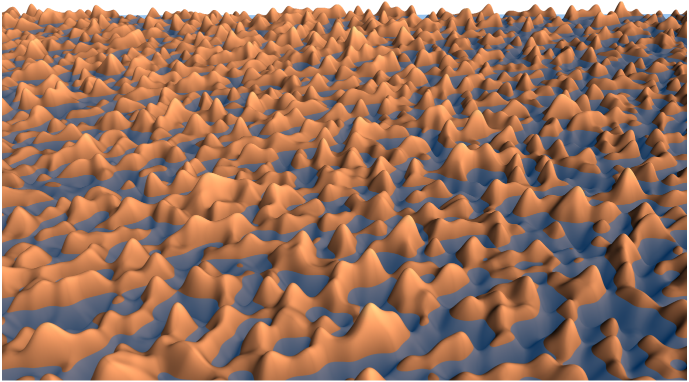
Percolation and Gaussian fields

Workshop on Random Real Algebraic Geometry

Middle East Technical University North Cyprus

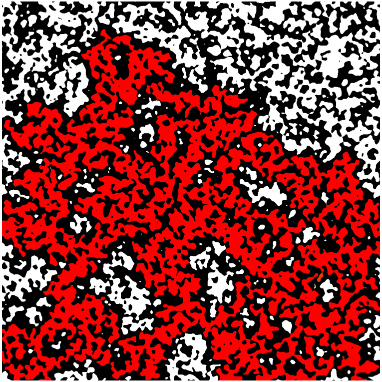


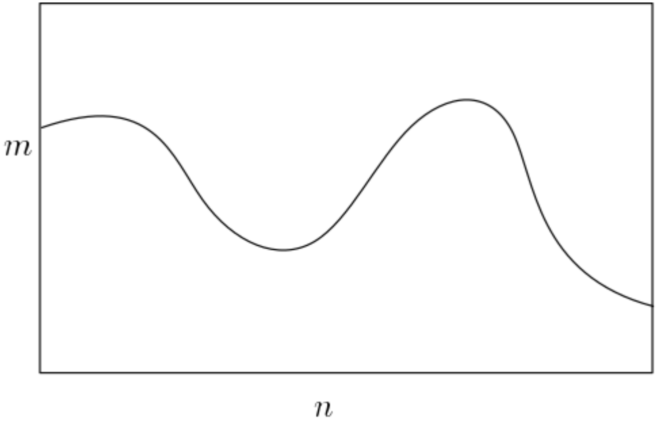
Damien Gayet (Institut Fourier, Grenoble)

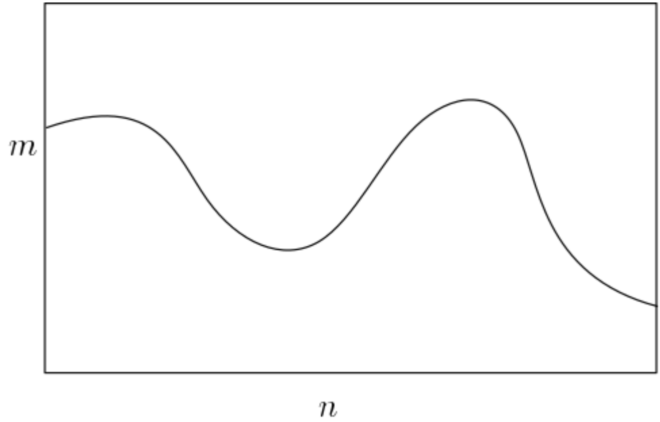
Lectures based on a common work with Vincent Beffara

Image: Alejandro Rivera

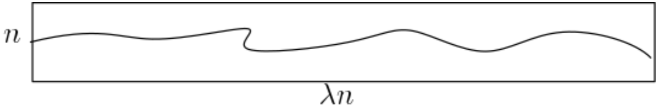
Introduction



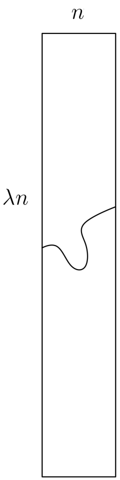

m
n



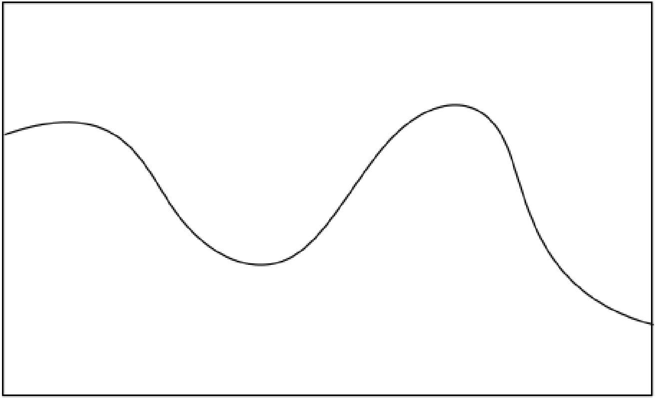
$$\liminf_{n,m\to\infty} \mathbb{P}(\text{crossing}) > c > 0?$$



