Three-dimensional stability of Burgers vortices

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Abstract

Burgers vortices are explicit stationary solutions of the Navier-Stokes equations which are often used to describe the vortex tubes observed in numerical simulations of three-dimensional turbulence. In this model, the velocity field is a two-dimensional perturbation of a linear straining flow with axial symmetry. The only free parameter is the Reynolds number $\mathrm{Re} = \Gamma/\nu$, where Γ is the total circulation of the vortex and ν is the kinematic viscosity. The purpose of this paper is to show that Burgers vortices are asymptotically stable with respect to small three-dimensional perturbations, for all values of the Reynolds number. This general result subsumes earlier studies by various authors, which were either restricted to small Reynolds numbers or to two-dimensional perturbations. Our proof relies on the fact that the linearized operator at Burgers vortex has a simple and very specific dependence upon the axial variable. This allows to reduce the full linearized equations to a vectorial two-dimensional problem, which can be treated using an extension of the techniques developped in earlier works. Although Burgers vortices are found to be stable for all Reynolds numbers, the proof indicates that perturbations may undergo an important transient amplification if Re is large, a phenomenon that was indeed observed in numerical simulations.

1 Introduction

The axisymmetric Burgers vortex is an explicit solution of the three-dimensional Navier-Stokes equations which provides a simple and widely used model for the vortex tubes or filaments that are observed in turbulent flows [1, 27]. Despite obvious limitations, due to oversimplified assumptions, this model describes in a correct way the fundamental mechanisms which are responsible for the persistence of coherent structures in three-dimensional turbulence, namely the balance between vorticity amplification due to stretching and vorticity dissipation due to viscosity. If one believes that vortex tubes play a significant role in the dynamics of turbulent flows, it is an important issue to determine their stability with respect to perturbations in the largest possible class. So far, this problem has been studied only for the axisymmetric Burgers vortex and for a closely related family of asymmetric vortices [24, 18].

As was shown by Leibovich and Holmes [17], one cannot hope to prove energetic stability of Burgers vortex even if the circulation Reynolds number is very small. To tackle the stability problem, it is therefore necessary to have a closer look at the spectrum of the linearized operator.

This is a relatively easy task if we restrict ourselves to *two-dimensional* perturbations. Assuming that the vortex tube is aligned with the vertical axis, this means that the perturbed velocity field lies in the horizontal plane and does not depend on the vertical variable. Under such conditions, Burgers vortex is known to be stable for any value of the Reynolds number. This result was first established by Giga and Kambe [14] for $\mathrm{Re} \ll 1$ and then in the general case by Gallay and Wayne [9], see also [2, 12, 13]. Moreover, a lot is known about the spectrum of the linearized operator, which turns out to be purely discrete in a neighborhood of the origin in the complex plane. Using perturbative expansions, Robinson and Saffman [24] showed that all linear modes are exponentially damped for small Reynolds numbers. This property was then numerically verified by Prochazka and Pullin [22] for $\mathrm{Re} \le 10^4$, and finally rigorously established in [9].

The situation is much more complicated if we allow for arbitrary *three-dimensional* perturbations. In that case, it was shown by Rossi and Le Dizès [25] that the linearized operator does not have any eigenfunction with nontrivial dependence in the vertical variable. While this result precludes the existence of unstable eigenvalues, it also implies that stability cannot be deduced from such a simple analysis, and that continuous spectrum necessarily plays an important role. Unfortunately, the vertical dependence of the perturbed solutions is not easy to determine, as can be seen from the note [3] where a few attempts are made in that direction. The only rigorous result so far is due to Gallay and Wayne [10], who proved that Burgers vortex is asymptotically stable with respect to three-dimensional perturbations in a fairly large class provided that the Reynolds number is sufficiently small. For larger Reynolds numbers, up to $\mathrm{Re} = 5000$, an important numerical work by Schmid and Rossi [26] indicates that all modes are exponentially damped by the linearized evolution, although significant short-time amplification can occur.

In this paper, we prove that the axisymmetric Burgers vortex is asymptotically stable with respect to small *three-dimensional* perturbations for *arbitrary values* of the Reynolds number. As in [10], we assume that the perturbations are nicely localized in the horizontal variables, but we do not impose any decay with respect to the vertical variable. Our approach is based on the fact that the linearized operator has a very simple dependence upon the vertical variable: the only term involving x_3 is the dilation operator $x_3\partial_{x_3}$, which originates from the background straining field. This crucial property was already exploited in [25, 3, 26], but we shall show that it allows to reduce the three-dimensional stability problem to a two-dimensional one, which can then be treated using an extension of the techniques developped in [9]. Although the spectrum of the linearized operator remains stable for all Reynolds numbers, the estimates we have on the associated semigroup deteriorate as Re increases, in full agreement with the amplification phenomena observed in [26].

We now formulate our results in a more precise way. We start from the three-dimensional incompressible Navier-Stokes equations:

$$\partial_t V + (V,\nabla)V \;=\; \nu\Delta V - \frac{1}{\rho}\nabla P~, \qquad \nabla\cdot V \;=\; 0~, \tag{1.1}$$

where $V = V(x,t) \in \mathbb{R}^3$ denotes the velocity field, $P = P(x,t) \in \mathbb{R}$ is the pressure field, and $x = (x_1,x_2,x_3)^\top \in \mathbb{R}^3$ is the space variable. The parameters in (1.1) are the kinematic viscosity $\nu > 0$ and the density $\rho > 0$. To obtain tubular vortices, we assume that the velocity V can be decomposed as follows:

$$V(x,t) \;=\; V^s(x) + U(x,t)~, \tag{1.2}$$

where V^s is an axisymmetric straining flow given by the explicit formula

$$V^s(x) \;=\; \frac{\gamma}{2}\begin{pmatrix} -x_1 \\ -x_2 \\ 2x_3 \end{pmatrix} \;\equiv\; \gamma M x~, \qquad \text{where} \quad M \;=\; \begin{pmatrix} -\frac12 & 0 & 0 \\ 0 & -\frac12 & 0 \\ 0 & 0 & 1 \end{pmatrix}~. \tag{1.3}$$

Here $\gamma > 0$ is a parameter which measures the intensity of the strain. Note that $\nabla \cdot V^s = 0$, and that V^s is a stationary solution of (1.1) with the associated pressure $P^s = -\frac{1}{2}\rho|V^s|^2$. Our goal is to study the evolution of the perturbed velocity field $U(x,t)$.

To simplify the notations, we shall assume henceforth that $\gamma = \nu = \rho = 1$. This can be achieved without loss of generality by replacing the variables x, t and the functions V, P with the dimensionless quantities

$$\tilde{x} = \Big(\frac{\gamma}{\nu}\Big)^{1/2} x\ , \qquad \tilde{t} = \gamma t\ , \qquad \tilde{V} = \frac{V}{(\gamma\nu)^{1/2}}\ , \qquad \tilde{P} = \frac{P}{\rho\gamma\nu}\ .$$

For further convenience, instead of considering the evolution of V or U, we prefer working with the vorticity field $\Omega = \nabla \times V = \nabla \times U$. Taking the curl of (1.1) and using (1.2), (1.3), we obtain for Ω the evolution equation

$$\partial_t \Omega + (U,\nabla)\Omega - (\Omega,\nabla)U \;=\; L\Omega\ , \qquad \nabla\cdot\Omega \;=\; 0\ , \tag{1.4}$$

where L is the differential operator defined by

$$L\Omega \;=\; \Delta\Omega - (Mx,\nabla)\Omega + M\Omega\ . \tag{1.5}$$

Under mild assumptions that will be specified below, the velocity field U can be recovered from the vorticity Ω via the three-dimensional Biot-Savart law

$$U(x) \;=\; -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\Omega(y)}{|x-y|^3}\,\mathrm{d}y \;=:\; (K_{3D} * \Omega)(x)\ . \tag{1.6}$$

In what follows we shall often encounter the particular situation where the velocity U is two-dimensional and horizontal, namely $U(x) = (U_1(x_h), U_2(x_h), 0)^\top$ where $x_h = (x_1,x_2)^\top \in \mathbb{R}^2$. In that case the vorticity satisfies $\Omega(x) = (0,0,\Omega_3(x_h))^\top$, and the relation (1.6) reduces to the two-dimensional Biot-Savart law

$$U_h(x_h) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x_h-y_h)^\perp}{|x_h-y_h|^2}\,\Omega_3(y_h)\,\mathrm{d}y_h \;=:\; (K_{2D} \star \Omega_3)(x_h)\ , \tag{1.7}$$

where $U_h = (U_1,U_2)^\top$ and $x_h^\perp = (-x_2,x_1)^\top$.

We can now introduce the *Burgers vortices*, which are explicit stationary solutions of (1.4) of the form $\Omega = \alpha G$, where $\alpha \in \mathbb{R}$ is a parameter. The vortex profile is given by

$$G(x) \;=\; \begin{pmatrix} 0 \\ 0 \\ g(x_h) \end{pmatrix}\ , \qquad \text{where} \quad g(x_h) \;=\; \frac{1}{4\pi}\, e^{-|x_h|^2/4}\ . \tag{1.8}$$

The associated velocity field $U = \alpha U^G$ can be obtained from the Biot-Savart law (1.7) and has the following form

$$U^G(x) \;=\; u^g(|x_h|^2)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}\ , \qquad \text{where} \quad u^g(r) \;=\; \frac{1}{2\pi r}\Big(1 - e^{-r/4}\Big)\ . \tag{1.9}$$

If $\Omega = \alpha G$, it is easy to verify that $\alpha = \int_{\mathbb{R}^2} \Omega_3(x_h)\,\mathrm{d}x_h$. This means that the parameter $\alpha \in \mathbb{R}$ represents the *total circulation* of the Burgers vortex αG. In the physical literature, the quantity $|\alpha|$ is often referred to as the (circulation) Reynolds number.

