\nabla^L_{\partial/\partial x_k} \sigma_{l|x})_{2 \le k \le l \le n}} \right| = |\det B + P_{n-1}(B, a_1 r(d,x) + s(d,x,B))|\, |dx_2 \wedge \cdots \wedge dx_n|, \tag{2.39}$$

where $P_{n-1}(X,Y)$ is a polynomial of degree less than n in the coefficients of the matrices X, Y such that $P_{n-1}(X,0) = 0$. Moreover by (2.31) and (2.32),

$$\frac{1}{\sqrt{\pi d}} \frac{\nabla^L \sigma}{\sigma_0(x)} = a_1(1 + O(d^{-6}))dx_1,$$

so that from (2.25),

$$\frac{1}{(\pi d)^n} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}} = |a_1(1 + O(d^{-6}))|^2 |\det B + P_{n-1}(B, a_1 r(d,x) + s(d,x,B))|^2 d\mathrm{vol}_{h|x}, \tag{2.40}$$

which converges to $|a_1|^2|\det B|^2 d\,\mathrm{vol}_{h|x}$ as d grows to infinity, where the convergence is dominated by the limit density plus $|a_1|^2$ times a polynomial function in $|a_1|\,\|\nabla^X dx_1\|$ and $\|B\|$ of degree less than $2n-1$.

The integration space in (2.28). Now, we decompose $\pi_1(\pi_2^{-1}(x))$ (see (2.18)) as

$$\pi_1(\pi_2^{-1}(x)) = (H_{3x} \cap \pi_1(\pi_2^{-1}(x))) \oplus (H_{3x} \cap \pi_1(\pi_2^{-1}(x)))^{\perp}$$
$$(\text{resp. } \pi_1(\pi_2^{-1}(x)) = (\mathbb{R}H_{3x} \cap \pi_1(\pi_2^{-1}(x))) \oplus (\mathbb{R}H_{3x} \cap \pi_1(\pi_2^{-1}(x)))^{\perp})$$

(see (2.13) and (2.14)). Denote by H' the vector space spanned by σ_1 and σ_{kl}, $1 \le k \le l \le n$, and denote by

$$\pi' : (H_{3x} \cap \pi_1(\pi_2^{-1}(x)))^{\perp} \to H'$$

the projection onto H' along H_{3x} (resp. $\mathbb{R}H_{3x}$). The coordinates on H' in this basis are a_1 and $B' = (b_{kl})_{1\le k\le l\le n}$ (while $B = (b_{kl})_{2\le k\le l\le n}$) (see (2.29)). We deduce that

$$\frac{1}{d^n}\int_{\pi_1(\pi_2^{-1}(x))} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}}\, d\mu_{\mathbb{C}}(\sigma) = \frac{1}{d^n}\int_{(H_{3x}\cap\pi_1(\pi_2^{-1}(x)))^{\perp}} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}}\, d\mu_{\mathbb{C}}(\sigma)$$
$$= \pi^n \int_{H'} |a_1(1+O(d^{-6}))|^2 |\det B + P_{n-1}(B, a_1 r(d,x) + s(d,x,B))|^2$$
$$\cdot (\pi'_* d\mu_{\mathbb{C}})(a_1, B'))\, d\,\mathrm{vol}_{h|x}\,.$$

Asymptotical estimation of the measure. Let us compare the push-forward measure $\pi'_* d\mu_{\mathbb{C}}$ with the Gaussian measure given by the coordinates a_1 and $B' = (b_{kl})_{1\le k\le l\le n}$. We decompose $\sigma_1 = \hat\sigma_1 + \check\sigma_1$ and for every $1\le k\le l\le n$, $\sigma_{kl} = \hat\sigma_{kl} + \check\sigma_{kl}$, where $\hat\sigma_1, \hat\sigma_{kl}$ are the orthogonal projections of σ_1, σ_{kl} onto $(H_{3x}\cap\pi_1(\pi_2^{-1}(x)))^{\perp}$ so that $\check\sigma_1, \check\sigma_{kl} \in H_{3x}$. By Lemma 2.6, $\|\check\sigma_1\| = \langle \sigma_1, \check\sigma_1/\|\check\sigma_1\|\rangle = O(d^{-3/2})$ and likewise,

$$\forall 1 \le k \le l \le n, \qquad \|\check\sigma_{kl}\| = \langle \sigma_{kl}, \check\sigma_{kl}/\|\check\sigma_{kl}\|\rangle = O(d^{-3/2}).$$

As a consequence,

$$\|\hat\sigma_1\|^2 = \langle \sigma_1 - \check\sigma_1, \sigma_1 - \check\sigma_1\rangle = 1 + O(d^{-3/2}), \tag{2.41}$$

and for all $1 \le k \le l \le n$, $\|\hat\sigma_{kl}\|^2 = 1 + O(d^{-3/2})$. Moreover, the scalar product between two of these elements is an $O(d^{-1})$ as follows from Lemma 2.6. Let P be the matrix of the basis $\hat\sigma_1, (\hat\sigma_{kl})_{1\le k\le l\le n}$ written in an orthonormal basis of $(H_{3x}\cap\pi_1(\pi_2^{-1}(x)))^{\perp}$. Then ${}^tPP$ is the Gram matrix of these vectors and it follows from (2.41) that

$$G = {}^tPP - \mathrm{Id} = O(d^{-1}), \tag{2.42}$$

where the $O(d^{-1})$ does not depend on $x \in X$. The Gaussian measure of $(H_{3x} \cap \pi_1(\pi_2^{-1}(x)))^{\perp}$ reads then, in the coordinates a_1, $B' = (b_{kl})_{1\le k\le l\le n}$,

$$e^{-{}^t(a_1,B'){}^tPP(a_1,B')}|\mathrm{Jac}(P)|\frac{da_1\, dB'}{\sqrt{\pi}^{n(n+1)/2+1}}$$
$$= e^{-{}^t(a_1,B')G(a_1,B')}|\mathrm{Jac}(P)|e^{-\|a_1\|^2-\|B'\|^2}\frac{da_1\, dB'}{\sqrt{\pi}^{n(n+1)/2+1}} = f(a_1,B')\, d\mu_{\mathbb{C}}(a_1,B').$$

By (2.42), $f(a_1, B') = |\mathrm{Jac}(P)|e^{-{}^t(a_1,B')G(a_1,B')} \in \mathbb{R}_+$ converges to 1 as d grows to infinity and is dominated by a function with the same property which does not depend on $x \in X$. We deduce that

$$\begin{aligned}\frac{1}{d^n}\int_{\pi_1(\pi_2^{-1}(x))} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}}\, d\mu_{\mathbb{C}}(\sigma) = \pi^n \int_{H'} &|a_1(1+O(d^{-6}))|^2 \\ &\cdot |\det B + P_{n-1}(B, a_1 r(d,x) + s(d,x,B))|^2 \\ &\cdot f(a_1, B')\, d\mu_{\mathbb{C}}(a_1, B')\, d\mathrm{vol}_{h|x}\,.\end{aligned}$$

Since

$$\int_{H'} |a_1|^2 |\det B|^2\, d\mu_{\mathbb{C}}(a_1, B') = \int_{\mathrm{Sym}(n-1,\mathbb{C})} |\det B|^2\, d\mu_{\mathbb{C}}(B) = e_{\mathbb{C}}(n-1),$$

we finally get

$$\lim_{d\to\infty} \frac{1}{d^n}\int_{\pi_1(\pi_2^{-1}(x))} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}}\, d\mu_{\mathbb{C}}(\sigma) = \pi^n e_{\mathbb{C}}(n-1)\, d\mathrm{vol}_{h|x}, \tag{2.43}$$

where the convergence is dominated by a polynomial function in $\|\nabla^X dx_1\|$ of degree less than $2n-1$.

This result remains unchanged if a different normalized volume form dx is used on X to define the L^2 scalar product, since from Remark 2.7(1), this asymptotically has the effect of dilating the scalar product on the subspace $(H_{3x} \cap \pi_1(\pi_2^{-1}(x)))^{\perp}$, while from Remark 2.11 and Proposition 2.12, such a dilation does not affect the integral $\int_{(H_{3x}\cap\pi_1(\pi_2^{-1}(x)))^{\perp}} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{\sigma,x}}\, d\mu_{\mathbb{C}}(\sigma)$.