$$\mathbb{P}(\text{crossing}) \underset{n,\lambda\to\infty}{\to} 0$$



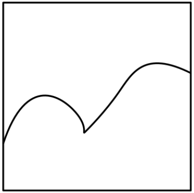
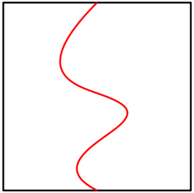
$$\mathbb{P}(\text{crossing}) \underset{n,\lambda\to\infty}{\longrightarrow} 1$$



nR

$$\liminf_{n\to\infty} \mathbb{P}(\text{crossing}) \geq c > 0 \ ?$$

Squares

Prob  + Prob  = 1

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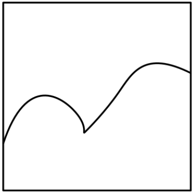
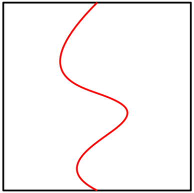
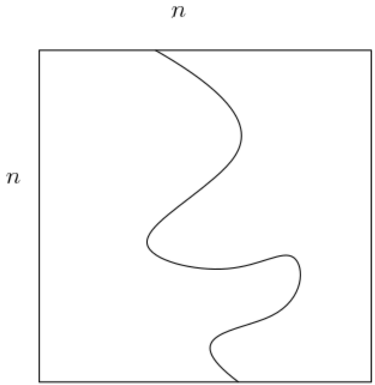
Prob  + Prob  = 1

With

- symmetry between + and -
- symmetry between x_1 and x_2

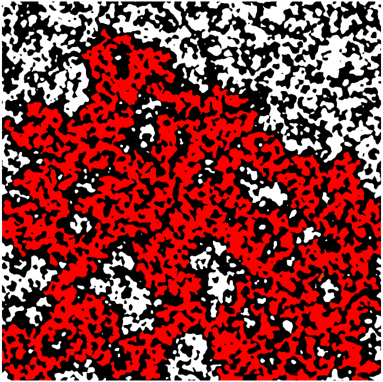
then both probabilities are equal...



$$\forall n,\ \mathbb{P}(\text{crossing}) = 1/2.$$

Theorem (Russo, Seymour-Welsh 1978) Let $R \subset \mathbb{R}^2$ be a fixed rectangle. Then there exists $c > 0$,

$$\liminf_{n\to\infty} \mathbb{P}(\text{crossing of } nR) > c.$$



Question: Let $f : \mathbb{R}^2 \to \mathbb{R}$ a be random smooth function and fix $R \subset \mathbb{R}^2$. Does it exist $c > 0$,

$$\liminf_{n\to\infty} \mathbb{P}\big(\{f > 0\} \text{ crosses } nR\big) > c?$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

- a centered Gaussian field, that is $\forall x_1, \cdots x_N \in \mathbb{R}^2$ any linear combination of the $(f(x_i))_{i=1,\cdots,N}$ is a centered Gaussian variable.
- We assume in this course that its covariant function is symmetric:

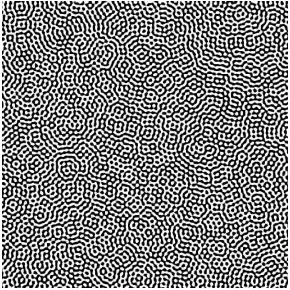
$$e(x, y) := \mathbb{E}\big(f(x)f(y)\big) = k(\|x - y\|).$$

- Almost surely, f is C^2. This is true if is e is C^3.

Two universal models with geometric origin

- The random wave model (Riemannian)
- The Bargmann-Fock model (algebraic)

The random wave model



$$g(r,\theta) = \sum_{m=-\infty}^{\infty} a_m J_{|m|}(r) e^{im\theta},$$

$(a_m)_m$ are i.i.d. following $N(0,1)$ and J_m is the m-th Bessel function.

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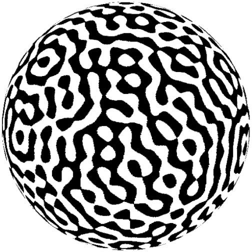

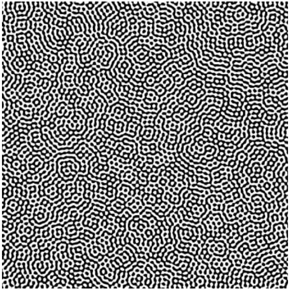

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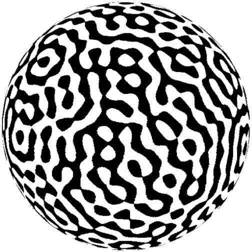
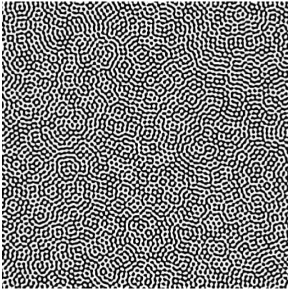
- Limit model for the rescaled **spherical harmonics**.

