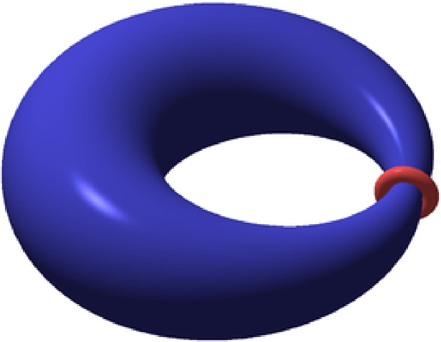
Systoles and Lagrangians of random projective hypersurfaces

Recent developments in microlocal analysis

MSRI, 17th october 2019



Damien Gayet (Institut Fourier, Grenoble)

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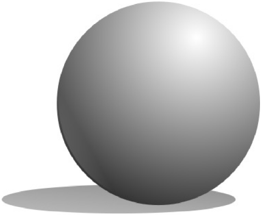
- is generically an orientable compact smooth Riemann surface ;
- connected ;
- with a constant genus $\frac{1}{2}(d-1)(d-2)$.



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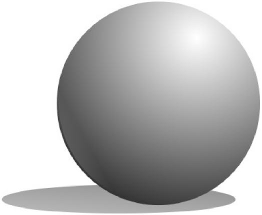

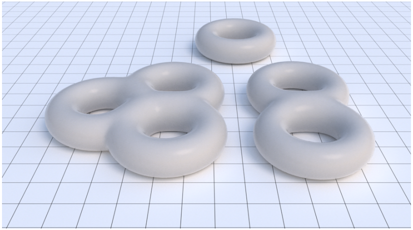

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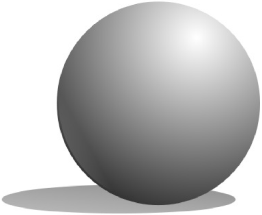

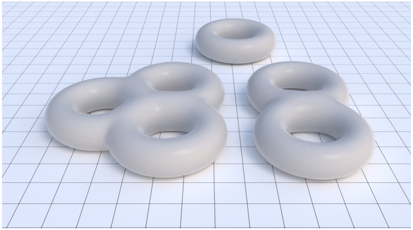

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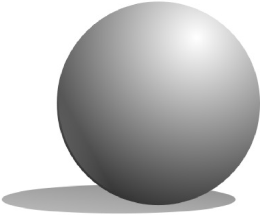 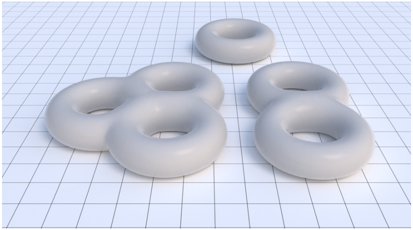

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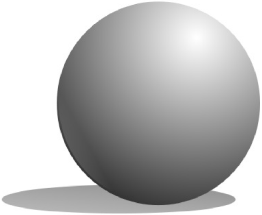 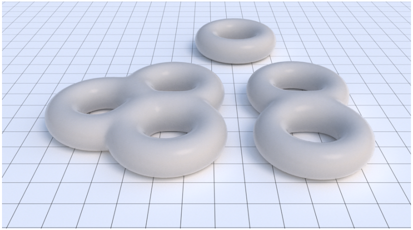

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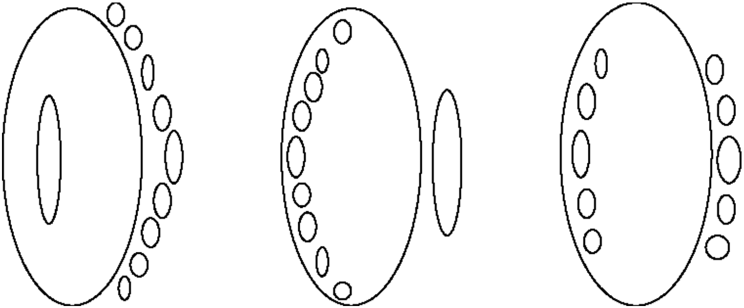
- $\dim \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2] \sim_d g$.
- Same for the moduli space of projective curves

Very different in the real case : various number of components...



