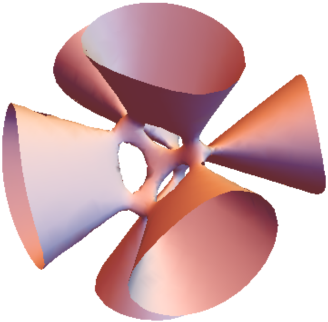
Local topology of random complex algebraic projective hypersurfaces

Quantum Chaos and nodal random waves

King's College London 25-28 July 2022



Damien Gayet (Institut Fourier, Grenoble, France)

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$Z(P)$ is invariant under complex homotheties. Better idea : consider the complex projective space

$$\mathbb{C}P^n = \mathbb{C}^{n+1}/(z \sim \lambda z, \lambda \in \mathbb{C}^*)$$

and study

$$Z(P) = \{[Z] \in \mathbb{C}P^n, P(Z) = 0\}.$$

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- $\mathbb{C}P^n = \mathbb{S}^{2n+1}/(z \sim \lambda z', \lambda \in \mathbb{U}(1))$.
- The standard metric over $\mathbb{S}^{2n+1}$ descents onto $\mathbb{C}P^n$ in the Fubini-Study metric g_{FS}.

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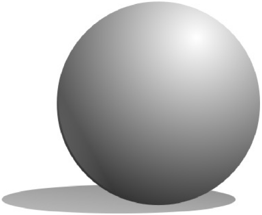
- For $n = 2$ $Z(P)$ is a real surface and has a constant genus

$$g = \frac{1}{2}(d-1)(d-2).$$



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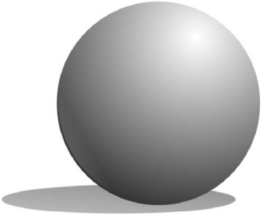 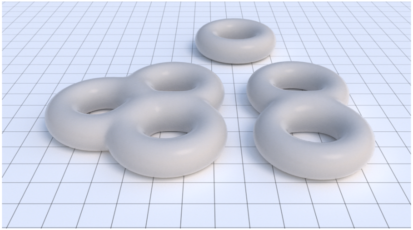

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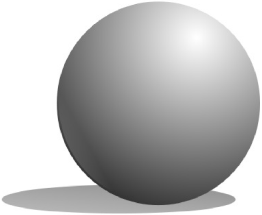

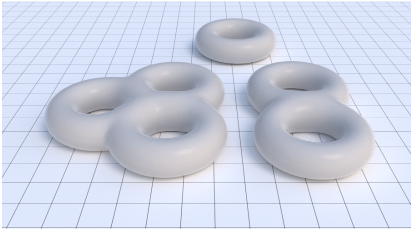

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- $d = 4$: genus $g = 3$

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Theorem (B. Shiffman-S. Zelditch 1998) Almost surely, a sequence $(Z(P_d))_{d\in\mathbb{N}}$ of increasing degree random complex curves gets equidistributed in $\mathbb{C}P^n$.

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  where $\Re a_{i_0\cdots i_n}, \Im a_{i_0\cdots i_n}$ are i.i.d. standard normal variables.
- This is the Gaussian measure associated to the Fubini-Study L^2-scalar product on the space of polynomials :

$$\langle P, Q\rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} dvol_{FS}.$$

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Strong contrast with the real setting :

Theorem (G.-Welschinger 2015) Take P with real coefficients. Then, there exist positive c, C such that for any $i \in \{0, \cdots, n-1\}$,

$$\forall d \gg 1,\ c \leq \frac{1}{\sqrt{d}^n}\mathbb{E}b_i(Z(P) \cap \mathbb{R}P^n) \leq C.$$

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- This provides the d^n in $\mathbb{C}P^n$ and the $\sqrt{d}^n$ in $\mathbb{R}P^n$.
- This heuristic argument fails for the number of connected compontents in the holomorphic case.

First ingredient of the proof : Morse theory

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  For instance, $b_1 \geq \mathrm{Crit}_1(f) - \mathrm{Crit}_0(f)$.

Second ingredient : Kac-Rice formula

Fix $p : \mathbb{C}P^n \to \mathbb{R}$ a Morse function and $U \subset \mathbb{C}P^n$ an open subset. We apply Morse theory to

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There is a Kac-Rice formula for critical points of $p_{|Z(P)}$:

$$\mathbb{E}(\#\mathrm{Crit}_i(p_{|Z(P)\cap U}))$$

because

$$x \in \mathrm{Crit}(p_{|Z(P)}) \Leftrightarrow (P(x), dP_{|\ker dp(x)}) = (0,0).$$

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$$= \int_{x\in U} \int_{\substack{\alpha\in\mathcal{L}_{onto}(T_xM,E_x)\\ \ker\alpha\subset\ker dp(x)}} \left|\det \alpha_{|\ker^\perp \alpha}\right|$$

$$\mathbb{E}\Big[\mathbf{1}_{\{\mathrm{Ind}\ (\nabla^2 p_{|Z(P)})=i\}}\Big|\det\Big(\langle\nabla^2 P(x)_{|\ker\alpha}, \epsilon(x,\alpha)\rangle$$

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Third tool : complex geometry

Let

$$\begin{aligned} p : \mathbb{C}^2 &\to \mathbb{R}, \\ (x+iy, s+it) &\mapsto x. \end{aligned}$$

This is a Morse function (with no critical points)

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- Let $Z \subset \mathbb{C}^2$ a smooth complex curve. Then if $p_{|Z}$ has a local minimum at any $x \in Z$, then Z is locally flat.

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- If $n = 2$, the dominant Kac-Rice formula is the one for $i = 1$.
- The horrible Kac-Rice formula can be computed when $i = n - 1$ and $d \to \infty$.

Related problem : holomorphic percolation

Let P be as before, $U \subset \mathbb{C}P^2$ be a small ball, $V \subset \partial U$ and $W \subset \partial U$ two open subsets with disjoint closure. Prove that there exists $c > 0$, such that for any large enough d,

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- None of the tools of classical percolation work in the complex setting.