The random wave model





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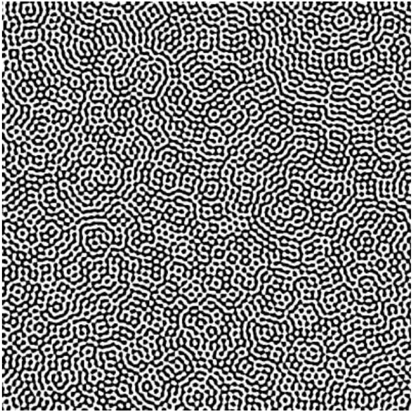
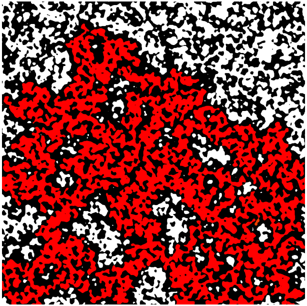
- Limit model for the rescaled **spherical harmonics**.
- Universal from compact Riemannian manifolds.



Conjecture (Bogomolny-Schmidt 2007) RSW should hold for this model.

The Bargmann-Fock model



$$\forall (x_1, x_2) \in \mathbb{R}^2, \; f(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} \frac{x_1^i x_2^j}{\sqrt{i! j!}} e^{-\frac{1}{2}\|x\|^2},$$

$(a_{ij})_{i,j \geq 0}$ are i.i.d. following $N(0,1)$.

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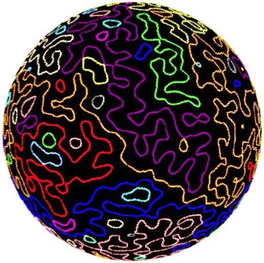

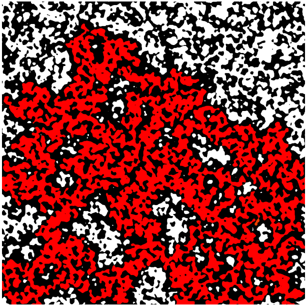

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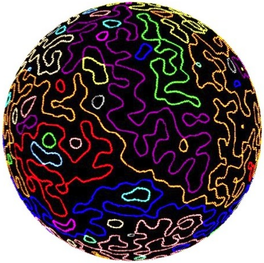

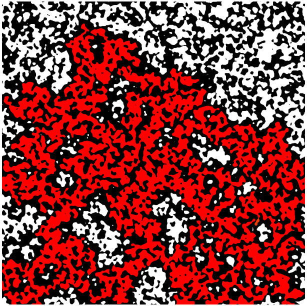

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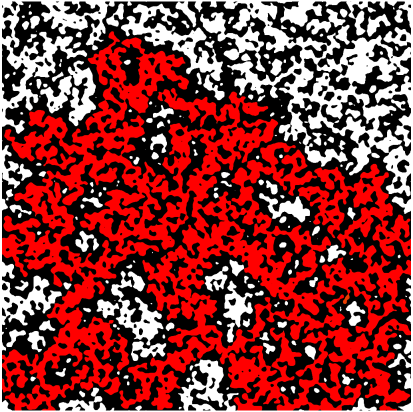
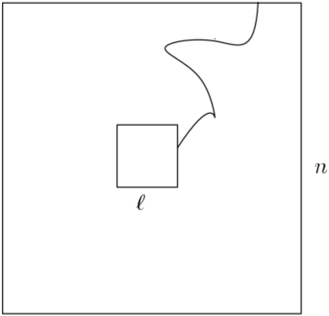
- Limit model for the rescaled **polynomials**.
- Universal from (complex) algebraic geometry.



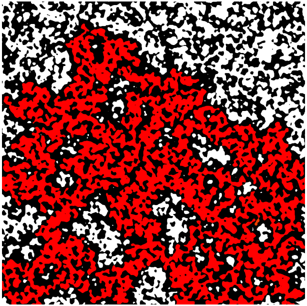
Theorem (Beffara-G 2016) RSW holds for Bargmann-Fock: for any rectangle R, there exists $c>0$ such that

$$\liminf_{n\to\infty} \mathbb{P}\big(\{f>0\} \text{ crosses } nR\big) > c.$$

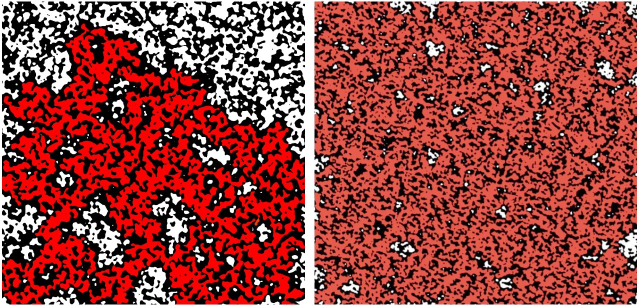


Corollary For Bargmann-Fock,

$$\exists \alpha > 0,\ \forall \ell, n,\ \mathbb{P}(\text{one arm}) < \left(\frac{\ell}{n}\right)^{\alpha}.$$

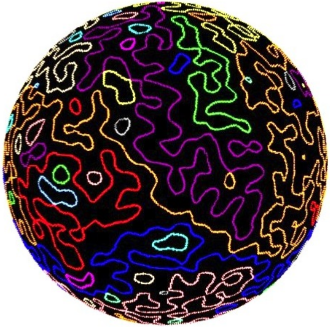


Corollary (Alexander 1996) Almost surely there is no infinite component of $\{f > 0\}$.



