



Deformations of associative submanifolds with boundary

Damien Gayet ^a, Frederik Witt ^{b,*}

^a *Université de Lyon, Université Lyon 1, CNRS, UMR5208, Institut Camille Jordan, 43,
boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France*

^b *Mathematisches Institut der Universität Münster, Einsteinstraße 62, D-48149 Münster, Germany*

Received 23 September 2009; accepted 21 September 2010

Available online 20 October 2010

Communicated by Daniel S. Freed

Abstract

Let M be a topological G_2 -manifold. We prove that the space of infinitesimal associative deformations of a compact associative submanifold Y with boundary in a coassociative submanifold X is the solution space of an elliptic problem. For a connected boundary ∂Y of genus g , the index is given by $\int_{\partial Y} c_1(\nu_X) + 1 - g$, where ν_X denotes the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$ and $c_1(\nu_X)$ the first Chern class of ν_X with respect to its natural complex structure. Further, we exhibit explicit examples of non-trivial index.

© 2010 Elsevier Inc. All rights reserved.

MSC: primary 53C38; secondary 35J55, 53C29, 58J32

Keywords: G_2 -manifolds; Calibrated submanifolds; Elliptic boundary problems on manifolds

1. Introduction

The deep and rich interplay between geometry and algebra on manifolds with G_2 -structure is reflected in the existence of special submanifolds, namely *associative* ones of dimension 3 and *coassociative* ones of dimension 4. These are particular instances of Harvey's and Lawson's *calibrated submanifolds* [19], a notion which also embraces complex submanifolds of a Kähler manifold or special Lagrangian submanifolds of a Calabi–Yau manifold. McLean [27] proved

* Corresponding author.

E-mail addresses: gayet@math.univ-lyon1.fr (D. Gayet), frederik.witt@uni-muenster.de (F. Witt).

that the infinitesimal coassociative deformations of a closed coassociative X is an unobstructed elliptic problem. The dimension of the moduli space is $b_+^2(X)$, i.e. the dimension of self-dual harmonic 2-forms on X . For associative submanifolds, the problem, though still elliptic, is more involved. Firstly, existence of smooth deformations is, as in the case of complex submanifolds, in general obstructed. Secondly, the *virtual* dimension, that is, the index of the elliptic equation, is always zero on dimensional grounds, so that no prediction on the existence of infinitesimal deformations can be made. This result was extended to arbitrary manifolds with topological G_2 -structure (whose holonomy is not necessarily contained in G_2) by Akbulut and Salur [3], who also address smoothness and compactness issues of the deformation spaces.

On the other hand, on symplectic manifolds one is naturally led to study the moduli space of (pseudo-)holomorphic curves with boundary in a Lagrangian submanifold [16,18]. In physics, Aganagic and Vafa translated this boundary problem for special Lagrangians of a Calabi–Yau into an open string problem [1], following Witten’s use of the moduli space of complex curves in the stringy world [30]. Now taking a Calabi–Yau 3-fold K times a circle yields a natural Riemannian manifold $M = K \times S^1$ with holonomy contained in G_2 . Moreover, holomorphic curves and special Lagrangians times a circle give examples of associative and coassociative submanifolds in M . In this way, the duality of complex versus special Lagrangian submanifolds is matched by the duality of associative versus coassociative submanifolds in a holonomy G_2 -manifold. On one side, this hints at the existence of a G_2 -analogue of Floer theory as conjectured in [13]. Further, it makes (co-)associatives play a key rôle in attempts to set up topological M -theory [4,12]. It is therefore natural to study deformations of (co-)associatives with boundary. Inspired by the work of Butscher [10], who investigated deformations of special Lagrangians with boundary on a symplectic, codimension 2 submanifold inside some compact Calabi–Yau, Kovalev and Lotay investigated in a recent paper the analogous problem for manifolds with closed G_2 -structures, where a compact coassociative has its boundary in a fixed, codimension 1 submanifold [24].

In this paper, we consider an associative Y inside topological G_2 -manifolds and study the space of infinitesimal associative deformations of Y with boundary inside a fixed coassociative X . We identify this space with the solutions of an elliptic boundary value problem whose index is given for connected boundary ∂Y of genus g by

$$\text{index}(X, Y) = \int_{\partial Y} c_1(\nu_X) + 1 - g.$$

Here, ν_X denotes the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$ and $c_1(\nu_X)$ the first Chern class of ν_X with respect to a natural complex structure we are going to define below. Further, we extend this result to 4-dimensional submanifolds X which do not contain any associative. In a sense, this class of submanifolds inside a (topological) G_2 -manifolds forms the natural counterpart of totally real submanifolds inside (almost) complex manifolds. Finally, assuming that Y is an embedded 3-disk, we associate with Y an element $\mu_{G_2}(\partial Y) \in \pi_2(G_2/SO(4)) \cong \mathbb{Z}_2$, which is best thought of as a G_2 -analogue of the Maslov index. Under suitable identifications, we show that

$$\mu_{G_2}(X, Y) = \int_{\partial Y} c_1(\nu_X) \bmod 2 = (\text{index}(X, Y) + 1) \bmod 2.$$

Explicit examples of pairs (X, Y) with non-trivial index will be given in Section 6. In particular, we shall construct compact pairs (X, Y) inside compact holonomy G_2 -manifolds using Joyce’s method [22].

The techniques we use are the standard ones from PDE theory; our reference is [6] whose conventions we shall follow throughout this paper.

2. The group G_2

We start by recalling some classical facts about G_2 (cf. for instance [3,8,19,22]).

2.1. Linear algebra

The octonions define a real 8-dimensional, non-associative division algebra $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$ generated by $\langle \mathbf{1}, i, j, k, e, e \cdot i, e \cdot j, e \cdot k \rangle$. Taking these generators as an orthonormal basis induces an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{O} compatible with the algebra structure. Further, we obtain a *vector cross product* taking values in the imaginary octonions $\text{Im } \mathbb{O} = \langle \mathbf{1} \rangle^\perp \cong \mathbb{R}^7$ defined by

$$u \times v = \text{Im}(\bar{v} \cdot u).$$

Here, \bar{v} is the natural conjugation which sends $v \in \text{Im } \mathbb{O}$ to $-v$. The term cross product is justified by the properties $u \times v = -v \times u$ and $|u \times v| = |u \wedge v|$. Over \mathbb{R}^7 , this yields the 3-form

$$\varphi_0(u, v, w) = \langle u \times v, w \rangle,$$

which with respect to the orthonormal basis $e_1 = i, e_2 = k, \dots, e_7 = e \cdot k$ can be written as¹

$$\varphi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56}). \tag{1}$$

We refer to any basis $\{e_j\}$ such that φ_0 is of the form (1) as a G_2 -frame, since the stabiliser of φ_0 inside $GL(7)$ is the real algebraic Lie group G_2 , which is of dimension 14. Conversely, any G_2 -invariant form $\varphi \in \Lambda^3 \mathbb{R}^{7*}$ induces a positive definite inner product $\langle \cdot, \cdot \rangle_\varphi$ and a cross product \times_φ as follows. Firstly, with φ we can associate a volume form μ_φ (which is somehow difficult to write down explicitly, cf. the appendix in [21]). Then we define

$$\langle u, v \rangle_\varphi = ((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi) / 6\mu_\varphi, \quad \langle u \times_\varphi v, w \rangle_\varphi = \varphi(u, v, w). \tag{2}$$

Next, we consider the *associator*

$$[u, v, w] = \frac{1}{2}((u \cdot v) \cdot w - u \cdot (v \cdot w))$$

which is totally skew-symmetric. The associated $\text{Im } \mathbb{O}$ -valued 3-form in $\Lambda^3 \mathbb{R}^{7*} \otimes \mathbb{R}^7$ will be written χ_0 . For this form, we have the important identity

$$\chi_0(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v. \tag{3}$$

In particular, we find $u \times (u \times a) = -|u|^2 \cdot a$ if a is orthogonal to u .

¹ This is the convention adopted in [3,8] and [22].

Finally we define a 4-form over \mathbb{R}^7 by

$$\psi_0(u, v, w, x) = \frac{1}{2} \langle u, [v, w, x] \rangle.$$

This form actually coincides with the Hodge dual of φ_0 , so that in a G_2 -frame $\{e_j\}$,

$$\psi_0 = \star\varphi_0 = -e^{12} \wedge (e^{47} + e^{56}) - e^{13} \wedge (e^{46} - e^{57}) + e^{23} \wedge (e^{45} + e^{67}) + e^{4567}.$$

2.2. G_2 -manifolds

Next consider a 7-dimensional manifold M . If the structure group $GL(7)$ reduces to G_2 , we say that M carries a G_2 -structure and refer to M as a *topological G_2 -manifold* or simply as a G_2 -manifold for sake of brevity. In this case, there exists a 3-form φ on M such that the associated G_2 -principal frame bundle consists of isomorphisms between $(T_x M, \varphi_x)$ and $(\mathbb{R}^7, \varphi_0)$ for $x \in M$. By an *abus de langage*, we refer to the defining 3-form φ itself as the G_2 -structure. Then there exists a vector cross product $\times = \times_\varphi$ on TM , inducing the structure of $\text{Im } \mathbb{O}$ on any tangent space $T_x M$. Moreover, formulae (2) give rise to a globally defined Riemannian metric $g = g_\varphi$ with Levi-Civita connection ∇^g . Similarly, there are global counterparts $\psi = \star_\varphi \varphi \in \Omega^4(M)$ and $\chi \in C^\infty(M, \Lambda^3 T^* M \otimes TM)$ of ψ_0 and χ_0 respectively. A G_2 -manifold is said to be *torsion-free* if $\nabla^g \varphi = 0$. By a result of Fernández and Gray [15], torsion-freeness is equivalent to $d\varphi = d\psi = 0$. Note that in this case, the holonomy of g is contained in G_2 . This is tantamount to saying that there exist coordinates around each $x \in M$ such that $\varphi(x) = \varphi_0 + O(|x|^2)$. In the sequel, we say that a torsion-free G_2 -manifold is a *holonomy G_2 -manifold*, if the holonomy actually equals² G_2 .

