What does a random complex hypersurface look like?

Algebraic Geometry and Complex Geometry

CIRM 28 november- 2 december 2022

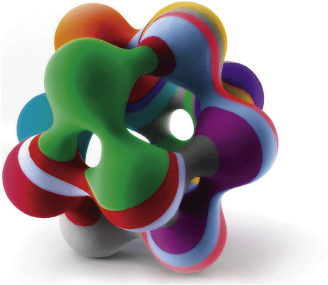

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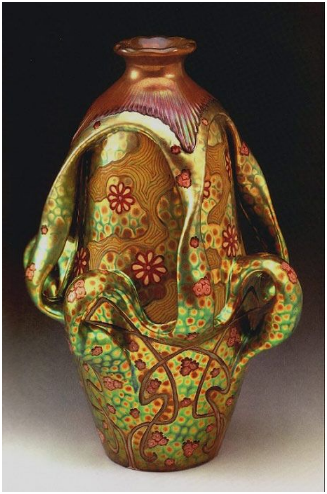
Image : M. Hajij T.K. Dey et X. Li

Introduction



Szolnay ceramic

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, \cdots, Z_n]$. Then

$$Z(P) = \{P = 0\} \subset \mathbb{C}P^n$$

- is generically a smooth complex hypersurface,
- with a constant diffeomorphism type :
  1. $n = 1$ $Z(P)$ is the union of d points.
  2. $n = 2$ $Z(P)$ is connected compact smooth Riemann surface of genus $\frac{1}{2}(d-1)(d-2)$.

Lefschetz theorem (1929)

$$\forall k \in \{0 \cdots, n-2\},\ H_k(Z(P), \mathbb{R}) = H_k(\mathbb{C}P^n, \mathbb{R}).$$

By Poincaré duality,

$$\forall k \in \{n, \cdots, 2n-2\},\ H_k(Z(P), \mathbb{R}) = H_{2n-2-k}(\mathbb{C}P^n, \mathbb{R}).$$

Chern computation

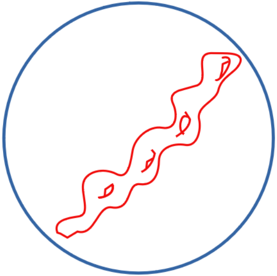
$$b_{n-1}(Z(P)) \sim d^n.$$

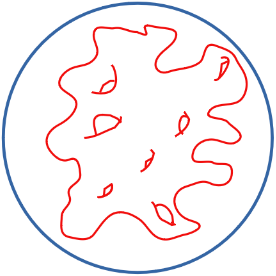
Conclusion : the only proper homology of $Z(P)$ is $H_{n-1}(Z(P))$.

Wirtinger theorem

$$\forall P \in \mathbb{C}_d^{hom}[Z],\ \mathrm{Vol}(Z(P)) = d\frac{1}{(n-1)!}.$$

Same topology and volumes but different shapes





$Z(Z_0^d + \epsilon Q)$ and $Z(Z_0^{d_0} \cdots Z_n^{d_0} + \epsilon Q)$

Local volume

Let $U \subset \mathbb{C}P^n$ be an open subset with smooth boundary.

$$\text{Vol}\,(Z(P) \cup U) \in [0, d].$$

Local topology

$$b_0(Z(P) \cap U) \in [0, +\infty[.$$

For a fixed U and large d, are they bounds for the local Betti numbers ?

Theorem (Milnor 1963). Let $U \subset \mathbb{C}P^n$ be an open set defined by real polynomials. Then, there exists C_U such that

$$\sum_{i=0}^{2n-2} b_i(Z(P) \cap U) \leq C_U d^{2n}.$$

Recall : $\sum_{i=0}^{2n-2} b_i(Z(P)) \sim d^n$.

Random hypersurfaces

If P is taken at random in $\mathbb{C}_d^{hom}[Z_0, \cdots, Z_n]$ and $U \subset \mathbb{C}P^n$,

1. What is the statistic of Vol $(Z(P) \cap U)$?
2. What are the statistics of $b_i(Z(P) \cap U)$?
3. Can we describe generators of $H_{n-1}(Z(P) \cap U)$?
4. Is there a local echo of the global rigid constraints ? In particular, could be the Milnor bound d^{2n} be amended ?

Random local volume

Recall that for any complex hypersurface $Z \subset \mathbb{C}P^n$, $[Z]$ denotes its *current of integration*, that is for any smooth $(2n-2)$-form φ ;

$$\langle [Z], \varphi \rangle = \int_Z \varphi.$$

- If $n = 1$, then

$$[Z(P)] = \sum_{x \in \mathbb{C}P^1,\ P(x)=0} \delta_x.$$

- If φ is closed and $P \in \mathbb{C}_d^{hom}[Z]$, then

$$\langle [Z(P)], \varphi \rangle = d \int_{\mathbb{C}P^n} \omega_{FS} \wedge \varphi.$$

- Moreover,

$$\text{"vol}(Z(P) \cup U) \quad = \quad \langle [Z(P)], \frac{\mathbf{1}_U \omega_{FS}^{n-1}}{(n-1)!} \rangle\text{"}$$

Theorem (Shiffman-Zelditch 1998)

$$\frac{1}{d}\mathbb{E}[Z(P)] \underset{d\to\infty}{\to} \omega_{FS}.$$

In particular, for $U \subset \mathbb{C}P^n$,

$$\mathbb{E}\left[\mathrm{vol}\ (Z(P) \cap U)\right] \underset{d\to\infty}{\simeq} \frac{d}{(n-1)!}\frac{\mathrm{vol}\ U}{\mathrm{vol}\ \mathbb{C}P^n}.$$

Random local topology

Theorem (G. 2022) Let $U \subset \mathbb{C}P^n$ be an open set with smooth boundary. Then,

$$\forall i \in \{0, 2n-2\} \setminus \{n-1\},\ \mathbb{E}\left[b_i(Z(P) \cap U)\right] \underset{d\to\infty}{=} o(d^n)$$

$$\mathbb{E}\left[b_{n-1}(Z(P) \cap U)\right] \underset{d\to\infty}{\sim} d^n \frac{\mathrm{vol}(U)}{\mathrm{vol}(\mathbb{C}P^n)}.$$

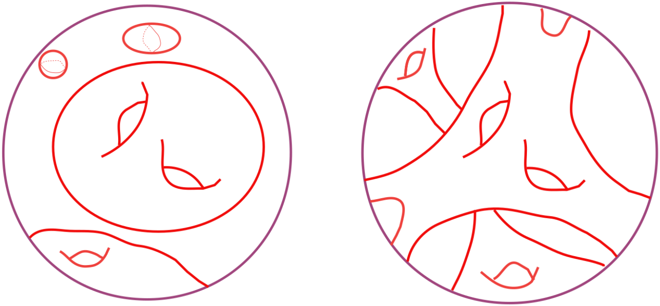
Known *deterministic* Corollary

$$b_{n-1}(Z(P)) \underset{d\to\infty}{\sim} d^n.$$

Random real algebraic geometry

Theorem (G.-Welschinger 2015) : If P is a *real* random polynomial, $Z(P) \subset \mathbb{R}P^n$, then

$$\forall i \in \{0, n-1\},\ \mathbb{E}\left[b_i(Z(P) \cap U)\right] \underset{d\to\infty}{\asymp} \sqrt{d}^n \mathrm{Vol}(U)$$