... and various possible configurations :
16th Hilbert problem
(here the maximal degree 6 possible curves)

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  - if P is close to Z_0^d, Z is concentrated near a round sphere,
  - if P is close to the product of equidistributed d lines, then Z is equidistributed.

Random projective curves

If P is taken at random, what can be said more ?

Theorem (B. Shiffman-S. Zelditch 1998) Almost surely, a sequence of increasing degree random complex curves gets equidistributed in $\mathbb{C}P^2$.

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$$P = \sum_{i_0+i_1+i_2=d} a_{i_0 i_1 i_2} \frac{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}}{\sqrt{i_0! i_1! i_2!}},$$

where $a_{i_0 i_1 i_2}$ are i.i.d. normal variables $\sim N_{\mathbb{C}}(0,1)$.

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- This is the Gaussian measure associated to the Fubini-Study L^2-scalar product on the space of polynomials :

$$\langle P, Q \rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} dvol_{FS}.$$

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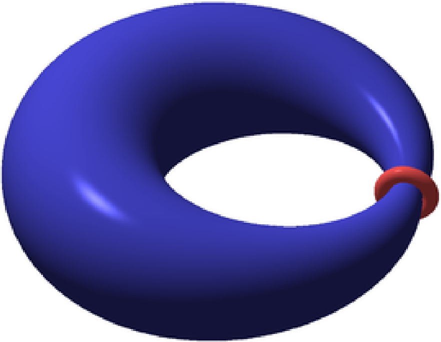
- Generalizes for random sections of high powers of an ample line bundle over a compact Kähler manifold.



What about the length of the **systole** of the random complex curve : its shortest non-contractible real loop ?

The origins : hyperbolic surfaces

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- There exists a natural probability measure Prob_{WP} on $\mathcal{M}_g$.

Theorem (M. Mirzakhani 2013). There exist $C > 0$ such that for all $g \geq 2$, $0 < \epsilon \leq 1$,

$$\frac{1}{C}\epsilon^2 \leq \mathrm{Prob}_{WP}\left[\text{Length of the systole } \leq \epsilon\right] \leq C\epsilon^2.$$

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Theorem 1. There exists $C > 0$, for all $0 < \epsilon \leq 1$,

$$\forall d \gg 1,\ e^{-\frac{C}{\epsilon^6}} \leq \mathrm{Prob}_{FS}\left[\mathrm{Length}_{\sqrt{d}g_{FS}} \text{ of the systole } \leq \epsilon\right].$$

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Theorem 1' There exists $c > 0$,

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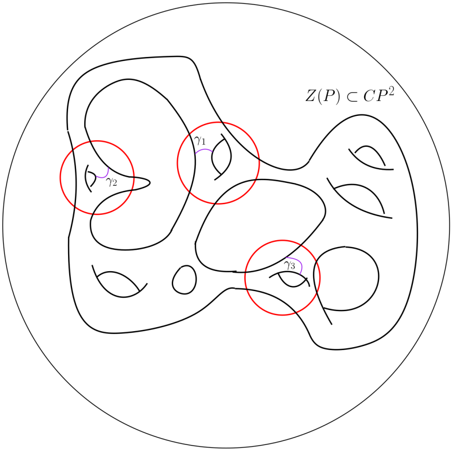
Theorem (M. Mirzakhani - B. Petri 2017) There exists $C > 0$,

$$\forall g \geq 2,\ \mathbb{E}_{WP}\Big[\text{number of simple geodesics of length} \leq 1\Big] \leq C.$$



For every d, there exists a basis of $H_1(Z)$ such that a uniform proportion of its elements are represented by small loops with uniform probability

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- complex curves become complex hypersurfaces ;
- non-contractible loops become Lagrangian submanifolds ;
- the useless deterministic bound becomes an non-trivial estimate for homological (Lagrangian) representatives.

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- $d = 1$: complex hyperplane

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- $\Rightarrow$ For $n = 3$, $Z \subset \mathbb{C}P^3$ is a connected and simply connected complex surface and its interesting homology lies in $H_2(Z)$, that is for real surfaces inside it.