The aim of this paper is to study the asymptotic stability of the Burgers vortices. We thus consider solutions of (1.4) of the form $\Omega = \alpha G + \omega$, $U = \alpha U^G + u$, and obtain the following evolution equation for the perturbation:

$$\partial_t \omega + (u, \nabla)\omega - (\omega, \nabla)u \;=\; (L - \alpha\Lambda)\omega \;, \qquad \nabla \cdot \omega \;=\; 0 \;, \tag{1.10}$$

where Λ is the integro-differential operator defined by

$$\Lambda\omega \;=\; (U^G, \nabla)\omega - (\omega, \nabla)U^G + (u, \nabla)G - (G, \nabla)u \;. \tag{1.11}$$

Here and in the sequel, it is always understood that $u = K_{3D} * \omega$.

An important issue is now to fix an appropriate function space for the admissible perturbations. Since the Burgers vortex itself is essentially a two-dimensional flow, it is natural to choose a functional setting which allows for perturbations in the same class, but we also want to consider more general ones. Following [10], we thus assume that the perturbations are nicely localized in the horizontal variables, but merely bounded in the vertical direction. As we shall see below, this choice is more or less imposed by the particular form of the linear operator (1.5).

To specify the horizontal decay of the admissible perturbations, we first introduce two-dimensional spaces. Given $m \in [0, \infty]$, we denote by $\rho_m : [0, \infty) \to [1, \infty)$ the weight function defined by

$$\rho_m(r) \;=\; \begin{cases} 1 & \text{if} \quad m = 0\,, \\ (1 + \frac{r}{4m})^m & \text{if} \quad 0 < m < \infty\,, \\ e^{r/4} & \text{if} \quad m = \infty\,. \end{cases} \tag{1.12}$$

We introduce the weighted L^2 space

$$L^2(m) \;=\; \Big\{ f \in L^2(\mathbb{R}^2) \;\Big|\; \|f\|^2_{L^2(m)} = \int_{\mathbb{R}^2} |f(x_h)|^2 \rho_m(|x_h|^2)\,\mathrm{d}x_h < \infty \Big\} \;, \tag{1.13}$$

which is a Hilbert space with a natural inner product. Using Hölder's inequality, it is easy to verify that $L^2(m) \hookrightarrow L^1(\mathbb{R}^2)$ if $m > 1$. In that case, we also define the closed subspace

$$L^2_0(m) \;=\; \Big\{ f \in L^2(m) \;\Big|\; \int_{\mathbb{R}^2} f(x_h)\,\mathrm{d}x_h = 0 \Big\} \;. \tag{1.14}$$

Next, we define the three-dimensional space $X(m)$ as the set of all $\phi : \mathbb{R}^3 \to \mathbb{R}$ for which the map $x_h \mapsto \phi(x_h, x_3)$ belongs to $L^2(m)$ for any $x_3 \in \mathbb{R}$, and is a bounded and continuous function of x_3. In other words, we set

$$X(m) \;=\; BC(\mathbb{R}\,; L^2(m)) \;, \qquad X_0(m) \;=\; BC(\mathbb{R}\,; L^2_0(m)) \;, \tag{1.15}$$

where "$BC(\mathbb{R}\,; Y)$" denotes the space of all bounded and continuous functions from $\mathbb{R}$ into Y. Both $X(m)$ and $X_0(m)$ are Banach spaces equipped with the norm

$$\|\phi\|_{X(m)} \;=\; \sup_{x_3 \in \mathbb{R}} \|\phi(\cdot, x_3)\|_{L^2(m)} \;. \tag{1.16}$$

Our goal is to study the stability of the Burgers vortex $\Omega = \alpha G$ with respect to perturbations $\omega \in X(m)^3$. In fact, we can assume without loss of generality that ω belongs to the subspace

$$\mathbb{X}(m) \;=\; X(m) \times X(m) \times X_0(m) \;\subset\; X(m)^3 \;, \tag{1.17}$$

which is invariant under the evolution defined by (1.10). Indeed, we have the following result, whose proof is postponed to Section 6.1:

Lemma 1.1 *Fix $m \in (1,\infty]$. If $\tilde\omega \in X(m)^3$ satisfies $\nabla \cdot \tilde\omega = 0$ in the sense of distributions, then there exists $\tilde\alpha \in \mathbb{R}$ such that*

$$\int_{\mathbb{R}^2} \tilde\omega_3(x_h, x_3)\,\mathrm{d}x_h \;=\; \tilde\alpha\ , \quad \textit{for all } x_3 \in \mathbb{R}\ . \tag{1.18}$$

At a formal level, this is a direct consequence of the divergence-free assumption, since

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\int_{\mathbb{R}^2} \tilde\omega_3(x_h,x_3)\,\mathrm{d}x_h \;=\; \int_{\mathbb{R}^2} \partial_{x_3}\tilde\omega_3(x_h,x_3)\,\mathrm{d}x_h \;=\; -\int_{\mathbb{R}^2} \nabla_h \cdot \tilde\omega_h(x_h,x_3)\,\mathrm{d}x_h \;=\; 0\ .$$

In view of Lemma 1.1, if $\Omega = \alpha G + \tilde\omega$ for some $\tilde\omega \in X(m)^3$, we can write $\Omega = (\alpha + \tilde\alpha)G + \omega$, where $\tilde\alpha$ is given by (1.18) and $\omega = \tilde\omega - \tilde\alpha G$. Then $\omega \in \mathbb{X}(m)$ by construction, and we are led back to the stability analysis of the Burgers vortex $(\alpha+\tilde\alpha)G$ with respect to perturbations in $\mathbb{X}(m)$.

In what follows we always consider the solutions $\omega(x,t)$ of (1.10) as $\mathbb{X}(m)$-valued functions of time, and we often denote by $\omega(\cdot,t)$ or simply $\omega(t)$ the map $x \mapsto \omega(x,t)$. A minor drawback of our functional setting is that we cannot expect the solutions of (1.10) to be continuous in time in the strong topology of $\mathbb{X}(m)$. This is because the operator L defined in (1.5) contains the dilation operator $-x_3\partial_{x_3}$, see Section 2.1 below. To restore continuity, it is thus necessary to equip $\mathbb{X}(m)$ with a weaker topology. Following [10], we denote by $X_{loc}(m)$ the space $X(m)$ equipped with the topology defined by the family of seminorms

$$\|\phi\|_{X_n(m)} \;=\; \sup_{|x_3|\le n} \|\phi(\cdot,x_3)\|_{L^2(m)}\ , \qquad n \in \mathbb{N}\ .$$

In analogy with (1.17), we set $\mathbb{X}_{loc}(m) = X_{loc}(m) \times X_{loc}(m) \times X_{0,loc}(m)$, where $X_{0,loc}(m)$ is of course the space $X_0(m)$ equipped with the topology of $X_{loc}(m)$.

We are now able to formulate our main result:

Theorem 1.2 *Fix $m \in (2,\infty]$ and $\alpha \in \mathbb{R}$. Then there exist $\delta = \delta(\alpha,m) > 0$ and $C = C(\alpha,m) \ge 1$ such that, for any $\omega_0 \in \mathbb{X}(m)$ with $\nabla\cdot\omega_0 = 0$ and $\|\omega_0\|_{\mathbb{X}(m)} \le \delta$, Eq. (1.10) has a unique solution $\omega \in L^\infty(\mathbb{R}_+\,;\mathbb{X}(m)) \cap C([0,\infty)\,;\mathbb{X}_{loc}(m))$ with initial data ω_0. Moreover,*

$$\|\omega(t)\|_{\mathbb{X}(m)} \;\le\; C\|\omega_0\|_{\mathbb{X}(m)}\, e^{-t/2}\ , \qquad \textit{for all } t \ge 0\ . \tag{1.19}$$

Theorem 1.2 shows that the Burgers vortex αG is *asymptotically stable* with respect to small perturbations in $\mathbb{X}(m)$, for any value of the circulation $\alpha \in \mathbb{R}$. If one prefers to consider perturbations in the larger space $X(m)^3$, then our result means that the family $\{\alpha G\}_{\alpha\in\mathbb{R}}$ of all Burgers vortices is asymptotically stable *with shift*, because the perturbations may then modify the circulation of the underlying vortex. The key point in the proof is to show that the linearized operator $L - \alpha\Lambda$ has a *uniform spectral gap* for all $\alpha \in \mathbb{R}$. This implies a uniform decay rate in time for the perturbations, as in (1.19). However, it should be emphasized that the constants C and δ in Theorem 1.2 depend on α, in such a way that $C(\alpha,m) \to \infty$ and $\delta(\alpha,m) \to 0$ as $|\alpha| \to \infty$. This is in full agreement with the amplification phenomena numerically observed in [26].