Real versus complex

$b_0 \asymp b_1$ versus $b_0 \ll b_1$

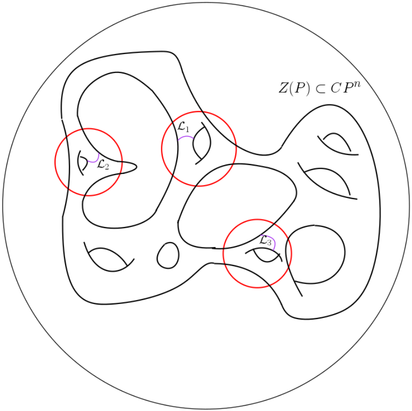
Lagrangian representatives

Theorem (G. 2021) Assume n is odd. Let $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth real hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0,\ \forall d \gg 1,\ c \leq \mathbb{P}\Big[\exists\ \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \text{ pairwise disjoint, Lagrangian}, \forall i, \mathcal{L}_i \sim_{\text{diff}} \mathcal{L}, \text{ and } [\mathcal{L}_1], \cdots, [\mathcal{L}_{cd^n}] \text{ are independent in } H_{n-1}\big(Z(P) \cap U, \mathbb{R}\big)\Big].$$

Lagrangian : $\omega_{FS|T\mathcal{L}} = 0$. In particular, $\mathcal{L}$ is totally real, that is

$$JT\mathcal{L} \cap T\mathcal{L} = \{0\}.$$



$Z(P) \subset CP^n$
$\mathcal{L}_1$
$\mathcal{L}_2$
$\mathcal{L}_3$

Symplectic fact. For any generic $P, Q \in \mathbb{C}_d^{hom}[Z]$,

$$\big(Z(P), \omega_{FS|Z(P)}\big) \sim_{sympl} \big(Z(Q), \omega_{FS|Z(Q)}\big)\,.$$

Deterministic symplectic Corollary. Under the same hypotheses, there exists $c > 0$ such that *for any* generic polynomial P of large enough degree d,

$$\begin{aligned} \exists\ \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \text{ pairwise disjoint,} \\ \text{Lagrangian}, \forall i, \mathcal{L}_i \sim_{\text{diff}} \mathcal{L}, \\ \text{and } [\mathcal{L}_1], \cdots, [\mathcal{L}_{cd^n}] \text{ are independent in } H_{n-1}\big(Z(P), \mathbb{R}\big). \end{aligned}$$

Older results in any dimensions :

- Andreotti-Frenkel 1968 : Lagrangian spheres
- Mikhalkin 2002 : Lagrangian spheres and tori
- Corollary of G-Welschinger 2014 : $\sqrt{d}^n$ instead of d^n.
- Ancona 2022 : d^n Lagrangians in $Z(P) \cap \mathbb{R}P^n$.

The natural measure

- The *Fubini-Study* measure μ_d on $\mathbb{C}_d^{hom}[Z_0, \cdots, Z_n]$:

$$P = \sqrt{(n+d)!} \sum_{i_0+\cdots+i_n=d} a_{i_0\cdots i_n} \frac{Z_0^{i_0}\cdots Z_n^{i_n}}{\sqrt{i_0!\cdots i_n!}},$$

  where $\Re a_{i_0\cdots i_n}, \Im a_{i_0\cdots i_n}$ are i.i.d. standard normal variables.
- These monomials form an ONB for the Fubini-Study L^2-scalar product :

$$\langle P, Q\rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} \frac{\omega_{FS}^n}{n!}.$$

- Then, for any Borelian $A \subset \mathbb{C}_d^{hom}[Z]$,

$$\mu_d(A) = \int_{P \in A} e^{-\frac{1}{2}\|P\|^2_{L^2(h_{FS})}} \frac{dP}{(2\pi)^{N_d}} = \int_{P(a) \in A} e^{-\frac{1}{2}\|a\|^2} \frac{da}{(2\pi)^{N_d}}$$

- One can use the uniform measure over the sphere $\mathbb{S}\mathbb{C}_d^{hom}$ of L^2-normalized polynomials.

Example : Let $z \in \mathbb{C}P^n$. What is the average

$$\mathbb{E}\left[\|P(z)\|_{h_{FS}}\right]?$$

By symmetries, one can assume that $z = [1 : 0 : \cdots : 0]$. Then the mean equals

$$\sqrt{(n+d)!}\, \mathbb{E}\left[\frac{|a_0 Z_0^d|}{\sqrt{d!}|Z_0|^d}\right].$$

Since

$$\mathbb{E}[|a_0|] = \int_{a_0 \in \mathbb{C}} |a_0| e^{-\frac{1}{2}|a_0|^2} \frac{da_0}{2\pi} = \int_{r>0} r^2 e^{-\frac{1}{2}r^2} dr = 1,$$

we obtain

$$\mathbb{E}\left[\|P(z)\|_{h_{FS}}\right] \underset{d \to \infty}{\sim} d^{\frac{n}{2}}. \square$$

General Kähler framework

Let

- X^n be a compact complex manifold, and
- $L \to X$ be an ample holomorphic line bundle equipped with
- a Hermitian metric h with positive curvature ω, that is locally if e is a holomorphic trivialization,

$$\omega = \frac{1}{i\pi}\partial\bar{\partial}\log\|e\|_h$$

is a Kähler form, that is for any $z \in X$, $\omega(z)$ is positive over any complex line in T_zX.

Topology of hypersurfaces

For $d \gg 1$ and any generic $s \in H^0(X, L^{\otimes d})$,

- Lefschetz :

$$\forall i < n-1, \; H_i(Z(s), \mathbb{R}) = H_i(X, \mathbb{R}).$$

- Chern :

$$b_{n-1}(Z(s)) \underset{d \to \infty}{\sim} d^n \int_X \omega^n.$$

- Wirtinger :

$$\text{vol } (Z(s)) = d \frac{\int_X \omega^n}{(n-1)!}.$$

In this general setting :

Theorem (S-Z) $\frac{1}{d}\mathbb{E}([Z(s)] \to \omega.$

Theorem (G) The two random topological theorems extend in this context.

The natural measure

The measure μ_d chosen on $H^0(X, L^d)$ is the Gaussian one associated to

- the scalar product

$$\forall s, t \in H^0(X, L^d),\ \langle s, t \rangle = \int_X h_d(s,t) \frac{\omega^n}{n!}.$$

- For any Borelian $A \subset H^0(X, L^d)$,

$$\mu_d(A) = \int_{s \in A} e^{-\frac{1}{2}\|s\|^2_{L^2}} \frac{ds}{(2\pi)^{N_d}}$$

- Other saying, if $(S_i)_{1\leq i\leq N_d}$ is an ONB of $(H^0(X,L^d),\langle\ ,\ \rangle_{FS})$,

$$s=\sum_i a_i S_i$$

where $\Re a_{i_0\cdots i_n}, \Im a_{i_0\cdots i_n}$ are i.i.d. standard normal variables.
- Again, one can use the uniform measure over $\mathbb{S}H^0(X,L^d)$.

