How curved is a random complex curve?

MICHELE ANCONA AND DAMIEN GAYET

February 15, 2024

Abstract

In this paper, we study the curvature properties of random complex plane curves. We bound from below the probability that a uniform proportion of the area of a random complex degree d plane curve has a curvature smaller than $-d/8$. Our lower bound is uniform, in the sense that it does not depend on d. We also provide uniform upper bounds for similar probabilities. These results extend to random complex curves of projective surfaces equipped with an ample line bundle. This paper can be viewed as a sequel of [1], where other metric statistics were given. On a larger time scale, it joins the general program initiated in [11] of understanding random complex hypersurfaces of projective manifolds.

This paper is dedicated to the memory of Steve Zelditch

Contents

1 Introduction

1.1 The standard projective setting

Smooth complex projective curves of the complex projective plane $\mathbb{C}P^2$ have the remarkable property that their topology depends only on the degree d of a defining homogeneous polynomial. More precisely, they all are compact connected Riemann surfaces of genus $\frac{1}{2}(d-1)(d-2)$. When equipped with the restriction of the Fubini-Study metric g_{FS}, these complex curves become Riemannian real surfaces. By the Wirtinger theorem, their area depends only on their degree:

$$\forall d \geq 1, \forall P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2],\ \text{area}_{g_{\mathrm{FS}|Z(P)}}(Z(P)) = d,$$

where $Z(P) := \{P = 0\} \subset \mathbb{C}P^2$. However, all the other Riemannian quantities strongly depend on the complex curve. This paper deals with the statistics of *curvature* properties of such surfaces, when they are chosen at random, for a fixed large degree.

Let us recall some deterministic facts about the curvature of plane complex curves. For any $P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2]$, and any $x \in Z(P)$, denote by $K(x)$ the Gaussian curvature for the restriction of g_{FS} on $Z(P)$. The curvature satisfies the upper bound

$$K \leq 2\pi,$$

which is achieved by complex lines in $\mathbb{C}P^2$, see for instance [10]. The points where $K = 2\pi$ are called *inflexion* points. In fact, for any $d \geq 2$ and any generic polynomial $P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2]$,

$$\#\{x \in Z(P), K(x) = 2\pi\} = 3d(d-2).$$

In particular, the degree 2 curves are the only one for which the curvature is strictly less than 2π. Moreover,

$$\forall d \geq 2, \inf_{P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2]} \inf_{x \in Z(P)} K(x) = -\infty.$$

Indeed, the smoothing the union of a degree $d-1$ smooth complex curve and a complex line provides a degree d curve whose curvature at the $d-1$ intersection points becomes infinitely negative when the smoothing becomes smaller and smaller.

Finally, by the Gauss–Bonnet theorem and the genus formula,

$$\int_{Z(P)} K(x) \mathrm{d}\,\mathrm{vol}(x) = 2 - (d-1)(d-2), \tag{1.1}$$

so that the average of K over the complex curve is asymptotic to $-d$.

In this paper, we study the curvature of these complex curves when the defining polynomial is taken at random. Let

$$P = \sum_{i+j+k=d} a_{i,j,k} \sqrt{\frac{(d+2)!}{2i!j!k!}} X_0^i X_1^j X_2^k, \tag{1.2}$$

be a degree d random polynomial, where $(a_I)_{|I|=d} \in \mathbb{C}P^{N_d-1}$ are random coefficients chosen uniformly on the projective space $\mathbb{C}P^{N_d-1}$ equipped with its Fubini-Study metric (the quotient metric induced by the standard metric on $\mathbb{C}^{N_d}$) and $N_d = \dim \mathbb{C}_d^{hom}[X_0, X_1, X_2]$. We can also choose $(a_I)_I$ in the standard sphere $\mathbb{S}^{2N_d-1} \subset \mathbb{C}^{N_d}$, or being independent complex standard Gaussian random variables. We denote by μ_d the measure associated to (1.2). This measure is naturally associated to the unique $U(3)$-invariant Hermitian product on $\mathbb{C}_d^{hom}[X_0, X_1, X_2]$, see Example 1.6.

In general, our results will estimate the proportion of a complex curve where its curvature is controlled. Hence, for any real surface Z equipped with a metric g with finite area and any subset $A \subset \mathbb{R}$, let us define

$$\kappa(Z, g, A) = \frac{\mathrm{area}_g\{x \in Z, K(x) \in A\}}{\mathrm{area}_g Z}. \tag{1.3}$$

We will write $\kappa(Z, A)$ when the metric is obvious. Our first result asserts that statistically, a uniform part of the area of a large degree random curve is very curved.

Theorem 1.1 *There exists $c > 0$ such that*

$$\forall d \gg 1,\ \mu_d \left[P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2],\ \kappa \left(Z(P), g_{\mathrm{FS}|Z(P)}, [-4d, -\frac{d}{8}] \right) > c \right] \geq c.$$

Recall that by the Gauss–Bonnet formula, the mean value of K over a degree d curve is $-d$. Note also that if we smooth d different complex lines in $\mathbb{C}P^2$ passing through one common point, an arbitrary large proportion of the area of the resulting degree d smooth curve have a curvature close to the one of a line, which implies that

$$\forall \varepsilon < 2\pi,\ \forall d \geq 2,\ \inf_{P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2]} \kappa(Z(P),]-\infty, \varepsilon]) = 0$$

Hence, Theorem 1.1 cannot be deduced from any deterministic fact. In a different direction, in [1] it was proven that

$$\mu_d \left[\inf_{x \in Z(P)} K(x) \geq -d^9 \right] \underset{d \to \infty}{\to} 1.$$

Our second theorem computes exactly the average of the proportion of the area of the random complex curve with prescribed curvature.

Theorem 1.2 *Let $0 < r < R$. There exists $\varphi_{r,R} \in]0, 1[$ defined below by (2.8) such that*

$$\forall d \geq 2,\ \mathbb{E}_{\mu_d} \left[\kappa \Big(Z(P), [2\pi - Rd, 2\pi - rd] \Big) \right] = \varphi_{r,R},$$

where κ is defined above by (1.3).

We now collect several consequences of Theorem 1.2.