The computation near critical and base points. Since ∇^X is not defined at the critical and base points of p, we have now to estimate the singularities of $\nabla^X(dx_1)$ near these loci. In the coordinates $(x_1, \dots, x_n)$ around x, let us write $dp = \sum_{i=1}^n \alpha_i dx_i$, so that at the point x, $\alpha_2(x) = \dots = \alpha_n(x) = 0$ and $|\alpha_1(x)| = \|dp_x\|$. Then

$$0 = \nabla^X(dp)_{|K_x} = \alpha_1(\nabla^X dx_1)_{|K_x} + \sum_{i=1}^n (d\alpha_i \otimes dx_i)_{|K_x}$$

so that $\|\nabla^X dx_{1|K_x}\| = (1/\|dp_{|x}\|)\|\sum_{i=1}^n d\alpha_i \otimes dx_{i|K_x}\|$ has a pole of order one at x, since by definition of a Lefschetz pencil, dp vanishes transversely at x. As a consequence, the domination in (2.43), which is polynomial in $\|\nabla^X dx_1\|$ of degree less than $2n-1$, has poles at order at most $2n-2$ near the critical points.

Near the base points, these are poles of order at most 2. Indeed, the normal form for p near a base point reads $p : (y_1, \dots, y_n) \in \mathbb{C}^n \mapsto y_1/y_2 \in \mathbb{C}$, so that

$$dp_{|(y_1,\dots,y_n)} = \frac{y_2\, dy_1 - y_1\, dy_2}{y^2}$$

is not well defined along $y_2 = 0$. Denote by β the numerator one-form $y_2 dy_1 - y_1 dy_2$, which is well defined everywhere. When the point x lies in such a chart, there is no obstruction to finding local coordinates around x which are orthonormal at x and such that in these coordinates, β reads $\alpha_1 dx_1 + \alpha_2 dx_2$ where α_1, α_2 only depend on x_1, x_2, $\alpha_2(x) = 0$ and $\alpha_1(x)$ is a function of x having a simple zero along the base locus. Then, by hypothesis on ∇^X,

$$0 = \nabla^X \beta_{|K_x} = \alpha_1 \nabla^X (dx_1)_{|K_x} + \frac{\partial \alpha_2}{\partial x_2} dx_2 \otimes dx_{2|K_x}.$$

It follows that $\nabla^X (dx_1)_{|K_x}$ has a simple pole along the base locus, and by (2.36) the matrix of $\nabla^2 \sigma_{1|K_x}$ in the basis $(\partial/\partial x_2, \dots, \partial/\partial x_n)$ is elementary, with only one diagonal coefficient having a simple pole along the base locus. After developing the determinant, we deduce that the polynomial P_{n-1} in (2.39) is only of degree one, so that the dominating function in (2.43) has a pole of order at most two near the base locus. $\square$

2.5.2. Proof of Proposition 2.13 in the real case. In the real case, we get likewise

$$\frac{1}{\sqrt{\pi d}^n} |\lambda_{(\sigma,x)}| = |a_1(1 + O(d^{-6}))|\, |\det B + P_{n-1}(B, a_1 r(d,x) + s(d,x,B))|\, d\mathrm{vol}_{h|x}\,. \tag{2.44}$$

After integration, we deduce that

$$\begin{aligned} \frac{1}{\sqrt{d}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\lambda_{(\sigma,x)}|\, d\mu_{\mathbb{R}}(\sigma) &= \sqrt{\pi}^n \int_{H'_i} |a_1(1 + O(d^{-6}))| \\ &\quad \cdot |\det B + P_{n-1}(B, a_1 r(d,x) + s(d,x,B))| (\pi'_* d\mu_{\mathbb{R}}(a_1, B'))\, d\mathrm{vol}_{h|x}, \end{aligned}$$

where from Lemma 2.9, $H'_i = \{\sigma \in H' \mid \phi_x \circ \nabla^2 \sigma_{|K_x} \text{ is of index } i\}$. Again, we deduce from Lemma 2.6 that

$$\begin{aligned} \lim_{d\to\infty} \frac{1}{\sqrt{d}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\lambda_{(\sigma,x)}|\, d\mu_{\mathbb{R}}(\sigma) &= \sqrt{\pi}^n \int_{\mathbb{R}} |a_1|\, d\mu_{\mathbb{R}}(a_1) \\ &\quad \cdot \int_{\mathrm{Sym}(i,n-1-i,\mathbb{R})} |\det B|\, d\mu_{\mathbb{R}}(B) |d\mathrm{vol}_{h|x}| \\ &= \sqrt{\pi}^{n-1} e_{\mathbb{R}}(i, n-1-i) |d\mathrm{vol}_{h|x}|, \end{aligned}$$

where the convergence is dominated by a polynomial function in $\|\nabla^X dx_1\|$ of degree less than n, so that it has poles of order less than n near the critical points of p. This result remains unchanged if a different normalized volume form dx is used on X. $\square$

2.5.3. Proofs of the theorems. Proof of Theorems 1.3 and 1.2. From Proposition 2.13 it follows that under the hypotheses of Theorem 1.3 (resp. Theorem 1.2), the measure

$$\frac{1}{d^n} \int_{\pi_1(\pi_2^{-1}(x))} \lambda_{(\sigma,x)} \wedge \overline{\lambda_{(\sigma,x)}}\, d\mu_{\mathbb{C}}(\sigma) \quad \Big(\text{resp. } \frac{1}{\sqrt{d}^n} \int_{\pi_1(\pi_2^{-1}(x))} |\lambda_{(\sigma,x)}|\, d\mu_{\mathbb{R}}(\sigma)\Big)$$

weakly converges to the measure

$$\pi^n e_{\mathbb{C}}(n-1) d\mathrm{vol}_h \quad (\text{resp. } \sqrt{\pi}^{n-1} e_{\mathbb{R}}(i, n-1-i) d\mathrm{vol}_h\,).$$

Theorems 1.3 and 1.2 then follow from Propositions 2.10, 2.12 and 3.8, for any normalized volume form dx on X chosen to define the L^2 scalar product $\langle \cdot, \cdot \rangle$ in (2.1). $\square$

Proof of Theorem 1.1. By definition, for every $\sigma \in \mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_p^d$,

$$\frac{1}{\sqrt{d}^n} m_i(\mathbb{R}C_\sigma) \leq \int_{\mathbb{R}X} \nu_i(\mathbb{R}C_\sigma),$$

with equality when $n = 1$ (and $i = 0$), where ν_i is the empirical measure defined by (1.5) and m_i is the ith Morse number (see (1.1)). By integration over $\mathbb{R}H^0(X, L^d) \setminus \mathbb{R}\Delta_p^d$, we deduce that

$$\frac{1}{\sqrt{d}^n} E(m_i) \leq \int_{\mathbb{R}X} E(\nu_i),$$

with equality when $n = 1$. Now, from Theorem 1.2,

$$\int_{\mathbb{R}X} E(\nu_i) \xrightarrow[d\to\infty]{} \frac{1}{\sqrt{\pi}} e_{\mathbb{R}}(i, n-1-i) \int_{\mathbb{R}X} d\,\mathrm{vol}_h,$$

hence the result. □

Remark 2.14. (1) Theorem 1.1 substantially improves [16, Theorem 4].

(2) When X is the Riemann sphere $\mathbb{C}P^1$, $L = \mathcal{O}_{\mathbb{C}P^1}(1)$ and h is the Fubini–Study metric, X equipped with its Kählerian metric is isometric to the round sphere of radius $1/(2\sqrt{\pi})$ in the Euclidean three-space, so that its volume equals 1. It follows that $\mathrm{Vol}_{\mathrm{FS}}(\mathbb{R}X) = \sqrt{\pi}$, and Theorem 1.1 then gives $E(b_0) \sim_{d\to\infty} \sqrt{d}$, which is consistent with Kostlan's [21] and Shub–Smale's [36] results.