Hypersurfaces as symplectic manifolds

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- Hence, if you prove that a property of symplectic nature is true with positive probability, then it is true for *any* hypersurface.

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- The cotangent bundle T^*M of a manifold is naturally symplectic.

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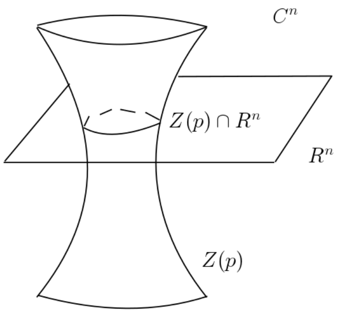
- Very easy to deform a Lagrangian : locally as much as the differentials of real functions over it.



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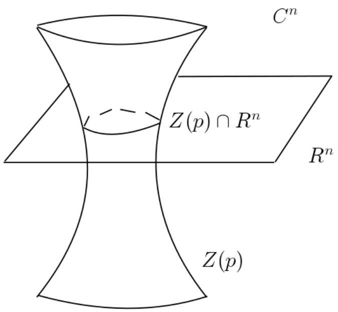

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Theorem 2. Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be *any* compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0,\ \forall d \gg 1,\ \forall P \in \mathbb{C}^d_{hom},\ \exists \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \subset Z(P)$$

- pairwise disjoint,
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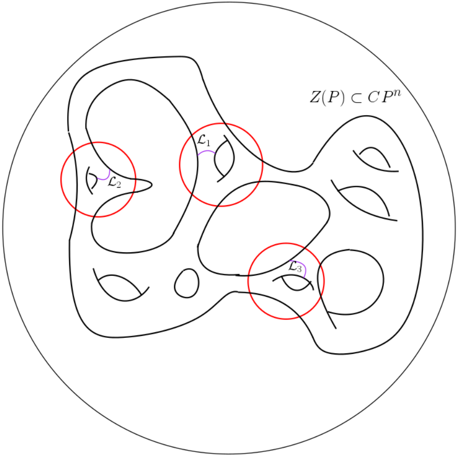
Proof : probabilistic !



For any real hypersurface $\mathcal{L}$ with non-vanishing Euler characteristic and every large enough degree, there exists a basis of $H_{n-1}(Z)$ such that a uniform proportion of its elements are represented by Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.

Topological Corollary Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be *any* compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

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Universal phenomenon : Same holds for zeros of sections of high powers of an ample line bundle over a compact Kähler manifold.

Former results

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From tropical arguments :

Theorem (G. Mikhalkin 2004). There exists cd^n disjoint Lagrangian spheres and cd^n Lagrangian tori, whose classes in $H_{n-1}(Z(P))$ are independent, with c explicit and natural.

From random real algebraic geometry :

Theorem (with J.-Y. Welschinger 2014). Let $\mathcal{L} \subset \mathbb{R}^n$ as before. Then there exists (an ugly but explicit and universal) $c > 0$, such that for $d \gg 1$,

$$c < \text{Prob}_{FS,\mathbb{R}}\Big[\exists \text{ at least } c\sqrt{d}^n \text{ components of } Z(P) \cap \mathbb{R}P^n \text{ diffeomorphic to } \mathcal{L}\Big].$$

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Corollary. At least $c\sqrt{d}^n$ disjoint Lagrangians diffeomorphic to $\mathcal{L}$ in any $Z(P)$.

Proof of Theorem 1 (systoles)

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Fact : Enough to prove that there exists a non-contractible curve with length ≤ 1 with uniform probability.

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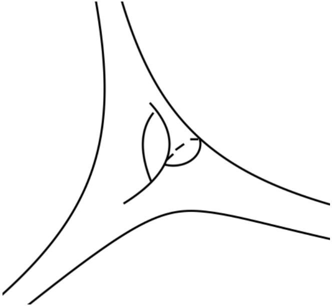
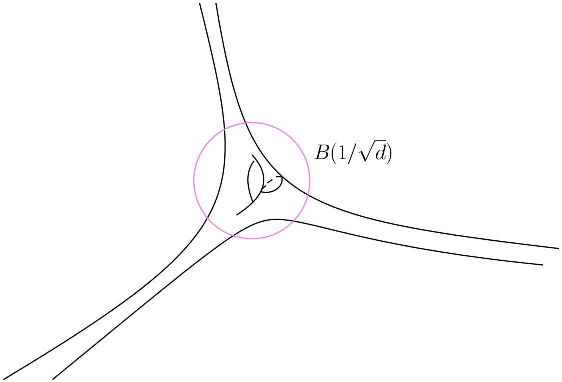