Unrealistic plan of the mini-course

1. Current of integration
2. Betti numbers
3. Local representatives of the homology
4. Annexes

Part I
The mean current of integration

Image : Barnett

Theorem (B. Shiffman-S. Zelditch 1998) Let X, L, h, ω and $(\mu_d)_d$ as before. Then

$$\frac{1}{d}\mathbb{E}[Z(s)] \underset{d\to\infty}{\to} \omega.$$

Proof Recall that by Poincaré-Lelong formula, for any local holomorphic function f,

$$[Z(f)] = \frac{i}{\pi}\partial\bar{\partial}\log|f|.$$

Hence, for any $s \in H^0(X, L^d)$, if locally $s = fe^d$,

$$\begin{aligned}[Z(f)] &= \frac{i}{\pi}\partial\bar{\partial}\log\|s\|_{h^d} - \frac{i}{\pi}\partial\bar{\partial}\log\|e^d\|_h \\ &= d\omega + \frac{i}{\pi}\partial\bar{\partial}\log\|s\|_{h^d}\end{aligned}$$

Write $s = \sum_{i=1}^{N_d} a_i S_i$, where $(S_i)_i$ is an ONB of $H^0(X, L^d)$. Then

$$\mathbb{E}[\log \|s\|_{h^d}^2] = \log \sum_i \|S_i\|_{h_d}^2 + \mathbb{E}\left[\log \frac{\|s\|^2}{\sum_i \|S_i\|^2}\right].$$

If $\forall i,\ S_i = f_i e^d$ and $F = (f_i)_i \in \mathbb{C}^{N_d}$,

$$\mathbb{E}\left[\log \frac{\|s\|^2}{\sum_i \|S_i\|^2}\right] = \mathbb{E}\left[\log |\langle a, \frac{F}{\|F\|}\rangle|^2\right]$$

with a standard Gaussian vector in $\mathbb{C}^{N_d}$. Using a rotation, this is equal to

$$\mathbb{E}\left[\log |a_1|^2\right]$$

which is killed by the $\partial\bar{\partial}$.

Hence,

$$\frac{1}{d}\mathbb{E}[Z(f)] = \omega + \frac{i}{2\pi d}\partial\bar{\partial}\mathbb{E}\left[\log \sum_i \|S_i\|^2_{h^d}\right].$$

Standard case :

$$\sum_i \|S_i\|^2_{h^d} = \frac{(n+d)!}{\|Z\|^{2d}} \sum_{i_0+\cdots+i_n=d} \frac{|Z_0|^{2i_0}\cdots|Z_n|^{2i_n}}{i_0!\cdots i_n!} = \frac{(n+d)!}{d!}$$

Hence,

$$\frac{1}{d}\mathbb{E}[Z(f)] = \omega_{FS}. \ \square$$

General case :

$$\frac{1}{d}\mathbb{E}[Z(f)] = \omega + \frac{i}{2\pi d}\partial\bar{\partial}\mathbb{E}\left[\log \sum_i \|S_i\|^2_{h^d}\right].$$

Tian Theorem : For any $x \in X$,

$$\sum_i \|S_i(x)\|^2_{h^d} = d^n + O(d^{n-1}).$$

Consequently weakly

$$\frac{1}{d}\mathbb{E}[Z(f)] \underset{d\to\infty}{\to} \omega. \ \square$$

Part 2 - Betti numbers

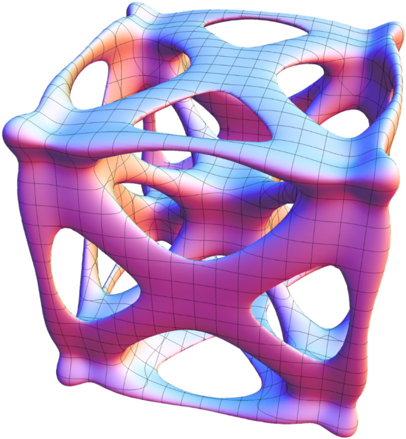


Image : Leon Lampret

For a generic $P \in \mathbb{C}_d^{hom}[Z_0, \cdots, Z_n]$,

$$\begin{array}{llll} \textbf{Lefschetz :} & \forall k \neq n-1, & b_i(Z(P)) & \underset{d\to\infty}{=} O(1) \\ \textbf{Chern :} & & b_{n-1}(Z(P)) & \underset{d\to\infty}{\sim} d^n. \end{array}$$

Random polynomial :

$$P = \sqrt{(n+d)!} \sum_{i_0+\cdots+i_n=d} a_{i_0\cdots i_n} \frac{Z_0^{i_0}\cdots Z_n^{i_n}}{\sqrt{i_0!\cdots i_n!}},$$

where $\Re a_{i_0\cdots i_n}, \Im a_{i_0\cdots i_n}$ are i.i.d. standard normal variables.

Theorem Let $U \subset \mathbb{C}P^n$ be an open set with smooth boundary. Then,

$$\begin{aligned} \forall i \neq n-1,\ \mathbb{E}\left[b_i(Z(P) \cap U)\right] & \underset{d \to \infty}{=} o(d^n) \\ \mathbb{E}\left[b_{n-1}(Z(P) \cap U)\right] & \underset{d \to \infty}{\sim} d^n \frac{\text{vol }(U)}{\text{vol }(\mathbb{C}P^n)}. \end{aligned}$$

Theorem (Milnor). Let $U \subset \mathbb{C}P^n$ be an open set defined by polynomials. Then, there exists C_U such that

$$\sum_{i=0}^{2n-2} b_i(Z(P) \cap U) \leq C_U d^{2n}.$$

"Proof" of Milnor's theorem

Simplification : assume $U = \mathbb{B}^n \subset \mathbb{R}^n$,

$$P(x) = x_n - Q(x_1, \cdots, x_{n-1}).$$

Then

$$T_x Z(P) = \text{vect}\left(\frac{\partial}{\partial x_i} + \frac{\partial Q}{\partial x_i}\frac{\partial}{\partial x_n}\right)_{1 \le i \le n-1}.$$

Let

$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \|x\|^2.$$

Fact. For a generic P, $f_{|Z(P)}$ is a *Morse function*, that is all its critical points are non-degenerate, i.e the Hessian is non-degenerate.

Weak Morse inequalities :

$$\sum_i b_i(Z(P)) \leq \#\mathrm{crit}(f_{|Z(P)}).$$

Now

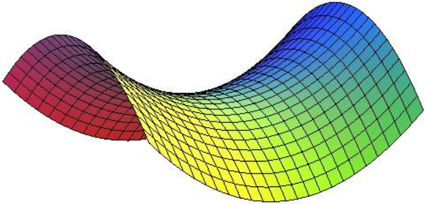
$$x \in \mathrm{crit}(f_{|Z(P)}) \Leftrightarrow \begin{cases} x_n = Q(x_1, \cdots, x_{n-1}) \\ \forall i \leq n-1, \\ \langle \nabla \|x\|^2, \frac{\partial}{\partial x_i} + \partial_i Q \frac{\partial}{\partial x_n} \rangle = 0. \end{cases}$$

- So, x is critical if it satisfies n algebraic equations in $\mathbb{R}^n$ of degree less than deg Q.
- **Van der Waerden Theorem (1949) :** there exists at most $(\deg Q)^n$ solutions.
- Hence,

$$\sum_i b_i(Z(P)) \leq d^n.$$

- The constant C_U appears when taking in account the boundary of U. $\square$

Holomorphic specificities ?