Corollary 1.3 *Let $0 < r < R$. Then, for any $d \geq 2$,*

$$\forall \eta \in]\varphi_{r,R}, 1[, \quad \mu_d \left[\kappa \Big(Z(P), [2\pi - Rd, 2\pi - rd] \Big) > \eta \right] \quad \leq \quad \tfrac{1}{\eta} \varphi_{r,R}\ .$$

Corollary 1.4 *Let $0 < r < R$. Then, for any $d \geq 2$,*

$$\forall \eta \in]0, \varphi_{r,R}[, \quad \mu_d \left[\kappa \Big(Z(P), [2\pi - Rd, 2\pi - rd] \Big) < \eta \right] \quad \leq \quad \tfrac{1}{1-\eta}(1 - \varphi_{r,R}).$$

Note that by the Gauss–Bonnet formula (1.1), for any $r \geq 1$ and any $P \in \mathbb{C}_d^{hom}[X_0, X_1, X_2]$,

$$\kappa \Big(Z(P),]-\infty, -rd] \Big) \leq \frac{1}{r} + O(\frac{1}{d}).$$

Corollary 1.5 *Let $\ell \in]-\infty, 2\pi[$. Then,*

$$\forall \eta \in]0, 1[, \quad \mu_d \left[\kappa \Big(Z(P), [\ell, 2\pi] \Big) > \eta \right] \underset{d \to \infty}{\to} 0.$$

The last assertion shows that, in particular, the event that a uniform proportion of the area of the curve has a non negative curvature becomes rarer and rarer for large degrees. We stress again that for any complex plane curve, there always exist points with positive curvature.

1.2 The general setting

The previous results can be extended in a much more general setting that we now introduce. Let S be a complex projective surface equipped with a Hermitian ample holomorphic line bundle $(L,h) \to S$ with positive curvature ω, that is, locally

$$\omega = \frac{1}{2i\pi}\partial\bar\partial \log \|s\|_h^2 > 0,$$

where s is any local non vanishing holomorphic section of L. Let $g_\omega = \omega(\cdot, i\cdot)$ be the associated Kähler metric. The space $H^0(S,L^d)$ of holomorphic sections of $L^d := L^{\otimes d}$ is non trivial for d large enough, more precisely

$$N_d := \dim_{\mathbb{C}} H^0(S,L^d) \underset{d\to\infty}{\sim} d^2 \operatorname{vol}_{g_\omega}(S).$$

Let $\Delta_d \subset H^0(S,L^d)$ be the discriminant subset, that is, the set of sections s such that there exists x in $Z(s)$ where $\nabla s(x)$ vanishes. Recall that Δ_d is a complex hypersurface, and that for any $s \in H^0(S,L^d)\setminus\Delta_d$, the zero set $Z(s) \subset S$ is a compact smooth complex curve of S. Moreover, since $H^0(S,L^d)\setminus\Delta_d$ is connected, for s outside Δ_d the diffeomorphism class of $Z(s)$ depends only on d. For any s we equip $Z(s)$ with the restriction $g_{\omega|Z(s)}$ of the Kähler metric g_ω. By Wirtinger's theorem,

$$\forall d\geq 1, \forall s \in H^0(S,L^d)\setminus\Delta_d,\ \operatorname{area}_{g_{\omega|Z(s)}}(Z(s)) = \int_{Z(s)} \omega = 2d \operatorname{vol}_{g_\omega}(S).$$

The space $H^0(S,L^d)$ can be equipped with the L^2 Hermitian product

$$(s,t) \in H^0(S,L^d)^2 \mapsto \langle s,t\rangle = \int_S \langle s(x), t(x)\rangle_{h^d} \frac{\omega^2}{2}. \tag{1.4}$$

This product induces a Gaussian measure μ_d over $H^0(S,L^d)$, that is, for any Borelian $U \subset H^0(S,L^d)$,

$$\mu_d(U) = \int_{s\in U} e^{-\frac{1}{2}\|s\|^2} \frac{\mathrm{d}s}{(2\pi)^{N_d}}, \tag{1.5}$$

where $\mathrm{d}s$ denotes the Lebesgue measure associated to the Hermitian product (1.4). If $(S_i)_{i\in\{1,\cdots,N_d\}}$ is an orthonormal basis of this space, then

$$s = \sum_{i=1}^{N_d} a_i S_i$$

follows the law μ_d if the random complexes $\sqrt{2}a_i$ are i.i.d standard complex Gaussians, that is, $\Re a_i$ and $\Im a_i$ are independent centered Gaussian variables with variance equal to $1/2$. Note that for any event depending only on the vanishing locus $Z(s)$ of $s \in H^0(S,L^d)$, the probability measure μ_d can be replaced by the invariant measure over the unit sphere $\mathbb{S}H^0(S,L^d)$ for the product (1.4), or equivalently the Fubini-Study measure on the linear system $\mathbb{P}H^0(S,L^d)$.

Example 1.6 *When $S = \mathbb{C}P^2$ and $(L,h) = (\mathcal{O}(1), h_{\mathrm{FS}})$ is the degree 1 holomorphic line bundle equipped with the standard Fubini-Study metric, then the vector space $H^0(S,L^d)$ is isomorphic to the space $\mathbb{C}_d^{hom}[X_0,X_1,X_2]$ of degree d homogeneous polynomials in 3 variables, and we recover the standard Fubini-Study measure given by (1.2).*

Then, $(Z(s), g_{\omega|Z(s)})$ can be seen as a fixed real surface with a random metric. The following theorem is the generalization of Theorem 1.1 in this context.

Theorem 1.7 *Let S be a compact smooth complex surface equipped with an ample holomorphic line bundle $(L,h) \to S$ endowed with a Hermitian metric h with positive curvature ω and g_ω be the induced Kähler metric. Then, there exist a universal constant $c > 0$ such that*

$$\forall d \gg 1,\ \mu_d \left[s \in H^0(S, L^d),\ \kappa\Big(Z(s), g_{\omega|Z(s)}, [-4d, -\frac{d}{8}]\Big) > c \right] \geq c,$$

where κ is defined by (1.3).

The deterministic facts in the standard setting extend in this general one. First,

$$\exists C > 0,\ \liminf_{d\to\infty} \inf_{s \in H^0(S,L^d)} \kappa\Big(Z(s),]-\infty, -C]\Big) = 0.$$

Indeed, let d_0 be such that $\dim_{\mathbb{C}} H^0(S, L^{d_0}) \geq 2$ and $F \subset H^0(S, L^{d_0}) \setminus \Delta_{d_0}$ be a compact subset. Then, there exists $C > 0$ such that for any $s \in F$, the curvature of $Z(s)$ is larger than $-C$. Now for any integer $k \geq 1$, let $s_1, \cdots, s_k$ be k sections in F such that for $i \neq j$, $Z(s_i)$ intersects transversally with $Z(s_j)$. Then, for any positive ε, we can find a small perturbation s of $s_1 \otimes \cdots \otimes s_k \in H^0(S, L^{kd_0})$ such that the area of $Z(s)$ where the curvature is less than $-2C$ is smaller than ε.