$$Z\Big[Q(1, z_1, z_2)\Big]$$

Rescaling



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Homogenization

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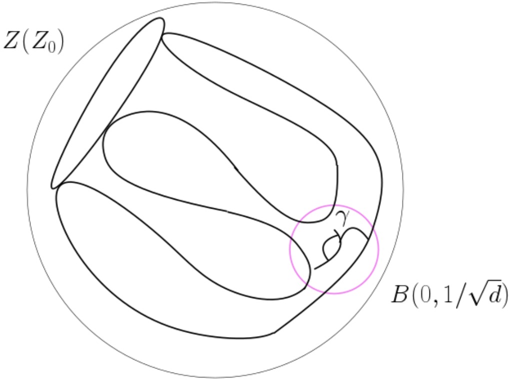

$Z(Q_d)$

Barrier method

The random P writes

$$\begin{aligned} P &= aQ_d + R, \\ \text{with } a \sim N_{\mathbb{C}}(0,1) \quad &\text{and} \quad R \in Q_d^{\perp} \text{ random independent} \end{aligned}$$

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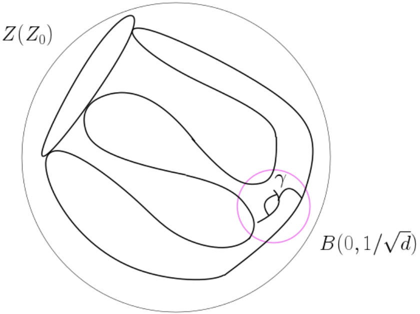

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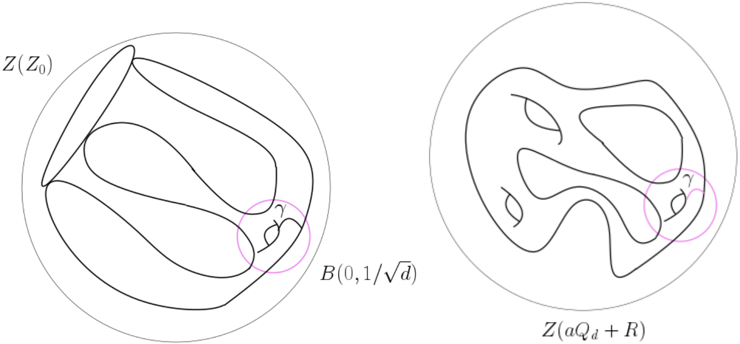

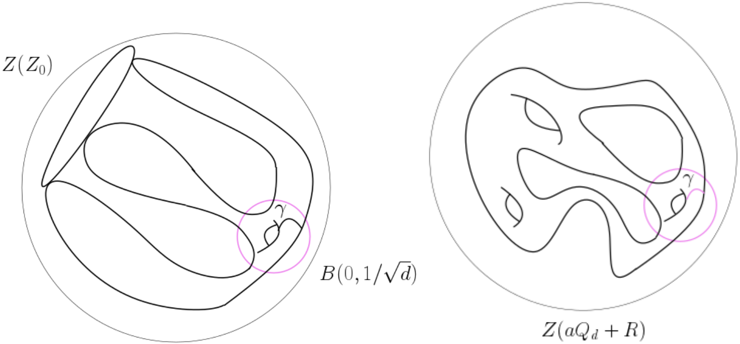

Proposition. With uniform probability in d, R does not destroy the toric shape of $Z(Q_d)$ in $B(x, 1/\sqrt{d})$.

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Everything is asymptotically independent of d !