(3) When $X = \mathbb{C}P^n$, $L = \mathcal{O}_{\mathbb{C}P^n}(1)$ and h is the Fubini–Study metric, the geodesics $\mathbb{R}P^1$ of $\mathbb{R}P^n$ have length $\sqrt{\pi}$, so that $\mathbb{R}P^n$ is isometric to the quotient of the sphere of radius $1/\sqrt{\pi}$ by the antipodal relation. Hence,

$$\mathrm{Vol}_{\mathrm{FS}}(\mathbb{R}P^n) = \frac{1}{2\sqrt{\pi}^n} \mathrm{Vol}(S^n) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{n+1}{2}\right)},$$

where S^n denotes the unit sphere in $\mathbb{R}^{n+1}$.

3. Expected determinant of random symmetric matrices

In §3.1 we study the asymptotic distribution of $e_{\mathbb{R}}(p, q)$ for large $n = p + q$. We then compute $e_{\mathbb{C}}(n)$ in §3.2.1 and $e_{\mathbb{R}}(n)$ in §§3.2.2 and 3.2.3. We also give in §3.2.4 the values of $e_{\mathbb{R}}(p, q)$ for $p + q \leq 3$.

3.1. Large random real symmetric matrices

3.1.1. The energy functional. Let

$$f : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}, \qquad (x, y) \mapsto \begin{cases} \frac{1}{2}(x^2 + y^2) - \log|x - y| & \text{if } x \neq y, \\ \infty & \text{if } x = y. \end{cases}$$

Let $\mathcal{M}_1^+(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$, and H be the energy functional defined by

$$H : \mathcal{M}_1^+(\mathbb{R}) \to \mathbb{R} \cup \{\infty\},$$
$$\mu \mapsto \begin{cases} \frac{1}{2}\iint_{\mathbb{R}^2} f(x,y)\, d\mu(x)\, d\mu(y) & \text{if } \int_{\mathbb{R}} \log(|x|+1)\, d\mu(x) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

This functional is lower semicontinuous, strictly convex and reaches its unique minimum at the semicircle law μ_W of Wigner (see [1, §2.6.1]). Moreover, $H(\mu_W) = \frac{1}{4}(3/2+\log 2)$.

For every $0 \le \alpha \le 1$, define

$$\mathcal{M}^+_{\alpha,1-\alpha}(\mathbb{R}) = \{\mu \in \mathcal{M}_1^+(\mathbb{R}) \mid \mu(\mathbb{R}^*_-) = \alpha \text{ and } \mu(\mathbb{R}^*_+) = 1-\alpha\}.$$

Since the functional H is strictly convex and equals ∞ on atomic measures, its restriction to $\mathcal{M}^+_{\alpha,1-\alpha}$ reaches its minimum at a unique measure $\mu_\alpha \in \mathcal{M}^+_{\alpha,1-\alpha}$ which has no atom. In particular, $\mu_{1/2} = \mu_W$. For every $0 \le \alpha \le 1$, we set

$$M_\alpha = \min_{\mathcal{M}^+_{\alpha,1-\alpha}(\mathbb{R})} H = H(\mu_\alpha).$$

Lemma 3.1. *The function $M : \alpha \in [0,1] \mapsto M_\alpha \in \mathbb{R}_+$ is strictly decreasing over $[0,1/2]$ and strictly increasing over $[1/2,1]$. More precisely, for every $\alpha \in [0,1] \setminus \{1/2\}$, there exists $c_\alpha > 0$ such that for all $t \in [0,1]$, $M_{t\alpha+(1-t)\cdot 1/2} \le M_\alpha + (t^2-1)c_\alpha$.*

Proof. Let $\alpha \in [0,1] \setminus \{1/2\}$ and $f^\alpha \in L^1(\mathbb{R}, dx)$ be the density of μ_α with respect to the Lebesgue measure dx. We decompose $f^\alpha = f_e + f_o$ into even and odd functions, so that $f_e = \frac{1}{2}(f^\alpha + f^\alpha \circ (-\mathrm{Id}))$ and $f_o = \frac{1}{2}(f^\alpha - f^\alpha \circ (-\mathrm{Id}))$. Likewise, we set $\mu_e = f_e dx$ and $\mu_o = f_o dx$, so that $\mu^\alpha = \mu_e + \mu_o$. Then, for every $t \in [-1,1]$,

$$\begin{aligned} H(\mu_e + t\mu_o) &= \frac{1}{2}\iint_{\mathbb{R}^2} f(x,y)\big(d\mu_e(x)d\mu_e(y) + t d\mu_e(x)d\mu_o(y) + t d\mu_o(x)d\mu_e(y)\big) \\ &\quad + \frac{t^2}{2}\iint_{\mathbb{R}^2} f(x,y)d\mu_o(x)d\mu_o(y) = H(\mu_e) + t^2 H(\mu_o) \end{aligned}$$

from Fubini's theorem, since $\int_{\mathbb{R}} \frac{1}{2}(x^2+y^2)\, d\mu_o(x) = \int_{\mathbb{R}} \frac{1}{2}(x^2+y^2)\, d\mu_o(y) = 0$ while likewise $\int_{\mathbb{R}} \log|y-x|\, d\mu_e(x)$ and $\int_{\mathbb{R}} \log|y-x|\, d\mu_e(y)$ are even functions of y and x respectively. Since H is strictly convex, so is its restriction to $\{\mu_e + t\mu_o \mid t \in [-1,1]\}$, so that $H(\mu_o)$ has to be positive. However, for every $t \in [0,1]$, $\mu_e + t\mu_o$ belongs to $\mathcal{M}^+_{(t\alpha+(1-t)\cdot 1/2,(1+t)\cdot 1/2-t\alpha)}(\mathbb{R})$ and we deduce that $M_{t\alpha+(1-t)\cdot 1/2} \le H(\mu_e) + t^2 H(\mu_o) = M_\alpha + (t^2-1)H(\mu_o)$, hence the result. □

Remark 3.2. It would be of interest to explicitly compute the function $\alpha \in [0,1] \mapsto M_\alpha \in [M_{1/2}, \infty]$ and the measure μ_α. With the help of some physical considerations, the case $\alpha = 0$ has been performed in [6] and the asymptotic of M_α near $\alpha = 1/2$ has been obtained in [26, (10)]. It reads

$$M_\alpha - M_{1/2} \underset{\alpha \to 1/2}{\sim} -\frac{\pi^2}{2}\frac{(\alpha-1/2)^2}{\log|\alpha-1/2|}.$$

3.1.2. Measure concentration around matrices of vanishing signature. Let us now prove Theorem 1.6, closely following the proof of [3, Theorem 3.2].

Proof of Theorem 1.6. The orthogonal group $O_n(\mathbb{R})$ acts by conjugation on real symmetric matrices and a fundamental domain for this action is given by diagonal matrices with stabilizer $\{\pm 1\}^n$. From the coarea formula (see [12, Theorem 3.2.3] or [36, Theorem 1]), we deduce that for every $0 \leq i \leq n$,

$$e_{\mathbb{R}}(i, n-i) = \frac{\mathrm{Vol}(O_n(\mathbb{R}))}{2^n \sqrt{\pi}^{n(n-1)/2}} \int_{\substack{\lambda_1 < \cdots < \lambda_i < 0 \\ 0 < \lambda_{i+1} < \cdots < \lambda_n}} \left| \prod_{i=1}^n \sqrt{2}\, \lambda_i \right| \prod_{1 \leq i < j \leq n} \sqrt{2}\, |\lambda_j - \lambda_i| \, d\mu(\lambda),$$

where the volume of $O_n(\mathbb{R})$ is computed with respect to the right invariant metric for which the basis $(E_{ij} - E_{ji})_{1 \leq i < j \leq n}$ of its Lie algebra is orthonormal (see §3.1.3), $\sqrt{2}\,\lambda_1, \ldots, \sqrt{2}\,\lambda_n$ denote the eigenvalues of the diagonal matrices and $d\mu(\lambda)$ denotes the Gaussian measure on $\mathbb{R}^n$. As a consequence, for every $0 \leq i \leq n$,