Second,

$$\forall d \gg 1,\quad \inf_{s\in H^0(S,L^d)} \inf_{x \in Z(s)} K(x) = -\infty.$$

Indeed, let Z be a degree d curve with a nodal singularity at x. Then, for any $C > 0$, there exists a smoothing of C such that the curvature near x is less than $-C$.

Third, by [6, Proposition 9.2], see also Theorem 2.2, for any complex curve Z in a complex Kähler manifold (S, g), the curvature K of Z for $g_{|Z}$ is bounded above by the holomorphic sectional curvature of S, so that there exists $C > 0$ such that

$$K \leq C.$$

Finally, by Gauss–Bonnet, for any generic $s \in H^0(S, L^d)$,

$$\int_{Z(s)} K(x) \mathrm{d}\,\mathrm{vol}(x) = -2\,\mathrm{vol}(S) d^2 + O(d),$$

so that for any $r \geq 1$,

$$\kappa\Big(Z(s),]-\infty, -rd]\Big) \leq \frac{1}{r} + O(\frac{1}{d}).$$

Also, observe that by [1],

$$\mu_d \left[s \in H^0(S, L^d),\ \inf_{x\in Z(s)} K(x) \geq -d^9 \right] \to 1.$$

The following theorem generalizes Theorem 1.7:

Theorem 1.8 *Under the hypotheses of Theorem 1.7, let $0 < r < R$ and $\varphi_{r,R} \in]0,1[$ be defined below by (2.8). Then,*

$$\mathbb{E}_{\mu_d} \left[\kappa\Big(Z(s), [-Rd, -rd]\Big) \right] \underset{d\to\infty}{\to} \varphi_{r,R},$$

where κ is defined by (1.3).

The following corollary generalizes Corollaries 1.3, 1.4 and 1.5:

Corollary 1.9 *Under the hypotheses of Theorem 1.7, let $0 < r < R$. Then,*

$$\begin{aligned} \forall \eta \in]\varphi_{r,R}, 1[, \quad \limsup_{d\to\infty} \mu_d \left[\kappa\Big(Z(s), [-Rd, -rd]\Big) > \eta\right] \quad &\leq \quad \tfrac{1}{\eta}\varphi_{r,R} \\ \textit{and } \forall \eta \in]0, \varphi_{r,R}[, \quad \limsup_{d\to\infty} \mu_d \left[\kappa\Big(Z(s), [-Rd, -rd]\Big) < \eta\right] \quad &\leq \quad \tfrac{1}{1-\eta}(1 - \varphi_{r,R}), \end{aligned}$$

where $\varphi_{r,R}$ is defined by (2.8). Moreover let $a, b \in \mathbb{R}$, $a < b$. Then,

$$\forall \eta \in]0, 1[, \quad \mu_d \left[\kappa\Big(Z(s), [a, b]\Big) > \eta\right] \underset{d\to\infty}{\to} 0.$$

We emphasize that our results in this general setting are universal, in the sense that the constants involved are universal and in particular do not depend on S. This is due to the fact that the Bergman kernel has itself a local universal behaviour, see appendix A.

2 Proofs of the theorems

2.1 Deterministic preliminaries

For any Riemannian surface (Z, g) of finite area, any $A \subset \mathbb{R}$, let

$$T(Z, g, A) = \{x \in Z, K_g(x) \in A\}, \tag{2.1}$$

where K_g denotes the Gauss curvature. Note that

$$\kappa(Z, g, A) = \frac{\text{area}_g(T(Z, g, A))}{\text{area}_g(Z)},$$

where κ is defined by (1.3).

Lemma 2.1 *There exists $f_0 : \mathbb{C}^2 \to \mathbb{C}$ a holomorphic function vanishing transversally such that*

$$\text{area}_{g_0}\left(T\Big(Z(f_0) \cap \mathbb{B}, g_0, [-2, -\frac{1}{4}]\Big)\right) > 1,$$

where g_0 denotes the standard metric over $\mathbb{C}^2$ and $\mathbb{B} \subset \mathbb{C}^2$ is the standard unit ball.

Proof. Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be the holomorphic function defined

$$\forall (z, w) \in \mathbb{C}^2, \; f_0(z, w) = zw - \frac{1}{4}.$$

Then, $Z(f_0) \cap \frac{1}{2}\mathbb{B} \neq \emptyset$, so that

$$\text{area}(Z(g) \cap \mathbb{B}) > 0.$$

By [12, Proposition 1], for any holomorphic function $f : \mathbb{B} \to \mathbb{C}$, for any $x \in Z(f)$, the Gaussian curvature $K_{g_0}(x)$ of $Z(f)$ at x for the standard metric equals

$$K_{g_0}(x) = -\frac{|2f_{zw}f_zf_w - f_{zz}f_w^2 - f_{ww}f_z^2|^2}{(|f_z|^2 + |f_w|^2)^3}(x). \tag{2.2}$$

One can check that the curvature of $Z(f_0)\cap\mathbb{B}$ belongs to $[-2,-1/4]$, and that the area of $Z(f_0)\cap\mathbb{B}$ satisfies

$$\mathrm{area}_{g_0}(Z(f_0)\cap\mathbb{B}) = \int_{|z|^2+\frac{1}{16|z|^2}\leq 1}(1+\frac{1}{16|z|^4})\mathrm{d}z\wedge\mathrm{d}\bar{z} \geq \int_{\frac14\leq|z|^2\leq\frac12}\mathrm{d}z\wedge\mathrm{d}\bar{z} > 1.$$

In conclusion, $\mathrm{area}_{g_0}\{x\in Z(f_0)\cap\mathbb{B}, -2<K_{g_0}(x)<-1/4\}>1$. $\square$

In the sequel, we want to prove that the area of T (defined by (2.1)) is continuous in the $\mathscr{C}^2$-norm as a map of the ambient metric and the defining function of the curve Z. For this, let us recall some general facts about the curvature. Let Z be a submanifold of a Riemannian manifold (S,g), $x\in Z$ and let

$$\begin{array}{rcl}\sigma : T_xZ\times T_xZ & \to & N_xZ \\ (X,Y) & \mapsto & (\nabla_X Y)^{\perp},\end{array} \tag{2.3}$$

where $NZ\subset TS$ denotes the normal bundle over Z, ∇ the Levi-Civita connection associated to g and $(\nabla_X Y)^{\perp}$ the g-orthogonal projection of $\nabla_X Y$ onto N_xZ.

Theorem 2.2 *Let Z be a submanifold of the Riemannian manifold S and x be in Z.*

1. *(Gauss's equations [5, Theorem 3.6.2]) For any tangent vector X,Y,V,W in T_xZ,*

$$\begin{array}{rcl}\langle R^Z(X,Y)V,W\rangle_g & = & \langle R^S(X,Y)V,W\rangle_g + \\ & & \langle\sigma(Y,V),\sigma(X,W)\rangle_g - \langle\sigma(X,V),\sigma(Y,W)\rangle_g,\end{array}$$

   where σ is defined by (2.3) and R^Z (resp. R^S) denotes the Riemannian curvature of $g_{|Z}$ on Z (resp. g on S).