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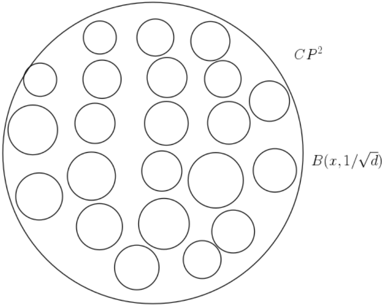
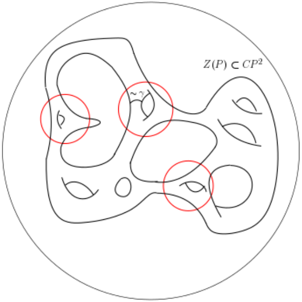
- Random sums of eigenfunctions of the Laplacian with eigenvalues less than L : $1/\sqrt{L}$ is the natural scale of the geometry of zeros of the random sums. Reason : universal behavior of the spectral kernel.



There is at least $\sim d^2$ disjoint small balls



With uniform probability, a uniform proportion of these d^2 balls contain the affine torus

Ideas of the proof of Theorem 2

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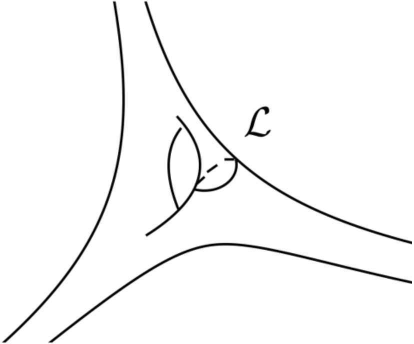
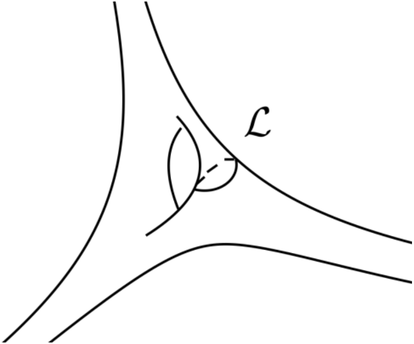
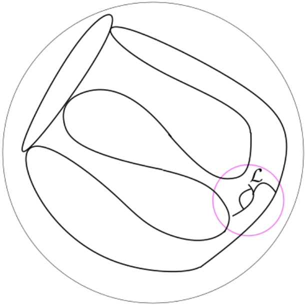
Theorem (Alexander 1936). Every compact smooth real hypersurface $\mathcal{L}$ in $\mathbb{R}^n$ can be C^1-perturbed into a component $\mathcal{L}'$ of an algebraic hypersurface.

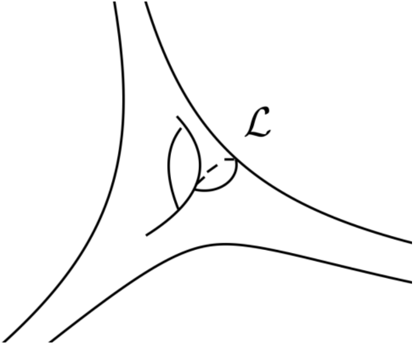


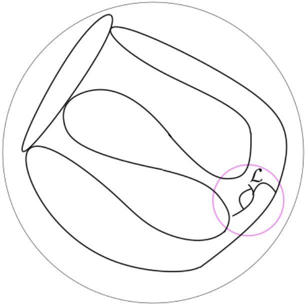
- Choose q such that $\mathcal{L} \subset Z(q)$;



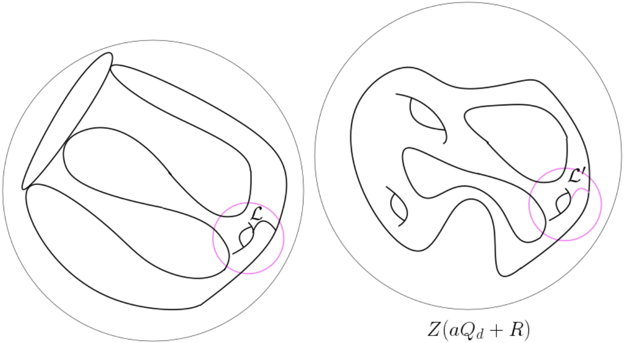


- Choose q such that $\mathcal{L} \subset Z(q)$;
- homogeneize and rescale q into Q_d ;