An oriented 3-dimensional submanifold Y is called *associative* if the pull-back of φ to Y is equal to the induced Riemannian volume form. Equivalently, the pull-back of χ to Y is identically zero. Furthermore, an oriented 4-dimensional submanifold X is called *coassociative* if the pull-back of ψ to X is equal to the induced Riemannian volume form. Equivalently, the pull-back of φ to X is identically zero. As shown by Harvey and Lawson, (co-)associative manifolds have the important property of being homologically volume minimising if the form φ (respectively ψ) is closed. In particular, following their language, φ and ψ define *calibrations*, and (co-)associative submanifolds are *calibrated*.

3. The geometry of the deformation problem

Let M be a G_2 -manifold and $Y \subset M$ a compact associative whose boundary ∂Y is contained in a fixed 4-submanifold $X \subset M$. We wish to describe the space of infinitesimal deformations of Y in the class of associatives with boundary in X , that is, the *Zariski tangent space* $T^{\text{Zar}} \mathfrak{M}_{X,Y}$ of

$$\mathcal{M}_{X,Y} = \{Y' \mid Y' \text{ associative isotopic to } Y \text{ with } \partial Y' \subset X\}.$$

3.1. The closed case

Our starting point is the closed case $\partial Y = \emptyset$, that is we first discuss the Zariski tangent space $T^{\text{Zar}} \mathfrak{M}_Y$ of $\mathcal{M}_Y = \{Y' \mid Y' \text{ associative isotopic to } Y \text{ with } \partial Y' = \emptyset\}$. In the case of associatives

² Note that some authors do not make this distinction and refer to any torsion-free G_2 -manifold as a holonomy G_2 -manifold or even simply as a G_2 -manifold.

inside *torsion-free* G_2 -manifolds, this problem was settled by McLean [27]. Later on, his result was generalised by Akbulut and Salur [2,3] to arbitrary G_2 -manifolds. We outline these results following the approach of [3].

Let (M, φ, ∇) be a G_2 -manifold endowed with a compatible connection ∇ , i.e. $\nabla\varphi \equiv 0$. Such connections always exist (cf. for instance [8]), but are not necessarily unique. In particular, ∇ preserves the metric and therefore induces a connection on the normal bundle $\nu \rightarrow Y$, which we also denote by ∇ . Since Y is compact, any deformation $Y \rightarrow Y_t$, $t \in (-\epsilon, \epsilon)$, can be assumed to be normal: Reparametrising by a time-dependent diffeomorphism if necessary, one can achieve that $\partial Y_t(p)/\partial t \in \nu_p$. Consequently, nearby (i.e. C^1 -close) Y_t can be identified with sections $s_t \in C^\infty(Y, \nu)$ via the exponential map. The associative Y then corresponds to $s_0 =$ zero-section of ν . Pulling TY_t back to $TM|_Y$ through parallel transport with respect to ∇ along the curves $\lambda \mapsto \exp_p(\lambda s_t(p))$, $p \in Y$, we obtain a curve in the fibre $G_3(TM)_p = G_3(T_pM)$. For $t = 0$, it passes through $E_p := T_pY$, which is an element in $G_\varphi(TM)_p$, the subset of calibrated 3-planes in $G_3(TM)_p$. The derivative of this curve at 0 can be identified with $\nabla \dot{s}_0(p)$, and the linearised condition is thus $\nabla \dot{s}_0(p) \in T_{E_p}G_\varphi(M) \subset T_{E_p}G_3(M) \cong E_p^* \otimes \nu_p$. Moreover, the vector cross product gives rise to an exact sequence

$$0 \rightarrow T_{E_p}G_\varphi(TM) \rightarrow T_{E_p}G_3(TM) \cong E_p^* \otimes \nu_p \xrightarrow{\times} \nu_p \rightarrow 0.$$

Picking an orthonormal basis e_1, e_2, e_3 of E_p we obtain the condition $\nabla \dot{s}_0(p) \in T_{E_p}G_3^\varphi(M)$ if and only if $\sum e_i \times \nabla_{e_i} \dot{s}_0(p) = 0$. An invariant formulation of this equation can be given in terms of a *Dirac operator*.³ The fibre bundles TY and ν are associated with an $SO(4)$ -representation. Further, since Y is spinnable the principal $SO(4)$ -bundle can be lifted to a principal $Spin(4)$ -bundle. Then ν is associated with the tensor product of a spin representation and some other representation of $Spin(4)$. As a result we may regard ν as a *twisted spinor bundle*. Under this identification, the operator $\sum e_i \times \nabla$ becomes the Dirac operator of (ν, ∇) . Summing up, we arrive at a generalised version of McLean’s result.

Theorem 3.1. (See [3,27].) *Let (M, φ) be a G_2 -manifold and $Y \subset M$ an associative. Then $T^{\text{Zar}}\mathfrak{M}_Y$ can be identified with the kernel of a twisted Dirac operator $\mathbf{D}^\nabla : C^\infty(Y, \nu) \rightarrow C^\infty(Y, \nu)$ taken with respect to a connection ∇ induced by a compatible connection of (M, φ) . In particular, we obtain the natural Dirac operator on ν with respect to ∇^g if (M, φ) is torsion-free.*

In the sequel, we denote the Dirac operator \mathbf{D}^∇ of Y simply by \mathbf{D} .

Remark. The *Dirac equation* $\mathbf{D}s = 0$ is elliptic, and as a consequence, the *virtual dimension* of \mathfrak{M}_Y , that is, $\dim T^{\text{Zar}}\mathfrak{M}_Y$, is finite. Furthermore, Y being odd-dimensional, we have $\text{index}(\mathbf{D}) = \dim \ker(\mathbf{D}) - \dim \text{coker}(\mathbf{D}) = 0$. In generic situations where one expects the cokernel to vanish, we would get $\dim T^{\text{Zar}}\mathfrak{M}_Y = 0$, and associatives would be rigid objects. However, we have no a priori control on the virtual dimension in terms of topological datum. This stands in sharp contrast to the deformations of a coassociative X , where McLean showed that the (actual) dimension of the (smooth) moduli space \mathfrak{M}_X is $b_+^2(X)$, the dimension of harmonic self-dual 2-forms [27].

³ For sake of brevity, we refer to any operator \mathbf{D} as a Dirac operator if it is of *Dirac type*, that is, the principal symbol of \mathbf{D}^2 satisfies $\sigma(\mathbf{D}^2)(p, \xi) = \|\xi\|^2$, cf. also [6, Section 3].

3.2. The geometry on the boundary

Next assume that the associative Y has a non-empty boundary ∂Y inside a fixed coassociative X . We first need to understand the geometry on the boundary of Y .

Fix a collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon)$ of ∂Y and let u denote the inward pointing unit vector field defined on \mathcal{C} . As before, $\nu \rightarrow Y$ denotes the normal bundle, as well as its restriction to ∂Y . As remarked in [2] and [3], $\nu|_{\mathcal{C}}$ carries a hermitian structure near the boundary induced by u , namely

$$J : \nu \rightarrow \nu, \quad J(x) = u \times x.$$

This acts indeed as an isometry with respect to g , as

$$g(Ja, Jb) = \varphi(u, a, u \times b) = -g(u \times (u \times b), a) = g(a, b)$$

for any $a, b \in \nu|_{\mathcal{C}}$. Let $\nu_X \subset TX|_{\partial Y}$ denote the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$, i.e.

$$TX|_{\partial Y} = T\partial Y \oplus \nu_X.$$

Lemma 3.2. *For the bundle $\nu \rightarrow \partial Y$ the following holds:*

1. *The bundle ν_X is contained in ν and is stable under J .*
2. *The orthogonal complement μ_X of ν_X in ν is also stable under J .*
3. *Viewing $T\partial Y$, ν_X and μ_X as J -complex bundles, we have*

$$\bar{\mu}_X \cong \nu_X \otimes_{\mathbb{C}} T\partial Y,$$

that is $\mu_X^{0,1} \cong \nu_X^{1,0} \otimes T^{1,0}\partial Y \cong \nu_X^{1,0} \otimes \bar{K}_{\partial Y}$, where $K_{\partial Y}$ is the canonical line bundle over ∂Y .

Proof. Let us fix a local orthonormal basis (u, v, w) of $TY|_{\partial Y}$ by choosing a local unit vector field $v \in T\partial Y$, i.e.

$$TY|_{\partial Y} \stackrel{\text{loc}}{=} \langle u, v, w \rangle.$$

The vector field w is defined by $w = u \times v$, which lies in $T\partial Y$ in virtue of the associativity of Y . If $a \in \nu_X$, then $g(a, u) = 0$, for $v \times w = u$ and $\varphi(v, w, a) = 0$, X being coassociative. Clearly, the vectors $a \times v$ and $a \times w$ are orthogonal to v and w as well as to u , since

$$g(a \times v, u) = \varphi(a, v, u) = -g(u \times v, a) = -g(w, a) = 0,$$

and similarly for $a \times w$. Hence $a \times v, a \times w \in \nu$. Further, these vectors are orthogonal to TX , for $a, v, w \in TX$ and X is coassociative, so that for $n \in \nu_X$ we find $g(a \times v, n) = \varphi(a, v, n) = 0$, etc. Hence

$$\mu_X \stackrel{\text{loc}}{=} \langle a \times v, a \times w \rangle.$$

As a consequence, $u \times a \in v$ is orthogonal to μ_X (for $g(u \times a, a \times v) = \varphi(u, a, a \times v)$, etc.), so that

$$v_X \stackrel{\text{loc}}{=} \langle a, Ja = u \times a \rangle, \quad TX|_{\partial Y} \stackrel{\text{loc}}{=} \langle v, w, aJa \rangle.$$

This shows that v_X is stable under J . On the other hand, $g(u \times (a \times v), a) = \varphi(u, (a \times v), a) = 0$ and similarly $g(u \times (a \times v), u \times a) = 0$, hence $u \times (a \times v) \in \mu_X$ which shows that μ_X is also stable under J .