Affine real function. Let Z be a generic complex hypersurface such that

$$0 \in \text{crit } (x_{n|Z}).$$

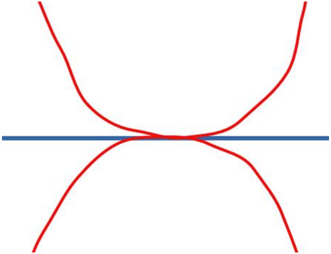
Then, Z is locally writes

$$Z = \{z_n = \sum_i k_i z_i^2 + O(3)\}.$$

Since

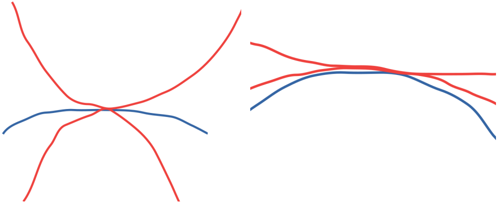
$$x_n(z_1, \cdots, z_{n-1}, \sum_i k_i z_i^2) = \sum_i k_i (x_i^2 - y_i^2),$$

- 0 is a critical point of $x_{n|Z}$ with index $n-1$
- the spectrum of the Hessian is even.



Conclusion : The Hessian of the restriction of a linear real function on Z at a critical point has an even spectrum. In particular, it has index $n-1$.

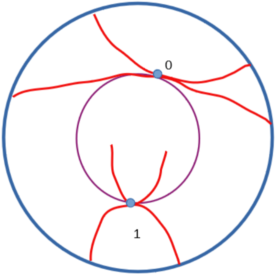
General function



Here : $n = 2$ and Z is a complex curve.

- Left, an index 1 critical point. The curve can be very curved.
- Right, an index 2 critical point. The curve cannot be locally very curved.
- If f is strictly (pseudo)convex, there is no index 0 critical point.

Revisiting Milnor's proof



- No maximum (index 2)
- The saddle points (index 1) are favored in comparison with minima (index 0).

Heuristic Proposition : Statistically,

$$\forall i < n-1,\ \frac{\#\mathrm{Crit}_i(f_{|Z(P)})}{\#\mathrm{Crit}_{n-1}(f_{|Z(P)})} \to_d 0$$

"Proof" :

- Near $[1:0\cdots:0]$, $p(z) := \sqrt{d}!\frac{P(Z)}{Z_0^d}$ equals

$$a_0 + \sqrt{d}\sum_{i=1}^{n} a_i z_i + d\sum_{i,j} a_{ij} z_i z_j + \text{etc } (z\sqrt{d}).$$

- Then, $p(\frac{z}{\sqrt{d}})$ becomes independent on d.

- Hence, the natural scale of $Z(P)$ is $\frac{1}{\sqrt{d}}$.
- After rescaling by $\times\sqrt{d}$ we should have a bounded geometry.
- Hence statistically the curvature $Z(P)$ has order d.
- However critical points with large curvature have index $n-1$.
- Hence $\frac{\#\mathrm{Crit}_i(f_{|Z(P)})}{\#\mathrm{Crit}_{n-1}(f_{|Z(P)})} \to_d 0$, statistically. $\square$

Proposition Let $U \subset X$ be an open set with smooth boundary. Then,

$$\begin{aligned} \forall i \neq n-1,\ \mathbb{E}\left[\#\mathrm{crit}_i(f_{|Z(P)} \cap U)\right] & \underset{d\to\infty}{=} o(d^n) \\ \mathbb{E}\left[\#\mathrm{crit}_{n-1}(f_{|Z(P)} \cap U)\right] & \underset{d\to\infty}{\sim} d^n \frac{\mathrm{vol}\ (U)}{\mathrm{vol}\ (\mathbb{C}P^n)}. \end{aligned}$$

Weak and strong Morse inequalities Let $f : Z \to \mathbb{R}$ be a Morse function. Then,

- (weak)

$$\forall i,\ b_i(Z) \leq \#\mathrm{crit}_i(f)$$

- (strong)

$$\forall i,\ \sum_{k=0}^{i} (-1)^{i-k} b_k(Z) \geq \sum_{k=0}^{i} (-1)^{i-k} \#\mathrm{crit}_k(f)$$

Consequence :

$$b_{n-1}(Z) \geq \#\mathrm{crit}_{n-1}(f) - 2 \sum_{i<n-1} \#\mathrm{crit}_i(f).$$

The Proposition for critical points and Morse inequalities imply :

$$\forall i \neq n-1,\ \mathbb{E}\left[b_i(Z(s)\cap U)\right] \underset{d\to\infty}{=} o(d^n)$$
$$\mathbb{E}\left[b_{n-1}(Z(s)\cap U)\right] \underset{d\to\infty}{\sim} d^n \int_U \omega^n.$$

How do we estimate $\mathbb{E}\left[\mathrm{crit}_i(f_{|Z(s)})\right]$?

With the help of *Kac-Rice formula*

Simplest Kac-Rice formula

Let $f : \mathbb{R} \to \mathbb{R}$ be random and $U \subset \mathbb{R}$. Then

$$\mathbb{E}\left[\#Z(f) \cap U\right] = \int_U \mathbb{E}\big(|f'(x)| \mid f(x) = 0\big)\phi_{f(x)}(0)dx,$$

where $\phi_{f(x)}$ denotes the density of $f(x)$.

"Proof".

- If f vanishes transversally,

$$\#Z(f) \cap U = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\mathbb{R}} |f'(x)| \mathbf{1}_{|f| \leq \epsilon} dx,$$

- hence

$$\mathbb{E}\left[\#Z(f) \cap U\right] = \int_{\mathbb{R}} \mathbb{E}\Big(|f'(x)| \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathbf{1}_{|f| \leq \epsilon}\Big) dx. \ \square$$

A friendly Kac-Rice formula

Proposition (G.-Welschinger 2015, G. 2022)

$$\mathbb{E}\#\left(\mathrm{crit}_i(p_{|Z(P)})\cap U\right)$$

is equal to

$$\int_{x\in U}\int_{\substack{\alpha\in\mathcal{L}_{onto}(T_xM,E_x)\\ \ker\alpha\subset\ker dp(x)}}\left|\det\alpha_{|\ker^\perp\alpha}\right|$$

$$\mathbb{E}\Big[\mathbf{1}_{\{\mathrm{Ind}\ (\nabla^2 p_{|Z(P)})=i\}}\Big|\det\Big(\langle\nabla^2P(x)_{|\ker\alpha},\epsilon(x,\alpha)\rangle$$

$$-\langle\alpha(\nabla p(x)),\epsilon(x,\alpha)\rangle\frac{\nabla^2p(x)_{|\ker\alpha}}{\|dp(x)\|^2}\Big)\Big|\ \Big|\ P(x)=0,\nabla P(x)=\alpha\ \Big]$$

$$\rho_{X(x)}(0,\alpha)d\mathrm{vol}(\alpha)d\mathrm{vol}(x).$$

How is it possible to compute such a thing ?