$$\begin{aligned} e_{\mathbb{R}}(i, n-i) &= \frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n \sqrt{\pi}^{n(n+1)/2}} \\ &\quad \cdot \int_{\substack{\lambda_1 < \cdots < \lambda_i < 0 \\ 0 < \lambda_{i+1} < \cdots < \lambda_n}} \exp\Big(-\sum_{j=1}^n \lambda_j^2 + \sum_{1 \leq j < k \leq n} \log|\lambda_j - \lambda_k| \Big) \prod_{j=1}^n (|\lambda_j| d\lambda_j) \\ &= c_n n! \int_{\substack{\gamma_1 < \cdots < \gamma_i < 0 \\ 0 < \gamma_{i+1} < \cdots < \gamma_n}} \exp\bigg(\sum_{1 \leq j < k \leq n} \log|\gamma_j - \gamma_k| - \frac{1}{2} \sum_{1 \leq j < k \leq n} (\gamma_j^2 + \gamma_k^2) \bigg) \\ &\quad \cdot \exp\Big(-\frac{1}{2} \sum_{j=1}^n \gamma_j^2 \Big) \prod_{j=1}^n (|\gamma_j| d\gamma_j) \end{aligned}$$

where

$$c_n = \frac{\sqrt{2}^{n(n-1)/2}\, \mathrm{Vol}(O_n(\mathbb{R}))}{n! \sqrt{2}^n \sqrt{\pi}^{n(n+1)/2}} \left(\frac{n}{2} \right)^{n(n-1)/4+n},$$

and where we wrote, for every $1 \leq j \leq n$, $\lambda_j = \sqrt{n/2}\, \gamma_j$. We now proceed as in [3, §3.1] (or [1, §2.6.1]). Define, for every $\gamma_1 < \cdots < \gamma_i < 0 < \gamma_{i+1} < \cdots < \gamma_n$,

$$\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{\gamma_j} \in \mathcal{M}^+_{i/n, 1-i/n}(\mathbb{R}),$$

so that

$$\sum_{1 \leq j < k \leq n} \log|\gamma_j - \gamma_k| - \frac{1}{2} \sum_{1 \leq j < k \leq n} (\gamma_j^2 + \gamma_k^2) = -n^2 \iint_{x < y} f(x, y)\, d\mu_n(x)\, d\mu_n(y).$$

For $R \in \mathbb{R}$, let $f_R = \min(f, R)$ and

$$\forall \mu \in \mathcal{M}_1^+(\mathbb{R}), \qquad H_R(\mu) = \frac{1}{2} \iint_{\mathbb{R}^2} f_R(x, y)\, d\mu(x)\, d\mu(y).$$

We set
$$M_{\alpha,R} = \min_{\bigcup_{\beta\leq\alpha}\mathcal{M}^+_{\beta,1-\beta}(\mathbb{R})} H_R.$$
For every $0 \leq i \leq \lfloor \alpha n \rfloor$,
$$\iint_{x<y} f_R(x,y)\,d\mu_n(x)\,d\mu_n(y) = H_R(\mu_n) - \frac{R}{2n} \geq M_{\alpha,R} - \frac{R}{2n}.$$
Moreover, by Lemma 3.5 (see §3.1.3 below) and Stirling's formula, $\log(c_n) = n^2 M_{1/2} + O(n)$. Hence, there exists a constant $D > 0$ such that
$$\sum_{i=0}^{\lfloor \alpha n\rfloor} e_{\mathbb{R}}(i, n-i) \leq \exp\big(-n^2(M_{\alpha,R} - M_{1/2}) + (D + R/2)n\big)\Big(\int_{\mathbb{R}} |\gamma| e^{-\gamma^2/2}\,d\gamma\Big)^n.$$
From [3, Property 2.1], $\lim_{R\to\infty} M_{\alpha,R} = M_\alpha$ since H_R weakly converges to H. Hence, Theorem 1.6 follows once R is chosen large enough, since $M_\alpha > M_{1/2}$ and by Lemma 3.1, $M_\alpha = \min_{\bigcup_{\beta\leq\alpha}\mathcal{M}^+_{\beta,1-\beta}(\mathbb{R})} H$. □

Remark 3.3. (1) Note that under the hypotheses of Theorem 1.6, Theorem 3.2 in [3] likewise implies that
$$\mu_{\mathbb{R}}\Big(\bigcup_{i=0}^{\lfloor \alpha n\rfloor} \mathrm{Sym}(i, n-i, \mathbb{R})\Big) \leq \exp(-c_\alpha n^2).$$

(2) An estimation of how the R in the proof of Theorem 1.6 grows to infinity as $\alpha \to 1/2$, combined with an asymptotic of M_α near $\alpha = 1/2$ (see Remark 3.2), would make it possible to improve Theorem 1.6. However, Proposition 3.9 seems to imply that we cannot hope for a better estimate than up to an $O(n \log n)$ error term using this approach. More precisely, [26, (10)] leads, for any diverging sequence δ_n, to
$$\log\Big(\sum_{i=0}^{n/2-\delta_n} e_{\mathbb{R}}(i, n-i)\Big) \leq \frac{\pi^2}{2}\,\frac{\delta_n^2}{\log(\delta_n/n)} + O(n\log n)$$
(see Remark 3.2).

3.1.3. Volume of the orthogonal group. Let us equip the vector space of real antisymmetric matrices with the scalar product turning the basis $(E_{ij} - E_{ji})_{1\leq i<j\leq n}$ into an orthonormal one. This scalar product on the Lie algebra of $O_n(\mathbb{R})$ induces on $O_n(\mathbb{R})$ a Riemannian metric for which multiplications on the right by elements are isometries. In the following lemma we recall the value of the total volume of $O_n(\mathbb{R})$ for this metric.

Lemma 3.4. *For every positive integer n,*
$$\mathrm{Vol}(O_n(\mathbb{R})) = \prod_{k=0}^{n-1} \mathrm{Vol}(S^k) = \frac{n!\sqrt{\pi}^{\,n(n+1)/2}}{\prod_{j=1}^n \Gamma(1+j/2)}.$$

Proof. The orthogonal group $O_n(\mathbb{R})$ acts by isometries on the unit sphere S^{n-1} with stabilizers conjugate to $O_{n-1}(\mathbb{R})$. From the coarea formula, we deduce that $\mathrm{Vol}(O_n(\mathbb{R})) = \mathrm{Vol}(O_{n-1}(\mathbb{R})) \times \mathrm{Vol}(S^{n-1})$, and the result follows by induction. □

Lemma 3.5. *The following asymptotic development holds:*

$$\log\left(\frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n\sqrt{\pi}^{n(n-1)/2}}\right) = -\frac{n^2\log n}{4} + n^2\left(\frac{3}{8}+\frac{\log 2}{2}\right) + \frac{1}{4}n\log n + O(n).$$

Proof. From Lemma 3.4, when $n = 2m$ is even,

$$\frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n\sqrt{\pi}^{n(n-1)/2}} = \frac{n!\sqrt{\pi}^n\sqrt{2}^{n(n-3)/2}}{\prod_{j=1}^{n/2}(j!\Gamma(j+1/2))}.$$

From Stirling's formula, $n! \sim n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$ and $\Gamma(j+1/2) \sim (j-1)!\sqrt{j-1}$. It follows that

$$\begin{aligned}
&\log\Big(\prod_{j=1}^{m}(j!\Gamma(j+1/2))\Big) = 2\sum_{j=1}^{m-1}\Big(j\log j - j + \tfrac{3}{4}\log j\Big) + m\log m + O(n)\\
&= \sum_{j=1}^{m-1}\Big((j+1)^2\log(j+1) - j^2\log j - 3j + \tfrac{1}{2}(j+1)\log(j+1) - \tfrac{1}{2}j\log j\Big) + m\log m + O(n)\\
&= \frac{n^2}{4}\log\left(\frac{n}{2}\right) - \frac{3}{8}n^2 + \frac{3n}{4}\log\left(\frac{n}{2}\right) + O(n).
\end{aligned}$$

Finally, we obtain

$$\log\left(\frac{\mathrm{Vol}(O_n(\mathbb{R}))}{\sqrt{2}^n\sqrt{\pi}^{n(n-1)/2}}\right) = n\log n + \frac{n^2}{4}\log 2 + \frac{n^2}{4}\log 2 - \frac{n^2}{4}\log n + \frac{3}{8}n^2 - \frac{3n}{4}\log n + O(n),$$

which gives the result when n is even. When $n = 2m+1$, we have

$$\frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n\sqrt{\pi}^{n(n-1)/2}} = \frac{\sqrt{2}^{m+n}\pi^m 2^{m^2}}{\prod_{j=0}^{m-1}(j!\Gamma(j+3/2))}.$$