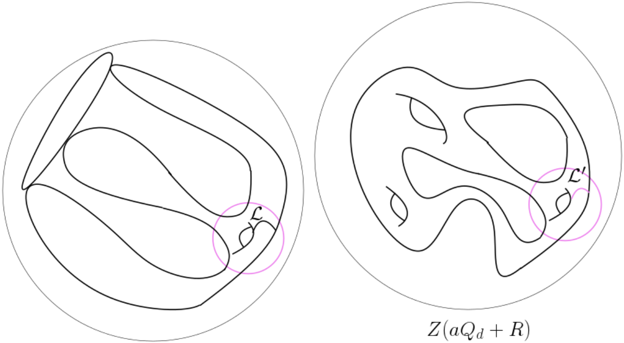


- Choose q such that $\mathcal{L} \subset Z(q)$;
- homogeneize and rescale q into Q_d ;
- decompose $P = aQ_d + R$.



Proposition. With uniform probability, in $B(1/\sqrt{d})$,

- R does not kill the shape of $Z(Q_d)$,



Proposition. With uniform probability, in $B(1/\sqrt{d})$,

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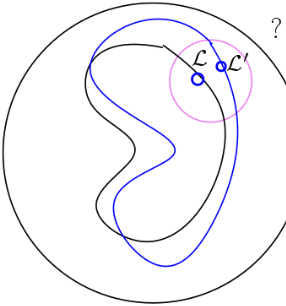
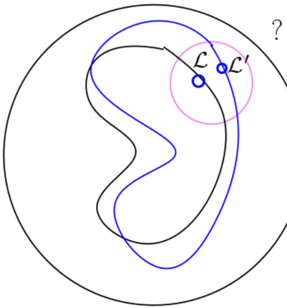
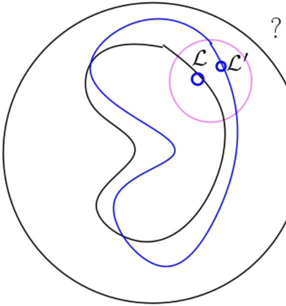
- there exists $\mathcal{L}' \subset Z(P)$ Lagrangian for ω_{FS}.



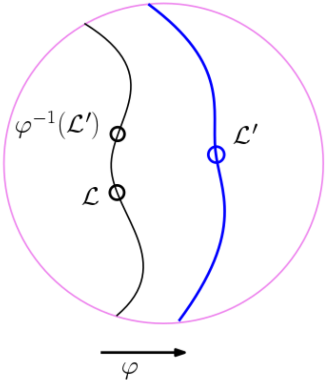
- $\mathcal{L} \subset Z(Q_d)$

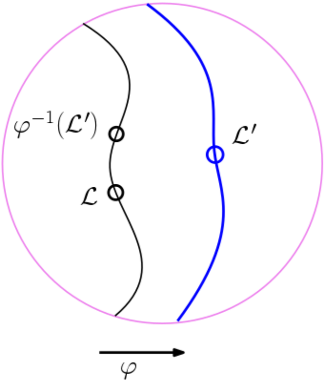


- $\mathcal{L} \subset Z(Q_d)$is Lagrangian for ω_0

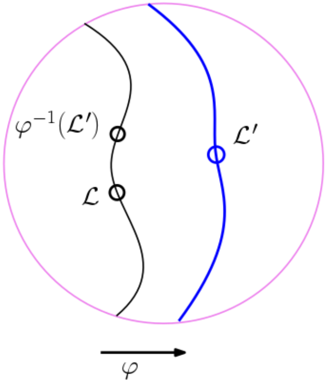


- $\mathcal{L} \subset Z(Q_d)$is Lagrangian for ω_0 ;
- how to find $\mathcal{L}' \subset Z(P)$ Lagrangian for ω_{FS} ?