- **General fact for Gaussian fields.** Kac-Rice formula can be expressed in terms of the sole covariance of f :

$$\mathrm{cov}\,(f(x), f(y)) = \mathbb{E}\,[f(x)f(y)]$$

  and its $(2,2)$-jet on the diagonal.
- The covariance of $P \in \mathbb{C}_d^{hom}[Z]$ or $s \in H^0(X, L^d)$ is the *Bergman kernel* which is known to converge to a universal covariance, after rescaling by $\sqrt{d}$.

Covariance and Bergman kernel

Define the *Bergman kernel*

$$\forall x, y \in X,\ k_d(x, y) := \sum_{i=1}^{n} S_i(x) \otimes S_i(y)^* \in L_x^d \otimes L_y^{d*}$$

where for any $s \in L_y$,

$$\forall t \in L,\ s^*(t) = h_d(s, t).$$

Fact. k_d is the kernel of the projection :

$$\pi_d : L^2(X, L) \to H^0(X, L^d).$$

Proof : For any $s \in L^2(X, L^d)$,

$$\begin{aligned} \pi_d s(x) &= \sum_i \langle S_i, s \rangle_{L^2} S_i(x) \\ &= \int_X k_d(x, y) s(y) d\text{vol } (y). \ \square \end{aligned}$$

Now the *covariance* of the random section $s \in H^0(X, L^d)$ is defined by

$$\text{cov }(s(x), s(y)) := \mathbb{E}[s(x) \otimes s^*(y)].$$

Fact :

$$\text{cov}(s(x), s(y)) = k_d(x, y).$$

Proof :

$$\mathbb{E}[s(x) \otimes s^*(y)] = \sum_{i,j} \mathbb{E}\left[a_i \bar{a_j}\right] S_i(x) \otimes S_j(y)^*.$$

Since a_i and a_j are independent,

$$\mathbb{E}\left[a_i \bar{a_j}\right] = \delta_{ij},$$

hence the result. $\square$

Propreties of the covariance :

- If $x = y$,

$$\text{var}(s(x)) = \text{cov}\ (s(x), s(x)) = \sum_i \|S_i(x)\|_{h_d}^2.$$

- If $s(x)$ is independent of $s(y)$, the we would have

$$\text{cov}(s(x), s(y)) = 0.$$

- Hence, the covariance measures the dependency between $s(x)$ and $s(y)$.
- The intuition is that $\text{cov} \to 0$ when $\text{dist}(x, y)$ becomes large.
- The distance where $\text{cov} \approx 0$ should be the natural scale of $Z(s)$.

Is there a simplification of the Bergman kernel when $d \to \infty$?

Theorem (Tian 1988) For any $x \in X$, for any $d \gg 1$, there exists

$$S_d^x \in \mathbb{S}H^0(X, L^d),$$

such that

$$\|S_d^x(y)\|_{h_d} \underset{d\to\infty}{\sim} d^{\frac{n}{2}} e^{-d\|y-x\|^2}$$

and $\{s \in \mathbb{S}H^0(X, L^d), s(x) = 0\}$ is asymptotically orthogonal to S_d^x.

Corollary : The Bergman kernel has a universal limit shape at scale $\frac{1}{\sqrt{d}}$..

Proof. Fix $x \in X$. Choose as an ONB of $H^0(X, L^d)$

$$S_1 = S_d^x \text{ and } (S_i)_{2 \leq N_d} \in (S_d^x)^{\perp}.$$

Then,

$$\sum_i \|S_i(x)\|_{h_d}^2 \sim d^n$$

and

$$\begin{aligned} \|k_d(x, x + \frac{y}{\sqrt{d}})\|_{h_d} &= \|\sum_{i=1}^{n} S_i(x) \otimes S_i(x + \frac{y}{\sqrt{d}})^* \|_{h_d} \\ &\underset{d \to \infty}{\sim} d^n \exp(-\|y\|^2). \ \square \end{aligned}$$

Standard example : $X = \mathbb{C}P^n$, $L = \mathcal{O}(1)$, $h = h_{FS}$. Then,
Let $x = [1 : 0... : 0]$. Then

$$S_d^x = \sqrt{\frac{(d+n)!}{d!}} Z_0^d.$$

Indeed,

$$\|S_x^d\|_{L^2(FS)} = 1$$

and pointwise

$$\|S_x^d\|_{FS} \sim d^{n/2} \frac{1}{\sqrt{1+\|z\|^2}^d} \sim_d d^{n/2} e^{-\frac{1}{2}d\|z\|^2}.$$

Moreover

$$\left(\frac{Z_0^{i_0}\cdots Z_n^{i_n}}{\sqrt{i_0!\cdots i_n!}}\right)_{i_0+\cdots+i_n=d}$$

is an ONB of $\mathcal{O}(d)$. Hence

$$\begin{aligned} k_d([Z],[W]) &= \frac{1}{\|W\|^d}\sum_{i_0+\cdots+i_n=d}\frac{(Z_0\overline{W_0})^{i_0}\cdots(Z_n\overline{W_n})^{i_n}}{i_0!\cdots i_n!} \\ &= \frac{(\langle Z,W\rangle)^d}{\|W\|^d} \end{aligned}$$

Hence,

$$\|k_d\|_{h_{FS}} = \frac{(\langle Z, W\rangle)^d}{(\|Z\|\|W\|)^d}.$$

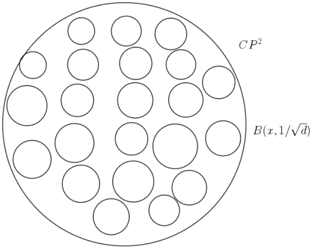
In coordinates near $[1 : 0 \cdots : 0]$, with $Z = (1, 0)$ and $W = (1, w_1, \cdots, w_n)$, we have

$$\|k_d(0, \frac{w}{\sqrt{d}})\|_{h_{FS}} = \frac{1}{\sqrt{1 + \frac{\|w\|^2}{d}}^d} \underset{d\to\infty}{\sim} \exp(-\|w\|^2). \ \square$$

Why d^n in complex versus $\sqrt{d}^n$ in real ?

Since the natural scale is $\frac{1}{\sqrt{d}}$,

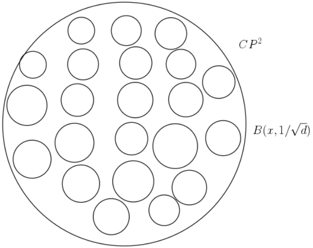
- $Z(s) \cap B_{x,\frac{1}{\sqrt{d}}}$, after rescaling $\times\sqrt{d}$, should look like a uniform random Z in $\mathbb{B}(0,1)$.
- So the topology should be uniform in such a ball. In particular, the Betti numbers of $Z(s) \cap B_{x,\frac{1}{\sqrt{d}}}$ should be bounded.



At least $\asymp d^n$ disjoint small balls in X

- Since vol $B_{x,\frac{1}{\sqrt{d}}} \asymp (\frac{1}{\sqrt{d}})^{2n}$, there are around d^n such balls.
- The total topology should be of order d^n.