But

$$\begin{aligned}
\log\Big(\prod_{j=0}^{m-1}(j!\Gamma(j+3/2))\Big) &= 2\sum_{j=1}^{m-1}\left(j\log j - j + \frac{3}{4}\log j\right) + O(n)\\
&= \frac{(n-1)^2}{4}\log\left(\frac{n-1}{2}\right) - \frac{3}{8}(n-1)^2 + \frac{n-1}{4}\log\left(\frac{n-1}{2}\right) + O(n).
\end{aligned}$$

We deduce that

$$\log\left(\frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n\sqrt{\pi}^{n(n-1)/2}}\right) = \frac{n^2}{4}\log 2 - \frac{n^2}{4}\log n + \frac{n}{2}\log n + \frac{n^2}{4}\log 2 + \frac{3}{8}n^2 - \frac{n}{4}\log n + O(n),$$

proving the result. □

3.2. Determinants of random symmetric matrices

3.2.1. Complex symmetric matrices. For every $n \in \mathbb{N}^*$, denote by S_n the group of permutations of $\{1, \dots, n\}$, and for every $\sigma \in S_n$, let $\mathrm{Cycles}(\sigma)$ be the set of cycles appearing in the decomposition of σ into a product of cycles with disjoint supports. For instance, if σ denotes the permutation $\left(\begin{smallmatrix}1&2&3&4&5\\3&2&1&5&4\end{smallmatrix}\right)$ of $\{1, \dots, 5\}$, then $\mathrm{Cycles}(\sigma) = \{(13), (2), (45)\}$.

Lemma 3.6. *For every $n \in \mathbb{N}^*$, $e_{\mathbb{C}}(n) = \sum_{\sigma\in S_n} 2^{\#\mathrm{Cycles}(\sigma)}$.*

Proof. For every $A \in \mathrm{Sym}(n, \mathbb{C})$, write $A = \sum_{1\le i\le j\le n} a_{ij}\widetilde{E}_{ij}$ and set $a_{ji} = a_{ij}$ if $i > j$. By definition,

$$\begin{aligned} e_{\mathbb{C}}(n) &= \int_{\mathrm{Sym}(n,\mathbb{C})} (\det A)(\overline{\det A})\, d\mu_{\mathbb{C}}(A) \\ &= \sum_{\sigma\in S_n} (-1)^{\epsilon(\sigma)} \sum_{\tau\in S_n} (-1)^{\epsilon(\tau)} \int_{\mathrm{Sym}(n,\mathbb{C})} \sqrt{2}^{\#\mathrm{Fix}(\sigma)+\#\mathrm{Fix}(\tau)} \cdots \\ &\qquad\qquad \cdots a_{1\sigma(1)}\overline{a_{1\tau(1)}} \cdots a_{n\sigma(n)}\overline{a_{n\tau(n)}}\, d\mu_{\mathbb{C}}(A), \end{aligned}$$

since the diagonal entries of A have weight $\sqrt{2}$. Now, the integral $\int_{\mathbb{C}} z^\alpha \bar{z}^\beta\, d\mu_{\mathbb{C}}(z)$ vanishes when $\alpha \ne \beta$, so that for every $\sigma \in S_n$, the only permutations $\tau \in S_n$ which contribute to the integral are the ones for which $\{a_{1\sigma(1)}, \dots, a_{n\sigma(n)}\} = \{a_{1\tau(1)}, \dots, a_{n\tau(n)}\}$. This implies that $\{a_{j\sigma(j)}, a_{\sigma^{-1}(j)j}\} = \{a_{j\tau(j)}, a_{\tau^{-1}(j)j}\}$ for every $j \in \{1, \dots, n\}$.

If j belongs to a cycle of length 1 or 2 of σ, we deduce that $\sigma(j) = \tau(j)$. More generally, if $\tilde\sigma \in \mathrm{Cycles}(\sigma)$, then either $\tilde\sigma$ or $\tilde\sigma^{-1}$ is in $\mathrm{Cycles}(\tau)$. In particular, $\epsilon(\sigma) = \epsilon(\tau)$. Conversely, every permutation τ which can be written as a product $\prod_{\tilde\sigma\in\mathrm{Cycles}(\sigma)} \tilde\sigma^{\pm 1}$ contributes to the integral. There are $2^{\#\mathrm{Cycles}_{\ge 3}(\sigma)}$ such permutations, where $\mathrm{Cycles}_{\ge 3}(\sigma)$ denotes the set of elements of $\mathrm{Cycles}(\sigma)$ having length ≥ 3. As a consequence,

$$e_{\mathbb{C}}(n) = \sum_{\sigma\in S_n} 2^{\#\mathrm{Cycles}_{\ne 2}(\sigma)} \int_{\mathrm{Sym}(n,\mathbb{C})} \prod_{i=1}^n |a_{i\sigma(i)}|^2\, d\mu_{\mathbb{C}}(A),$$

where $\mathrm{Cycles}_{\ne 2}(\sigma)$ denotes the subset in $\mathrm{Cycles}(\sigma)$ of elements having length different from 2. Now, $\int_{\mathbb{C}} |z|^2\, d\mu_{\mathbb{C}}(z) = 1$ whereas $\int_{\mathbb{C}} |z|^4\, d\mu_{\mathbb{C}}(z) = 2$. Every transposition of $\mathrm{Cycles}(\sigma)$ produces an element of this second type whereas the other elements of $\mathrm{Cycles}(\sigma)$ give rise to products of the first type. Hence the result. □

Lemma 3.7. *For every $n \in \mathbb{N}^*$, $\sum_{\sigma \in S_n} 2^{\#\mathrm{Cycles}(\sigma)} = (n+1)!$.*

Proof. When $n = 1$, the equality is satisfied. Assume that it is satisfied up to a rank n and let us prove it for rank $n+1$. Let $\sigma \in S_{n+1}$ and let $\sigma = \tilde{\sigma}_1 \cdots \tilde{\sigma}_k$ be its decomposition into a product of cycles with disjoint supports. If we remove $n+1$ from the cycle which contains it, we get a permutation τ of S_n together with its decomposition into a product of cycles with disjoint supports. We deduce from this an $(n+1)$-to-1 forgetful map $f_n : \sigma \in S_{n+1} \mapsto \tau \in S_n$ such that $\#\mathrm{Cycles}(\sigma) = \#\mathrm{Cycles}(f_n(\sigma))$ if $n+1$ is not fixed by σ and $\#\mathrm{Cycles}(\sigma) = \#\mathrm{Cycles}(f_n(\sigma)) + 1$ otherwise. Hence,

$$\sum_{\sigma \in S_{n+1}} 2^{\#\mathrm{Cycles}(\sigma)} = \sum_{\tau \in S_n} \sum_{\sigma \in f_n^{-1}(\tau)} 2^{\#\mathrm{Cycles}(\sigma)} = (n+2) \sum_{\tau \in S_n} 2^{\#\mathrm{Cycles}(\tau)} = (n+2)!$$

by induction. □

Proposition 3.8. *For every $n \in \mathbb{N}$, $e_{\mathbb{C}}(n) = (n+1)!$.*

Proof. This is a consequence of Lemmas 3.6 and 3.7 when $n > 0$ and of our convention when $n = 0$. □

3.2.2. Real symmetric matrices of odd size. We recall here the values of $e_{\mathbb{R}}(n)$, $n > 0$, distinguishing between the cases of n even and n odd (see [27, §25.5 and §26.6]). The odd-dimensional case turns out to be easier:

Proposition 3.9 ([27, formula 26.5.2]). *For every odd integer n,*

$$e_{\mathbb{R}}(n) = \frac{2\sqrt{2}}{\pi} \Gamma\left(\frac{n+2}{2}\right).$$

Let us briefly recall the proof of this proposition as given in [27].