The Riemann surface structure on ∂Y is induced by the hermitian structure $J = u \times$, for $Jv = u \times v = w$ and $Jw = u \times w = -v$ (to keep notation tight we abuse notation and also write J for the endomorphism on $T\partial Y$ induced by $u \times$). The map

$$a \otimes y \in v_X \otimes_{\mathbb{C}} T\partial Y \mapsto a \times y \in \bar{\mu}_X,$$

where we now view v_X and $\bar{\mu}_X$ as complex line bundles via J , is well-defined and a *real* bundle isomorphism. It remains to see that it is complex-linear, i.e.

$$Ja \times y = a \times Jy = -J(a \times y).$$

This is equivalent to $(u \times a) \times y$ and $a \times (u \times y)$ being equal to $-u \times (a \times y)$. But this follows from (3) and the skew-symmetry of χ . \square

Let $\mathbf{B}: C^\infty(\partial Y, v) \rightarrow C^\infty(\partial Y, \mu_X) \subset C^\infty(\partial Y, v)$ be the orthogonal projector⁴ taking smooth sections of $v = v_X \oplus \mu_X$ to μ_X . As a corollary to the generalised version of McLean’s theorem as given in Theorem 3.1 and Lemma 3.2, we obtain:

Corollary 3.3. *The Zariski tangent space of $\mathcal{M}_{X,Y}$ can be identified with solutions of the system*

$$\mathbf{D}s = 0, \quad \mathbf{B}(s|_{\partial Y}) = 0, \quad s \in C^\infty(Y, v). \tag{4}$$

4. Ellipticity and index

In view of applying the standard machinery of index theory for manifolds with boundary, we consider the complexification of Eq. (4), namely

$$(\mathbf{D}^{\mathbb{C}} \oplus \mathbf{B}^{\mathbb{C}})(\sigma \oplus \sigma|_{\partial Y}) = 0, \quad s \in C^\infty(Y, v^{\mathbb{C}}) \tag{5}$$

with operators $\mathbf{D}^{\mathbb{C}}$ and $\mathbf{B}^{\mathbb{C}}$ extended to the complexified bundles $v^{\mathbb{C}} = v \otimes \mathbb{C}$ and $\mu_X^{\mathbb{C}} = \mu_X \otimes \mathbb{C}$. However, we shall not distinguish between the original operators and their complexification in the sequel for sake of keeping notation tight. To show that the kernel is finite-dimensional requires a suitable ellipticity condition. We first introduce the *Calderón projector* $\mathbf{Q}_{\mathbf{D}}$ associated with a Dirac operator \mathbf{D} (cf. [6, Theorem 12.4]). This is a zero order pseudo-differential operator

⁴ By an *orthogonal projector* we understand an operator \mathbf{P} of order 0 satisfying $\mathbf{P} = \mathbf{P}^2 = \mathbf{P}^*$.

$$\mathbf{Q_D} : C^\infty(\partial Y, \nu) \rightarrow \mathcal{H}(\mathbf{D}) = \{s|_{\partial Y} \mid s \in C^\infty(Y, \nu), \mathbf{D}s = 0\} \subset C^\infty(\partial Y, \nu)$$

mapping smooth sections of ν over ∂Y to the space of Cauchy data⁵ of \mathbf{D} .

Definition 4.1. (Cf. [6, Definition 18.1].) Let Y be an arbitrary smooth manifold with boundary, $\nu \rightarrow Y$ a (twisted) spinor bundle and $\mu \rightarrow \partial Y$ a vector bundle. A pseudo-differential operator $\mathbf{B} : C^\infty(\partial Y, \nu) \rightarrow C^\infty(\partial Y, \mu)$ of order 0 is said to define an *elliptic boundary condition* (abbreviated e.b.c.) if and only if

1. the extension $\mathbf{B}^{(s)} : H^s(\partial Y, \nu) \rightarrow H^s(\partial Y, \mu)$ to the chain of Sobolev spaces $H^s(\partial Y, \nu)$ and $H^s(\partial Y, \mu)$, $s \geq 0$, has closed range;
2. the restriction of the principal symbol $b = \sigma(\mathbf{B})|_{\text{range}(\sigma(\mathbf{Q_D}))} : \text{range}(\sigma(\mathbf{Q_D})) \rightarrow \text{range}(b)$ is an isomorphism.

Furthermore, an e.b.c. is said to be *local*, if in addition $\text{range}(p, \xi) = \nu_p$ holds for all $p \in \partial Y$ (in this case, 1. is automatically satisfied, cf. [6, Remark 18.2]).

The following theorem summarises the main properties of local e.b.c.

Theorem 4.2. (See [6, Theorems 19.1, 20.12, 20.13 and 21.5].) If \mathbf{B} defines a local e.b.c., then

1. regularity holds, that is, $s \in H^s(Y, \nu) \cap \ker(\mathbf{D} \oplus \mathbf{B})$ implies $s \in C^\infty(Y, \nu)$;
2. the operator $\mathbf{D} \oplus \mathbf{B}$ is Fredholm with index

$$\text{index}(\mathbf{D} \oplus \mathbf{B}) = \text{index}(\mathbf{BQ_D} : \mathcal{H}(\mathbf{D}) \rightarrow C^\infty(\partial Y, \nu));$$

3. $\text{index}(\mathbf{D} \oplus \mathbf{B})$ depends only on the homotopy type of the principal symbols involved.

Furthermore, if Y is odd-dimensional, the orthogonal projector \mathbf{P}^\pm onto the space of positive and negative half-spinors ν^\pm over the (even-dimensional) boundary always defines a local e.b.c. with $\text{index}(\mathbf{D} \oplus \mathbf{P}^\pm) = 0$.

Remark. For even-dimensional manifolds the existence of local e.b.c. is topologically obstructed (e.g. [5, Section II.7.B]).

The following result is a valuable tool for explicit index computations.

Proposition 4.3. (See [6, Theorem 21.2].) Let $\mathbf{D} : C^\infty(Y, \nu) \rightarrow C^\infty(Y, \nu)$ be a Dirac operator on some (twisted) spinor bundle ν over an odd-dimensional manifold Y with boundary. Further, consider two orthogonal projectors onto subbundles $\nu_{1,2}$ of $\nu|_{\partial Y}$, $\mathbf{B}_1 : C^\infty(\partial Y, \nu) \rightarrow C^\infty(\partial Y, \nu_1)$ and $\mathbf{B}_2 : C^\infty(\partial Y, \nu) \rightarrow C^\infty(\partial Y, \nu_2)$, and suppose they define a local e.b.c. Then

$$\text{index}(\mathbf{D} \oplus \mathbf{B}_2) - \text{index}(\mathbf{D} \oplus \mathbf{B}_1) = \text{index}(\mathbf{B}_2 \mathbf{Q_D} \mathbf{B}_1^* : C^\infty(\partial Y, \nu_1) \rightarrow C^\infty(\partial Y, \nu_2)).$$

⁵ We are glossing over some technical details such as the passing to the “closed double” $M = Y \cup_{\partial Y} Y$, cf. [6, Chapters 9, 11 and 12].

Coming back to Eq. (5), we shall write $\text{index}(X, Y)$ for $\text{index}(\mathbf{D} \oplus \mathbf{B})$. Let $\bar{\partial}_{\nu_X}$ denote the Cauchy–Riemann operator associated with the natural complex structure (or equivalently, the natural orientation) of ν_X (cf. Lemma 3.2). Finally, we are in a position to prove the central theorem of this paper.

Theorem 4.4. *The operator \mathbf{B} in (5) defines a local e.b.c. with index*

$$\text{index}(X, Y) = \text{index}(\bar{\partial}_{\nu_X}).$$

Proof. Let us fix some collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon)$ of ∂Y for which we may assume the Riemannian structure to be a product (possibly after homotopically deforming the metric). Further, we complete the inward pointing coordinate vector u to a local orthonormal basis $(v(y, t), w(y, t))$ of $T_y \partial Y \times \{t\}$ such that $u \times v = w$. Near the boundary, we have the decomposition $\mathbf{D} = u \times (\nabla_u + \mathbf{R})$ with $\nabla_u = \partial_u$ and

$$\mathbf{R} = w \times \nabla_v - v \times \nabla_w, \tag{6}$$

as follows from $(a \times b) \times c = -a \times (b \times c)$ valid whenever $\{a, b, c\}$ is an orthogonal family, cf. (3). Note that the bundles of positive and negative half-spinors ν^\pm inside $\nu^\mathbb{C}$ are just the eigenspaces of $J = u \times$.

Locally, we will work with the following basis of $\nu^\mathbb{C}$: Choose a nowhere vanishing local section $a \in C^\infty(\partial Y, \nu_X)$ so that

$$\nu_X^\mathbb{C} \stackrel{\text{loc}}{=} \langle \alpha = a - iJa, \bar{\alpha} = a + iJa \rangle,$$

cf. Lemma 3.2. Consider then the sections $\beta = -v \times \bar{\alpha}$ and $\bar{\beta} = -v \times \alpha$. Again, the lemma implies that

$$\mu_X^\mathbb{C} \stackrel{\text{loc}}{=} \langle \beta, \bar{\beta} \rangle.$$

Furthermore,

$$\nu^+ \stackrel{\text{loc}}{=} \langle \alpha, \beta \rangle, \quad \nu^- \stackrel{\text{loc}}{=} \langle \bar{\alpha}, \bar{\beta} \rangle.$$

As an example, take $J\alpha = Ja + ia = i\alpha$ and $J\bar{\beta} = v \times (u \times \alpha) = -i\bar{\beta}$, etc. For any subsequent matrix representation over $\nu^\mathbb{C}$, the ordered basis $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$ shall be used.