The proof of Theorem 1.2 gives a more detailed information on the solutions of (1.10) than what is summarized in (1.19). First of all, we can prove stability in $\mathbb{X}(m)$ for any $m > 1$, but the exponential factor $e^{-t/2}$ in (1.19) should then be replaced by $e^{-\eta t}$, where $\eta < (m-1)/2$ if $m \le 2$. Next, thanks to parabolic smoothing, we can obtain decay estimates not only for ω but also for the spatial derivatives $\partial_{x_j}\omega \equiv \partial\omega/\partial x_j$, $j = 1,2,3$. For convenience, we shall often use the multi-index notation $\partial_x^\beta = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_{x_3}^{\beta_3}$ for $\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{N}^3$. Finally, due to the particular structure of the linear operator $L - \alpha\Lambda$, it turns out that the horizontal part $\omega_h = (\omega_1,\omega_2)^\top$ of the vorticity vector has a faster decay than the vertical component ω_3 as $t \to \infty$. Thus, a more complete (but less readable) version of our result is as follows:

Theorem 1.3 *Fix $m \in (1,\infty]$, $\alpha \in \mathbb{R}$, and take $\mu \in (1,\frac{3}{2})$, $\eta \in (0,\frac{1}{2}]$ such that $2\mu < m+1$ and $2\eta < m-1$. Then there exist $\delta = \delta(\alpha,m) > 0$ and $C = C(\alpha,m,\mu,\eta) \ge 1$ such that, for all initial data $\omega_0 \in \mathbb{X}(m)$ with $\nabla\cdot\omega_0 = 0$ and $\|\omega_0\|_{\mathbb{X}(m)} \le \delta$, Eq.* (1.10) *has a unique solution $\omega \in L^\infty(\mathbb{R}_+;\mathbb{X}(m)) \cap C([0,\infty);\mathbb{X}_{loc}(m))$. Moreover, for all $t > 0$,*

$$\|\partial_x^\beta \omega_h(t)\|_{X(m)^2} \le \frac{C\|\omega_0\|_{\mathbb{X}(m)}}{a(t)^{|\beta|/2}}\, e^{-\mu t} , \tag{1.20}$$

$$\|\partial_x^\beta \omega_3(t)\|_{X(m)} \le \frac{C\|\omega_0\|_{\mathbb{X}(m)}}{a(t)^{|\beta|/2}}\, e^{-\eta t} , \tag{1.21}$$

where $a(t) = 1 - e^{-t}$ and $\beta \in \mathbb{N}^3$ is any multi-index of length $|\beta| = \beta_1+\beta_2+\beta_3 \le 1$.

The decay rates (1.20), (1.21) are optimal when $\beta = 0$, but it turns out that vertical derivatives such as $\partial_{x_3}\omega_h(t)$ or $\partial_{x_3}\omega_3(t)$ have a faster decay as $t \to \infty$, see Sections 4 and 5 for more details. In any case, we believe that the optimal rates are those provided by the linear stability analysis, as in Proposition 4.1 below.

The rest of this paper is devoted to the proof of Theorems 1.2 and 1.3. Before giving the details, we explain here the important ideas in an informal way. As was already mentioned, the main difficulty is to obtain good estimates on the solutions of the linearized equation

$$\partial_t \omega = (L - \alpha\Lambda)\omega , \qquad \nabla\cdot\omega = 0 . \tag{1.22}$$

Once this is done, the nonlinear terms in (1.10) can be controlled using rather standard arguments, which are recalled in Section 5. To study (1.22), we use the fact that the operator $L-\alpha\Lambda$ depends on the vertical variable in a simple and very specific way. Indeed, it is easy to verify that $[\partial_{x_3}, L] = -\partial_{x_3}$ and $[\partial_{x_3},\Lambda] = 0$, where $[A,B] = AB - BA$ denotes the commutator of the operators A and B. This key observation, which already plays a crucial role in the previous works [25, 3, 26], implies the following identity:

$$\partial_{x_3}^k\, e^{t(L-\alpha\Lambda)}\omega_0 = e^{-kt}\, e^{t(L-\alpha\Lambda)}\partial_{x_3}^k\omega_0 , \tag{1.23}$$

for all $k \in \mathbb{N}$ and all $t \ge 0$. If we take $k \in \mathbb{N}$ sufficiently large, depending on $|\alpha|$, we can use (1.23) to show that $\partial_{x_3}^k\omega(t)$ decays exponentially as $t \to \infty$ if $\omega(t)$ is a solution of (1.22). Then, by an interpolation argument, we deduce that all expressions involving at least one vertical derivative play a negligible role in the long-time asymptotics, see Section 4 for more details. This "smoothing effect" in the vertical direction is due to the stretching properties of the linear flow (1.2).

As a consequence of these remarks, we can restrict our attention to those solutions of (1.22) which are independent of the vertical variable x_3. We call this particular situation the *vectorial 2D problem*, and we study it in Section 3. Note that the perturbations we allow here are two-dimensional in the sense that $\partial_{x_3}u = \partial_{x_3}\omega = 0$, but that all three components of u or ω are possibly nonzero. This is in contrast with the purely two-dimensional case considered in [9, 10], where in addition $u_3 = \omega_1 = \omega_2 = 0$. Extending the techniques developped in [9, 10], it is possible to show that all solutions of (1.22) with $\partial_{x_3}\omega = 0$ converge exponentially to zero as $t \to \infty$, and that the decay rate is uniform in α. This is done using spectral estimates and a detailed study of the eigenvalue equation $(L - \alpha\Lambda)\omega = \lambda\omega$. It is then a rather straightforward task to complete the proof of Theorem 1.2 using the arguments outlined above.

Remark. The vortex tubes observed in numerical simulations are usually not axisymmetric: in general, they rather exhibit an elliptical core region [18, 6]. A simple model for such asymmetric vortices is obtained by replacing the straining flow V^s in (1.3) with the nonsymmetric strain $V^s_\lambda(x) = \gamma M_\lambda x$, where $\lambda \in (0,1)$ is an asymmetry parameter and

$$M_\lambda \;=\; \begin{pmatrix} -\frac{1+\lambda}{2} & 0 & 0 \\ 0 & -\frac{1-\lambda}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.24}$$

Asymmetric Burgers vortices are then stationary solutions to (1.4), where the operator L in the right-hand side is defined by (1.5) with M replaced by M_λ. Unlike in the symmetric case $\lambda = 0$, no explicit formula is available and proving the existence of stationary solutions is already a nontrivial task, except perhaps in the perturbative regime where either the asymmetry parameter λ or the circulation number α is very small. In view of these difficulties, asymmetric Burgers vortices were first studied using formal asymptotic expansions and numerical calculations, see e.g. [24, 18, 23]. The mathematical theory is more recent, and includes several existence results which cover now the whole range of parameters $\lambda \in (0,1)$ and $\alpha \in \mathbb{R}$ [10, 11, 19, 20]. In addition, the stability with respect to two-dimensional perturbations is known to hold at least for small values of the asymmetry parameter [11, 19]. However, the only result so far on three-dimensional stability is restricted to the particular case where the circulation number α is sufficiently small, depending on λ [10].

Using Theorem 1.2 and a simple perturbation argument, it is easy to show that asymmetric Burgers vortices are stable with respect to small three-dimensional pertubations in the space $\mathbb{X}(m)$, provided that the asymmetry parameter λ is small enough depending on the circulation number α. This follows from the fact the linearized operator at the symmetric Burgers vortex has a uniform spectral gap for all $\alpha \in \mathbb{R}$, and that the asymmetric Burgers vortex is $O(\lambda)$-close to the corresponding symmetric vortex in the topology of $\mathbb{X}(m)$, uniformly for all $\alpha \in \mathbb{R}$ [11]. Although this stability result is new and not covered by [10], it is certainly not optimal, and we prefer to postpone the study of the three-dimensional stability of asymmetric Burgers vortices to a future investigation.

2 Preliminaries

In this preliminary section we collect a few basic estimates which will be used throughout the proof of Theorems 1.2 and 1.3. They concern the semigroup generated by the linear operator (1.5), and the Biot-Savart law (1.6) relating the velocity field to the vorticity. Most of the results were already established in [10, Appendix A], and are reproduced here for the reader's convenience.

As in [10], we introduce the following generalization of the function spaces (1.13) and (1.15). Given $m \in [0,\infty]$ and $p \in [1,\infty)$, we define the weighted L^p space

$$L^p(m) \;=\; \Bigl\{ f \in L^p(\mathbb{R}^2) \;\Big|\; \|f\|^p_{L^p(m)} = \int_{\mathbb{R}^2} |f(x_h)|^p \rho_m(|x_h|^2)^{p/2}\,\mathrm{d}x_h < \infty \Bigr\} ,$$

and the corresponding three-dimensional space

$$X^p(m) \;=\; BC(\mathbb{R}\,; L^p(m)) , \qquad \|\phi\|_{X^p(m)} \;=\; \sup_{x_3\in\mathbb{R}} \|\phi(\cdot,x_3)\|_{L^p(m)} .$$

If $m > 2-\frac{2}{p}$, we also denote by $L^p_0(m)$ the subspace of all $f \in L^p(m)$ such that $\int_{\mathbb{R}} f \,\mathrm{d}x_h = 0$. In analogy with (1.17), we set $\mathbb{X}^p(m) = X^p(m)\times X^p(m)\times X^p_0(m)$, where $X^p_0(m) = BC(\mathbb{R}\,; L^p_0(m))$.

2.1 The semigroup generated by L

If we decompose the vorticity ω into its horizontal part $\omega_h = (\omega_1,\omega_2)^\top$ and its vertical component ω_3, it is clear from (1.3) and (1.5) that the linear operator L has the following expression:

$$L\omega \;=\; \begin{pmatrix} L_h\omega_h \\ L_3\omega_3 \end{pmatrix} \;=\; \begin{pmatrix} (\mathcal{L}_h + \mathcal{L}_3 - \frac{3}{2})\omega_h \\ (\mathcal{L}_h + \mathcal{L}_3)\omega_3 \end{pmatrix} , \tag{2.1}$$

where $\mathcal{L}_h$ is the two-dimensional Fokker-Planck operator

$$\mathcal{L}_h \;=\; \Delta_h + \frac{x_h}{2}\cdot\nabla_h + 1 \;=\; \sum_{j=1}^{2}\partial^2_{x_j} + \sum_{j=1}^{2}\frac{x_j}{2}\partial_{x_j} + 1 \;, \tag{2.2}$$

and $\mathcal{L}_3 = \partial^2_{x_3} - x_3\partial_{x_3}$ is a convection-diffusion operator in the vertical variable.