Corollary (Alexander 1996) Almost surely there is no infinite component of $\{f > 0\}$.

Theorem (Rivera-Vanneuville 2017) For any $\epsilon > 0$, almost surely $\{f > -\epsilon\}$ as an infinite component.



Theorem (Belyaev-Muirhead-Wigman 2017) RSW holds for polynomials with the complex Fubini-Study measure.

Prequel: random real polynomials

Kostlan or complex Fubini-Study measure:

$$P = \sum_{i+j+k=d} a_{ijk} \frac{X_0^i X_1^j X_2^k}{\sqrt{i!j!k!}},$$

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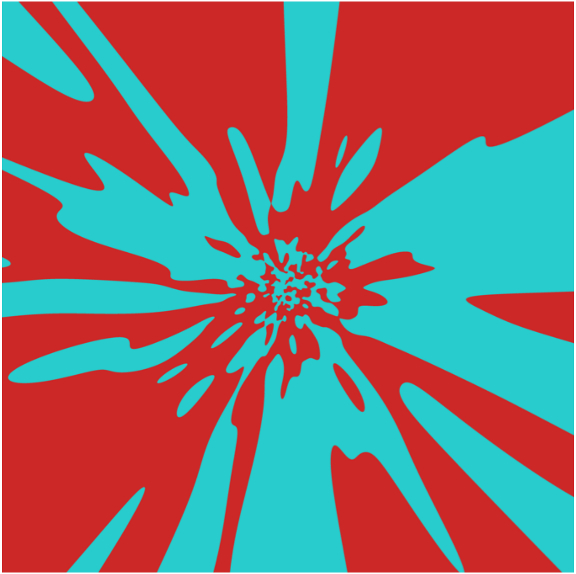
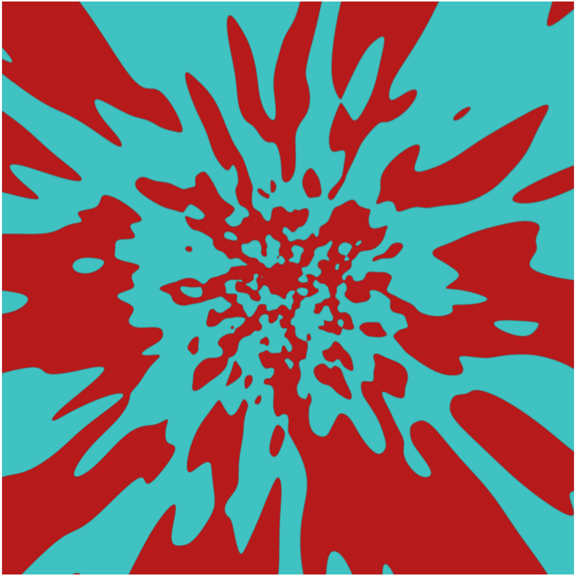
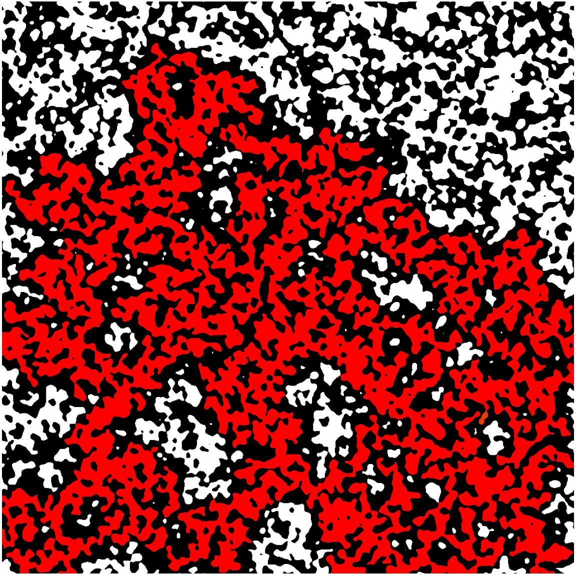
Rescaling: For every $(x_1, x_2) \in \mathbb{R}^2$,

$$P\Big(1, \frac{(x_1, x_2)}{\sqrt{d}}\Big)$$







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$$P\big(1, \frac{(x_1, x_2)}{\sqrt{d}}\big) = \sum_{i+j \leq d} a_{i,j} \frac{1}{\sqrt{d-(i+j))!i!j!}} \frac{x_1^i x_2^j}{\sqrt{d}^{i+j}}$$

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$$\underset{d\to\infty}{\sim} \frac{1}{\sqrt{d}!} \sum_{i,j=0}^{\infty} a_{i,j} \frac{x_1^i x_2^j}{\sqrt{i!j!}}.$$