2. *(Kähler version [6, Proposition 9.2]) Assume furthermore that S is a Kähler manifold and Z is a complex submanifold of S. Then, for any tangent vector $X\in T_xZ$,*

$$\langle R^Z(X,JX)JX,X\rangle_g = \langle R^S(X,JX)JX,X\rangle_g - \|\sigma(X,X)\|_g^2,$$

   where J denotes the complex structure of S and g is the Kähler metric.

In particular, as said in the introduction, the Gauss curvature of a complex curve is bounded by the holomorphic sectional curvature of its ambient space along the complex direction provided by the curve.

Lemma 2.3 *Let $0<r<R$ and $f_0 : 2\mathbb{B}\subset\mathbb{R}^4\to\mathbb{R}^2$ be a $\mathscr{C}^2$-function. Assume that f_0 vanishes transversely and that*

$$\mathrm{area}_{g_0}\Big(T\Big(Z(f_0),g_0,[-R,-r]\Big)\Big)>0,$$

where k is defined by (2.1). Then, there exists $\delta>0$ such that for any $\mathscr{C}^2$ metric g and $\mathscr{C}^2$ function $f : 2\mathbb{B}\to\mathbb{R}^2$ satisfying $\|g-g_0\|_{\mathscr{C}^2(2\mathbb{B})}+\|f-f_0\|_{\mathscr{C}^2(2\mathbb{B})}\leq\delta$,

$$\mathrm{area}_g\Big(T(Z(f),g,[-2R,-\frac{r}{2}])\Big) > \frac12\,\mathrm{area}_{g_0}\Big(T\Big(Z(f_0),g_0,[-R,-r]\Big)\Big).$$

Proof. By hypothesis, by the implicit function theorem and by compactness, there exist $\eta > 0$ and a finite set of cubes of the form $C_i = x_i + Q^1_\eta \times Q^2_\eta$, where $x_i \in \mathbb{B}$, $Q^i_\eta =]0, \eta[^2$, $i = 1, 2$, and

$$\mathbb{B} \subset \cup_i C_i \subset 2\mathbb{B},$$

such that for any $\mathscr{C}^2$-function $f : 2\mathbb{B} \to \mathbb{R}^2$ $\mathscr{C}^1$-close enough to f_0, for any i, $Z(f) \cap C_i$ is the graph of a $\mathscr{C}^2$-function $\psi_i(f)$ over the translated of Q^1_η or Q^2_η. Fix i and assume that the graph $\psi_i(f)$ is defined over Q^1_η in C_i. Let

$$h_i(f) : x \in Q^1_\eta \mapsto (x, \psi_i(f)(x)) \in C_i.$$

By the implicit function theorem with parameters, $h_i(f)$ depends on f continuously in the $\mathscr{C}^2$-norm. Then, for any $A \subset \mathbb{R}$,

$$\mathrm{area}_g\{x \in Z(f) \cap C_i, K_g(x) \in A\} = \mathrm{area}_{h_i(f)^* g}\{x \in Q^1_\eta, K_g(h_i(f)(x)) \in A\}. \tag{2.4}$$

In order to estimate the continuity of the right-hand side in g and f, let us recall how to compute K with respect to g and f. Let $x \in Q^1_\eta$ and let

$$\begin{array}{rcl} s : (T_x Q^1_\eta)^2 & \to & N_{h_i(f)(x)} Z(f) \\ (V, W) & \mapsto & \sigma(\mathrm{d}h_i(f)(x)(V), \mathrm{d}h_i(f)(x)(W)), \end{array}$$

where σ is defined by (2.3). Recall that $\mathbb{R}^2 \times \{0\} = T_x Q^1_\eta$ and $\mathrm{d}h_i(f)(x)(V) \in T_{h_i(f)(x)} Z(f)$. Then by [1, Proposition 3.2], for any pair $(V, W) \in (T_x Q^1_\eta)^2$,

$$s(V, W) = -\sum_{k=1}^{2} \Phi^{-1} \Big((\langle W_i, \nabla^g_{V_i} \nabla f_j \rangle_g)_{j \in \{1, \cdots, 2\}} \Big)_i \nabla f_k,$$

where everything is computed at $h_i(f)(x) \in Z(f)$, where ∇f_k denotes the g-gradient of the k-th coordinate f_k of f and where

$$\Phi = (\langle \nabla f_k, \nabla f_j \rangle_g)_{1 \le k, j \le 2}\,.$$

In particular, $s(V, W)$ is continuous in the $\mathscr{C}^1$-norm in g, and in the $\mathscr{C}^2$-norm in f. Now by the Gauss equations (Theorem 2.2), for any $x \in Q^1_\eta$ and and any $X, Y, Z, W \in T_x Q^1_\eta = \mathbb{R}^2 \times \{0\}$, writing $X_i = \mathrm{d}h_i(f)(x)(X) \in T_{h_i(f)(x)} Z(f)$ etc.,

$$\begin{array}{rcl} \langle R^{Z(f)}(X_i, Y_i) Z_i, W_i \rangle_g & = & \langle R_g(X_i, Y_i) Z_i, W_i \rangle_g + \\ & & \langle s(Y, Z), s(X, W) \rangle_g - \langle s(X, Z), s(Y, W) \rangle_g, \end{array}$$

where everything is computed at $h_i(f)(x)$. Now, the Riemannian curvature R_g depends continuously of g in the $\mathscr{C}^2$-norm. Consequently, if $K_{g_0}(h_i(f_0)(x)) \in]-R, -r[$, then uniformly in $x \in Q^1_\eta$, for g $\mathscr{C}^2$-close to g_0 and f $\mathscr{C}^2$-close to f_0, then $K(h_i(f)(x)) \in]-2R, -r/2[$.