As is shown in [8, appendix A], the operator $\mathcal{L}_h$ is the generator of a strongly continuous semigroup in $L^2(m)$ given by the explicit formula

$$(e^{t\mathcal{L}_h}f)(x_h) \;=\; \frac{e^t}{4\pi a(t)}\int_{\mathbb{R}^2} e^{-\frac{|x_h-y_h|^2}{4a(t)}} f(y_h e^{t/2})\,\mathrm{d}y_h \;, \qquad t>0 \;, \tag{2.3}$$

where $a(t) = 1-e^{-t}$. Similarly, the operator $\mathcal{L}_3$ generates a semigroup of contractions in $BC(\mathbb{R})$ given by

$$(e^{t\mathcal{L}_3}f)(x_3) \;=\; \frac{1}{\sqrt{2\pi a(2t)}}\int_{\mathbb{R}} e^{-\frac{|x_3e^{-t}-y_3|^2}{2a(2t)}} f(y_3)\,\mathrm{d}y_3 \;, \qquad t>0 \;, \tag{2.4}$$

see [10, Appendix A]. Note that the semigroup $e^{t\mathcal{L}_3}$ is not strongly continuous in the space $BC(\mathbb{R})$ equipped with the supremum norm. This is mainly due to the dilation factor e^{-t} in (2.4). However, if we equip $BC(\mathbb{R})$ with the (weaker) topology of uniform convergence on compact sets, then the map $t \mapsto e^{t\mathcal{L}_3}f$ is continuous for any $f \in BC(\mathbb{R})$. This observation is the reason for introducing the space $X_{loc}(m)$ in Section 1.

Since the operators $\mathcal{L}_h$ and $\mathcal{L}_3$ act on different variables, it is easy to obtain the semigroup generated by $L_3 = \mathcal{L}_h + \mathcal{L}_3$ by combining the formulas (2.3) and (2.4). We find

$$(e^{tL_3}\phi)(x) \;=\; \frac{1}{\sqrt{2\pi a(2t)}}\int_{\mathbb{R}} e^{-\frac{|x_3e^{-t}-y_3|^2}{2a(2t)}} \Big(e^{t\mathcal{L}_h}\phi(\cdot,y_3)\Big)(x_h)\,\mathrm{d}y_3 \;, \qquad t>0 \;. \tag{2.5}$$

In [10, Proposition A.6], it is shown that this expression defines a uniformly bounded semigroup in $X(m)$ for any $m > 1$, and that the map $t \mapsto e^{tL_3}$ is strongly continous in the topology of $X_{loc}(m)$. Moreover, the subspace $X_0(m)$ is left invariant by e^{tL_3} for any $t \ge 0$. Using these results and the relation (2.1), we conclude that the three-dimensional operator L generates a uniformly bounded semigroup in the space $\mathbb{X}(m)$, given by

$$e^{tL}\omega \;=\; \Big(e^{-3t/2}e^{tL_3}\omega_1\,,\, e^{-3t/2}e^{tL_3}\omega_2\,,\, e^{tL_3}\omega_3\Big)^\top , \qquad t \ge 0 \;. \tag{2.6}$$

As is easily verified, if $\nabla\cdot\omega = 0$, then $\nabla\cdot e^{tL}\omega = 0$ for all $t \ge 0$.

The asymptotic stability of the Burgers vortices relies heavily on the decay properties of the semigroup e^{tL} as $t\to\infty$. In the proof of Theorems 1.2 and 1.3, we also use the smoothing properties of the operator e^{tL} for $t > 0$, and in particular the fact that e^{tL} extends to a bounded operator from $\mathbb{X}^p(m)$ into $\mathbb{X}^2(m)$ for all $p \in [1,2]$. All the needed estimated are collected in the following statement.

Proposition 2.1 *Let $m \in (1,\infty]$, $p \in [1,2]$, and take $\eta \in (0,\frac{1}{2}]$ such that $2\eta < m-1$. For any $\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{N}^3$, there exists $C > 0$ such that the following estimates hold:*

$$\|\partial_x^\beta e^{tL_h}\omega_h\|_{X(m)^2} \;\le\; \frac{Ce^{-(\frac{3}{2}+\beta_3)t}}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|\omega_h\|_{X^p(m)^2}\ , \tag{2.7}$$

$$\|\partial_x^\beta e^{tL_3}\omega_3\|_{X(m)} \;\le\; \frac{Ce^{-(\eta+\beta_3)t}}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|\omega_3\|_{X^p(m)}\ , \tag{2.8}$$

for any $\omega \in \mathbb{X}^p(m)$ and all $t > 0$. Here $a(t) = 1-e^{-t}$ and $|\beta| = \beta_1+\beta_2+\beta_3$.

Proof. We first assume that $m \in (1,\infty)$. If $p \in [1,2]$ and $\beta_h = (\beta_1,\beta_2) \in \mathbb{N}^2$, it is proved in [8, Appendix A] that

$$\|\partial_{x_h}^{\beta_h} e^{t\mathcal{L}_h} f\|_{L^2(m)} \;\le\; \frac{C}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta_h|}{2}}}\|f\|_{L^p(m)}\ , \qquad t > 0\ , \tag{2.9}$$

for all $f \in L^p(m)$. If in addition $f \in L^p_0(m)$, we have the stronger estimate

$$\|\partial_{x_h}^{\beta_h} e^{t\mathcal{L}_h} f\|_{L^2(m)} \;\le\; \frac{Ce^{-\eta t}}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta_h|}{2}}}\|f\|_{L^p(m)}\ , \qquad t > 0\ , \tag{2.10}$$

where $\eta > 0$ is as in Proposition 2.1. On the other hand, using (2.4), we find by direct calculation

$$\|\partial_{x_3}^{\beta_3} e^{t\mathcal{L}_3} f\|_{L^\infty(\mathbb{R})} \;\le\; \frac{Ce^{-\beta_3 t}}{a(t)^{\frac{\beta_3}{2}}}\|f\|_{L^\infty(\mathbb{R})}\ , \qquad t > 0\ . \tag{2.11}$$

Here, as in (1.23), the stabilizing factor $e^{-\beta_3 t}$ comes from the dilation operator $-x_3\partial_{x_3}$ which enters the definition of $\mathcal{L}_3$. Now, if we start from the representation (2.5) and use the estimates (2.9)–(2.11), we easily obtain (2.7), (2.8) by a direct calculation, see [10, Proposition A.6].

To complete the proof of Proposition 2.1, it remains to show that (2.9), (2.10) still hold when $m = \infty$. If $t \in (0,1)$, estimate (2.9) is easily obtained by a direct calculation, based on the representation (2.3). Using this remark and the semigroup property of $e^{t\mathcal{L}_h}$, we conclude that it is sufficient to establish (2.9), (2.10) in the particular case where $p = 2$ and $\beta_h = 0$. This in turns follows easily from the spectral properties of the generator $\mathcal{L}_h$. Indeed, it is well-known that $\mathcal{L}_h$ is a self-adjoint operator in $L^2(\infty)$ with purely discrete spectrum $\sigma(\mathcal{L}_h) = \{-\frac{k}{2}\,|\,k = 0,1,2,\dots\}$. Moreover, the subspace $L^2_0(\infty)$ is precisely the orthogonal complement of the eigenspace corresponding to the zero eigenvalue, see for example [9, Lemma 4.7]. It follows that $e^{t\mathcal{L}_h}$ is a semigroup of contractions in $L^2(\infty)$, and that $\|e^{t\mathcal{L}_h}f\|_{L^2(\infty)} \le e^{-t/2}\|f\|_{L^2(\infty)}$ for all $t \ge 0$ if $f \in L^2_0(\infty)$. This proves (2.9) and (2.10), with $\eta = 1/2$. $\square$

2.2 Estimates for the velocity fields

If the velocity u and the vorticity ω are related by the Biot-Savart law (1.6), we have $|u| \le J(|\omega|)$, where J is the Riesz potential defined by

$$J(\phi)(x) \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\,\phi(y)\,\mathrm{d}y\ , \qquad x \in \mathbb{R}^3\ . \tag{2.12}$$

Since ω will typically belong to the Banach space $\mathbb{X}(m)$, we need estimates on the Riesz potential $J(\phi)$ for $\phi \in X(m)$. We start with a preliminary result:

Lemma 2.2 *Let $p_1 \in [1,2)$, $p_2 \in [1,2]$, and assume that $\phi \in X^{p_1}(0) \cap X^{p_2}(0)$. If $q_1, q_2 \in [1,\infty]$ satisfy*

$$\frac{2p_1}{2-p_1} \;<\; q_1 \;\le\; \infty \;, \qquad p_2 \;<\; q_2 \;<\; \frac{2p_2}{2-p_2} \;, \tag{2.13}$$

then $J(\phi) = J_1(\phi) + J_2(\phi)$ with $J_i(\phi) \in X^{q_i}(0)$ for $i = 1,2$, and we have the following estimates

$$\|J_1(\phi)\|_{X^{q_1}(0)} \;\le\; C(p_1,q_1)\|\phi\|_{X^{p_1}(0)} \;, \tag{2.14}$$

$$\|J_2(\phi)\|_{X^{q_2}(0)} \;\le\; C(p_2,q_2)\|\phi\|_{X^{p_2}(0)} \;. \tag{2.15}$$