The natural scale for degree d polynomials is $1/\sqrt{d}$

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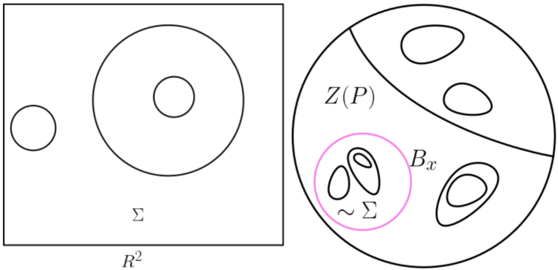

Theorem (G-Welschinger 2014) Let $x \in S^2$ and $\Sigma \in \mathbb{R}^2$ be any nested union of circles. Then with uniform probability in d, $\{P = 0\} \cap B(x, \frac{1}{\sqrt{d}})$ is a diffeomorphic copy of Σ.

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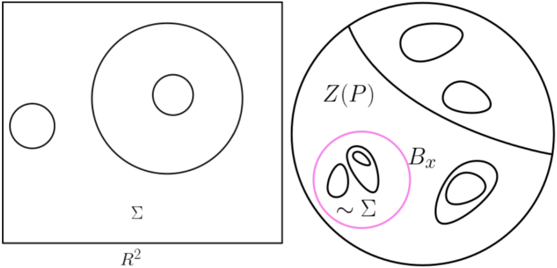

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Every topology happens at the natural scale

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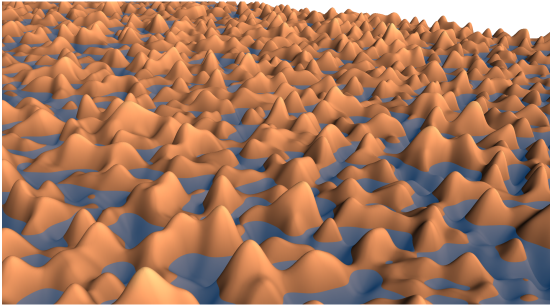

Theorem (Nazarov-Sodin 2016)

$$\mathbb{E}\big(\#\text{connected components of } \{f=0\} \text{ in } B_R\big) \underset{R\to\infty}{\sim} aR^2.$$

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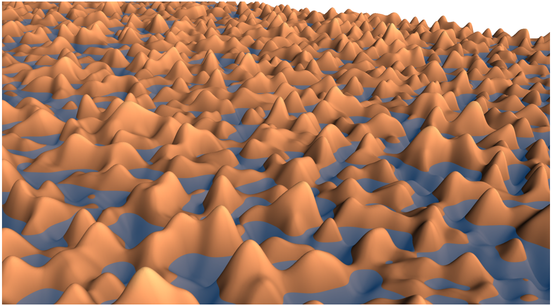

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There is a uniform density of components of size one.

Sketch of the proof of the BF-RSW theorem

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simplifications & improvements
provided by
Belyaev-Muirhead and Rivera-Vanneuville

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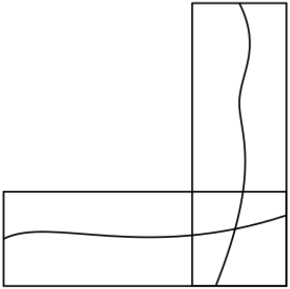
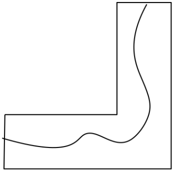
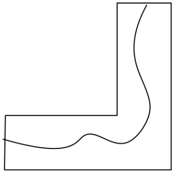
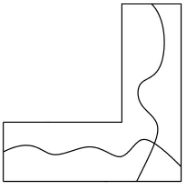
FKG

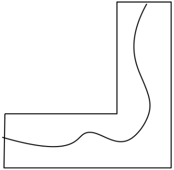
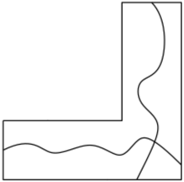


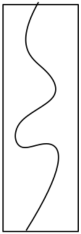
FKG (Fortuin-Kasteleyn-Ginibre) implies

$$\mathbb{P}\big(\text{crossing of } R \quad \text{and} \quad \text{crossing of } S\big)$$
$$\geq$$
$$\mathbb{P}(\text{crossing of } R) \qquad \mathbb{P}(\text{crossing of } S).$$

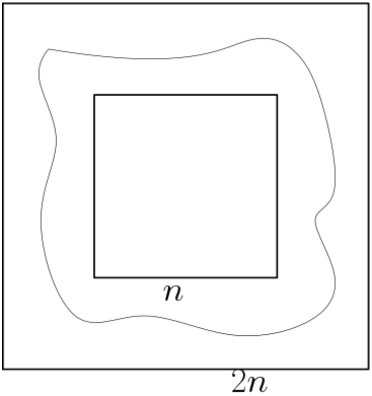
Prob  $\geq$

Prob  $\geq$ Prob  $\geq$

Prob  $\geq$ Prob  $\geq$

Prob  $\times$ Prob 

$$= \text{Prob (crossing the rectangle)}^2$$



$$\mathbb{P}(\text{circuit in the annulus}) \geq \mathbb{P}(\text{crossing the rectangle})^4$$

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- Uniform crossing of squares
- Asymptotic independence
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Theorem (Tassion 2016) If $f : \mathbb{R}^2 \to \{-1, 1\}$ is random and satisfies these conditions, then it satisfies RSW.