Now, on Q^1_η the pull-back metric $h_i(f)^* g$ converges uniformly to $h_i(f_0)^* g_0$ when g converges in $\mathscr{C}^0$-norm to g_0 and f to f_0 in the $\mathscr{C}^1$-norm, so the quotient of their area form is bounded by 2 for (g, f) close enough to (g_0, f_0). Hence, by (2.4) the area of $T(Z(f) \cap C_i, g, r/2, 2R)$ is larger or equal to half of the same for $g = g_0$ and $f = f_0$. Finally, using a partition of the unity associated to the C_i's, we obtain the result. $\square$

2.2 The Bargmann-Fock field

The proofs of Theorems 1.7 and 1.8 rely on similar probabilistic estimates for the universal algebraic rescaled model case, namely the Bargmann-Fock field over $\mathbb{C}^2$, see Theorem A.1. The Bargmann-Fock field is defined by the following measure:

$$\forall z = (z_1, z_2) \in \mathbb{C}^2,\ f(z) = \sum_{(i_1,i_2)\in\mathbb{N}^2} a_{i_0,i_2}\sqrt{\frac{\pi^{i_1+i_2}}{i_1!i_2!}} z_1^{i_1} z_2^{i_2} e^{-\frac{1}{2}\pi\|z\|^2}, \tag{2.5}$$

where the a_I's are independent normal complex Gaussian random variables. We denote this measure by μ_{BF}. note that, up to the exponential factor, its support is the set of holomorphic functions over $\mathbb{C}^2$.

2.3 Proof of Theorem 1.7

Theorem 1.7 is a consequence of the following local estimate:

Proposition 2.4 *Under the hypotheses of Theorem 1.7, there exists a universal constant $c > 0$ such that for any d large enough and any $x \in S$,*

$$\mu_d\left[s \in H^0(S, L^d),\ \operatorname{area}_{g_{\omega|Z(s)}}\left\{y \in Z(s) \cap B(x, \frac{1}{\sqrt{d}}), -4d < K(y) < -\frac{d}{8}\right\} > \frac{1}{2d}\right] \geq c.$$

We emphasize that the constant c does not depend on S.

Proof. Let $x \in S$ and $R > 0$ be such that $2R$ is less than the radius of injectivity of S at x. Note that since S is compact, R can be chosen independently of x. Then the exponential map based at x induces a chart near x with values in $B_{T_xS}(0, 2R)$. We identify a point in S with its coordinates. The tangent space (T_xS, g_ω) is identified with $(\mathbb{C}^2, g_0)$. For any degree $d \geq 1$, let

$$\begin{array}{rcl} \psi_d : B(0,1) & \to & B(0, 1/\sqrt{d}), \\ y & \mapsto & y/\sqrt{d}. \end{array}$$

For $d \geq 1$, let

$$g_d = d\psi_d^* g_\omega.$$

Then g_d converges to the standard metric g_0 in the $\mathscr{C}^2$-topology. For any d, the function ψ_d and the trivialization given by the parallel transport and explained in the appendix A provide a sequence of Gaussian complex functions $(f_d)_{d\geq 1}$ defined on $\mathbb{B}$ induced by the measure μ_d defined by (1.5). Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be the map given by Lemma 2.1. Let $\delta > 0$ given by Lemma 2.3 applied to f_0 and $(r, R) = (\frac{1}{4}, 2)$ and A be the event

$$A = \left\{f, \|f - f_0\|_{C^2(2\mathbb{B})} < \delta/2\right\}.$$

By Lemma 2.3, for d large enough,

$$f \in A \Rightarrow \operatorname{area}_{g_d}\left(T\Big(Z(f), g_d, [-\frac{1}{8}, 4]\Big)\right) > \frac{1}{2}, \tag{2.6}$$

where T is defined by (2.1). Since by Theorem A.1, the kernel of f_d converges in the $\mathscr{C}^\infty$-topology to the kernel of the Bargmann-Fock field, by [7, Theorem 4],

$$\liminf_{d\to\infty} \mu_d[f_d \in A] \geq \mu_{BF}[f \in \operatorname{Int} A].$$

Now $\operatorname{Int} A = A$ and since f_0 lies in the support of μ_{BF}, there exists $c > 0$ such that

$$\liminf_{d \to \infty} \mu_d[f_d \in A] \geq c.$$

By (2.6), this implies that for d large enough,

$$\mu_d \left[\operatorname{area} \left(T\Big(Z(f), g_d, [-\frac{1}{8}, 4]\Big) \right) > \frac{1}{2} \right] \geq c,$$

hence the result after dilation by $1/\sqrt{d}$. $\square$

Proof of Theorem 1.7. For any $d \geq 1$, let Λ_d be a maximal finite subset of points $x_i \in S$ such that the balls $B(x_i, \frac{1}{\sqrt{d}})$ are disjoint. Then, it is easy to see that for any $\varepsilon > 0$,

$$\forall d \gg 1,\ |\Lambda_d| \geq \frac{1}{16 + \varepsilon} \frac{\operatorname{vol} S}{\operatorname{vol} \mathbb{B}} d^2,$$

see [4, 2.5]. Since $\operatorname{vol} \mathbb{B} = \pi^2/2$,

$$\forall d \gg 1,\ |\Lambda_d| \geq \frac{\operatorname{vol} S}{80} d^2. \tag{2.7}$$

For any $x \in \Lambda$, we define the following event $A(x)$:

$$A(x) = \left\{ s \in H^0(S, L^d), \operatorname{area}\{y \in Z(s) \cap B(x, \frac{1}{\sqrt{d}}), -8d < K(y) < -\frac{d}{8}\} \geq \frac{1}{2d} \right\}.$$

By Proposition 2.4, there exists a universal constant $c > 0$ such that

$$\forall d \gg 1,\ \forall x \in \Lambda_d,\ \mu_d[A(x)] \geq c.$$

Let N be the random variable defined by

$$N := \#\{x \in \Lambda_d, A(x)\}.$$

Then,

$$\begin{aligned}
c|\Lambda_d| &\leq \sum_{x \in \Lambda_d} \mu_d[A(x)] = \mathbb{E} \sum_{x \in \Lambda_d} \mathbf{1}_{s \in A(x)} \\
&\leq \sum_{j=1}^{|\Lambda_d|} j \mu_d[N = j] \\
&\leq \frac{c}{2} |\Lambda_d| \mu_d \left[N \leq \frac{c}{2} |\Lambda_d| \right] + |\Lambda_d| \mu_d \left[N \geq \frac{c}{2} |\Lambda_d| \right]
\end{aligned}$$

which implies that $\mu_d \left[N \geq \frac{c}{2} |\Lambda_d| \right] \geq \frac{c}{2}$ and hence by (2.7),

$$\mu_d \left[\operatorname{area}\{x \in Z(s), -4d < K(x) < -\frac{d}{8}\}) > c \frac{\operatorname{vol} S}{320} d \right] \geq \frac{c}{2},$$

hence the result. $\square$

2.4 Proof of Theorem 1.8

For Theorem 1.8, we will use the following Bargmann-Fock estimate:

Proposition 2.5 *Let $U \subset \mathbb{C}^2$ be a bounded open subset with smooth boundary and $f : \mathbb{C}^2 \to \mathbb{C}$ be the Bargmann-Fock field defined by (2.5). Then, for any $r, R > 0$,*

$$\mathbb{E}_{\mu_{BF}}\left[\text{area}\{x \in Z(f) \cap U, -R < K(x) < -r\}\right] = 2\,\text{vol}(U)\varphi_{r,R},$$

where

$$\varphi_{r,R} := \int_{(a,b,\alpha,\beta,\gamma)\in\mathbb{C}^5} (|a|^2+|b|^2)e^{-|(a,b,\alpha,\beta,\gamma)|^2/2}\mathbf{1}_{\{\pi\frac{|2\gamma ab-\sqrt{2}\alpha b^2-\sqrt{2}\beta a^2|^2}{(|a|^2+|b|^2)^3}\in[r,R]\}}\frac{\mathrm{d}(a,b,\alpha,\beta,\gamma)}{4(2\pi)^5}. \tag{2.8}$$

Note that

$$\varphi_{0,\infty} := 2\int_{a\in\mathbb{C}} |a|^2 e^{-|a|^2/2}\frac{\mathrm{d}a}{8\pi} = 1,$$

so that $\varphi_{r,R} \in]0,1[$.

Proof. Let $0 < r < R$. By the Kac–Rice formula, see for instance [2],

$$\mathbb{E}_{\mu_{BF}}[\text{area}\{x \in Z(f) \cap U, -R < K(x) < -r\}]$$

equals

$$\int_U \mathbb{E}\left[\mathbf{1}_{\{-R<K(x)<-r\}}|\mathrm{d}f(x)|^2 \mid f(x) = 0\right]\rho_{f(x)}(0)\mathrm{d}x,$$

where $\rho_{f(x)}(0)$ is the density of $f(x)$ at 0, that is $(2\pi)^{-1}$. By invariance of the covariance function $\mathcal{P}$ of the BF field, see appendix A, we can assume that $x = 0$. Then, $f(0)$ and $\mathrm{d}f(0)$ are independent, and

$$\forall (z,w) \in \mathbb{C}^2,\ f(z,w) = \sqrt{\pi}(az+bw) + \pi\frac{\alpha}{\sqrt{2}}z^2 + \pi\frac{\beta}{\sqrt{2}}w^2 + \pi\gamma zw + O(3),$$

where a, b, α, β and γ are independent complex standard Gaussians, so that by (2.2) the curvature of $Z(f)$ at $(0,0)$ equals $\pi\frac{|2\gamma ab-\sqrt{2}\alpha b^2-\sqrt{2}\beta a^2|^2}{(|a|^2+|b|^2)^3}$, hence the result. $\square$

For the convenience's reader, we prove now Theorem 1.2, that is Theorem 1.8 in the standard projective setting, which is easier and clearer than the general one.

Proof of Theorem 1.2. Let $A \subset \mathbb{R}$ be a Borel subset. By the Kac–Rice formula, see for instance [2],

$$\mathbb{E}\left[\text{area}\left\{x \in Z(P), K_{g_{\text{FS}}}(x) \in A\right\}\right] = \int_{\mathbb{C}P^2} \mathbb{E}\left[\mathbf{1}_{K_{g_{\text{FS}}}(x)\in A}|\det{}^{\perp}\nabla P| \mid P(x) = 0\right]\rho_{P(x)}(0)\mathrm{d}\,\text{vol}(x),$$

where $\rho_{P(x)}(0)$ denotes the density at 0 of the Gaussian field $P(x)$. By symmetry, we can assume that $x = [1 : 0 \cdots : 0] \in \mathbb{C}P^2$. For $X_0 \neq 0$, let $f : \mathbb{C}^2 \to \mathbb{C}$ be the random holomorphic function defined by

$$f := \sqrt{\frac{2}{(d+2)!}}\frac{P}{X_0^d}$$

Then, by (1.2),

$$\forall (z,w) \in \mathbb{C}^2,\ f(z,w) = \sqrt{d}(az+bw) + d(\frac{1}{\sqrt{2}}\alpha z^2 + \frac{1}{\sqrt{2}}\beta w^2 + \gamma zw) + R_d, \tag{2.9}$$

where R_d is a random polynomial vanishing at order 3 at 0 and independent of the coefficients a, b, α, β and γ, which are standard complex Gaussians. If $d = 1$, $Z(P)$ has constant curvature equal to 2π. Assume that $d \geq 2$. By [10, p. 60], the curvature of $Z(f)$ at 0 is

$$K_{g_{\rm FS}}(0) = 2\pi - \pi d \frac{|2\gamma ab - \sqrt{2}\alpha b^2 - \sqrt{2}\beta a^2|^2}{(|a|^2 + |b|^2)^3}. \tag{2.10}$$

The presence here of π compared to [10] is due to the difference choice of the Fubini-Study metric in her paper. By (2.9) and (2.10), using the fact that at $[1 : 0 : \cdots : 0]$, $g_{\rm FS} = \sqrt{\pi} g_0$, and assuming that $A = [2\pi - Rd, 2 - rd]$,

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{K_{g_{\rm FS}}(x)\in A} |\det{}^{\perp}\nabla P| \,\middle|\, P(x) = 0\right] &= d \int_{(a,b,\alpha,\beta,\gamma)\in\mathbb{C}^5} \pi(|a|^2 + |b|^2) e^{-|a,b,\alpha,\beta,\gamma|^2/2,} \times \\
&\quad \times \mathbf{1}_{\{2\pi - \pi d \frac{|2\gamma ab - \sqrt{2}\alpha b^2 - \sqrt{2}\beta a^2|^2}{(|a|^2+|b|^2)^3} \in A\}} \frac{\mathrm{d}(a, b, \alpha, \beta, \gamma)}{(2\pi)^5} \\
&= 2\pi d \varphi_{r,R}.
\end{aligned} \tag{2.11}$$

Consequently, for any $r, R > 0$, since $\mathrm{vol}(\mathbb{C}P^2) = \frac{1}{2}$, and $\rho_{f(0)}(0) = (2\pi)^{-1}$, one has

$$\forall d \geq 2, \ \frac{1}{d}\mathbb{E}\left[\mathrm{area}\left\{x \in Z(P), K_{g_{\rm FS}}(x) \in [2\pi - Rd, 2\pi - rd]\right\}\right] = \varphi_{r,R} \in]0, 1[,$$

where $\varphi_{r,R}$ is defined by (2.8). □

In the sequel, for any $r, R > 0$, let

$$\mathrm{area}_{r,R} := \mathrm{area}\left\{x \in Z(s), K_{g_{\rm FS}}(x) \in [2\pi - Rd, 2\pi - rd]\right\}.$$