Checking that \mathbf{B} defines a local e.b.c. requires the principal symbol $q = \sigma(\mathbf{Q}_\mathbf{D})$ of the associated Calderón operator. By the Calderón–Seeley theorem (cf. [6, Theorem 12.4]), q is the projector onto the eigenspace of $\sigma(\mathbf{R})$ corresponding to eigenvalues with positive real part. With respect to our fixed local ordered basis of $\nu^\mathbb{C}$ around $p \in \partial Y$, v and w act as

$$v \times = \begin{pmatrix} \mathbf{0} & 0 & 1 \\ 0 & 1 & \mathbf{0} \\ -1 & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad w \times = \begin{pmatrix} \mathbf{0} & 0 & -i \\ 0 & i & \mathbf{0} \\ -i & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

This follows from $v \times \alpha = -\bar{\beta}$, $w \times \alpha = -u \times (v \times \alpha) = -i\bar{\beta}$, etc. For $(\eta_v, \eta_w) \in T_p^* \partial Y \setminus \{0\}$ of unit norm, we deduce from (6) (with $\eta = \eta_v + i\eta_w$) that

$$\sigma(\mathbf{R})(p, \eta) = i(\eta_v \cdot w \times -\eta_w \cdot v \times) = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} 0 & \bar{\eta} \\ -\bar{\eta} & 0 \end{smallmatrix} \\ \eta & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & r_-(p, \eta) \\ r_+(p, \eta) & \mathbf{0} \end{pmatrix}.$$

Now $r_+(p, \eta)^* = r_-(p, \eta)$ and $r_+(p, \eta) = r_-(p, \eta)^{-1}$, so that $\sigma(\mathbf{R})(p, \eta)^* = \sigma(\mathbf{R})(p, \eta) = \sigma(\mathbf{R})(p, \eta)^{-1}$. Consequently, the eigenvalues are ± 1 , and the projector on the eigenspace associated with 1 is given by

$$q(p, \eta) = \frac{1}{2} \begin{pmatrix} \text{Id}_2 & r_-(p, \eta) \\ r_+(p, \eta) & \text{Id}_2 \end{pmatrix}.$$

On the other hand, \mathbf{B} is the orthogonal projector onto $\mu_X^{\mathbb{C}}$, so that its principal symbol is the matrix (taken with respect to the fixed basis of $v^{\mathbb{C}}$ and $\{\beta, \bar{\beta}\}$ of $\mu_X^{\mathbb{C}}$)

$$\sigma(\mathbf{B})(p, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is of full rank. Since

$$\sigma(\mathbf{B}) \circ q(p, \eta) = \frac{1}{2} \begin{pmatrix} 0 & 1 & -\bar{\eta} & 0 \\ \eta & 0 & 0 & 1 \end{pmatrix}$$

is also of full rank, the boundary condition defined by \mathbf{B} is local elliptic according to Definition 4.1.

It remains to compute the index. As in Theorem 4.2, let \mathbf{P}^+ denote the orthogonal projector onto v^+ . We recall that according to this theorem, \mathbf{P}^+ defines a local e.b.c. with vanishing index. In virtue of Proposition 4.3 and the established local ellipticity of \mathbf{B} ,

$$\text{index}(X, Y) = \text{index}(\mathbf{D} \oplus \mathbf{B}) = \text{index}(\mathbf{BQ}_D\mathbf{P}^+ : C^\infty(\partial Y, v^+) \rightarrow C^\infty(\partial Y, \mu_X^{\mathbb{C}})).$$

But the symbol of $\mathbf{BQ}_D\mathbf{P}^+$ is just

$$\begin{aligned} \sigma(\mathbf{BQ}_D\mathbf{P}^+)(p, \eta) &= \sigma(\mathbf{B}) \circ q \circ \sigma(\mathbf{P}^+)(p, \eta) \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 & -\bar{\eta} & 0 \\ \eta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} : v_p^+ \rightarrow (\mu_X^{\mathbb{C}})_p, \end{aligned}$$

where the matrix is taken with respect to the basis $\{\alpha, \beta\}$ of S^+ and $\{\beta, \bar{\beta}\}$ of $\mu_X^{\mathbb{C}}$. In particular, the symbol sends β to β and therefore acts as the identity on $\mu_X^{1,0} = v^+ \cap \mu_X^{\mathbb{C}}$. On the other hand, the induced map $v_X^{1,0} = v^+ \cap v_X^{\mathbb{C}} \rightarrow \mu_X^{0,1} = v^- \cap \mu_X^{\mathbb{C}}$ is up to $-i$ the symbol of the Cauchy–Riemann operator $\bar{\partial}_{v_X}$ on $v_X^{1,0}$ after the identification $\mu_X^{0,1} \cong v_X^{1,0} \otimes \bar{K}_{\partial Y}$ (cf. Lemma 3.2). Indeed, on a

trivialisation of $v_X^{\mathbb{C}}$ it acts as $\bar{\partial}_{v_X} = (\partial_1 + i\partial_2)/2$, where (x_1, x_2) are coordinates such that $\partial_1(p) = v(p)$ and $\partial_2(p) = w(p)$. Hence $\sigma(\bar{\partial}_{v_X})(p, \eta) = i(\eta_v + i\eta_w)/2$ which is what we wanted. \square

As a consequence of Riemann–Roch, we obtain

Corollary 4.5. *Let Σ_{g_j} be the connected components of ∂Y of genus g_j , and $c_1(v_{X|\Sigma_{g_j}})$ be the first Chern classes of $v_{X|\Sigma_{g_j}}$. Then*

$$\text{index}(X, Y) = \sum_j \int_{\Sigma_{g_j}} c_1(v_{X|\Sigma_{g_j}}) + 1 - g_j.$$

5. ψ -Positive boundary conditions

For the deformation problem (5) the bundle v_X is the only non-intrinsic piece of datum attached to Y , and its properties were derived using the coassociativity of X . In view of the previous theorem, we may relax the coassociativity condition as follows:

Definition 5.1. Let (M, φ) be a G_2 -manifold and $\psi = \star\varphi$. An orientable 4-submanifold of M is said to be ψ -positive at $p \in X$ if and only if its tangent space at p is ψ -positive for some orientation, i.e. $\psi_p|_{T_p X} > 0$. Further, X is called ψ -positive if for a suitable orientation X is ψ -positive at all points $p \in X$.

Obvious examples of ψ -positive submanifolds are coassociatives. An alternative characterisation of ψ -positivity is the notion of φ -freeness going back to recent work of Harvey and Lawson [20].

Lemma 5.2. *A suitably oriented 4-plane $F \in G_4(\mathbb{R}^7)$ is ψ -positive at $p \in X$ if and only if F is φ -free, i.e. F contains no associative 3-plane.*

Proof. If $F \in G_4(T_x M)$ is not φ -free, then F contains an associative subplane E . Hence $\varphi|_{E^\perp} \equiv 0$ and in particular $\varphi|_{F^\perp} = \psi|_F \equiv 0$, which proves necessity. Conversely, assume that $\psi|_F \equiv 0$. We write $F^\perp = u \wedge v \wedge w$ and choose a vector a in the orthogonal complement to the linear span of $u, v, w, u \times v, u \times w, v \times w$, which at most is six-dimensional. As a result, the 4-plane E^\perp spanned by F^\perp and a is coassociative, for $\varphi|_{E^\perp} \equiv 0$. Hence $E \subset F$ is associative. \square

Put differently, for any $a, b \in T_p X$, we have $a \times b \notin T_p X$ if X is ψ -positive at p . From this point of view the class of ψ -positives inside G_2 -manifolds naturally matches the class of totally real submanifolds inside Kähler manifolds (which, in particular, comprises Lagrangians).

Next we wish to investigate the Zariski tangent space of $\mathfrak{M}_{X,Y}$ under the assumption that X is merely ψ -positive. To that end, let $N_X \rightarrow \partial Y$ be the orthogonal complement of $T\partial Y$ in $TX|_{\partial Y}$, and define v_X to be the image of N_X under the orthogonal projection $\mathbf{P}: TM|_Y \rightarrow v|_Y$. As before, $\mu_X \rightarrow \partial Y$ denotes the orthogonal complement of v_X in $v|_{\partial Y}$. The geometry on the boundary is specified by the following

Lemma 5.3. *If X is ψ -positive, then*

1. *The restriction of \mathbf{P} to N_X defines an isomorphism onto $\nu_X = \mathbf{P}(N_X)$.*
2. *For any non-zero $b \in \mu_X$, $Jb = u \times b \notin \nu_X$, that is for the orthogonal projection \mathbf{p}_{μ_X} on μ_X we have $\mathbf{p}_{\mu_X}(Jb) \neq 0$.*

Proof. The kernel of $\mathbf{P}|_{TM|_{\partial Y}}$ is $TY|_{\partial Y}$. Since TX is ψ -positive, $N_X \cap \ker \pi = \{0\}$ according to the previous proposition, whence the first assertion. Next, suppose there is a $b_0 \in \mu_X$ of unit norm such that $Jb_0 \in \nu_X$. Let $b_1 \in \mu_X$ be a vector orthogonal to b_0 . Then Jb_1 lies in ν_X , for it is orthogonal to b_1 and $g(b_0, Jb_1) = -g(Jb_0, b_1) = 0$. By 1. there exist uniquely determined $n_{0,1} \in N_X$ with $\mathbf{P}(n_{0,1}) = Jb_{0,1}$. Since $N_X \perp T\partial Y$, we have in fact $n_{0,1} = Jb_{0,1} + \lambda_{0,1}u$ for $\lambda_{0,1} \in \mathbb{R}$ (where u denotes again the inward pointing normal vector on the boundary). It is straightforward to check that the orthogonal complement of $T_{\pi(b_0)}X$ in $T_{\pi(b_0)}M$ is spanned by $A = b_0$, $B = b_1$ and $C = u - \lambda_0 Jb_0 - \lambda_1 Jb_1$. Moreover, $v = A \times B \in TY|_{\partial Y}$ belongs in fact to $T\partial Y$, for $g(u, v) = \varphi(b_0, b_1, u) = -g(Jb_1, b_0) = 0$. Consequently, $\varphi(A, B, C) = g(A \times B, C) = 0$ which contradicts the ψ -positiveness of TX . \square