Proof. As in the proof of Theorem 1.6, it follows from the coarea formula that

$$e_{\mathbb{R}}(n) = \frac{\mathrm{Vol}(O_n(\mathbb{R}))}{2^n \sqrt{\pi}^{n(n-1)/2}} \int_{\lambda_1 < \cdots < \lambda_n} \left| \prod_{i=1}^{n} \sqrt{2}\, \lambda_i \right| \prod_{1 \le i < j \le n} \sqrt{2}\, |\lambda_j - \lambda_i| \, d\mu(\lambda),$$

where as before the volume of $O_n(\mathbb{R})$ is computed with respect to the right invariant metric for which the basis $(E_{ij} - E_{ji})_{1 \le i < j \le n}$ of its Lie algebra is orthonormal (see §3.1.3), and where $d\mu(\lambda)$ denotes the Gaussian measure on $\mathbb{R}^n$. The integrand is a Vandermonde determinant. Integrating the odd lines of this determinant and then expanding by pairs of rows in the Laplace manner, we get the relation

$$e_{\mathbb{R}}(n) = \frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n \sqrt{\pi}^{n(n-1)/2}} \det B.$$

Here, writing $n = 2m+1$, B denotes an $(m+1) \times (m+1)$ matrix with entries $(b_{ij})_{0 \le i < j \le m}$ defined by

$$\forall 0 \le i \le m,\ \forall 0 \le j < m, \qquad b_{ij} = 2(\psi_{ij} + \eta_{2i}\eta_{2j+1}),$$

and $b_{im} = 2\eta_{2i}$ where

$$\psi_{ij} = \int_{0\le x<y<\infty} |xy|(x^{2i}y^{2j+1} - y^{2i}x^{2j+1})\, d\mu(x)\, d\mu(y), \tag{3.1}$$
$$\eta_k = \int_0^\infty x^{k+1}\, d\mu(x) = \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{k+2}{2}\right).$$

Using linear combinations of rows and columns of B with the help of the relations

$$\forall i, j \ge 0, \quad \psi_{i+1,j} = (i+1)\psi_{ij} - \frac{1}{\pi 2^{i+j+7/2}}\Gamma(i+j+5/2)$$

and $\eta_{2i+2} = (i+1)\eta_{2i}$, we get

$$\det B = \frac{1}{\sqrt{\pi}^n\sqrt{2}^m 2^{m(m+1)}} \det\left(\Gamma(i+j+5/2)\right)_{0\le i,j\le m-1}$$
$$= \frac{1}{\sqrt{\pi}^n\sqrt{2}^m 2^{m(m+1)}} \prod_{j=0}^{m-1}(j!\Gamma(5/2+j))$$

(see [27, formula A.18.7]). When $n = 2m+1$ we deduce from this that

$$e_{\mathbb{R}}(n) = \frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{\pi}^{n(n+1)/2}\sqrt{2}^{m+1}2^{m(m+2)}} \prod_{j=0}^{m-1}(j!\Gamma(5/2+j)).$$

The result now follows from Lemma 3.4. □

The proof of Proposition 3.9 may also provide an alternative proof of Lemma 3.4 for odd n's, as suggested in [27].

Alternative proof of Lemma 3.4 in odd dimensions. Proceeding as in the proof of Proposition 3.9, we get

$$1 = \frac{\mathrm{Vol}(O_n(\mathbb{R}))}{2^n\sqrt{\pi}^{n(n-1)/2}} \int_{\lambda_1<\cdots<\lambda_n} \prod_{1\le i<j\le n} \sqrt{2}(\lambda_j - \lambda_i)\, d\mu(\lambda)$$
$$= \frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{2^n\sqrt{\pi}^{n(n-1)/2}} \det B',$$

where B' denotes an $(m+1)\times(m+1)$ matrix $(b'_{ij})_{0\le i,j\le m}$ defined by

$$\forall 0 \le i \le m, \forall 0 \le j < m, \quad b'_{ij} = 2(\psi'_{ij} + \eta_{2i-1}\eta_{2j})$$

and $b'_{im} = 2\eta_{2i-1}$, whereas

$$\psi'_{ij} = \int_{0\le x<y\le\infty} (x^{2i}y^{2j+1} - y^{2i}x^{2j+1})\, d\mu(x)\, d\mu(y) = -\psi_{j(i-1)}.$$

From linear combinations and the relation

$$\forall i,j\geq 0,\qquad \psi'_{i+1,j}=\frac{2i+1}{2}\psi'_{ij}-\frac{1}{\pi 2^{i+j+5/2}}\Gamma(i+j+3/2),$$

we get

$$\det B'=\frac{1}{\pi^m\sqrt{2}^m 2^{m^2}}\prod_{j=0}^{m-1}(j!\Gamma(3/2+j)).$$

Finally,

$$\mathrm{Vol}(O_n(\mathbb{R}))=\frac{\sqrt{\pi}^{m(n+2)}\sqrt{2}^m 2^{m^2+n}}{\prod_{j=0}^{m-1}(j!\Gamma(3/2+j))\sqrt{2}^{n(n-1)/2}} \tag{3.2}$$

and $e_{\mathbb{R}}(n)=\frac{2\sqrt{2}}{\pi}\Gamma\big(\frac{n+2}{2}\big)$, since $\Gamma(1/2)=\sqrt{\pi}$. □

Remark 3.10. The first values given by Proposition 3.9 are

$$e_{\mathbb{R}}(1)=\sqrt{\frac{2}{\pi}},\qquad e_{\mathbb{R}}(3)=\frac{3}{\sqrt{2\pi}},\qquad e_{\mathbb{R}}(5)=\frac{15}{2\sqrt{2\pi}}.$$

Moreover, from Stirling's formula, $e_{\mathbb{R}}(n)$ is equivalent to $\frac{2\sqrt{2}}{\pi}\sqrt{m}m!$ as $n=2m+1\to\infty$.

3.2.3. Real symmetric matrices of even size. When the dimension $n=2m$ is even, the value of $e_{\mathbb{R}}(n)$ is given by the following proposition.

Proposition 3.11. *For every even positive integer $n=2m$,*

$$e_{\mathbb{R}}(n)=(-1)^m\frac{n!}{m!2^n}+(-1)^{m-1}\frac{4\sqrt{2}\,n!}{\sqrt{\pi}\,m!2^n}\sum_{k=0}^{m-1}(-1)^k\frac{\Gamma(k+3/2)}{k!}.$$

This expression can be rewritten as

$$e_{\mathbb{R}}(n)=(-1)^m\frac{4\sqrt{2}\,n!}{\sqrt{\pi}\,m!2^n}\int_0^\infty\sqrt{t}\bigg(e^{-t}-\sum_{k=0}^{m-1}\frac{(-t)^k}{k!}\bigg)e^{-t}\,dt.$$

The first term on the right-hand side of the expression in Proposition 3.11 is alternating and negligible for large values of n with respect to the second one which is always nonnegative. The latter can be checked by pairing the terms of the sum (see the proof of Corollary 3.13).

Remark 3.12. The first values of $e_{\mathbb{R}}(n)$ for even n's are

$$e_{\mathbb{R}}(0)=1,\qquad e_{\mathbb{R}}(2)=\sqrt{2}-1/2,\qquad e_{\mathbb{R}}(4)=\tfrac{3}{4}(\sqrt{2}+1),$$
$$e_{\mathbb{R}}(6)=\tfrac{165}{32}\sqrt{2}-\tfrac{15}{8},\qquad e_{\mathbb{R}}(8)=\tfrac{3\times5\times7}{16}\big(\tfrac{13}{8}\sqrt{2}+1\big).$$

Note that $e_{\mathbb{R}}(n)$ is algebraic in $\mathbb{Q}[\sqrt{2}]$ for even n and transcendental for odd n.