In the real world



At least $\asymp \sqrt{d}^n$ disjoint small balls in $\mathbb{R}X$

- In $\mathbb{R}X$, there are around $\sqrt{d}^n$ balls.
- The total topology should be $\sqrt{d}^n$

Part 3 - Topology

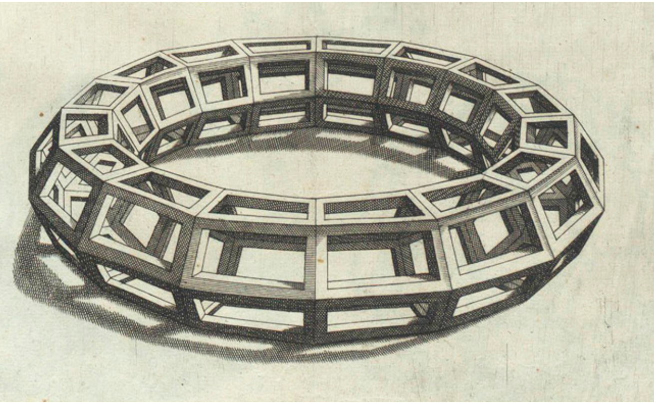
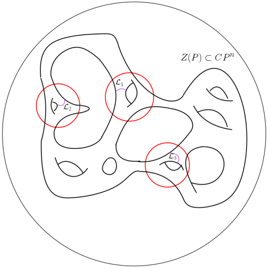


Image : Lorenzo Sirigatti, 1596



Theorem (G. 2021) Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be any compact hypersurface with $\chi(\mathcal{L}) \neq 0$, and $U \subset X$ an open subset with smooth bounary. Then

$$\exists c > 0,\ \forall d \gg 1,\ c \leq \mathbb{P}\Big[\exists\, \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \text{ pairwise disjoint, } \text{Lagrangian}, \forall i, \mathcal{L}_i \sim_{\text{diff}} \mathcal{L},$$

$$\text{and } [\mathcal{L}_1], \cdots, [\mathcal{L}_{cd^n}] \text{ are independent in } H_{n-1}\big(Z(s) \cap U, \mathbb{Z}\big)\Big].$$

At microscopical scale

Proposition. Let $x \in X$ and $\mathcal{L} \subset \mathbb{R}^n$ any compact smooth hypersurface. Then,

$$\exists c_{\mathcal{L}} > 0, \forall d \gg 1, \ \mathbb{P} \quad \left[\exists \mathcal{L}' \sim_{\text{diff}} \mathcal{L}, \mathcal{L}' \text{ and totally real } | \right.$$
$$\left. \mathcal{L}' \subset Z(s) \cap B_{x, \frac{1}{\sqrt{d}}}, \right] \geq c_{\mathcal{L}}.$$

Proposition implies Theorem :
By Proposition :

$$\begin{aligned} c\mathrm{vol}\,(U)d^n &\leq \sum_{x\in\frac{2}{\sqrt{d}}\mathbb{Z}^n\cap U} \mathbb{P}\left[Z(s)\cap B_{x,\frac{1}{\sqrt{d}}} \supset \mathcal{L}\right] \\ &= \sum_{1}^{\mathrm{vol}\,(U)d^n} k\mathbb{P}[\#\text{ small balls containing }\mathcal{L}=k] \\ &\leq c\frac{1}{2}\mathrm{vol}\,(U)d^n\mathbb{P}[\#\text{ balls with }\mathcal{L}\leq c\frac{1}{2}\mathrm{vol}\,(U)d^n] \\ &\quad +\mathrm{vol}\,(U)d^n\mathbb{P}[\#\text{ balls with }\mathcal{L}\geq c\frac{1}{2}\mathrm{vol}\,(U)d^n], \end{aligned}$$

so that

$$\frac{c}{2}\leq\mathbb{P}\left[\#\text{ balls with }\mathcal{L}\geq c\frac{1}{2}\mathrm{vol}\,(U)d^n\right].$$

Proof of the proposition in the standard case

(Based on the real proof done with J.-Y. Welschinger)

Theorem (Seifert 1936). Every compact smooth real hypersurface $\mathcal{L}$ in $\mathbb{R}^n$ can be C^1-perturbed into a component $\mathcal{L}'$ of an algebraic regular hypersurface.

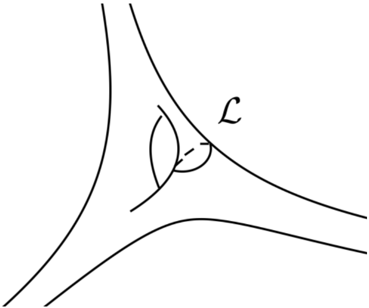
By symmetry one can assume that $x = [1 : 0 \cdots : 0]$. Recall that

$$S_x^d := d^{n/2} Z_0^d$$

has

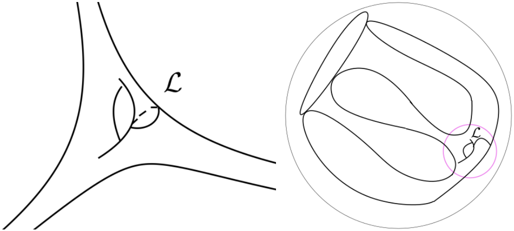
1. L^2 norm $\simeq 1$
2. is exponentially concentrated near x at scale $\frac{1}{\sqrt{d}}$.
3. On $B(x, \frac{1}{\sqrt{d}})$,

$$S_x^d \asymp_d d^{\frac{n}{2}}.$$



By Seifert Theorem, let $p \in \mathbb{R}[x_1, \cdots, x_n]$ be such that

1. p vanishes transversally onto $\Sigma := Z(p) \cap \mathbb{B} \subset \mathbb{C}^n$.
2. $\Sigma \cap \mathbb{R}^n$ contains a diffeomorphic copy of $\mathcal{L}$;
3. $\mathcal{L}$ is Lagrangian, hence totally real.



$Z(p)$ and $Z(P)$

For $i \geq 1$, let $z_i = \frac{Z_i}{Z_0}$, and define :

$$P := p(z\sqrt{d})S_x^d.$$

Then

1. $\|P\|_{L^2} \asymp 1$ since S_x^d has an exponential decay against a polynome.
2. P vanishes along $\Sigma' \sim \Sigma$, containing $\mathcal{L}' \sim_{\text{diff}} \mathcal{L}$ (and other things) in $B_{x,\frac{1}{\sqrt{d}}}$, and $\mathcal{L}'$ is totally real.

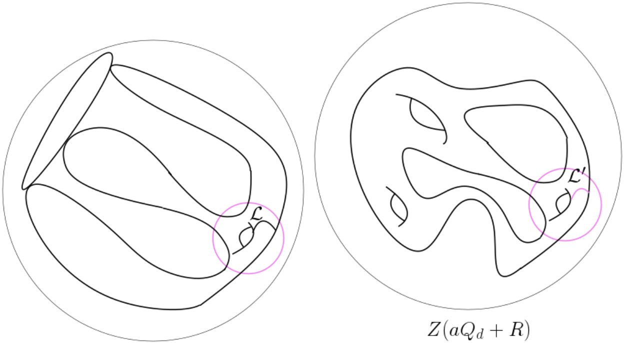
Now a random $Q \in \mathbb{C}_d^{hom}[Z]$ can be written as

$$Q = aP + R,$$

with

$$a \sim N_{\mathbb{C}}(0,1) \text{ and } R \in P^{\perp} \subset \mathbb{C}_d^{hom}[Z]$$

taken at random for the restriction of the Gaussian law on the hyperplane $R^{\perp}$. Then a and R are independent.