Proof. We proceed as in [10, Proposition A.9]. We first observe that

$$\begin{aligned} J(\phi)(x_h, x_3) \;&=\; \int_{|x_3-y_3|\ge 1} F(x_h; x_3, y_3)\,\mathrm{d}y_3 + \int_{|x_3-y_3|<1} F(x_h; x_3, y_3)\,\mathrm{d}y_3 \\ &=\; J_1(\phi)(x_h, x_3) + J_2(\phi)(x_h, x_3) \;, \end{aligned}$$

where

$$F(x_h; x_3, y_3) \;=\; \int_{\mathbb{R}^2} \frac{\phi(y_h, y_3)}{|x_h-y_h|^2 + (x_3-y_3)^2}\,\mathrm{d}y_h \;, \qquad x_h \in \mathbb{R}^2 \;, \quad x_3, y_3 \in \mathbb{R} \;.$$

For any $a \in \mathbb{R}$, let $f_a(y_h) = (a^2 + |y_h|^2)^{-1}$. Then $f_a \in L^r(\mathbb{R}^2)$ for any $r > 1$ and any $a \neq 0$, and there exists $C_r > 0$ such that

$$\|f_a\|_{L^r(\mathbb{R}^2)} \;\le\; \frac{C_r}{|a|^{2-\frac{2}{r}}} \;.$$

Moreover, we have $F(\cdot\,; x_3, y_3) = \phi(\cdot, y_3) \star f_{x_3-y_3}$ by construction. Thus, if we take $1 \le p, q, r \le \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, we obtain using Young's inequality

$$\|F(\cdot\,; x_3, y_3)\|_{L^q(\mathbb{R}^2)} \;\le\; \|\phi(\cdot, y_3)\|_{L^p(\mathbb{R}^2)} \|f_{x_3-y_3}\|_{L^r(\mathbb{R}^2)} \;\le\; \frac{C_r \|\phi(\cdot, y_3)\||_{L^p(\mathbb{R}^2)}}{|x_3-y_3|^{2-\frac{2}{r}}} \;.$$

To estimate $J_1(\phi)$, we choose $p = p_1$, $q = q_1$. In view of (2.13), the corresponding exponent $r = r_1$ satisfies $2 < r_1 \le \infty$, so that $2 - \frac{2}{r_1} \in (1,2]$. By Minkowski's inequality, we thus find

$$\|J_1(\phi)(\cdot, x_3)\|_{L^{q_1}(\mathbb{R}^2)} \;\le\; \int_{|x_3-y_3|\ge 1} \|F(\cdot\,; x_3, y_3)\|_{L^{q_1}(\mathbb{R}^2)}\,\mathrm{d}y_3 \;\le\; C(r_1) \sup_{y_3\in\mathbb{R}} \|\phi(\cdot, y_3)\|_{L^{p_1}(\mathbb{R}^2)} \;.$$

Taking the supremum over $x_3 \in \mathbb{R}$, we obtain (2.14). Similarly, to bound $J_2(\phi)$, we take $p = p_2$, $q = q_2$. Then $1 < r_2 < 2$, so that $2 - \frac{2}{r_2} \in (0,1)$. We thus obtain

$$\|J_2(\phi)(\cdot, x_3)\|_{L^{q_2}(\mathbb{R}^2)} \;\le\; \int_{|x_3-y_3|<1} \|F(\cdot\,; x_3, y_3)\|_{L^{q_2}(\mathbb{R}^2)}\,\mathrm{d}y_3 \;\le\; C(r_2) \sup_{y_3\in\mathbb{R}} \|\phi(\cdot, y_3)\|_{L^{p_2}(\mathbb{R}^2)} \;,$$

and (2.15) follows. Finally, the uniform continuity of $J_i(\phi)(\cdot, x_3)$ with respect to x_3 can be verified exactly as in the proof of [10, Proposition A.9]. □

As an immediate consequence, we obtain the following useful statements.

Proposition 2.3 *Let $\phi \in X(m)$ for some $m \in (1,\infty]$. Then $J(\phi) \in X^q(0)$ for all $q \in (2,\infty)$, and there exists a positive constant $C = C(m,q)$ such that*

$$\|J(\phi)\|_{X^q(0)} \;\le\; C\|\phi\|_{X(m)} \;. \tag{2.16}$$

Proof. If $m > 1$, we recall that $X(m) \hookrightarrow X^p(0)$ for all $p \in [1,2]$. Thus we can apply Lemma 2.2 with $p_1 = 1$, $p_2 = 2$, $q_1 = q_2 = q \in (2,\infty)$, and the result follows. $\square$

Corollary 2.4 *Let $\phi_1,\ \phi_2 \in X(m)$ for some $m \in (1,\infty]$. Then $\phi_1 J(\phi_2) \in X^p(m)$ for all $p \in (1,2)$, and there exists a positive constant $C = C(m,p)$ such that*

$$\|\phi_1 J(\phi_2)\|_{X^p(m)} \;\le\; C\|\phi_1\|_{X(m)}\|\phi_2\|_{X(m)}\ . \tag{2.17}$$

Proof. We proceed as in [10, Corollary A.10]. Let $p \in (1,2)$, and take $q \in (2,\infty)$ such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. For any $x_3 \in \mathbb{R}$, we have by Hölder's inequality

$$\begin{aligned}
\|\phi_1(\cdot,x_3)J(\phi_2)(\cdot,x_3)\|_{L^p(m)} \;&=\; \Big(\int_{\mathbb{R}^2} \rho_m(|x_h|^2)^{p/2}|\phi_1(x_h,x_3)|^p|J(\phi_2)(x_h,x_3)|^p\,\mathrm{d}x_h\Big)^{1/p} \\
&\le\; \Big(\int_{\mathbb{R}^2} \rho_m(|x_h|^2)|\phi_1(x_h,x_3)|^2\,\mathrm{d}x_h\Big)^{1/2}\Big(\int_{\mathbb{R}^2} |J(\phi_2)(x_h,x_3)|^q\,\mathrm{d}x_h\Big)^{1/q} \\
&=\; \|\phi_1(\cdot,x_3)\|_{L^2(m)}\|J(\phi_2)(\cdot,x_3)\|_{L^q(0)}\ .
\end{aligned}$$

Taking the supremum over $x_3 \in \mathbb{R}$ and using Proposition 2.3, we obtain (2.17). Finally, it is clear that the map $x_3 \mapsto \phi_1(\cdot,x_3)J(\phi_2)(\cdot,x_3)$ is continuous from $\mathbb{R}$ into $L^p(m)$. $\square$

We conclude this section with an estimate on the linear operator (1.11) which will be needed in Section 4.

Lemma 2.5 *Let $p \in [1,2]$ and $2 - \frac{2}{p} < m \le \infty$. For any $\beta \in \mathbb{N}^3$, there exists $C > 0$ such that*

$$\|\partial_x^\beta \Lambda\omega\|_{\mathbb{X}^p(m)} \;\le\; C\sum_{|\tilde\beta|\le|\beta|+1}\|\partial_x^{\tilde\beta}\omega\|_{\mathbb{X}^p(m)}. \tag{2.18}$$

Proof. It is sufficient to prove (2.18) for $\beta = 0$. The general case easily follows if we use the Leibniz rule to differentiate $\Lambda\omega$ (we omit the details).

Assume thus that ω belongs to $\mathbb{X}^p(m)$, together with its first order derivatives. Since the function U^G defined in (1.9) is smooth and bounded (together with all its derivatives), it is clear that

$$\|(U^G,\nabla)\omega\|_{\mathbb{X}^p(m)} + \|(\omega,\nabla)U^G\|_{\mathbb{X}^p(m)} \;\le\; C\sum_{|\tilde\beta|\le 1}\|\partial_x^{\tilde\beta}\omega\|_{\mathbb{X}^p(m)}\ .$$

We now estimate the term $(u,\nabla)G = (K_{3D}*\omega,\nabla)G$, using the fact that $|K_{3D}*\omega| \le J(|\omega|)$. Since $|\omega| \in X^1(0)\cap X^p(0)$ by assumption, we can apply Lemma 2.2 with $p_1 = 1$, $q_1 = \infty$, $p_2 = p$, and $q_2 \in (p, \frac{2p}{2-p})$. By Hölder's inequality, we easily find

$$\begin{aligned}
\|J_1(|\omega|)|\nabla G|\|_{X^p(m)} \;&\le\; C\|J_1(|\omega|)\|_{X^\infty(0)} \;\le\; C\||\omega|\|_{X^1(0)} \;\le\; C\|\omega\|_{\mathbb{X}^p(m)}\ , \\
\|J_2(|\omega|)|\nabla G|\|_{X^p(m)} \;&\le\; C\|J_2(|\omega|)\|_{X^{q_2}(0)} \;\le\; C\||\omega|\|_{X^p(0)} \;\le\; C\|\omega\|_{\mathbb{X}^p(m)}\ .
\end{aligned}$$

We conclude that $\|(u,\nabla)G\|_{\mathbb{X}^p(m)} = \|(K_{3D}*\omega,\nabla)G\|_{\mathbb{X}^p(m)} \le C\|\omega\|_{\mathbb{X}^p(m)}$. In a similar way, commuting the derivative and the convolution operator, we obtain the estimate $\|(G,\nabla)u\|_{\mathbb{X}^p(m)} \le \|(G,\nabla)(K_{3D}*\omega)\|_{\mathbb{X}^p(m)} \le C\|\nabla\omega\|_{\mathbb{X}^p(m)}$. This completes the proof. $\square$

3 The vectorial 2D problem

In this section we study the linearized equation $\partial_t\omega = (L-\alpha\Lambda)\omega$ in the particular case where the vorticity ω does not depend on the vertical variable. As was explained in the introduction, this preliminary step is an essential ingredient in the linear stability proof which will be presented in Section 4. Since the results established here will eventually be applied to the restriction of the 3D vorticity field to a horizontal plane $x_3 = \text{const.}$, we do not assume in this section that ω is divergence-free.