$\varphi^{-1}(\mathcal{L}')$
$\mathcal{L}$
$\mathcal{L}'$
φ

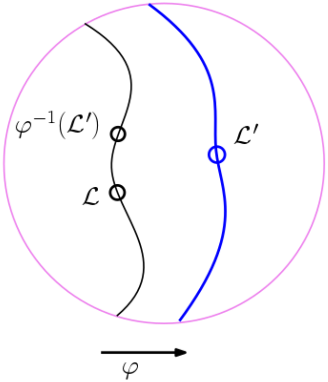


Facts :



Facts :

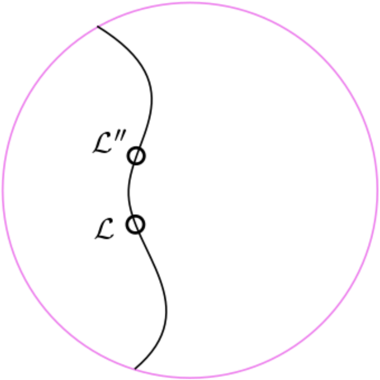
- $\exists \varphi,\ \varphi(Z(Q_d)) = Z(P)$.



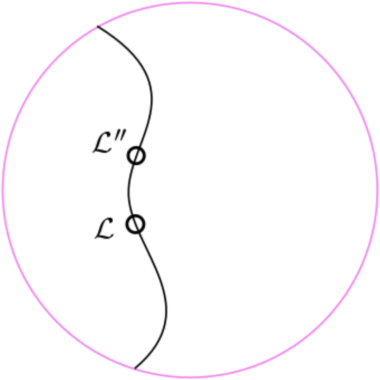
Facts :

- $\exists \varphi$, $\varphi(Z(Q_d)) = Z(P)$.
- Then

$$\mathcal{L}' \quad \text{Lagrangian for } \omega_{FS} \quad \text{in Z(P)}$$
$$\Leftrightarrow$$
$$\varphi^{-1}(\mathcal{L}') \quad \text{Lagrangian for } \varphi^*\omega_{FS} \quad \text{in } Z(Q_d)$$



- $\mathcal{L}$ Lagrangian for ω_0 in $Z(Q_d)$;



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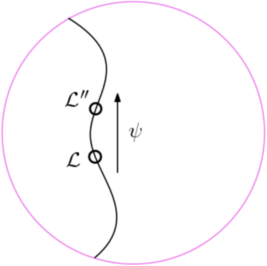
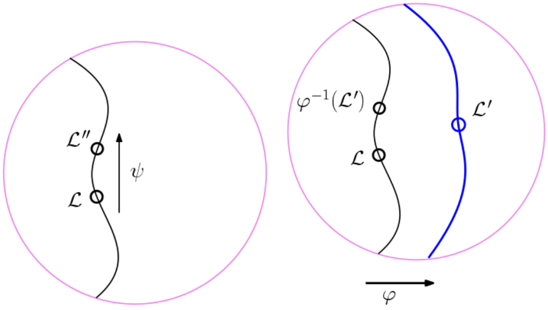


Moser Trick. Let ω symplectic and exact over $Z \cap \mathbb{B}$. Then, there exists $\psi : Z \cap \mathbb{B} \to Z$ such that $\psi^*\omega = \omega_0$.



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For us : $\omega = \phi^*\omega_{FS}$,

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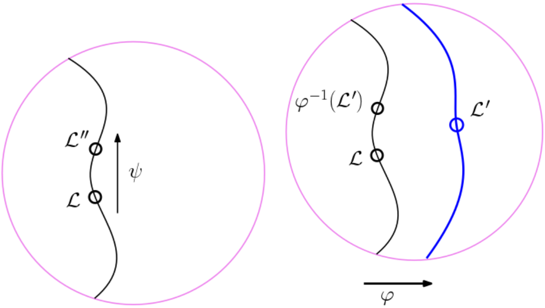

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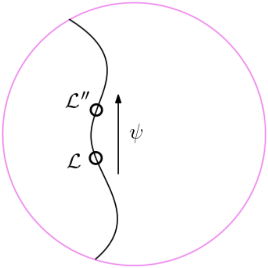
Objection ! It could happen that ψ or φ sends $\mathcal{L}''$ out of the ball !