Corollary 3.13. *Whatever the parity of n, $e_{\mathbb{R}}(n) \sim \frac{2\sqrt{2}}{\pi}\Gamma(\frac{n+2}{2})$ as $n \to \infty$.*

Proof. For every odd n, $e_{\mathbb{R}}(n) = \frac{2\sqrt{2}}{\pi}\Gamma\big(\frac{n+2}{2}\big)$ from Proposition 3.9. When $n = 2m$ is even, the first term on the right-hand side in Proposition 3.11 is equivalent to $(-1)^m m^m e^{-m}\sqrt{2}$, that is, $(-1)^m\Gamma\big(\frac{n+2}{2}\big)/\sqrt{\pi m}$ from Stirling's formula. Pairing the terms of the sum in the second one, we get

$$\sum_{k=0}^{m-1}(-1)^k\frac{\Gamma(k+3/2)}{k!} = -\frac{1}{2}\sum_{j=0}^{m/2-1}\frac{\Gamma(2j+3/2)}{(2j+1)!}$$

when m is even, and

$$\sum_{k=0}^{m-1}(-1)^k\frac{\Gamma(k+3/2)}{k!} = \frac{\Gamma(m+1/2)}{(m-1)!} - \frac{1}{2}\sum_{j=0}^{(m-1)/2-1}\frac{\Gamma(2j+3/2)}{(2j+1)!}$$

when m is odd. In both cases, this sum gets equivalent to $(-1)^{m-1}\sqrt{m}/2$ as $n \to \infty$, hence the result. □

In order to prove Proposition 3.11, we first compute $e_{\mathbb{R}}(n)$ in terms of a sequence $(b_m)_{m\in\mathbb{N}}$ which we now introduce (see Proposition 3.14). Let $(a_j)_{j=0}$ be the sequence defined by the relations $a_0 = (8\sqrt{2}-7)/3$ and

$$\forall j > 0, \quad a_j = \frac{4j+2}{2j+3}a_{j-1} + 1.$$

Let $b_1 = a_0 + 1 = \frac{4}{3}(2\sqrt{2}-1)$ and for every $m > 1$,

$$b_m = \sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}a_j.$$

Proposition 3.14. *For every even integer $n = 2m > 0$,*

$$e_{\mathbb{R}}(n) = \frac{n!\Gamma\big(\frac{n+3}{2}\big)b_m}{m!(m-1)!2^n\sqrt{\pi}}.$$

Proof. As in the proof of Proposition 3.9, we establish that

$$e_{\mathbb{R}}(n) = \frac{\mathrm{Vol}(O_n(\mathbb{R}))\sqrt{2}^{n(n-1)/2}}{\sqrt{2}^n\sqrt{\pi}^{n(n-1)/2}}\det C,$$

where C denotes an $m \times m$ matrix $(c_{ij})_{0\le i,j\le m-1}$ defined by

$$\forall 0 \le i, j \le m-1, \quad c_{ij} = 2(\psi_{ij} + \eta_{2i}\eta_{2j+1})$$

with

$$\psi_{ij} = \int_{0\le x<y\le\infty}|xy|(x^{2i}y^{2j+1} - y^{2i}x^{2j+1})\,d\mu(x)\,d\mu(y)$$

and $\eta_k = \int_0^\infty x^{k+1}\,d\mu(x) = \frac{1}{2\sqrt{\pi}}\Gamma\big(\frac{k+2}{2}\big)$. Following [27, §26.6], we get

$$
\begin{aligned}
\det C &= -2^m \begin{vmatrix} -1 & (\eta_{2j+1})_{0\le j\le m-1} \\ (\eta_{2i})_{0\le i\le m-1} & (\psi_{ij})_{0\le i,j\le m-1} \end{vmatrix} \\
&= -2^m \begin{vmatrix} -1 & (\eta_{2j+1})_{0\le j\le m-1} \\ \eta_0 & (\psi_{0j})_{0\le j\le m-1} \\ & \\ 0 & \left(\psi_{ij} - i\psi_{i-1j} = -\frac{\Gamma(i+j+3/2)}{\pi 2^{i+j+5/2}}\right)_{1\le i\le m-1,\, 0\le j\le m-1} \end{vmatrix} \\
&= \frac{(-1)^m}{\pi^m 2^{m^2}\sqrt{2}^{m-1}} \begin{vmatrix} -1 & 0 & \\ 2\sqrt{\pi} & & 2^j(4\pi\psi_{0j} + \Gamma(j+3/2))_{0\le j\le m-1} \\ 0 & \Gamma(5/2) & \Gamma(j+5/2)_{1\le j\le m-1} \\ 0 & 0 & \begin{array}{c}(\Gamma(i+j+3/2) - (i+1/2)\Gamma(i+j+1/2) \\ = j\Gamma(i+j+1/2))_{2\le i\le m-1,\, 1\le j\le m-1}\end{array} \end{vmatrix},
\end{aligned}
$$

which equals

$$
\frac{(-1)^{m-1}\prod_{j=0}^{m-1}(j!\Gamma(j+5/2))}{\pi^m 2^{m^2}\sqrt{2}^{m-1}} \begin{vmatrix} \frac{2^j}{j!\Gamma(j+5/2)} & & & (4\pi\psi_{0j} + \Gamma(j+3/2))_{0\le j\le m-1} \\ 1 & & & (1/j!)_{1\le j\le m-1} \\ 0 & 1 & & (1/(j-1)!)_{2\le j\le m-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0\ \ 1 & 1 \end{vmatrix},
$$

so that the (i, j) entry, $1 \le i, j \le m-1$, equals $1/(j-i+1)!$ if $j-i+1 \ge 0$ and 0 otherwise. Subtracting from the $m-1$ first lines multiples of the last one, we obtain zeros in the last column, whereas the entry of the penultimate column in the ith line, $1 \le i \le m-2$, equals $(m-1-i)/(m-i)!$. Then, subtracting from the $m-2$ first lines multiples of the penultimate one, we get zeros in the penultimate column whereas the $(i, m-2)$ entry, $1 \le i \le m-3$, equals $(m-1-i)(m-2-i)/(m-i)!$. By recurrence, we get a lower triangular matrix and

$$
\det C = \frac{(-1)^{m-1}\prod_{j=0}^{m-1}(j!\Gamma(j+5/2))}{\pi^m 2^{m^2}\sqrt{2}^{m-1}} \begin{vmatrix} \alpha & 0 & \dots & & 0 \\ 1 & \frac{1}{m-1} & \ddots & & \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \frac{1}{2} & 0 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}
$$

with

$$
\alpha = \sum_{j=0}^{m-1}(-1)^j \frac{2^j \prod_{k=m-j}^{m-1} k}{j!\Gamma(j+5/2)}(4\pi\psi_{0j} + \Gamma(j+3/2)).
$$

We then deduce from Lemma 3.4 the relation

$$
e_{\mathbb{R}}(n) = \frac{n!\Gamma(\frac{n+3}{2})}{m!(m-1)!2^n\sqrt{\pi}} 2\sqrt{2}\sum_{j=0}^{m-1}(-1)^{m-1-j}\frac{2^j\binom{m-1}{j}}{\Gamma(j+5/2)}(4\pi\psi_{0j} + \Gamma(j+3/2)).
$$

The result follows by setting, for every $j \in \{0, \dots, m-1\}$,

$$a_j = \frac{2^{j+1}\sqrt{2}}{\Gamma(j+5/2)}(4\pi\psi_{0j} + \Gamma(j+3/2)) - 1.$$

Indeed,

$$\psi_{00} = \frac{1}{8\sqrt{2\pi}}(\sqrt{2}-1) \tag{3.3}$$

so that $a_0 = (8\sqrt{2}-7)/3$ and the recurrence relation satisfied by $(a_j)_{j\geq 0}$ follows from

$$\forall j > 0, \quad \psi_{0j} = (j+1/2)\psi_{0j-1} + \frac{1}{\pi 2^{j+5/2}}\Gamma(j+3/2)$$

(compare [27, formula 26.4.13]). □

Remark 3.15. As in the alternative proof of Lemma 3.4, we may show that for every $n = 2m > 0$,

$$\mathrm{Vol}(O_n(\mathbb{R})) = \frac{2^{m(m+1)}\sqrt{2}^m\sqrt{\pi}^{n(n+1)/2}(m-1)!}{\sqrt{2}^{n(n-1)/2}\prod_{j=0}^{m-1}(j!\Gamma(j+3/2))b'_m},$$

where $b'_1 = a'_0 + 1$ and for every $m > 1$, $b'_m = \sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}a'_j$. The sequence $(a'_j)_{j\geq 0}$ is defined by $a'_0 = 1$ and $a_j = \frac{4j}{2j+1}a_{j-1} + 1$ for $j > 0$.