Intuitive fact : If R is C^1-small compared to aP, then

1. $Z(Q) \cap B_{x, \frac{1}{\sqrt{d}}} \sim_{\text{diff}} \Sigma \supset \mathcal{L}' \sim_{\text{diff}} \mathcal{L}$;
2. $\mathcal{L}'$ remains totally real.

Making the intuition quantitative :
Second, we saw in the introduction that

$$\mathbb{E}\left[\|R(x)\|_{FS}\right] \sim_d d^{\frac{n}{2}}.$$

Since the scale is $\frac{1}{\sqrt{d}}$,

$$\mathbb{E}\left[\max_{B_{x,\frac{1}{\sqrt{d}}}} \|R\|_{FS}\right] \asymp_d d^{\frac{n}{2}}.$$

Again because of $\frac{1}{\sqrt{d}}$ scale,

$$\mathbb{E}\left[\max_{B_{x,\frac{1}{\sqrt{d}}}} \frac{1}{\sqrt{d}}\|\nabla R\|_{FS}\right] \asymp_d d^{\frac{n}{2}}.$$

Since p vanishes transversally, there exists $\epsilon > 0$, tel que

$$\forall z \in \mathbb{B}, |p(z)| < \epsilon \Rightarrow |dp(z)| > \epsilon.$$

This implies that on $B_{x,1\sqrt{d}}$,

$$|P| < \epsilon d^{n/2} \Rightarrow |\nabla P| > \epsilon \sqrt{d} d^{n/2}.$$

Ehresmann Theorem : For any $M > 0$,

$$\left\{ \begin{array}{ccc} |a| & \geq & M \\ (\|R\| + \frac{1}{\sqrt{d}}\|\nabla R\|)_{L^\infty(B_{x,\frac{1}{\sqrt{d}}})} & < & \frac{M}{2}\epsilon d^{\frac{n}{2}} \end{array} \right.$$

implies that

$$Z(f) \cap B_{x,\frac{1}{\sqrt{d}}} \sim_{\text{diff}} \Sigma$$

with $\Sigma \supset \mathcal{L}' \sim_{diff} \mathcal{L}$ and $\mathcal{L}'$ totally real.

Hence, $\mathbb{P}[Z(Q) \cap B_{x,\frac{1}{\sqrt{d}}} \sim \Sigma]$ is larger than

$$\mathbb{P}[|a| > M] \; \mathbb{P}\left[\|R\|_{L^\infty} \text{ and } \frac{1}{\sqrt{d}} \|\nabla R\|_{L^\infty} < \frac{M}{4} \epsilon d^{\frac{n}{2}} \right].$$

Markov inequality : $\mathbb{P}[X > m] < \frac{\mathbb{E}X}{m}$.

Hence,

$$\mathbb{P}\left[\|R\|_{L^\infty} > \frac{M}{4} \epsilon d^{\frac{n}{2}} \right] \leq 4 \frac{\mathbb{E}\|R\|_{L^\infty}}{M \epsilon d^{n/2}} \underset{d \to \infty}{\asymp} \frac{4}{M\epsilon}.$$

Same for $\frac{1}{\sqrt{d}} \|\nabla R\|_{L^\infty}$.

Hence,

$$\mathbb{P}[Z(Q) \cap B_{x, \frac{1}{\sqrt{d}}} \sim \Sigma] \geq e^{-M^2}(1 - \frac{8}{M\epsilon}).$$

Hence, for $M = \frac{16}{\epsilon}$, we obtain a uniform positive probability. $\square$

For the Theorem, it remains to prove that totally real implies non trivial homology

Totally reality and homology

Facts : If $\mathcal{L} \subset (Z, J)$ is totally real, then

- $JT\mathcal{L} \cap T\mathcal{L} = \{0\}$, so that
$$N\mathcal{L} \sim T\mathcal{L}.$$
- If moreover $\chi(\mathcal{L}) \neq 0$ then
$$0 \neq [\mathcal{L}] \in H_{n-1}(Z).$$

Proof : for $\mathcal{L}$ orientable,

$$\begin{aligned} \chi(\mathcal{L}) &= \#\{\text{ zeros of a tangent vector field with signs}\}. \\ &= \#\{\text{ zeros of a normal vector field with signs}\} \\ &= [\mathcal{L}] \cdot [\mathcal{L}]. \ \square \end{aligned}$$

- If $\mathcal{L}_1, \cdots, \mathcal{L}_k$ are disjoint totally real submanifolds with $\chi(\mathcal{L}_i) \neq 0$, then they form an independent family.

Proof : Assume that

$$\sum_{i=1}^{k} \lambda_i [\mathcal{L}_i] = 0.$$

Intersecting with $\mathcal{L}_j$ gives

$$\lambda_j [\mathcal{L}_j]^2 = 0$$

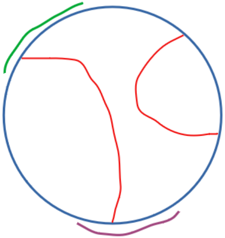
so that $\forall j, \lambda_j = 0 \ \square$.

See the annex for the general proof (for X, L, h, ω).

Annexes

1. An open question : holomorphic percolation
2. A proof in the general setting
3. Peak sections

Bonus : holomorphic percolation



Let P as before, $U \subset \mathbb{C}P^2$ a ball, $V \subset \partial U$ and $W \subset \partial U$ two open subsets of the sphere, whose adherence are disjoint.
Conjecture. There exists $c > 0$, such that for d large enough,

$$\mathbb{P}(\exists \text{ a c. c. of } Z(P) \cap U \text{ intersecting } V \text{ and } W) > c.$$

- Prove in real in $\mathbb{R}^2$ by G.-Beffara
- and in $\mathbb{R}P^2$ by Belyaev-Muirhead-Wigman.

Proof of the Proposition

- Let $p \in \mathbb{R}[x_1, \cdots, x_n]$ such that $Z(p) \subset \mathbb{R}^n$ contains a diffeomorphic copy of $\mathcal{L}$ and vanishing transversally.
- Fix $x \in X$ and let S_x^d be a peak section at x for L^d.
- Let
$$\chi : X \to \mathbb{R}$$
be a cut-off function, that is $\chi = 1$ in the ball $B(x, \delta)$ and $\chi = 0$ oustide $B(x, 2\delta)$, where $\delta > 0$ is small enough.
- Then,
$$s_x := \chi p(z\sqrt{d}) S_x^d(z) \in C^\infty(X, L^d)$$
is holomorphic over $B(x, \delta)$ and vanishes along $\mathcal{L}' \sim_{\text{diff}} \mathcal{L}$ (and other things).
- Since $\mathcal{L}' \subset \mathbb{R}^n$ in complex coordinates, it is totally real.