If $\partial_{x_3}\omega = 0$, then $\mathcal{L}_3\omega = 0$, and the expression (2.1) of the linear operator L becomes significantly simpler. On the other hand, we know from (1.11) that

$$\Lambda\omega \,=\, \Lambda_1\omega - \Lambda_2\omega + \Lambda_3\omega - \Lambda_4\omega\ , \tag{3.1}$$

where

$$\begin{aligned} \Lambda_1\omega &= (U^G,\nabla)\omega = (U_h^G,\nabla_h)\omega\ , & \Lambda_3\omega &= (u,\nabla)G = (u_h,\nabla_h)G\ ,\\ \Lambda_2\omega &= (\omega,\nabla)U^G = (\omega_h,\nabla_h)U^G\ , & \Lambda_4\omega &= (G,\nabla)u = g\partial_{x_3}u\ . \end{aligned} \tag{3.2}$$

Here $u = K_{3D} * \omega$ is the velocity field obtained from ω via the three-dimensional Biot-Savart law (1.6). Since $\partial_{x_3}\omega = 0$, we have $\partial_{x_3}u = 0$, hence $\Lambda_4\omega = 0$ in our case. Moreover, it is easy to verify that $u = (u_h, u_3)$, where $u_h = K_{2D} \star \omega_3$. Thus, we see that

$$(L-\alpha\Lambda)\omega \,=\, \mathscr{L}_\alpha\omega\ , \qquad \text{if } \partial_{x_3}\omega \,=\, 0\ ,$$

where $\mathscr{L}_\alpha$ is the two-dimensional differential operator defined by

$$\mathscr{L}_\alpha\omega \,=\, \begin{pmatrix} \mathscr{L}_{\alpha,h}\,\omega_h \\ \mathscr{L}_{\alpha,3}\,\omega_3 \end{pmatrix} \,=\, \begin{pmatrix} (\mathcal{L}_h - \frac{3}{2})\omega_h - \alpha(\Lambda_1 - \tilde\Lambda_2)\omega_h \\ \mathcal{L}_h\omega_3 - \alpha(\Lambda_1 + \tilde\Lambda_3)\omega_3 \end{pmatrix}\ . \tag{3.3}$$

Here $\tilde\Lambda_2\omega_h = (\omega_h,\nabla_h)U_h^G$ and $\tilde\Lambda_3\omega_3 = (K_{2D}\star\omega_3,\nabla_h)g$.

For any $\alpha \in \mathbb{R}$ and any $m \in (1,\infty]$, the operator $\mathscr{L}_\alpha$ defined by (3.3) is the generator of a strongly continuous semigroup in the space $L^2(m)^3$. This property can be established by a standard perturbation argument, see Lemma 3.2 below. Our main goal here is to obtain accurate decay estimates for the semigroup $e^{t\mathscr{L}_\alpha}$ as $t\to\infty$. As is clear from (3.3), the evolutions for ω_h and ω_3 are completely decoupled, so that we can consider the semigroups $e^{t\mathscr{L}_{\alpha,h}}$ and $e^{t\mathscr{L}_{\alpha,3}}$ separately. The main contribution of this section is:

Proposition 3.1 *Fix $m \in (1,\infty]$, $\alpha\in\mathbb{R}$, $\mu \in (0,\frac{3}{2})$, and take $\eta \in (0,\frac{1}{2}]$ such that $1+2\eta < m$. Then there exists $C > 0$ such that*

$$\|e^{t\mathscr{L}_{\alpha,h}}\omega_h\|_{L^2(m)^2} \,\le\, C\,e^{-\mu t}\|\omega_h\|_{L^2(m)^2}\ , \qquad t \ge 0\ , \tag{3.4}$$

$$\|e^{t\mathscr{L}_{\alpha,3}}\omega_3\|_{L^2(m)} \,\le\, C\,e^{-\eta t}\|\omega_3\|_{L^2(m)}\ , \qquad t \ge 0\ , \tag{3.5}$$

for all $\omega \in L^2(m)^2 \times L^2_0(m)$.

Estimate (3.5) was obtained in [9, Proposition 4.12] for $m < \infty$, and the proof given there extends to the limiting case $m = \infty$ without additional difficulty. We recall that the decay rate $e^{-\eta t}$ is obtained using the fact that $\omega_3 \in L^2_0(m)$: If we only assume that $\omega_3 \in L^2(m)$ for some $m > 1$, then (3.5) holds with $\eta = 0$.

From now on, we focus on the semigroup $e^{t\mathscr{L}_{\alpha,h}}$, which has not been studied yet. To prove (3.4), we use the same arguments as in [9, Section 4.2]. We first establish a short time estimate:

Lemma 3.2 *Fix $m \in (1,\infty]$, $\alpha \in \mathbb{R}$, and $T > 0$. There exists $C = C(T,m,|\alpha|) > 0$ such that*

$$\sup_{0\le t\le T}\Big(\|e^{t\mathscr{L}_{\alpha,h}}\omega_h\|_{L^2(m)^2} + a(t)^{\frac12}\|\nabla_h e^{t\mathscr{L}_{\alpha,h}}\omega_h\|_{L^2(m)^4}\Big) \le C\|\omega_h\|_{L^2(m)^2}\ , \tag{3.6}$$

for all $\omega_h \in L^2(m)^2$. Here $a(t) = 1 - e^{-t}$.

Proof. Given $\omega_h^0 \in L^2(m)^2$, the idea is to solve the integral equation

$$\omega_h(t) \,=\, e^{t(\mathcal{L}_h - \frac32)}\omega_h^0 - \alpha\int_0^t e^{(t-s)(\mathcal{L}_h-\frac32)}(\Lambda_1 - \tilde\Lambda_2)\omega_h(s)\,\mathrm{d}s\ , \qquad t\in[0,T]\ , \tag{3.7}$$

by a fixed point argument, in the space $X_T = \{\omega_h \in C([0,T], L^2(m)^2\,|\,\|\omega_h\|_{X_T} < \infty\}$ defined by the norm

$$\|\omega_h\|_{X_T} \,=\, \sup_{0\le t\le T}\|\omega_h(t)\|_{L^2(m)^2} + \sup_{0\le t\le T} a(t)^{\frac12}\|\nabla_h\omega_h(t)\|_{L^2(m)^4}\ .$$

From (2.9) we know that $\|e^{t(\mathcal{L}_h-\frac32)}\omega_h^0\|_{X_T} \le C_1\|\omega_h^0\|_{L^2(m)^2}$, for some $C_1 > 0$ independent of T. To estimate the integral term in (3.7), we first observe that the velocity field U^G defined by (1.9) satisfies

$$\sup_{x_h\in\mathbb{R}^2}(1+|x_h|)|U^G(x_h)| + \sup_{x_h\in\mathbb{R}^2}(1+|x_h|)^2|\nabla_h U^G(x_h)| \,<\, \infty\ . \tag{3.8}$$

In view of the definitions (3.2), we thus have

$$\|(1+|x_h|)\Lambda_1\omega_h\|_{L^2(m)^2} \,\le\, C\|\nabla_h\omega_h\|_{L^2(m)^4}\ , \tag{3.9}$$

$$\|(1+|x_h|)^2\tilde\Lambda_2\omega_h\|_{L^2(m)^2} \,\le\, C\|\omega_h\|_{L^2(m)^2}\ . \tag{3.10}$$

Using these estimates together with (2.9), we can bound

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)(\mathcal{L}_h-\frac32)}(\Lambda_1-\tilde\Lambda_2)\omega_h(s)\,\mathrm{d}s\Big\|_{L^2(m)^2}\\
&\qquad\le\, C\int_0^t e^{-\frac32(t-s)}\Big(\|\omega_h(s)\|_{L^2(m)^2} + \|\nabla_h\omega_h(s)\|_{L^2(m)^4}\Big)\,\mathrm{d}s\\
&\qquad\le\, C\|\omega_h\|_{X_T}\int_0^t e^{-\frac32(t-s)}a(s)^{-\frac12}\,\mathrm{d}s \,\le\, Ca(T)^{\frac12}\|\omega_h\|_{X_T}\ .
\end{aligned}$$

In a similar way,

$$\begin{aligned}
&\Big\|\nabla_h\int_0^t e^{(t-s)(\mathcal{L}_h-\frac32)}(\Lambda_1-\tilde\Lambda_2)\omega_h(s)\,\mathrm{d}s\Big\|_{L^2(m)^4} \\
&\qquad\le\, C\int_0^t \frac{e^{-\frac32(t-s)}}{a(t-s)^{\frac12}}\Big(\|\omega_h(s)\|_{L^2(m)^2} + \|\nabla_h\omega_h(s)\|_{L^2(m)^4}\Big)\,\mathrm{d}s \,\le\, C\|\omega_h\|_{X_T}\ .
\end{aligned} \tag{3.11}$$

Summarizing, we have shown that $\|\omega_h\|_{X_T} \le C_1\|\omega_h^0\|_{L^2(m)^2} + C_2|\alpha|a(T)^{1/2}\|\omega_h\|_{X_T}$, for some positive constants C_1, C_2. If we now take $T > 0$ small enough so that $C_2|\alpha|a(T)^{1/2} \le 1/2$, we see that the right-hand side of (3.7) is a strict contraction in X_T. We deduce that (3.7) has a unique solution, which satisfies $\|\omega_h\|_{X_T} \le 2C_1\|\omega_h^0\|_{L^2(m)^2}$. Since $\omega_h(t) = e^{t\mathscr{L}_{\alpha,h}}\omega_h^0$ by construction, this proves (3.6) for T sufficiently small, and the general case follows due to the semigroup property. This concludes the proof. □

We next consider the essential spectrum of the semigroup $e^{t\mathscr{L}_{\alpha,h}}$, and begin with a few definitions. If A is a bounded linear operator on a (complex) Banach space X, we define the essential spectrum $\sigma_{ess}(A\,;X)$ as the set of all $z \in \mathbb{C}$ such that $A - z$ is not a Fredholm operator with zero index, see [16] or [4]. The essential spectral radius of A in X is given by

$$r_{ess}(A\,;X) \;=\; \sup\Big\{|z|\,;\, z \in \sigma_{ess}(A\,;X)\Big\} \;<\; \infty\,.$$

If $|z| > r_{ess}(A\,;X)$, then either z is in the resolvent set of A, or z is an eigenvalue of A with finite multiplicity, see [4, Corollary IV.2.11]. In the latter case, we say that z belongs to the discrete spectrum of A.