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These are the symmetries needed by Tassion.

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Independence

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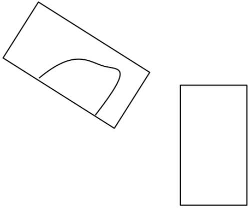
However...



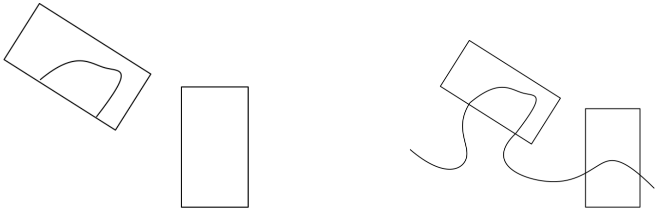
Independence

Correlation function for Bargmann-Fock:

$$e(x, y) = \exp(-\|x - y\|^2)$$

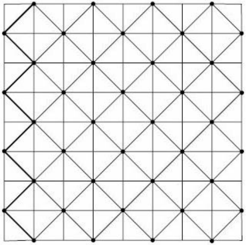
seems very decorrelating!

However...



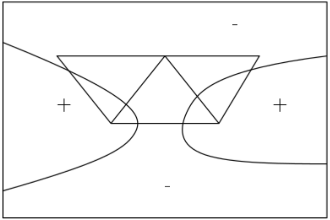
... because of the analytic continuation phenomenon.

Solution : blurring by discretization



- $\mathcal{T}$ = Union Jack lattice
- $\mathcal{V}$ = its vertices,
- sign $f_{|\mathcal{V}} : \mathcal{V} \to \{\pm 1\}$.
- Site percolation: the edge is positive iff its extremities are.

Is the discretization trustful?



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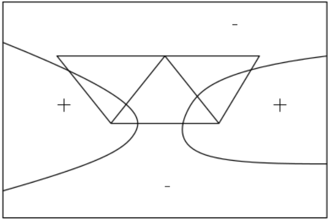

1. If $\mathcal{T}$ is too coarse, then no.

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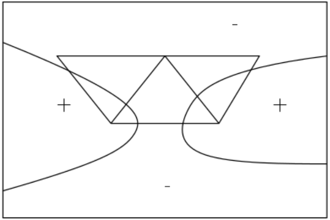

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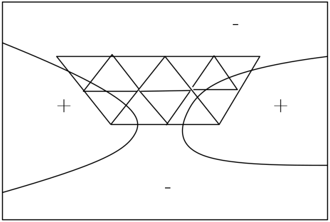

2. If $\mathcal{T}$ is very thin, then yes, but...

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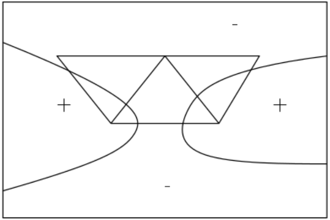

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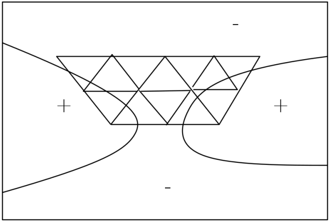

2. If $\mathcal{T}$ is very thin, then yes, but... dependence comes back.

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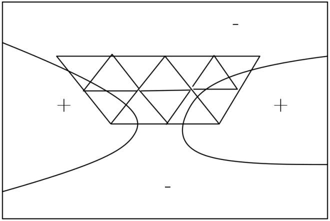

Topological Lemma If in a rectangle the nodal lines of f crosses only once every edge of the lattice, then

$\{f > 0\}$ crosses $R \Leftrightarrow$ the discretization site percolation crosses R.

Quantitative blurring

Hypotheses: f, $\mathcal{T}$, $\mathcal{V}$, e, k is C^1, $k'(0) \neq 0$, $B_n := [-n, n]^2$.

Theorem (Beffara-G 2016) There exists $C > 0$ such that for any $n > 1$,

$$\mathbb{P}\Big[\forall e \in \frac{1}{n^3}\mathcal{V} \cap B_n,\ \#\{f = 0 \cap e\} \leq 1\Big] \geq 1 - \frac{C}{n}.$$

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Corollary Discretization site percolation on $\frac{1}{n^3}\mathcal{V} \cap B_n$ is equivalent to the continuous one with the same probability.

Fear: This gives

$$\#(B_n \cap \frac{1}{n^3}\mathcal{V}) \sim_n n^8 \text{ points!}$$

This is a threat for independence. It must be counterbalanced by the decorrelation of the field.