Hörmander theorem : There exist $C > 0$ depending only on (X, L, h), and $u \in C^\infty(X, L^d)$, such that

$$\sigma_x := s_x + u \in H^0(X, L^d)$$

and

$$\|u\|_{L^2(h_d)} \leq C\|\bar{\partial} s_x\|_{L^2(h_d)}.$$

However

$$\begin{aligned} \|\bar{\partial} s_x\|_{h_d} &= |\bar{\partial}\chi|\|S_d^x \mathbf{1}_{\{|z|>\delta\}}\|_{h_d} \\ &\leq C\exp(-d\delta^2). \end{aligned}$$

Since u is holomorphic in $B_{x,\frac{1}{\sqrt{d}}}$, $\frac{u}{S_d^x}$ is a holomorphic function, and by the mean inequality,

$$\|\frac{u}{S_d^x}\|_{L^\infty(B_{x,\frac{1}{\sqrt{d}}})} \leq d^n C\exp(-d\delta^2).$$

Now a random $s \in H^0(X, L^d)$ can be written as

$$s = a \frac{\sigma_x}{\|\sigma_x\|_{L^2(h_d)}} + \tau,$$

with

$$a \sim N_{\mathbb{C}}(0,1) \text{ and } \tau \in \sigma_x^{\perp} \subset H^0(X, L^d)$$

taken at random for the restriction of the Gaussian law. Then a and τ are independent.

Intuitive fact : If τ is C^1-small compared to $a\frac{\sigma_x}{\|\sigma_x\|_{L^2(h_d)}}$, then

$$Z(s) \cap B_{x, \frac{1}{\sqrt{d}}} \supset \mathcal{L}' \sim_{\text{diff}} \mathcal{L},$$

and $\mathcal{L}'$ is totally real.

Making the intuition quantitative : First,

$$\|\sigma_x\|^2_{L^2} \underset{d\to\infty}{\asymp} \int_{B(0,\frac{\log d}{\sqrt{d}})} |p(z\sqrt{d})|^2 e^{-d\|z\|^2} dz$$

$$\sim_d \quad d^{-n} \int_{\mathbb{C}^n} |p|^2 e^{-|z|^2} dz.$$

Second, writing over $B_{x,\frac{1}{\sqrt{d}}}$

$$s = fS_x^d \text{ and } \tau = gS_x^d,$$

we have

$$f \asymp ap(z\sqrt{d})d^{\frac{n}{2}} + g.$$

Since p vanishes transversally, there exists $\epsilon > 0$, tel que

$$\forall z \in B_{x,\frac{1}{\sqrt{d}}}, |p(z\sqrt{d})| < \epsilon \Rightarrow |d(p(z\sqrt{d}))| > \epsilon\sqrt{d}.$$

Ehresmann Theorem : For any $M > 0$,

$$\left\{ \begin{array}{ccc} |a| & \geq & M \\ (\|g\| + \frac{1}{\sqrt{d}}\|dg\|)_{B_{x,\frac{1}{\sqrt{d}}}} & < & \frac{M\epsilon}{2} d^{\frac{n}{2}} \end{array} \right. \Rightarrow Z(f) \cap B_{x,\frac{1}{\sqrt{d}}} \sim \Sigma.$$

Now, since g is holomorphic, $|g|^2$ is plurisubharmonic so that

$$\begin{aligned} |g(z)|^2 &\leq \frac{1}{\mathrm{vol} B_{x,\frac{2}{\sqrt{d}}}} \int_{B_{x,\frac{2}{\sqrt{d}}}} |g|^2 d\mathrm{vol} \\ &\leq \frac{e^4}{\mathrm{vol} B_{x,\frac{2}{\sqrt{d}}}} \int_{B_{x,\frac{2}{\sqrt{d}}}} \|\tau\|^2_{h_d} d\mathrm{vol} \end{aligned}$$

This implies that

$$\mathbb{E}|g(z)|^2 \leq C \max \mathbb{E}\|\tau\|_{h_d}^2$$

Let $(S_i)_{i=1,\cdots,N_d}$ be an orthonormal basis of $H^0(X, L^d)$. Then,

$$\mathbb{E}\|\tau\|_{h_d}^2 = \sum_i \|S_i\|_{h_d}^2.$$

Theorem (Tian 1988) : For any $x \in X$,

$$\sum_i \|S_i(x)\|_{h^d}^2 = d^n + O(d^{n-1}).$$

Markov inequality : $\mathbb{P}[X > m] < \frac{\mathbb{E}X}{m}$.

Hence,

$$\mathbb{P}[|g| > \frac{M}{4}\epsilon d^{\frac{n}{2}}] \leq 4C\frac{\mathbb{E}\|\tau\|_{h_d}^2}{M^2\epsilon^2 d^n} \underset{d\to\infty}{\sim} \frac{4C}{M^2\epsilon^2}$$

Hence,

$$\begin{aligned}\mathbb{P}[Z(s) \cap B_{x,\frac{1}{\sqrt{d}}} \sim \Sigma] &\geq \mathbb{P}[|a| > M] \\ &\quad \mathbb{P}[|g| < \frac{M}{4}\epsilon d^{\frac{n}{2}} \text{ and} \\ &\quad |dg| < \frac{M}{4}\epsilon d^{\frac{n}{2}}\sqrt{d}] \\ &\geq e^{-M^2}(1 - \frac{8C}{M^2\epsilon^2}).\end{aligned}$$

Hence, for $M^2 = \frac{8C}{\epsilon^2}$, we obtain a uniform positive probability.

Lastly, $\mathcal{L} \subset \Sigma \cap \mathbb{R}^n$ is totally real, that is $T\mathcal{L} \cap iT\mathcal{L} = \{0\}$ (it is even Lagrangian). Hence, after C^1 perturbation, its copy $\mathcal{L}' \subset Z(s)$ remains totally real.

Existence of a peak section

Proposition (Existence of a local peak section) For any $x \in X$, there exists a local holomorophic trivizalization S_x of L such that

$$\|S_x(z)\|_h = \exp\left(-\|z\|^2 + O(\|z\|^3)\right)$$

Let $x \in X$ and e_x be a local trivialization.
Proof. Let e_x any local trivialization and write

$$\|e_x\|_h = \exp(-\varphi),$$

where φ is a plurisubharmonic function satisfying

$$i\partial\bar{\partial}\varphi = \omega.$$

The Taylor expansion of φ at x writes

$$\varphi(x+z) = \Re Q(z) + \sum_{i,j} \partial^2_{z_i\overline{z_j}} \varphi(x) z_i \overline{z_j} + O(\|z\|^3),$$

where

$$Q(z) = \varphi(x) + \sum_j \partial_{z_j} \varphi z_j + \sum_{i,j} \partial^2_{z_i z_j} \varphi z_i z_j,$$

so that

$$\|e_x e^{\varphi(x)+Q(z)}\| \leq \exp(-\|z\|^2_{g_\omega}),$$

where g_ω is the metric associated to $\omega(x)$. $\square$

Proof of the first part of Tian's theorem.
Hörmander estimate for S^d_x, as above. $\square$