In what follows, we consider the linear operator $\mathscr{L}_{\alpha,h}$ as acting on the complexified space $L^2(m)^2$, i.e. the space of all $\omega_h : \mathbb{R}^2 \to \mathbb{C}^2$ such that $\|\omega_h\|_{L^2(m)^2} < \infty$. Our first result shows that the essential spectral radius of the operator $e^{t\mathscr{L}_{\alpha,h}}$ in $L^2(m)^2$ does not depend on α.

Proposition 3.3 *Let $m \in (1,\infty]$ and $\alpha \in \mathbb{R}$. Then for each $t > 0$ we have*

$$r_{ess}\Big(e^{t\mathscr{L}_{\alpha,h}}\,; L^2(m)^2\Big) \;=\; r_{ess}\Big(e^{t\mathscr{L}_{0,h}}\,; L^2(m)^2\Big) \;=\; e^{-(\frac{m}{2}+1)t}\,. \tag{3.12}$$

Proof. Since $\mathscr{L}_{0,h} = \mathcal{L}_h - \frac{3}{2}$, the last equality in (3.12) follows from [8, Theorem A.1] if $m < \infty$. If $m = \infty$, then $e^{t\mathcal{L}_h}$ is a compact operator for any $t > 0$, hence $r_{ess}(e^{t\mathscr{L}_{0,h}}\,; L^2(\infty)^2) = 0$. To prove the first equality in (3.12), we fix $t > 0$. Our goal is to show that the linear operator $\Delta_\alpha(t) = e^{t\mathscr{L}_{\alpha,h}} - e^{t(\mathcal{L}_h - \frac{3}{2})}$ is compact in $L^2(m)^2$. By Weyl's theorem, this will imply that both semigroups have the same essential spectrum, hence the same essential spectral radius. In view of (3.7) we have, for all $\omega_h \in L^2(m)^2$,

$$\Delta_\alpha(t)\omega_h \;=\; -\alpha \int_0^t e^{(t-s)(\mathcal{L}_h - \frac{3}{2})}(\Lambda_1 - \tilde{\Lambda}_2)e^{s\mathscr{L}_{\alpha,h}}\omega_h \,\mathrm{d}s\,. \tag{3.13}$$

Let $w(x_h) = 1 + |x_h|$. If $m < \infty$, it follows from (2.9) and definition (1.13) that

$$\|w\, e^{t\mathcal{L}_h}\omega_h\|_{L^2(m)^2} \;\le\; C\|e^{t\mathcal{L}_h}\omega_h\|_{L^2(m+1)^2} \;\le\; C\|w\,\omega_h\|_{L^2(m)^2}\,, \tag{3.14}$$

for all $\omega_h \in L^2(m)^2$ and all $t \ge 0$. If $m = \infty$, we know from [11, Proposition 2.1] that $w(-\mathcal{L}_h + 1)^{-1/2}$ is a bounded operator in $L^2(\infty)^2$, and since $\mathcal{L}_h$ is the generator of an analytic semigroup we easily obtain

$$\|w\, e^{t\mathcal{L}_h}\omega_h\|_{L^2(\infty)^2} \;\le\; C\|(-\mathcal{L}_h + 1)^{1/2}e^{t\mathcal{L}_h}\omega_h\|_{L^2(\infty)^2} \;\le\; \frac{C}{a(t)^{1/2}}\,\|\omega_h\|_{L^2(\infty)^2}\,, \tag{3.15}$$

for all $t > 0$. Now, starting from (3.13) and using either (3.14) or (3.15) together with (3.9), (3.10), and Lemma 3.2, we find

$$\begin{aligned}\|w\,\Delta_\alpha(t)\omega_h\|_{L^2(m)^2} \;&\le\; C|\alpha| \int_0^t \frac{e^{-\frac{3}{2}(t-s)}}{a(t-s)^{1/2}}\Big(\|e^{s\mathscr{L}_{\alpha,h}}\omega_h\|_{L^2(m)^2} + \|\nabla_h e^{s\mathscr{L}_{\alpha,h}}\omega_h\|_{L^2(m)^4}\Big)\,\mathrm{d}s \\ &\le\; C|\alpha|\|\omega_h\|_{L^2(m)^2} \int_0^t \frac{e^{-\frac{3}{2}(t-s)}}{a(t-s)^{1/2}a(s)^{1/2}}\,\mathrm{d}s \;\le\; C|\alpha|\|\omega_h\|_{L^2(m)^2}\,.\end{aligned}$$

Moreover, proceeding as in (3.11), we find $\|\nabla_h\Delta_\alpha(t)\omega_h\|_{L^2(m)^4} \le C|\alpha|\|\omega_h\|_{L^2(m)^2}$. Thus we have shown that $w\Delta_\alpha(t)$ and $\nabla_h\Delta_\alpha(t)$ are bounded operators in $L^2(m)$. By Rellich's criterion, we conclude that $\Delta_\alpha(t)$ is a compact operator in $L^2(m)^2$, for any $t > 0$. This completes the proof. □

In view of Proposition 3.3, the spectrum of the semigroup $e^{t\mathscr{L}_{\alpha,h}}$ outside the disk of radius $e^{-(\frac{m}{2}+1)t}$ in the complex plane is purely discrete. By the spectral mapping theorem [4], to control that part of the spectrum it is sufficient to locate the eigenvalues of the generator $\mathscr{L}_{\alpha,h}$. Thus we look for nontrivial solutions of the eigenvalue problem

$$\mathscr{L}_{\alpha,h}\omega_h \,=\, \lambda\omega_h \;, \tag{3.16}$$

where $\omega_h \in L^2(m)^2$ and $\lambda \in \mathbb{C}$ satisfies $\mathrm{Re}\,\lambda > -\frac{m}{2} - 1$. The following auxiliary result shows that the eigenfunctions ω_h always have a Gaussian decay at infinity.

Proposition 3.4 *Let $m \in (1,\infty)$ and $\alpha \in \mathbb{R}$. If $\omega_h \in L^2(m)^2$ is a solution of* (3.16) *with* $\mathrm{Re}\,\lambda > -\frac{m}{2} - 1$, *then $\omega_h \in L^2(\infty)^2$.*

The proof of Proposition 3.4 is postponed to Section 6.2 below. Note that a similar result for the nonlocal operator $\mathscr{L}_{\alpha,3}$ has been obtained in [9, Lemma 4.5], and plays a key role in the derivation of estimate (3.5). Thanks to Proposition 3.4, we only need to control the eigenvalues of $\mathscr{L}_{\alpha,h}$ in the Gaussian space $L^2(\infty)^2$. This is the last important step in the proof of Proposition 3.1.

Proposition 3.5 *If λ is an eigenvalue of $\mathscr{L}_{\alpha,h}$ in $L^2(\infty)^2$, then* $\mathrm{Re}\,\lambda \le -\frac{3}{2}$.

Proof. Assume that $\omega_h \in L^2(\infty)^2$ is a nontrivial solution of the eigenvalue problem (3.16), for some $\alpha \in \mathbb{R}$ and some $\lambda \in \mathbb{C}$. Using (3.3), we thus have

$$\lambda\omega_h \,=\, \mathcal{L}_h\omega_h - \frac{3}{2}\omega_h - \alpha(U_h^G,\nabla_h)\omega_h + \alpha(\omega_h,\nabla_h)U_h^G \;, \tag{3.17}$$

where the velocity field U^G is defined in (1.9). Since $\mathscr{L}_{\alpha,h}$ is a relatively compact perturbation of $\mathscr{L}_{0,h} = \mathcal{L}_h - \frac{3}{2}$, both operators have the same domain, and it follows that ω_h belongs to the domain of $\mathcal{L}_h$. In particular, we have $\nabla_h\omega_h \in L^2(\infty)^4$ and $|x_h|\omega_h \in L^2(\infty)^2$, see e.g. [11, Section 2].

In the rest of the proof, we denote by $\langle\cdot,\cdot\rangle$ the inner product in the complexified space $L^2(\infty)^2$, namely

$$\langle\omega_h^1,\omega_h^2\rangle \,=\, \int_{\mathbb{R}^2} p(x_h)\omega_h^1(x_h)\cdot\overline{\omega_h^2(x_h)}\,\mathrm{d}x_h \;,$$

where $p(x_h) = \rho_\infty(|x_h|^2) = e^{|x_h|^2/4}$. We also denote $\|\omega_h\|^2 = \langle\omega_h,\omega_h\rangle$. We recall that $\mathcal{L}_h$ is a selfadjoint operator in $L^2(\infty)^2$ which satisfies $-\mathcal{L}_h \ge 0$ on $L^2(\infty)^2$ and $-\mathcal{L}_h \ge 1/2$ on $L_0^2(\infty)^2$. For later use, we observe that the (unbounded) operator $\omega_h \mapsto (U_h^G,\nabla_h)\omega_h$ is skew-symmetric in $L^2(\infty)^2$, because the vector field $p(x_h)U^G(x_h)$ is divergence-free.