Quantitative dependence

Theorem (, Beffara-G 2016) Let $f : \mathcal{V} \to \mathbb{R}$ be a centered symmetric Gaussian over $\mathcal{V}$ a lattice. Then, there exists $C > 0$, such that for any R, S two disjoint open sets in $\mathbb{R}^2$,

$$\text{dependence}(R, S) :=$$

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The Ultimate Fight: Information versus Oblivion

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Tassion's condition:

$$\text{dependence}\big(A(n,2n), A(3n, n\log n)\big) \to_{n\to\infty} 0,$$

where $A(n,n') = B_{n'} \setminus B_n$.

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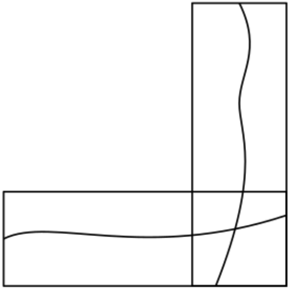
Oblivion wins!

- Symmetries ✓
- Uniform crossing of squares ✓
- Asymptotic independence ✓
- Positive correlation of positive crossings (FKG)

FKG



FKG (Fortuin-Kasteleyn-Ginibre) implies

$$\mathbb{P}\big(\text{positive crossing of } R \quad \text{and} \quad \text{positive crossing of } S\big)$$
$$\geq$$
$$\mathbb{P}(\text{positive crossing of } R) \quad . \quad \mathbb{P}(\text{positive crossing of } S).$$

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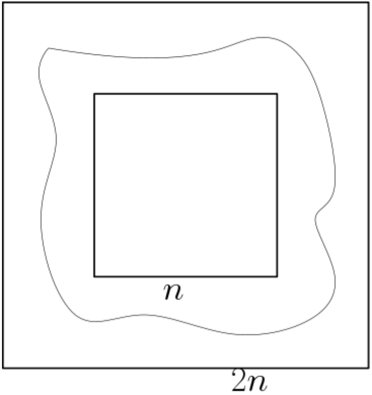
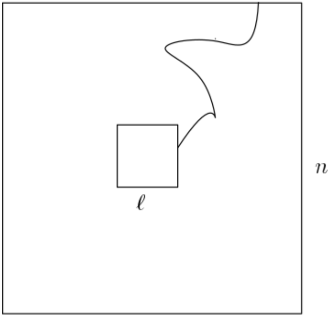
RSW+FKG

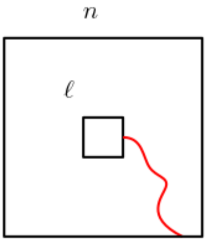


$$\mathbb{P}(\text{circuit in the annulus}) \geq \mathbb{P}(\text{crossing the rectangle})^4$$

RSW+FKG+weak dependence



$$\mathbb{P} < \left(\frac{\ell}{n}\right)^{\alpha}.$$

Prob


$\leq$

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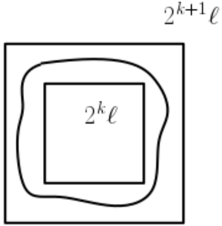
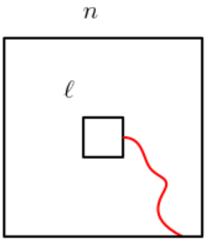


$\Big)$

Prob 

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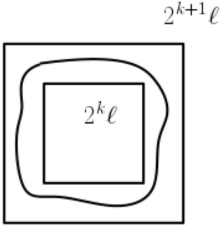
Prob $\left(\forall 0 \leq k \leq \log_2(\frac{n}{\ell}) \right.$ NO 

$\left. \right)$

$$\simeq (1-c)^{\log_2(\frac{n}{\ell})} = (\frac{\ell}{n})^{-\log_2(1-c)}.$$

Why Bargmann-Fock and not Random Waves?

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  2. slow decay $\rightarrow$ strong dependence

Tools and proofs

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1. Discretization scheme
2. Asymptotic independence
3. Pitt's theorem (FKG)
4. Tassion's theorem

Discretization scheme

Theorem There exists $C > 0$ such that for any $n > 1$,

$$\mathbb{P}\Big[\forall e \in \frac{1}{n^3}\mathcal{E} \cap B_n,\ \#\{f = 0 \cap e\} \leq 1\Big] \geq 1 - \frac{C}{n}.$$

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$$N_I := \#\{f = 0\} \cap I,$$

then

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Corollary If f is C^2 and $k'(0) \neq 0$, then

$$\mathbb{E}(N_I(N_I - 1)) \leq O(|I|^3).$$

Proof of the discretization theorem.
We want to prove that with high probability,

$$\forall e \in \frac{1}{n^3}\mathcal{E} \cap B_n,\ \ N_e \leq 1.$$

By Markov inequality and Kac-Rice,

$$\mathbb{P}\big[N_e > 1\big] = \mathbb{P}\big[N_e(N_e - 1) \geq 1\big] \leq C|e|^3.$$

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□

Proof of the Corollary. We have

$$\mathbb{E}(N(N-1)) = \int_{I^2} \mathbb{E}\big(|f'(x)||f'(y)| \bigm| f(x)=f(y)=0\big) \phi_{(f(x),f(y))}(0,0)dxdy.$$

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This gives the $|I|^3$.