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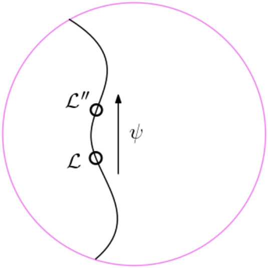
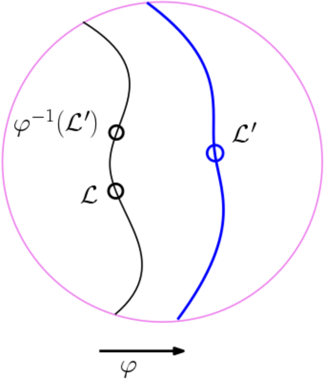


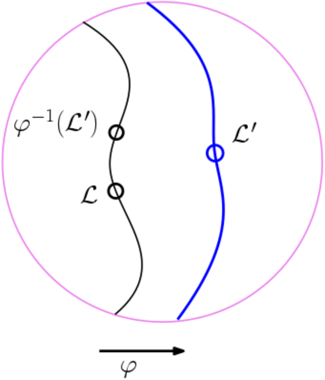
Quantitative Moser Trick. Let ω symplectic and exact over $Z \cap \mathbb{B}$. Then, there exists $\psi : Z \cap \mathbb{B} \to Z$ such that

- $\psi^* \omega = \omega_0$
- $|\psi - id|$ is controlled by $|\omega - \omega_0|$



Since

- ω_{FS} is close to ω_0,



Since

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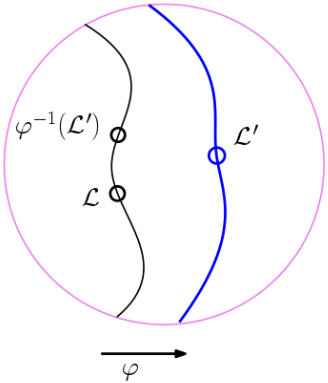

Since

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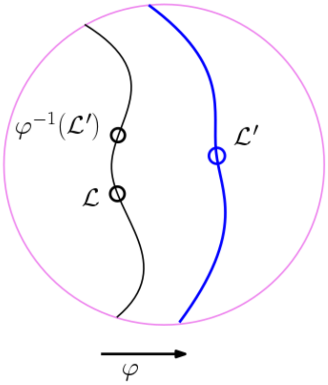
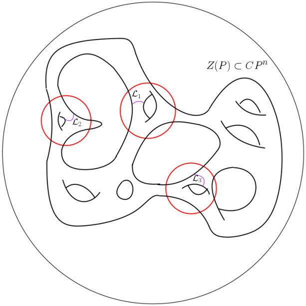


Since

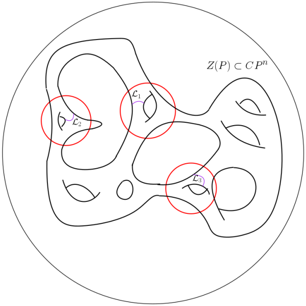
- ω_{FS} is close to ω_0,
- with uniform probability R is small,
- so that φ close to the identity,
- so that $\mathcal{L}''$ and $\mathcal{L}'$ stay in the ball. $\square$

From one to a lot of Lagrangians



- There exists $\sim d^n$ balls of size $1/\sqrt{d}$

From one to a lot of Lagrangians



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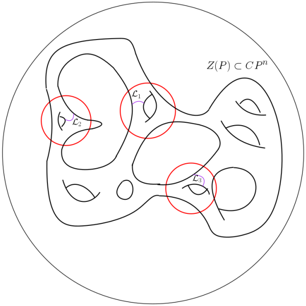

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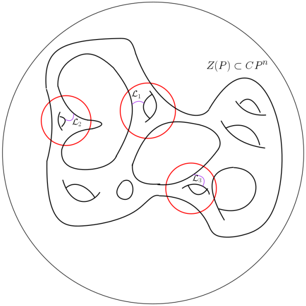

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- With uniform probability, a uniform proportion of them contains a Lagrangian copy of $\mathcal{L}$
- Deterministic conclusion : there exists at least one such hypersurface
- Hence, all of them have cd^n such Lagrangians.

□

Why non-vanishing Euler characteristics ?

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Corollary The only orientable compact Lagrangian in $\mathbb{R}^4$ is the torus.

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Since ω_t is non-degenerate, this has a solution $(X_t)_t$. $\square$