Proof of Proposition 3.11. When $m = 1$, Proposition 3.11 is a consequence of Proposition 3.14 since $b_1 = \frac{4}{3}(2\sqrt{2}-1)$. From Proposition 3.14, when $m > 1$, we have to compute the values of $b_m \in \mathbb{Q}[\sqrt{2}]$. For every $j \geq 0$, we deduce from the recurrence relation that

$$a_j = \frac{2^j}{2j+3}\left(8\sqrt{2} - 7 + 2\sum_{k=2}^{j}\frac{2k+1}{2^k}\right) + 1.$$

Writing $\sum_{k=0}^{j}\frac{2k+1}{2^k} = \left(\sum_{k=0}^{j}\frac{x^{2k+1}}{2^k}\right)'_{|x=1} = 6 - \frac{2j+5}{2^j}$, we deduce

$$\forall j > 0, \quad a_j = 8\sqrt{2}\frac{2^j}{2j+3} - \frac{4}{2j+3} - 1,$$

and as a consequence

$$b_m = 4\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\frac{2\sqrt{2}\,2^j - 1}{2j+3}.$$

Now,

$$\left(\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\frac{2\sqrt{2}\,2^j-1}{2j+3}x^{2j+3}\right)'$$
$$= \sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(2\sqrt{2}\,2^j - 1)x^{2j+2} = x^2(2\sqrt{2}(2x^2-1)^{m-1} - (x^2-1)^{m-1}),$$

so that

$$b_m = 8\sqrt{2}\int_0^1 x^2(2x^2-1)^{m-1}\,dx - 4\int_0^1 x^2(x^2-1)^{m-1}\,dx.$$

From the relations

$$\forall j>0,\quad \int_0^1 x^2(x^2-1)^j\,dx = \frac{-2j}{2j+3}\int_0^1 x^2(x^2-1)^{j-1},$$
$$\int_0^1 x^2(2x^2-1)^j\,dx = \frac{-2j}{2j+3}\int_0^1 x^2(2x^2-1)^{j-1} + \frac{1}{2j+3},$$

it follows that

$$b_m = (-1)^{m-1}\frac{4\sqrt{2}(m-1)!}{\Gamma(m+3/2)}\sum_{k=0}^{m-1}(-1)^k\frac{\Gamma(k+3/2)}{k!} + (-1)^m\frac{(m-1)!\sqrt{\pi}}{\Gamma(m+3/2)}.$$

Now, Proposition 3.11 follows from Proposition 3.14. □

Remark 3.16. An expression of $e_{\mathbb{R}}(n)$ in terms of hypergeometric series can be extracted from [8], whereas an equivalent in logarithmic scale can be deduced from [38].

3.2.4. Values of $e(p,q)$ for $p+q\le 3$. We have not been able to compute the numbers $e_{\mathbb{R}}(p,q)$ in general and only give their values for $p+q\le 3$. These values make the upper bounds given by Theorem 1.1 explicit for every smooth real projective manifold X of dimension less than or equal to four.

Lemma 3.17.

$$e_{\mathbb{R}}(1,0) = e_{\mathbb{R}}(0,1) = \frac{1}{\sqrt{2\pi}},$$
$$e_{\mathbb{R}}(2,0) = e_{\mathbb{R}}(0,2) = \frac{1}{4}(\sqrt{2}-1),\qquad e_{\mathbb{R}}(1,1) = \frac{1}{\sqrt{2}},$$
$$e_{\mathbb{R}}(p,3-p) = \frac{3}{4\sqrt{2\pi}} - \frac{(-1)^p}{2\sqrt{\pi}},\qquad \forall p\in\{0,\dots,3\}.$$

Proof. First note that $e_{\mathbb{R}}(1,0) = e_{\mathbb{R}}(0,1) = \frac{1}{2}e_{\mathbb{R}}(1) = 1/\sqrt{2\pi}$ from Proposition 3.9, since $\Gamma(3/2) = \sqrt{\pi}/2$. Proceeding as at the beginning of the proof of Proposition 3.9, we get

$$e_{\mathbb{R}}(2,0) = \frac{\mathrm{Vol}(O_2(\mathbb{R}))\sqrt{2}}{2\sqrt{\pi}}\int_{0<\lambda_1<\lambda_2<\infty}\det\begin{pmatrix}|\lambda_1| & \lambda_1^2\\ |\lambda_2| & \lambda_2^2\end{pmatrix}d\mu(\lambda_1)\,d\mu(\lambda_2) = 2\sqrt{2\pi}\,\psi_{00},$$

by (3.1) and Lemma 3.4 according to which $\mathrm{Vol}(O_2(\mathbb{R})) = 4\pi$. But $\psi_{00} = \frac{1}{8\sqrt{2\pi}}(\sqrt{2}-1)$ by (3.3), so that $e_{\mathbb{R}}(2,0) = e_{\mathbb{R}}(0,2) = \frac{1}{4}(\sqrt{2}-1)$. Likewise,

$$\begin{aligned} e_{\mathbb{R}}(1,1) &= \frac{\mathrm{Vol}(O_2(\mathbb{R}))\sqrt{2}}{2\sqrt{\pi}}\int_{-\infty}^0\int_0^\infty \det\begin{pmatrix}|\lambda_1| & \lambda_1^2\\ |\lambda_2| & \lambda_2^2\end{pmatrix}d\mu(\lambda_1)\,d\mu(\lambda_2)\\ &\overset{(3.1)}{=} 4\sqrt{2\pi}\,\eta_0\eta_1 = \frac{1}{\sqrt{2}}.\end{aligned}$$

Along the same lines we get

$$e_{\mathbb{R}}(3,0)=\frac{\mathrm{Vol}(O_3(\mathbb{R}))\sqrt{2}^3}{\sqrt{2\pi}^3}\Big(\eta_2\psi_{00}-\eta_0\psi_{10}-\eta_1\int_{0\leq x<y<\infty}|xy|(y^2-x^2)\,d\mu(x)\,d\mu(y)\Big).$$

From (3.2), $\mathrm{Vol}(O_3(\mathbb{R}))=16\pi^2$. From the recurrence relation in the proof of Proposition 3.9, we deduce that $\psi_{10}=\frac{1}{8\sqrt{\pi}}-\frac{7\sqrt{2}}{64\sqrt{\pi}}$. Finally, for all $i\geq 0$ and $j>0$,

$$\begin{aligned}\int_{0\leq x<y<\infty}&|xy|(x^{2i}y^{2j}-x^{2j}y^{2i})\,d\mu(x)\,d\mu(y)\\&=j\int_{0\leq x<y<\infty}|xy|(x^{2i}y^{2j-2}-x^{2j-2}y^{2i})\,d\mu(x)\,d\mu(y)+\frac{(i+j)!}{\pi 2^{i+j+2}},\end{aligned}$$

so that $\int_{0\leq x<y<\infty}|xy|(y^2-x^2)\,d\mu(x)\,d\mu(y)=\frac{1}{8\pi}$. It follows that

$$e_{\mathbb{R}}(3,0)=e_{\mathbb{R}}(0,3)=16\sqrt{\pi}\Big(\frac{(\sqrt{2}-1)}{16\sqrt{2\pi}}-\frac{1}{16\pi}+\frac{7}{64\sqrt{2\pi}}-\frac{1}{32\pi}\Big)=-\frac{1}{2\sqrt{\pi}}+\frac{3}{4\sqrt{2\pi}}.$$

Likewise,

$$\begin{aligned}e_{\mathbb{R}}(2,1)&=e_{\mathbb{R}}(1,2)\\&=\frac{\mathrm{Vol}(O_3(\mathbb{R}))\sqrt{2}^3}{\sqrt{2\pi}^3}\Big(-\eta_0\psi_{10}+\eta_2\psi_{00}+\eta_1\int_{0\leq x<y<\infty}|xy|(y^2-x^2)\,d\mu(x)\,d\mu(y)\Big)\\&=16\sqrt{\pi}\Big(\frac{(\sqrt{2}-1)}{16\sqrt{2\pi}}-\frac{1}{16\pi}+\frac{7}{64\sqrt{2\pi}}+\frac{1}{32\pi}\Big)=\frac{1}{2\sqrt{\pi}}+\frac{3}{4\sqrt{2\pi}}.\end{aligned}$$

□

Acknowledgements. We are grateful to Alice Guionnet and Gaëtan Borot for pointing out the reference [26] to us (see Remarks 3.2 and 3.3). The research leading to these results has received funding from the European Community's Seventh Framework Progamme ([FP7/2007-2013] [FP7/2007-2011]) under grant agreement n$^{\mathrm{o}}$ [258204].

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