□

Kac-Rice first moment formula

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2. Asymptotic independence
3. Pitt's theorem (FKG)
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The cost for independency

Gaussian reminder

A real valued random variable $X \sim N(0, \sigma)$ iff $\sigma = \mathbb{E}(X^2)$ and for any Borelian $A \subset \mathbb{R}$,

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If $X \in \mathbb{R}^N$ and $X' \in \mathbb{R}^{N'}$,

$$\mathrm{cov}(X, X') = \big(\mathbb{E}(X_i X_j')\big)_{1 \le i \le N, 1 \le j \le N'}.$$

$X = (X_i) \in \mathbb{R}^N$ centered non-degenerate Gaussian means that if

$$\Sigma := \big(\mathbb{E}(X_i X_j)\big)_{1 \le i,j \le N},$$

then for any Borelian $A \subset \mathbb{R}^N$,

$$\mathbb{P}(X \in A) = \int_A \phi_X(u) du,$$

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Fact. X and X' are independent iff $\operatorname{cov}(X, X') = 0$.

Proof of Plackett-Piterbarg theorem Let

$$U := \big(f(x)\big)_{x \in R \cap \mathcal{V}} \qquad V := \big(f(y)\big)_{y \in S \cap \mathcal{V}}$$
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Then X_t has variance

$$\Sigma_t = \begin{pmatrix} \mathrm{cov}(U,U) & t\ \mathrm{cov}(U,V) \\ t\ \mathrm{cov}(U,V)^T & \mathrm{cov}(V,V) \end{pmatrix}$$

with

$$\mathrm{cov}\ (U,V) = (e(x,y))_{x \in R \cap \mathcal{V}, y \in S \cap \mathcal{V}}.$$

Then

$$\mathbb{E}_{X_0}(\mathbf{1}_{A\cap B}) - \mathbb{E}_{X_1}(\mathbf{1}_{A\cap B}) \quad = \quad \int_0^1 \frac{d}{dt}\mathbb{E}_{X_t}(\mathbf{1}_{A\cap B})dt$$

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&= \sum_{i\le j} \int_0^1 dt \int_{A\times B} \frac{d\sigma_{t,ij}}{dt}\frac{\partial\phi_{X_t}}{\partial\sigma_{t,ij}}d(u,v)
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with

$$\frac{d\sigma_{t,ij}}{dt} = \begin{cases} e(x,y) \text{ if } i = x \in R\cap\mathcal{V} \text{ and } j = y \in S\cap\mathcal{V} \\ 0 \text{ in the other cases.} \end{cases}$$

A very Gaussian equality

$$\forall i \neq j,\ \ \frac{\partial \phi_X}{\partial \sigma_{ij}} = \frac{\partial^2 \phi_X}{\partial u_i \partial u_j}.$$

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Proof. Use

$$\phi_X(u) = \int_{\xi \in \mathbb{R}^N} e^{i\langle u,\xi\rangle} e^{-\frac{1}{2}\langle \Sigma\xi,\xi\rangle} \frac{d\xi}{\sqrt{2\pi}^N}.$$

Then

$$\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B] = \sum_{\substack{x \in R \\ y \in S}} e(x,y) \int_0^1 dt \int_{A \times B} \frac{\partial^2 \phi_{X_t}}{\partial u_x \partial v_y} d(u,v).$$

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- depending if the sign of f_x is crucial for the crossing or not.

Integrating by parts gives, with $N = \#\big[(R \cup S) \cap \mathcal{V}\big]$,

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Note that if $e(x,y) \geq 0$, we have proved that

$$\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B] \geq 0$$

which is Pitt's theorem.

In general,

$$\int_{Z \in A, Y=0} \phi_{Z,Y}(Z,0)dZ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{Z \in A, Y \in [0,\epsilon]} \phi_{Z,Y} dZ dY$$

In general,

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This gives

$$\left|\mathbb{P}[A\cap B] - \mathbb{P}[A]\mathbb{P}[B]\right| \leq (\#R\cap\mathcal{V})(\#S\cap\mathcal{V}) \max_{\substack{x\in R\\ y\in S}} \frac{|e(x,y)|}{\sqrt{2\pi(1-e(x,y)^2)}}.$$

□

1. Discretization scheme ✓
2. Asymptotic independence ✓
3. Pitt's theorem (FKG) ✓
4. Tassion's theorem

Theorem (Tassion) If $f : \mathbb{R}^2 \to \{-1, 1\}$ is random and satisfies

- Symmetries
- Uniform crossing of squares
- Asymptotic independence
- Positive correlation of positive crossings (FKG)

then it satisfies RSW.

Some open problems

Not too hard

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$\emptyset$

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$\emptyset$

Hard
RSW for fast decorrelating fields without positive correlation

Some open problems

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$\emptyset$

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Very hard
RSW for random waves

Some open problems

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Super very hard
Prove a Cardy formula/conformal invariance