Proof of Corollary 1.3 and 1.4. For any $\eta \in]0, 1[$, since $\mathrm{area}_{r,R} < d$, one has

$$\eta\mu_d(\mathrm{area}_{r,R} > \eta d) < \frac{1}{d}\mathbb{E}(\mathrm{area}_{r,R}) < \eta\mu_d(\mathrm{area}_{r,R} < \eta d) + 1 - \mu_d(\mathrm{area}_{r,R} < \eta d),$$

so that by Theorem 1.2, for any $\eta \in]\varphi_{r,R}, 1[$,

$$\mu_d[\mathrm{area}_{r,R} > \eta d] \leq \frac{1}{\eta}\varphi_{r,R}$$

and for any $\eta \in]0, \varphi_{r,R}[$,

$$\mu_d[\mathrm{area}_{r,R} < \eta d] \leq \frac{1 - \varphi_{r,R}}{1 - \eta}.$$

□

Proof of Corollary 1.5. Let $\ell \in] - \infty, 2\pi[$. Then, by (2.10)

$$K \in [\ell, 2\pi] \Leftrightarrow \frac{|2\gamma ab - \sqrt{2}\alpha b^2 - \sqrt{2}\beta a^2|^2}{(|a|^2 + |b|^2)^3} \leq \frac{2\pi - \ell}{\pi d}.$$

When d grows to infinity, the indicator function $\mathbf{1}_{\{K\in[\ell,2\pi]\}}$ converges to the indicator of the hypersurface

$$\{(a, b\alpha, \beta, \gamma) \in \mathbb{C}^5, \ 2\gamma ab - \sqrt{2}\alpha b^2 - \sqrt{2}\beta a^2 = 0\}.$$

By the dominated convergence theorem, the Kac–Rice formula (2.11) implies that

$$\mathbb{E}_{\mu_d}[\mathrm{area}\{x \in Z(s), \ \ell < K(x) \leq 2\}] = o(d).$$

Now by Markov inequality, for any $\eta > 0$,

$$\mu_d[\text{area}\{x \in Z(s), C < K(x) \leq 2\} > \eta d] = o(1),$$

hence the result. $\square$

In the general setting of random holomorphic sections of $H^0(S, L^d)$, we need to control more precisely the way the curvature of $Z(s)$ depends on s. For this, assume the hypotheses of Theorem 1.7 and let $x \in S$. Let $(t_i)_{i\in\{1,2\}}$ be a real local orthonormal (for h) frame of L given by parallel transport, and $G : TS^* \to TS$ be defined by

$$\forall \alpha \in TS^*,\ \langle G(\alpha), \cdot\rangle_g = \alpha,$$

where $g := g_\omega$ denotes the Kähler metric. Then [1, Proposition 3.2], the bilinear operator σ defined by (2.3) satisfies the formula: for any $s \in H^0(S, L^d) \setminus \Delta_d$,

$$\forall V, W \in T_x Z(s),\ \sigma(V, W) = -\sum_{i=1}^{2} \left(\Phi^{-1}\left(\langle \nabla^2_{VW} s, t_j\rangle_h)_{j\in\{1,2\}}\right)\right)_i G\langle \nabla s, t_i\rangle_h,$$

where $\Phi = (\langle\langle \nabla s, t_i\rangle_h, \langle \nabla s, t_j\rangle_h\rangle_{g^*})_{1\leq i,j\leq 2}$, and g^* denotes the scalar product on TS^* associated with g. In the sequel, we will use the following homogeneity property of σ:

$$\forall \alpha \in \mathbb{R}, \beta \in \mathbb{R}^*,\ \sigma(V, W)(\nabla s, \nabla^2 s) = \frac{\alpha}{\beta}\sigma(V, W)(\alpha\nabla s, \beta\nabla^2 s),$$

so that by (2.4),

$$\begin{aligned} \forall X \in T_x Z(s),\ \langle R^{Z(s)}(X, JX)JX, X\rangle(\nabla s, \nabla^2 s) &= \langle R^S(X, JX)JX, X\rangle - \\ &\quad d\|\sigma(X, X)\|^2\left(\frac{1}{\sqrt{d^{n+1}}}\nabla s, \frac{1}{\sqrt{d^{n+2}}}\nabla^2 s\right). \end{aligned} \tag{2.12}$$

Proof of Theorem 1.8. Let $A \subset \mathbb{R}$ be a Borel subset. By the Kac–Rice formula, for any $d \geq 1$,

$$\mathbb{E}\left[\text{area}\left\{x \in Z(s), K(x) \in A\right\}\right] = \int_S \mathbb{E}\left[\mathbf{1}_{K(x)\in A}|\det{}^\perp \nabla s|\,\middle|\, s(x) = 0\right]\rho_{s(x)}(0) \mathrm{d}\,\mathrm{vol}(x),$$

where $\rho_{s(x)}(0)$ denotes the density at 0 of the Gaussian field $s(x)$. Let $x \in S$. Under the hypotheses and trivializations above near x described in the appendix A in any orthonormal basis of $T_x S$ (see for instance [3, Corollary 4.7]):

$$\mathrm{Cov}\left(s, \nabla s, \nabla^2 s\right)_{|x} = d^2 \begin{pmatrix} (1 + O(\frac{1}{d})) & O(\frac{1}{\sqrt{d}}) & O(1) \\ O(\frac{1}{\sqrt{d}}) & \pi d I_2(1 + O(\frac{1}{d})) & O(\sqrt{d}) \\ O(1) & O(\sqrt{d}) & \pi^2 d^2 \Sigma_{\mathrm{GOE}}(1 + O(\frac{1}{d})) \end{pmatrix}, \tag{2.13}$$

where $I_2 \in M_2(\mathbb{R})$ is the identity matrix and Σ_{GOE} is defined by:

$$\Sigma_{\mathrm{GOE}} = \left(\delta_{(ij)(kl)} + \delta_{(ji)(kl)}\right)_{\substack{1\leq i\leq j\leq 2 \\ 1\leq k\leq l\leq 2}} \in M_{\frac{2(2+1)}{2}}(\mathbb{C}). \tag{2.14}$$

Let us define the random Gaussian variables

$$Q := \frac{1}{\sqrt{d^2}} s(x),\ S := \frac{1}{\sqrt{d^{2+1}}}\nabla s(x) \text{ and } T := \frac{1}{\sqrt{d^{2+2}}}\nabla^2 s(x).$$

Then, by (2.13) and [3, Corollary 4.3], $\mathrm{Cov}(R, S, T)$ converges to the covariance of $(f(x), \mathrm{d}f(x), \mathrm{d}^2 f(x))$ where f is the Bargmann Fock field defined by (2.5).