Consequently, $N = N_X \oplus \mu_X$ and $\nu_X \oplus \mu_X$ are isomorphic subbundles of $TM|_{\partial Y}$. Further, we can extend $N \rightarrow \partial Y$ to a new subbundle $N \subset TM|_Y$ transversal to TY , and adapt Corollary 3.3 to this more general situation: If N_X is orthogonal to $TY|_{\partial Y}$ (this does not imply the fibres of N to be coassociative), we can take $N = \nu$ and identify $T^{\text{Zar}}\mathcal{M}_{X,Y}$ with the space of solutions of (5). Otherwise first extend N_X trivially on some collar neighbourhood. Since being isomorphic is a stable property, we can homotope N_X over \mathcal{C} into a new bundle (still written N_X) with fibres in ν_X sufficiently far away from the boundary. Then extend $N = N_X \oplus \mu_X$ by ν . A section $s \in C^\infty(Y, N)$ induces a family of submanifolds Y_t with boundary in X if $s|_{\partial Y}$ lies in the kernel of $\mathbf{B}: C^\infty(\partial Y, N) \rightarrow C^\infty(\partial Y, \mu_X)$, the orthogonal projection onto μ_X in $N|_{\partial Y}$. Further, for Y_t to be associative we need the normal component $\mathbf{P}(s)$ of s to be harmonic. Thus $T^{\text{Zar}}\mathcal{M}_{X,Y}$ can be identified with the solution space of

$$\mathbf{DP}(s) = 0, \quad \mathbf{B}(s|_{\partial Y}) = 0, \quad s \in C^\infty(Y, N). \tag{7}$$

Proposition 5.4. *Let $Y \subset M$ be an associative with boundary inside a ψ -positive submanifold X . Then*

$$\text{index}(X, Y) = \text{index}(\tilde{\partial}_{\nu_X}).$$

Remark. Here, the Cauchy–Riemann operator on ν_X is defined by the complex structure coming from the induced orientation on $\nu_X \cong N_X$.

Proof of Proposition 5.4. For simplicity we assume $\mathbf{P} = \text{Id}$, i.e. N_X is orthogonal to $TY|_{\partial Y}$. We choose a nowhere vanishing local section a of ν_X and extend $b = -v \times a$ to a local trivialisation $\{b, \tilde{b}\}$ of μ_X . By the previous lemma we may take \tilde{b} to be the orthogonal projection of Jb to μ_X . Let $(0, 1, s, t)$ be the coordinates of \tilde{b} with respect to the local basis $\{b, Jb, v \times b, w \times b\}$ of ν , where v is a nowhere vanishing local section of $T\partial Y$ and $w = Jv = u \times v$. The latter basis gives

rise to the basis $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$ of $v^{\mathbb{C}}$ as given in the proof of Theorem 4.4 with respect to which the matrix of $\sigma(\mathbf{B}^{\mathbb{C}})$ can be written as

$$\sigma(\mathbf{B}^{\mathbb{C}})(x, \eta) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ z & -i & \bar{z} & i \end{pmatrix},$$

where $z = s + it$. For $\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+$, we therefore find

$$\sigma(\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+)(x, \eta) = \frac{1}{2} \begin{pmatrix} \eta & 1 \\ z + i\eta & -i - z\eta \end{pmatrix} : S_x^+ \rightarrow (\mu_X^{\mathbb{C}})_x. \tag{8}$$

The determinant of this matrix is $(-2i\eta - z - \bar{z}\eta^2)/4$, and multiplication with $\bar{\eta}$ shows this to vanish only if $\Re(\bar{T}\eta) = -i$. Hence the system (7) is still elliptic. Furthermore, $F(Jb, t) = t\mathbf{p}_{v_X}(Jb) \oplus \mathbf{p}_{\mu_X}(\tilde{J}b)$ is a global homotopy deforming $J\mu_X$ into μ_X and which in particular deforms Jb into \tilde{b} . Consequently, the symbol (8) is homotoped into the symbol of $\bar{\partial}_{v_X}$. By homotopy invariance we recover the same index as before. \square

6. Examples

In this section we construct explicit examples of pairs (X, Y) with $\text{index}(X, Y) \neq 0$. Throughout this section we denote by Σ_g a surface diffeomorphic to a compact oriented Riemann surface of genus g .

Local coassociative submanifolds. The first example will be local in nature, that is, the boundary of the compact associative will be contained in a local coassociative submanifold. Existence will be established by using Cartan–Kähler theory which requires all geometric objects involved (manifolds, boundaries, maps, etc.) to be real analytic. Note that a torsion-free G_2 -manifold (M, g) is Ricci-flat so that the underlying Riemannian metric is real analytic in harmonic coordinates [11]. Since φ is harmonic with respect to g (cf. Section 2.2), it will be real analytic in these coordinates, too. Consider then an associative Y with real analytic boundary ∂Y , and a nowhere vanishing real analytic section $a \in C^\infty(\partial Y, \nu)$. The geodesic flow $\gamma_a : \partial Y \times (-\epsilon, \epsilon) \rightarrow M$ induced by a is also real analytic in harmonic coordinates and therefore generates an analytical submanifold N of dimension 3. Further, $\varphi(v, w, a) = 0$ for $v, w \in T\partial Y$. We conclude that the pull-back of φ to N vanishes identically, for $\nabla\varphi = 0$. A Cartan–Kähler type argument invoked by Harvey and Lawson [19] (see also [7]) shows that N is contained in a local coassociative X . Further, ν_X is generated by a and $u \times a$, where u denotes again the inward pointing normal vector field of ∂Y . Thus $c_1(\nu_X) = 0$, whence $\text{index}(X, Y) = 1 - g$. For example, taking $M = \text{Im } \mathbb{H} \oplus \mathbb{H}$ and $\partial Y = \Sigma_g \subset \text{Im } \mathbb{H}$, where Σ_g is a g -handle body in $\text{Im } \mathbb{H} \cong \mathbb{R}^3$ and Y the relatively compact interior bounded by Σ_g , yields examples of arbitrary negative index. An actual deformation is given by moving Y along the straight line determined by a . In fact, taking for Σ_0 the standard 2-sphere bounding the unit ball, one obtains a smooth moduli space which is thus of actual dimension 1 (cf. [17]).

Local associative submanifolds. In the same vein, consider a real analytic surface Σ_g inside some real analytic coassociative X in M . Again, Cartan–Kähler theory yields the existence of a local associative Y containing Σ_g [19,28]. Using a collar neighbourhood of Σ_g inside Y we can construct an associative of the form $\Sigma_g \times [0, 1]$ which we keep on denoting by Y for simplicity.

Further, we can translate X into a submanifold X' containing $\Sigma_g \times \{1\}$ by a suitable diffeomorphism C^1 -close enough to the identity. Of course, there is no reason for this diffeomorphism to preserve the G_2 -structure, so that X' will not be coassociative in general. However, as ψ -positivity is a pointwise open condition, X' will be still ψ -positive for suitable choices, and the generalised deformation theorem applies. To compute the index we note that $N_{X'} \rightarrow \Sigma_g \times \{1\}$ is obtained by translating $\nu_X \rightarrow \Sigma_g \times \{0\}$, but the orientation flips as u which points inward at $\Sigma \times \{0\}$ points outward at $\Sigma_g \times \{1\}$. Hence $c_1(\nu_X) = -c_1(\nu_{X'})$. Since the orientation also flips on $\Sigma_g \times \{1\}$ we finally get $\text{index}(X \cup X', Y) = 2 \text{index}(\partial_{\Sigma_g})$.

The Bryant–Salamon construction. In [9] Bryant and Salamon constructed holonomy G_2 -metrics on (an open set of) the total space of the spinor bundle $S \rightarrow M^3$, where M is a three-dimensional space form. In particular, when M is taken to be the round sphere S^3 , there exists a *complete* holonomy G_2 -metric on the total space $S \cong S^3 \times \mathbb{H}$ such that the fibres are orthogonal to the horizontal distribution of the canonical spin connection induced by ∇^g . Furthermore, the zero section defines an associative. Any 3-ball Y inside S^3 is therefore associative with boundary $\partial Y = S^2 = \Sigma_0$. Let u be again the inward pointing normal vector field near the boundary. Assume that $S|_{\Sigma_0}$ has a subbundle which is a $J = u \times$ -complex line bundle over Σ_0 of degree n . If X_n denotes the total space, then X_n is coassociative at ∂Y (cf. Lemma 3.2), hence ψ -positive near ∂Y . Therefore, the index formula applies and yields $\text{index}(X_n, Y) = n + 1$. For $n = 0$, such a line bundle can be constructed by taking a nowhere vanishing section $a \in \Gamma(S|_{\Sigma_0})$. This gives the trivial complex line bundle spanned by a and $u \times a$, whence $\text{index}(X_0, Y) = 1$.