Now, by (2.12), for any $r, R > 0$,

$$K(s, \nabla s, \nabla^2 s) \in [-Rd, -rd] \Leftrightarrow K(Q, S, T) \in [-R, -r] + O(\frac{1}{d}).$$

Finally, when d grows to infinity,

$$\frac{1}{d}\mathbb{E}\left[\mathbf{1}_{K(x)\in[-Rd,rd]}|\det{}^{\perp}\nabla s(x)|\,|\, s(x)=0\right]\rho_{s(x)}(0)$$

converges to

$$\mathbb{E}_{\mu_{BF}}\left[\mathbf{1}_{K_{g_0}(x)\in[-R,-r]}|\det{}^{\perp}\nabla f|\,|\, f(x)=0\right]\rho_{f(x)}(0).$$

Hence,

$$\frac{1}{d}\mathbb{E}\left[\mathrm{area}\left\{x\in Z(s), K(x)\in A\right\}\right]\to 2\,\mathrm{vol}(S)\varphi_{r,R},$$

where φ is defined by (2.8). By Wirtinger theorem, $\mathrm{area}(Z(s)) = 2\,\mathrm{vol}(S)d$, which concludes the proof of the theorem. □

Proof of Corollary 1.9. The proof is the same as the one of the corollaries in the standard setting. □

A Asymptotics of the Bergman kernel

In this paragraph we assume that the setting and hypotheses of Theorem 1.7 are satisfied. The covariance function E_d for the Gaussian field generated by the holomorphic sections $s \in H^0(S, L^d)$ is defined by

$$\forall z, w \in S,\ \ E_d(z, w) = \mathbb{E}\left[s(z)\otimes(s(w))^*\right] \in L_z^d \otimes (L_w^d)^*,$$

where the averaging is made for the measure μ_d given by (1.5), where L^* is the (complex) dual of L and

$$\forall w \in S,\ \forall s, t \in L_w^d,\ \ s^*(t) = \langle s, t\rangle_{h^d(w)}.$$

The covariance E_d is the *Bergman kernel*, that is the kernel of the orthogonal projector from $L^2(M, L^d)$ onto $H^0(M, L^d)$. This fact can be seen through the equations

$$\forall z, w \in M,\ \ E_d(z, w) = \sum_{i=1}^{N_d} S_i(z)\otimes S_i^*(w),$$

where $(S_i)_i$ is an orthonormal basis of $H^0(M, L^d)$ for the Hermitian product (1.4). Recall that the metric g_ω is induced by the curvature form ω and the complex structure. It is now classical that the Bergman kernel has a universal rescaled (at scale $\frac{1}{\sqrt{d}}$) limit, the Bargmann-Fock kernel $\mathcal{P}$:

$$\forall z, w \in \mathbb{C}^n,\ \ \mathcal{P}(z, w) := \exp\left(-\frac{\pi}{2}(\|z\|^2 + \|w\|^2 - 2\langle z, w\rangle)\right). \tag{A.1}$$

Theorem A.1 below quantifies this phenomenon. For this, we need to introduce local trivializations and charts. Let $x \in S$ and $R > 0$ be such that $2R$ is less than the radius of injectivity of S at x. Then the exponential map based at x induces a chart near x

with values in $B_{T_xS}(0,2R)$. We identify a point in S with its coordinates. The parallel transport provides a trivialization

$$\varphi_x : B_{T_xS}(0,2R) \times L_x^d \to L^d_{|B_{T_xS}(0,2R)}$$

which induces a trivialization of $(L^d \boxtimes (L^d)^*)_{|B_{T_xS}(0,2R)^2}$. Under this trivialization, the Bergman kernel E_d becomes a map from T_xM^2 with values into $\mathrm{End}\left(L_x^d\right)$.

Theorem A.1 *([9, Theorem 1]) Under the hypotheses of Theorem 1.7, let $m \in \mathbb{N}$. Then, there exist $C > 0$, such that for any $k \in \{0, \cdots, m\}$, for any $x \in S$, $\forall z, w \in B_{T_xS}(0, \frac{1}{\sqrt{d}})$,*

$$\left\| \mathrm{d}^k_{(z,w)} \left(\frac{1}{d^n} E_d(z,w) - \mathcal{P}(z\sqrt{d}, w\sqrt{d})\, \mathrm{Id}_{L_x^d} \right) \right\| \leq C d^{\frac{k}{2}-1}.$$

The original reference is more general, see [8, Proposition 3.4] for the present simplification.

Acknowledgments. The second author thanks Igor Wigman for a discussion about this subject. The research leading to these results has received funding from the French Agence nationale de la ANR-20-CE40-0017 (Adyct).

References

[1] Michele Ancona and Damien Gayet, *Metric and spectral aspects of random complex divisors*, arXiv 2311.09679 (2023).

[2] Jean-Marc Azaïs and Mario Wschebor, *Level sets and extrema of random processes and fields*, John Wiley & Sons, 2009.

[3] Damien Gayet, *Expected local topology of random complex submanifolds*, J. Algebraic Geom. DOI: https://doi.org/10.1090/jag/817 (2022).

[4] Damien Gayet and Jean-Yves Welschinger, *Expected topology of random real algebraic submanifolds*, Journal of the Institute of Mathematics of Jussieu **14** (2015), no. 4, 673–702.

[5] Jürgen Jost, *Riemannian geometry and geometric analysis*, vol. 42005, Springer, 2008.

[6] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry, volume 2*, Intersciences Publishers, 1969.

[7] Antonio Lerario and Michele Stecconi, *Differential topology of Gaussian random fields*, arXiv 1902.03805 (2021).

[8] Thomas Letendre and Martin Puchol, *Variance of the volume of random real algebraic submanifolds II*, Indiana Univ. Math. J. **68** (2019), no. 6, 16491720.

[9] Xiaonan Ma and George Marinescu, *Remark on the off-diagonal expansion of the Bergman kernel on compact Kähler manifolds*, Communications in Mathematics and Statistics **1** (2013), no. 1, 37–41.

[10] Linda Ness, *Curvature on algebraic plane curves. I*, Compositio Mathematica **35** (1977), no. 1, 57–63.

[11] Bernard Shiffman and Steve Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Communications in Mathematical Physics **200** (1999), 661–683.

[12] Albert Vitter, *On the curvature of complex hypersurfaces*, Indiana University Mathematics Journal **23** (1974), no. 9, 813–826.

Michele Ancona
Laboratoire J.A. Dieudonné
UMR CNRS 7351
Université Côte d'Azur, Parc Valrose
06108 Nice, Cedex 2, France

Damien Gayet
Univ. Grenoble Alpes, Institut Fourier
F-38000 Grenoble, France
CNRS UMR 5208
CNRS, IF, F-38000 Grenoble, France