The Calabi–Yau extension. Let (K, ω, Ω) be a Calabi–Yau 3-fold with Kähler 2-form ω and holomorphic volume form Ω . Then $\omega^j / j!$ and $\cos(\theta)\text{Re } \Omega + i \sin(\theta)\text{Im } \Omega$ define calibrations on K which calibrate complex submanifolds (of real dimension $2l$) and *special Lagrangians of phase $e^{i\theta}$* respectively. On $M = K \times S^1$ we can define a torsion-free G_2 -structure by

$$\varphi = \text{Re } \Omega + \omega \wedge dt.$$

Further, if $C \subset K$ is a complex curve or $L \subset K$ a special Lagrangian of phase 1, then $C \times S^1$ and $L \times \{pt\}$ are associative. If $S \subset K$ is a complex surface or $L \subset K$ is a special Lagrangian of phase i , then $S \times \{pt\}$ and $L \times S^1$ are coassociative. Therefore, let L be a special Lagrangian of phase 1 with boundary $\partial L = \Sigma_g$ inside a complex surface S . The normal bundle of L is just $J(TL)$, for L is Lagrangian. On the other hand, S is complex, so that $J(T\partial L) \subset TS$. Lifting S and L to the coassociative $X = S \times \{pt\}$ and the associative $Y = L \times \{pt\}$ inside M , we find that the underlying real rank 2 bundle of ν_X coincides with $J(T\partial L) \oplus 0$. Now the normal bundle $\nu \rightarrow Y$ is generated by $J(TL)$ and ∂_t . Consequently, $\overline{\partial_t}$ is a section of μ_X , the orthogonal complement of ν_X inside ν . Lemma 3.2 then implies $\nu_X \cong \overline{T\partial L} = K_{\partial L}$ as complex line bundles so that $c_1(\nu_X) = c_1(K_{\partial L}) = 2(1 - g)$. We finally obtain $\text{index}(X, Y) = 3(1 - g)$ from the index formula. These considerations apply for instance to real, 3-dimensional submanifolds contained in the real part of a smooth quintic inside $\mathbb{C}P^4$, and whose boundaries are real analytic. The real part condition guarantees that such submanifolds are special Lagrangian, while the boundary condition ensures the boundary to be contained in a complex surface S .

Flat compact examples. On \mathbb{R}^7 we have the trivial G_2 -structure $\varphi_0 = dx^{123} + dx^1 \wedge (dx^{45} + dx^{67}) + dx^2 \wedge (dx^{46} - dx^{57}) + dx^3 \wedge (-dx^{47} - dx^{56})$. Since φ_0 is translation invariant, it de-

scends to a torsion-free G_2 -structure on the torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ still denoted φ_0 . Let (x_1, \dots, x_7) , $x_i \in \mathbb{R}/\mathbb{Z}$ be coordinates on T^7 . The isometric involutions

$$\begin{aligned} \sigma_0(x_1, \dots, x_7) &= (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7), \\ \tau_0(x_1, \dots, x_7) &= (-x_1, x_2, x_3, x_4, x_5, -x_6, -x_7) \end{aligned}$$

satisfy $\sigma_0^*\varphi_0 = \varphi_0$ and $\tau_0^*\varphi_0 = -\varphi_0$. In the sequel, let α_i , $i = 1, \dots, 7$ denote an element in $\{0, 1/2\}$. The fixed point locus of σ_0 consists of the sixteen 3-tori $\sigma_0 T_{\dots a_4 a_5 a_6 a_7}^3 = \{(x_1, x_2, x_3, a_4, a_5, a_6, a_7) \mid x_i \in \mathbb{R}/\mathbb{Z}\}$ which are associative as can be seen by direct inspection. Further, the fixed point locus of τ_0 consists of the eight coassociative 4-tori $\tau_0 T_{a_1 \dots a_6 a_7}^3 = \{(a_1, x_2, x_3, x_4, x_5, a_6, a_7) \mid x_i \in \mathbb{R}/\mathbb{Z}\}$. Then $Y = \{(t, x_2, x_3, 0, 0, 0, 0) \mid 0 \leq t \leq 1/2, x_i \in \mathbb{R}/\mathbb{Z}\} \cong [0, 1/2] \times \Sigma_1$ is an associative with boundary in the coassociative $X = \tau_0 T_{0 \dots 00}^4 \cup \tau_0 T_{\frac{1}{2} \dots 00}^4$. Since ν_X is trivial, we find $\text{index}(X, Y) = 0$. Of course, there exists actual deformations of Y along X induced by the flow of ∂_{x_4} and ∂_{x_5} .

Joyce manifolds. For the remaining examples we first invoke the following result which generalises the previous example.

Proposition 6.1. (See [22, Section 10.8].) *Let (M, φ) be a torsion-free G_2 -manifold.*

- (i) *If $\sigma : M \rightarrow M$ is a non-trivial isometric involution with $\sigma^*\varphi = \varphi$, then the fixed point locus of σ defines an associative submanifold.*
- (ii) *If $\tau : M \rightarrow M$ is a non-trivial isometric involution with $\tau^*\varphi = -\varphi$, then each connected component of the fixed point locus of τ is either a coassociative 4-fold or a single point.*

A method for the construction of compact holonomy G_2 -manifolds is also due to Joyce. We briefly outline his method to fix notation (cf. [22] for details and [23] for a quick introduction). One considers quotients T^7/Γ where Γ is a discrete group acting as a subgroup of φ_0 -preserving isometries of (T^7, g_0) . In particular, φ_0 descends to a G_2 -form φ on T^7/Γ . For instance choose the group $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^3$ given in [22, 12.2] which is generated by

$$\begin{aligned} \alpha(x_1, \dots, x_7) &= (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7), \\ \beta(x_1, \dots, x_7) &= \left(x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7\right), \\ \gamma(x_1, \dots, x_7) &= \left(-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7\right). \end{aligned}$$

There are no fixed points of $\delta \in \Gamma \setminus \{\text{Id}\}$ unless $\delta = \alpha, \beta$ or γ . For instance, α fixes the 3-tori $\alpha T_{\dots a_4 a_5 a_6 a_7}^3 = \{(x_1, x_2, x_3, a_4, a_5, a_6, a_7) \mid x_i \in \mathbb{R}/\mathbb{Z}\}$. The subgroup $\langle \beta, \gamma \rangle$ acts freely on these sixteen tori. For example, β maps $\alpha T_{\dots 0000}^3$ to $\alpha T_{\dots 00\frac{1}{2}0}^3$, etc. Hence, if we denote the image of these tori in T^7/Γ by $\alpha T_{[\dots a_4 a_5 a_6 a_7]}^3$, then $\alpha T_{[\dots 0000]}^3 = \alpha T_{[\dots 00\frac{1}{2}0]}^3$ so that α contributes four distinct copies of T^3 to the singular locus of T^7/Γ . A similar analysis applies to β and γ . In total, the singular locus consists of twelve distinct copies of T^3 which do not intersect. Near any singular 3-torus we can fix a neighbourhood homeomorphic to $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$

for instance by mapping $[x_1, x_2, x_3, a_4 + y_4, a_5 + y_5, a_6 + y_6, a_7 + y_7]$ near $\alpha T^3_{[...a_4a_5a_6a_7]}$ to $(x_1, x_4, x_5, \{z_1 = y_2 + iy_3, z_2 = y_6 + iy_7\})$ (with $\{z_1, z_2\} = \{-z_1, -z_2\}$ the coset of (z_1, z_2) inside $\mathbb{C}^2/\{\pm 1\}$). Blowing up the singular point in $\mathbb{C}^2/\{\pm 1\}$ yields a resolution R on which there exists a family h_t of asymptotically flat Calabi–Yau metrics. Together with the flat 3-torus (T^3, g_0) , the product metric $g_0 \oplus h_t$ on $T^3 \times R$ gives a torsion-free G_2 -structure which can be glued into T^7/Γ . Proceeding this way with the remaining singular tori yields a smooth compact resolution $\pi : M \rightarrow T^7/\Gamma$ on which we have a family of G_2 -structures $\tilde{\varphi}_t$ with “small” torsion. For t sufficiently small, Joyce’s deformation theorem ensures the existence of a torsion-free G_2 -structure $\tilde{\varphi}_t$ in a C^0 -vicinity of $\tilde{\varphi}_t$ controlled by t which induces a holonomy G_2 -metric for M is simply-connected.

As a warm-up we consider the example induced by the isometric involutions

$$\begin{aligned} \sigma_0(x_1, \dots, x_7) &= \left(x_1, \frac{1}{2} - x_2, \frac{1}{2} - x_3, x_4, x_5, -x_6, \frac{1}{2} - x_7\right), \\ \tau_0(x_1, \dots, x_7) &= \left(x_1, x_2, \frac{1}{2} - x_3, \frac{1}{2} - x_4, x_5, x_6, \frac{1}{2} - x_7\right) \end{aligned}$$

of (T^7, φ_0) . Clearly, $\sigma_0^* \varphi_0 = \varphi_0$ and $\tau_0^* \varphi_0 = -\varphi_0$. Further, σ_0 and τ_0 commute with Γ and thus descend to isometric involutions σ and τ of T^7/Γ . The fixed point set of σ_0 is given by the sixteen 3-tori $\sigma_0 T^3_{.b_2b_3..a_6b_7} = \{(x_1, b_2, b_3, x_4, x_5, a_6, b_7) \mid x_i \in \mathbb{R}/\mathbb{Z}\}$ where from now on, b_j denotes an element in $\{1/4, 3/4\}$. Furthermore, $\sigma_0 \circ \delta$ has no fixed points for $\delta \in \Gamma$ unless $\delta = \text{Id}$. As Γ acts transitively on the set of the tori $\sigma_0 T^3_{.b_2b_3..a_6b_7}$ the resulting fixed point set of σ consists of two tori, namely $\sigma T^3_{[. \frac{1}{4} \frac{1}{4} .. 0 \frac{1}{4}]}$ and $\sigma T^3_{[. \frac{1}{4} \frac{1}{4} .. \frac{1}{2} \frac{1}{4}]}$ which do not hit the singular locus of T^7/Γ .

Similarly, $\tau_0 \circ \delta$ has no fixed points unless $\delta = \text{Id}$ (giving eight 4-tori $\tau_0 T^4_{.b_3b_4..b_7}$), or $\delta = \beta\gamma$ (in which case the fixed point locus consists of the 128 fixed points $(a_1, a_2, b_3, b_4, b_5, b_6, b_7)$). The fixed point locus of τ is thus $\tau T^4_{[. \frac{1}{4} \frac{1}{4} .. \frac{1}{4}]}$ plus eight isolated points. Again, this does not hit the singular locus. Now as in 12.6 of [22] one can resolve T^7/Γ in a σ, τ -equivariant way such that σ and τ lift to isometric involutions preserving and reversing the torsion-free G_2 -form $\hat{\varphi}_t$ respectively. Since the fixed point loci of σ and τ are not modified by the resolution process, we obtain an associative Y coming from $\{[x_1, \frac{1}{4}, \frac{1}{4}, t, x_5, 0, \frac{1}{4}] \mid x_{1,5} \in \mathbb{R}/\mathbb{Z}, \frac{1}{4} \leq t \leq \frac{3}{4}\} \cong [1/4, 3/4] \times \Sigma_1$ which intersects the coassociative X coming from $T^4_{[. \frac{1}{4} \frac{1}{4} .. \frac{1}{4}]}$ in the 2-tori coming from $T^2_{[. \frac{1}{4} \frac{1}{4} \frac{1}{4} .. 0]}$ and $T^2_{[. \frac{1}{4} \frac{1}{4} \frac{1}{4} .. 0]}$. In particular, ν_X is trivial so we find $\text{index}(X, Y) = 2(1 - g) = 0$.

For the second example we start with the isometric involutions

$$\begin{aligned} \sigma_0(x_1, \dots, x_7) &= \left(x_1, \frac{1}{2} - x_2, \frac{1}{2} - x_3, x_4, x_5, -x_6, -x_7\right), \\ \tau_0(x_1, \dots, x_7) &= \left(\frac{1}{2} - x_1, x_2, x_3, x_4, x_5, -x_6, -x_7\right) \end{aligned}$$

which also satisfy $\sigma_0^* \varphi_0 = \varphi_0$ and $\tau_0^* \varphi_0 = -\varphi_0$ respectively. Now $\sigma_0 \circ \delta$ has no fixed points unless $\delta = \text{Id}$ or α . The fixed point locus of σ is therefore the image under the quotient map of the sixteen tori $\sigma_0 T^3_{.b_2b_3..a_6a_7}$ fixed by σ_0 and the sixteen tori $\sigma_0 \alpha T^3_{.b_2b_3a_4a_5..}$ fixed by $\sigma_0 \alpha$. Similarly, $\tau_0 \circ \delta$ has no fixed points unless $\delta = \text{Id}$ (in which case the fixed point locus consists of the eight tori $\tau_0 T^4_{b_1...a_6a_7}$), $\delta = \alpha$ (giving eight tori $\tau_0 \delta T^4_{b_1...a_4a_5..}$) or $\delta = \alpha \circ \beta$ (giving 128

points $(b_1, a_2, a_3, a_4, a_5, b_6, a_7)$). Consequently, the fixed point loci of both σ and τ hit the singular α -tori of T^7/Γ . To resolve T^7/Γ in a σ, τ -equivariant way and to compute their fixed point locus, we therefore proceed in two steps (cf. also 12.6.3 in [22]). Firstly, we consider the induced involutions on $T^7/\langle\alpha\rangle \cong T^3 \times (T^4/\langle\pm 1\rangle)$ also denoted by σ and τ , and resolve in a σ, τ -equivariant way. The result is diffeomorphic to $T^3 \times K3$, where $K3$ denotes the $K3$ surface. To resolve T^7/Γ completely, we lift the actions of β and γ to $T^3 \times K3$ and resolve the orbifold $(T^3 \times K3)/\langle\beta, \gamma\rangle$ in a σ, τ -equivariant way. First the fixed point locus of σ inside $T^7/\langle\alpha\rangle$ consists of the sets $T^1_{b_2b_3} \times S \cong \{(x_1, b_2, b_3) \mid x_1 \in \mathbb{R}/\mathbb{Z}\} \times S$. Here, S is the singular connected surface given by the union of $T^2_{..a_6a_7}/\langle\pm 1\rangle \cong \{(x_4, x_5, a_6, a_7) \mid x_i \in \mathbb{R}/\mathbb{Z}\}/\langle\alpha\rangle$ with $T^2_{a_4a_5..}/\langle\pm 1\rangle \cong \{(a_4, a_5, x_6, x_7) \mid x_i \in \mathbb{R}/\mathbb{Z}\}/\langle\alpha\rangle$ which intersect at the sixteen singular points (a_4, \dots, a_7) . Similarly, the fixed point set of τ inside T^7/Γ is $T^2_{b_1..} \times S = \{(b_1, x_2, x_3)\} \times S$. The lift of σ and τ to the first resolution yields $T^1_{b_2b_3} \times \Sigma$ and $T^2_{b_1..} \times \Sigma$ for some compact oriented Riemann surface Σ as the corresponding fixed point set. Working out the zeroth and first Betti number of Σ as in [22, 12.5.2] shows that $b_0(\Sigma) = 8$ and $b_1(\Sigma) = 0$, that is, Σ consists of eight copies of the 2-sphere $\Sigma_0 = S^2$. In essence, every $T^2_{a_4a_5..}/\langle\pm 1\rangle$ or $T^2_{..a_6a_7}/\langle\pm 1\rangle$ contributes one 2-sphere on the resolution which we label accordingly by $S^2_{a_4a_5..}$ and $S^2_{..a_6a_7}$. Now the action of β and γ on the fixed 32 copies of $T^1 \times S^2$ results in eight distinct copies inside $(T^3 \times K3)/\langle\beta, \gamma\rangle$. For $T^2_{b_1..} \times \Sigma$ we find four distinct copies of $(T^2 \times S^2)/\langle\beta\rangle$ and two distinct copies of $T^2 \times S^2$, for γ acts freely on the set of 2-tori $T^2_{b_1..}$, while β acts trivially and maps for instance $S^2_{a_4a_5..}$ into itself, but $S^2_{..a_6a_7}$ to $S^2_{..a_6a_7}$, etc. Consequently, the boundary of the associative Y coming from $\{[t, b_2, b_3] \mid t \in [1/4, 3/4]\} \times S^2_{[..a_6a_7]}$ inside $(T^3 \times K3)/\langle\beta, \gamma\rangle$ consists of two 2-spheres inside the coassociative X coming from the union $X_{\frac{1}{4}} \cup X_{\frac{3}{4}} = T^2_{[\frac{1}{4}..]} \times S^2_{[..a_6a_7]} \cup T^2_{[\frac{3}{4}..]} \times S^2_{[..a_6a_7]}$. Inspection of the construction and the resolution process shows that $\nu_{X_{\frac{1}{4}}}$ and $\nu_{X_{\frac{3}{4}}}$ are trivial whence $\text{index}(X, Y) = 2$.

7. A G_2 analogue of the Maslov index

Finally, we wish to introduce a G_2 analogue of the Maslov index. This index plays an important role in symplectic geometry, in particular for Floer homology, where it arises as a sort of relative Morse index [14,26,29].

Let us briefly recall its construction. We consider an almost complex manifold (M^{2m}, J) with an embedded (not necessarily holomorphic) 2-disk D with boundary in a totally real oriented submanifold X^m (i.e. for all $p \in X$, $T_p X$ does not contain any J -complex line). Since D is contractible, we can trivialise $TM|_D$ and regard the subbundle $TX|_{\partial D}$ as a closed curve in the set of totally real oriented m -planes in \mathbb{C}^m . On the other hand, this set is parametrised by⁶ $GL_m(\mathbb{C})/GL_m^+(\mathbb{R})$ and is homotopy equivalent to $U(m)/SO(m)$ – the set of oriented Lagrangian m -planes inside \mathbb{C}^m . By the exact homotopy sequence for fibrations $\pi_1(U(m)/SO(m)) \cong \mathbb{Z}$. The Maslov index $\mu(X, D)$ of D is the integer corresponding to the homotopy class induced by $TX|_{\partial D}$.

The natural counterpart in the G_2 -setting should be the following. Let Y be an embedded (not necessarily associative) 3-disk inside a topological G_2 -manifold M such that $\partial Y \cong S^2$ lies in

⁶ We denote by $GL_m^+(\mathbb{R})$ the set of invertible $m \times m$ -matrices with positive determinant.

some ψ -positive submanifold X . Trivialising $TM|_Y$ yields thus a map from S^2 to the set \mathcal{P}_+ of ψ -positive planes in \mathbb{R}^7 .

Proposition 7.1. *The set of ψ -positive planes $\mathcal{P}_+ \subset G_4(\mathbb{R}^7)$ is homotopy equivalent to $G_{\psi_0}(\mathbb{R}^7) \cong G_2/SO(4)$, the set of coassociatives. In particular, $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$.*

Proof. Instead of \mathcal{P}_+ we shall consider the dual set $\mathcal{P}_+^\perp \subset G_3(\mathbb{R}^7)$. Restricting φ_0 to $E \in G_3(\mathbb{R}^7)$ yields a multiple of the volume form induced by $g_{0|E}$, and thus a real number (recall that we always consider the Grassmannian of *oriented* subplanes). Since φ_0 is a calibration, we may regard φ_0 as a map $G_3(\mathbb{R}^7) \rightarrow [-1, 1]$. By convention we shall orient the orthogonal complements F^\perp , $F \in \mathcal{P}_+$, in such a way that $\varphi_0(F^\perp) < 0$, whence $\mathcal{P}_+^\perp = \varphi_0^{-1}([-1, 0])$. Any fibre $\varphi^{-1}(\cos \alpha)$ is acted on transitively by G_2 and contains an element of the form

$$E_\alpha = e_1 \wedge e_2 \wedge (\cos \alpha e_3 + \sin \alpha e_4)$$

with respect to the fixed G_2 frame e_1, \dots, e_7 of Section 2. To see this, write $E_\alpha = x \wedge y \wedge z$ for some unit vectors $x, y, z \in \mathbb{R}^7$. Since G_2 acts transitively on ordered orthonormal pairs with stabiliser $SU(2)$ [19], we may assume that $E_\alpha = e_1 \wedge e_2 \wedge z$ upon transformation by a suitable element in G_2 . The $SU(2)$ -action induced by the inclusion into G_2 gives rise to a decomposition $\mathbb{R}^7 = \text{Im } \mathbb{H} \oplus \mathbb{H}$, where $SU(2)$ acts trivially on $\text{Im } \mathbb{H}$ and $\mathbb{H} = \mathbb{C}^2$ becomes the standard vector representation. We are still free to modify E_α without changing e_1 and e_2 by an element in $SU(2)$. Since this group acts transitively on the unit sphere in \mathbb{C}^2 , we may transform the unit vector $z = \sum_{i=3}^7 z^i e_i$ into $\cos \alpha e_3 + \sin \alpha e_4$ with $\cos \alpha = z^3 = \varphi_0(E_\alpha)$. From this one easily deduces that (a) ± 1 are the only critical values and (b) that the Hessian of φ_0 is non-degenerate in directions transverse to the orbits $\varphi_0^{-1}(\pm 1)$ which are diffeomorphic to $G_2/SO(4)$ [19]. Consequently, φ_0 defines a G_2 -invariant Morse function in the sense of [25]. By a theorem in the same paper, one can conclude – in analogy with classical Morse theory – that $\varphi_0^{-1}((-\infty, 0])$ is (equivariantly) homotopy equivalent to the disk bundle $G_2 \times_{SO(4)} D^4$ (inside the normal bundle) over $\varphi^{-1}((-\infty, -1]) \cong G_2/SO(4)$ attached. Hence \mathcal{P}_+^\perp is homotopy equivalent to the open disk bundle which can be retracted to the base. In particular, \mathcal{P}_+ is of the same homotopy type as $G_2/SO(4)$. Since $\pi_k(G_2) = 0$ for $k = 1, 2$ and $\pi_1(SO(4)) = \mathbb{Z}_2$, the exact homotopy sequence for fibrations yields the asserted homotopy group. \square

Definition 7.2. Let D be an embedded associative 3-disk in some topological G_2 -manifold. We refer to the natural class of $TX|_{S^2}$ in $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$ as the G_2 -Maslov index of D and denote it by $\mu_{G_2}(X, D)$.

Proposition 7.3. *If $D \subset M$ is an embedded associative 3-disk with boundary in some coassociative X , then*

$$\mu_{G_2}(X, D) = \int_{S^2} c_1(\nu_X) \text{ mod } 2 = (\text{index}(X, D) + 1) \text{ mod } 2.$$

Proof. The natural complex structure of S^2 gives a natural identification with $\mathbb{C}P^1$ as a complex manifold. Consequently, any complex line bundle is of the form $\mathcal{O}(n)$ where $n = \int_{\mathbb{C}P^1} c_1(\mathcal{O}(n))$. For instance $TS^2 = \mathcal{O}(2)$, and in particular, we find $TX|_{S^2} = \mathcal{O}(2) \oplus \mathcal{O}(n)$. Let $G_{\psi_0}(\mathbb{R}^7)$ denote

the set of coassociative planes which is diffeomorphic to $G_2/SO(4)$. By assumption, there is a map

$$f : p \in \mathcal{S}^2 \mapsto F_p \in G_{\psi_0}(\mathbb{R}^7).$$

The homotopy class $[f]$ generates $\pi_2(G_2/SO(4))$ if and only if its boundary $\partial[f]$ generates $\pi_1(SO(4)) \cong \mathbb{Z}_2$. If $K : (D^2, S^1) \rightarrow (S^2, N)$ denotes the collapsing map sending S^1 to the north pole N of S^2 , a representative of $\partial[f]$ is obtained by restricting the lift $\tilde{f} : (D^2, S^1) \rightarrow G_2$ of the composition $f \circ K$ to S^1 . We can think of $\tilde{f}|_{S^1} \in SO(4)$ as taking values in the set of bases of F_N . Its action on F_N can thus be homotoped into the action of the maximal torus $T^2 = S^1 \times S^1$ of $SO(4)$ (seen as structure group of $TX|_{S^2}$), namely

$$t \in S^1 \mapsto \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{nit} \end{pmatrix}.$$

On the other hand, the inclusion of the torus induces an epimorphism $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \rightarrow \pi_1(SO(4)) = \mathbb{Z}_2$ which is mod 2 reduction of the second summand, whence $[\partial f] = n \pmod 2$. \square

Example. We consider again the local coassociative example of Section 6, where Y is the 3-disk D inside $\text{Im } \mathbb{H} \subset \text{Im } \mathbb{O}$. Then $\text{index}(X, D) = 1$. On the other hand, $TX|_{S^2} = \mathcal{O}(2) \oplus \mathcal{O}(0)$ from which we conclude $\mu_{G_2}(X, D) = 0$ in accordance with the proposition.

Acknowledgments

The first author was supported by the French Agence nationale pour la recherche whilst working on this project. He would like to thank Vincent Borrelli, Jean-Claude Sikorav and Jean-Yves Welschinger for their interest and comments on this work. The second author was partially supported by the SFB 647 ‘‘Space. Time. Matter’’ funded by the DFG. He thanks Bernd Ammann, Uli Bunke and Dominic Joyce for useful discussions and suggestions.

References

- [1] M. Aganagic, C. Vafa, Mirror symmetry, D-branes and counting holomorphic discs, arXiv:hep-th/0012041.
- [2] S. Akbulut, S. Salur, Calibrated manifolds and gauge theory, *J. Reine Angew. Math.* 625 (2008) 187–214.
- [3] S. Akbulut, S. Salur, Deformations in G_2 manifolds, *Adv. Math.* 217 (5) (2008) 2130–2140.
- [4] G. Bonelli, A. Tanzini, M. Zabzine, On topological M -theory, *Adv. Theor. Math. Phys.* 10 (2006) 239–260.
- [5] B. Booss, D. Bleecker, *Topology and Analysis. The Atiyah–Singer Index Formula and Gauge-Theoretic Physics*, Springer, New York, 1985.
- [6] B. Booss, K. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser Verlag, Boston, 1993.
- [7] R. Bryant, Calibrated embeddings in the special Lagrangian and coassociative cases, *Ann. Global Anal. Geom.* 18 (3–4) (2000) 405–435.
- [8] R. Bryant, Some remarks on G_2 -structures, in: *Gökova Geometry/Topology Conference (GGT)*, Gökova, 2006, pp. 75–109.
- [9] R. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy, *Duke Math. J.* 58 (3) (1989) 829–850.
- [10] A. Butscher, Deformations of minimal Lagrangian submanifolds with boundary, *Proc. Amer. Math. Soc.* 131 (6) (2003) 1953–1964.
- [11] D. DeTurck, J. Kazdan, Some regularity theorems in Riemannian geometry, *Ann. Sci. École Norm. Sup.* (4) 14 (3) (1981) 249–260.

- [12] R. Dijkgraaf, S. Gukov, A. Neitzke, C. Vafa, Topological M -theory as unification of form theories of gravity, *Adv. Theor. Math. Phys.* 9 (2005) 603–665.
- [13] S. Donaldson, E. Segal, Gauge theory in higher dimensions, II, arxiv:0902.3239 [math.DG].
- [14] S. Dostoglou, D. Salamon, Cauchy–Riemann operators, self-duality, and the spectral flow, in: *First European Congress of Mathematics*, vol. I, Paris, 1992, in: *Progr. Math.*, vol. 119, Birkhäuser, Basel, 1994, pp. 511–545.
- [15] M. Fernández, A. Gray, Riemannian manifolds with structure group G_2 , *Ann. Mat. Pura Appl.* (4) 132 (1982) 19–45.
- [16] A. Floer, Morse theory for Lagrangian intersections, *J. Differential Geom.* 28 (3) (1988) 513–547.
- [17] D. Gayet, Vanishing theorems for associative submanifolds, arXiv:0909.2233 [math.DG].
- [18] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* 82 (2) (1985) 307–347.
- [19] R. Harvey, H. Lawson, Calibrated geometries, *Acta Math.* 148 (1982) 47–157.
- [20] R. Harvey, H. Lawson, An introduction to potential theory in calibrated geometry, *Amer. J. Math.* 131 (4) (2009) 893–944.
- [21] N. Hitchin, Stable forms and special metrics, *Contemp. Math.* 288 (2001) 70–89.
- [22] D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Univ. Press, Oxford, 2000.
- [23] D. Joyce, The exceptional holonomy groups and calibrated geometry, in: *Gökova Geometry/Topology Conference (GGT)*, Gökova, 2006, pp. 110–139.
- [24] A. Kovalev, J. Lotay, Deformations of compact coassociative 4-folds with boundary, *J. Geom. Phys.* 59 (2009) 63–73.
- [25] K.-H. Mayer, G -invariant Morse-Funktionen, *Manuscripta Math.* 63 (1989) 99–114.
- [26] D. McDuff, D. Salamon, *J-holomorphic Curves and Quantum Cohomology, University Lecture Series, vol. 6, AMS, Providence, RI, 1994.*
- [27] R. McLean, Deformations of calibrated submanifolds, *Comm. Anal. Geom.* 6 (4) (1998) 705–747.
- [28] C. Robles, S. Salur, Calibrated associative and Cayley embeddings, *Asian J. Math.* 13 (3) (2009) 287–306.
- [29] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.* 45 (10) (1992) 1303–1360.
- [30] E. Witten, Two-dimensional gravity and intersection theory on moduli space, in: *Surveys in Differential Geometry*, Lehigh Univ., Bethlehem, 1991, pp. 243–310.