We now take the inner product of (3.17) with ω_h, and evaluate the real part of the result. Using the skew-symmetry of the operator (U_h^G,∇_h), we easily obtain

$$\begin{aligned}\mathrm{Re}\,\lambda\,\|\omega_h\|^2 \,&=\, \langle\mathcal{L}_h\omega_h,\omega_h\rangle - \frac{3}{2}\|\omega_h\|^2 + \alpha\,\mathrm{Re}\langle(\omega_h,\nabla_h)U_h^G,\omega_h\rangle \\ &=\, \langle\mathcal{L}_h\omega_h,\omega_h\rangle - \frac{3}{2}\|\omega_h\|^2 + 2\alpha\,\mathrm{Re}\int_{\mathbb{R}^2} p(x_h)(x_h\cdot\omega_h)(x_h^\perp\cdot\overline{\omega_h})(u^g)'(|x_h|^2)\,\mathrm{d}x_h \;,\end{aligned} \tag{3.18}$$

where $u^g(r)$ is defined in (1.9). On the other hand, it follows from (3.17) that the scalar function $x_h\cdot\omega_h \in L^2(\infty)$ satisfies

$$\lambda\, x_h\cdot\omega_h \,=\, \mathcal{L}_h(x_h\cdot\omega_h) - 2x_h\cdot\omega_h - \alpha(U_h^G,\nabla_h)(x_h\cdot\omega_h) - 2\nabla_h\cdot\omega_h \;.$$

Thus, proceeding as above and using the same notation $\langle\cdot,\cdot\rangle$ for the inner product in $L^2(\infty)$, we find

$$\operatorname{Re}\lambda\,\|x_h\cdot\omega_h\|^2 \;=\; \langle\mathcal{L}_h(x_h\cdot\omega_h),x_h\cdot\omega_h\rangle - 2\|x_h\cdot\omega_h\|^2 - 2\operatorname{Re}\langle\nabla_h\cdot\omega_h,x_h\cdot\omega_h\rangle\;. \tag{3.19}$$

Finally, the two-dimensional divergence $\nabla_h\cdot\omega_h\in L^2_0(\infty)$ satisfies

$$\lambda\,\nabla_h\cdot\omega_h \;=\; \mathcal{L}_h(\nabla_h\cdot\omega_h) - \nabla_h\cdot\omega_h - \alpha(U_h^G,\nabla_h)(\nabla_h\cdot\omega_h)\;, \tag{3.20}$$

hence

$$\operatorname{Re}\lambda\,\|\nabla_h\cdot\omega_h\|^2 \;=\; \langle\mathcal{L}_h(\nabla_h\cdot\omega_h),\nabla_h\cdot\omega_h\rangle - \|\nabla_h\cdot\omega_h\|^2\;. \tag{3.21}$$

Since $\nabla_h\cdot\omega_h\in L^2_0(\infty)$, it follows from (3.21) that $\operatorname{Re}\lambda\,\|\nabla_h\cdot\omega_h\|^2\le-\frac{3}{2}\|\nabla_h\cdot\omega_h\|^2$. Thus we must have $\operatorname{Re}\lambda\le-\frac{3}{2}$, unless $\nabla_h\cdot\omega_h\equiv 0$. In the latter case, we deduce from (3.19) that $\operatorname{Re}\lambda\,\|x_h\cdot\omega_h\|^2\le-2\|x_h\cdot\omega_h\|^2$, hence $\operatorname{Re}\lambda\le-2$ unless $x_h\cdot\omega_h\equiv 0$. But if this last condition is met, it follows from (3.18) that $\operatorname{Re}\lambda\,\|\omega_h\|^2\le-\frac{3}{2}\|\omega_h\|^2$, hence $\operatorname{Re}\lambda\le-\frac{3}{2}$ because ω_h is not identically zero. Summarizing, we conclude that $\operatorname{Re}\lambda\le-\frac{3}{2}$ in all cases. □

Remark. Actually the conclusions of Proposition 3.5 can be slightly strengthened. First, in the invariant subspace where $\nabla_h\cdot\omega_h=0$, one can show that all eigenvalues of $\mathscr{L}_{\alpha,h}$ satisfy $\operatorname{Re}\lambda\le-2$. This follows from the proof above if we use in addition the fact that $\omega_h\in L^2_0(\infty)^2$, due to the divergence-free condition. The result is clearly sharp, because if $g(x_h)$ is defined by (1.8) it is easy to verify that the function $\omega_h=x_h^\perp g(x_h)$ satisfies $\mathscr{L}_{\alpha,h}\omega_h=-2\omega_h$ for any $\alpha\in\mathbb{R}$. On the other hand, if ω_h is a solution of (3.17) such that $\nabla_h\cdot\omega_h\neq 0$, we have $\operatorname{Re}\lambda<-\frac{3}{2}$ if $\alpha\neq 0$. This follows from (3.21), because we know from [8, Appendix A] that

$$\langle\mathcal{L}_h(\nabla_h\cdot\omega_h),\nabla_h\cdot\omega_h\rangle \;<\; -\frac{1}{2}\|\nabla_h\cdot\omega_h\|^2\;,$$

unless $\nabla_h\cdot\omega_h=(a_1x_1+a_2x_2)g(x_h)$ for some $a_1,a_2\in\mathbb{C}$. But this ansatz is not compatible with (3.20) if $\alpha\neq 0$. In fact, using the techniques developped in [21] or [7], it is possible to show that, given any $M>0$, the eigenvalue equation (3.20) restricted to the orthogonal complement of the space of all radially symmetric functions in $L^2(\infty)$ has no nontrivial solution such that $\operatorname{Re}\lambda\ge-M$, if $|\alpha|$ is sufficiently large depending on M.

It is now easy to conclude the proof of Proposition 3.1. As was already mentioned, we only need to prove that estimate (3.4) holds for any $\mu<3/2$. If $\rho_\alpha(m)>0$ denotes the spectral radius of the operator $e^{\mathscr{L}_{\alpha,h}}$ in $L^2(m)^2$, this is equivalent to showing that $\log\rho_\alpha(m)\le-3/2$, see [4, Proposition IV.2.2]. But that inequality follows immediately from Propositions 3.3, 3.4, and 3.5, since $m>1$. The proof of Proposition 3.1 is now complete. □

4 Linear stability

Equipped with the results of the previous section, we now study the linearized equation (1.22) in its full generality. Using Proposition 2.1 and a perturbation argument, it is not difficult to verify that the linear operator $L-\alpha\Lambda$ generates a locally bounded semigroup in the space $\mathbb{X}(m)$ for any $\alpha\in\mathbb{R}$ and any $m\in(1,\infty]$, see Proposition 4.2 below. The goal of this section is to show that the semigroup $e^{t(L-\alpha\Lambda)}$ extends to a bounded operator from $\mathbb{X}^p(m)$ to $\mathbb{X}(m)$ for any $t>0$ and any $p\in[1,2]$, and satisfies the following uniform estimates:

Proposition 4.1 *Fix $m \in (1,\infty]$, $p \in [1,2]$, $\alpha \in \mathbb{R}$, and take $\mu \in (1,\frac{3}{2})$, $\eta \in (0,\frac{1}{2}]$ such that $2\mu < m+1$ and $2\eta < m-1$. For any $\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{N}^3$, there exists $C > 0$ such that*

$$\|\partial_x^\beta (e^{t(L-\alpha\Lambda)}\omega_0)_h\|_{X(m)^2} \;\le\; \frac{C\, e^{-(\mu+\beta_3)t}}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|\omega_0\|_{\mathbb{X}^p(m)}\;, \tag{4.1}$$

$$\|\partial_x^\beta (e^{t(L-\alpha\Lambda)}\omega_0)_3\|_{X(m)} \;\le\; \frac{C\, e^{-(\eta+\beta_3)t}}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|\omega_0\|_{\mathbb{X}^p(m)}\;, \tag{4.2}$$

for any $\omega_0 \in \mathbb{X}^p(m)$ and all $t > 0$. Moreover, if $\nabla\cdot\omega_0 = 0$, then $\nabla\cdot e^{t(L-\alpha\Lambda)}\omega_0 = 0$ for all $t > 0$.

The proof of this important result is divided into several steps.

4.1 Global existence and short time estimates

We first prove that the linearized equation (1.22) has a unique global solution in $\mathbb{X}(m)$.

Proposition 4.2 *Fix $m \in (1,\infty]$, $p \in [1,2]$, and $\alpha \in \mathbb{R}$. Then, for any $\omega_0 \in \mathbb{X}^p(m)$, Eq. (1.22) has a unique solution $\omega \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{X}(m)) \cap C([0,\infty);\mathbb{X}^p_{loc}(m))$ with initial data ω_0. Moreover, for any $\beta \in \mathbb{N}^3$, there exist positive constants C_1, C_2 (independent of α) such that*

$$\|\partial_x^\beta \omega(t)\|_{\mathbb{X}(m)} \le \frac{C_1}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|\omega_0\|_{\mathbb{X}^p(m)}\;, \qquad \textit{for} \quad 0 < t \le \frac{C_2}{|\alpha|^2+1}\;, \tag{4.3}$$

where $a(t) = 1-e^{-t}$. Finally, if $\nabla\cdot\omega_0 = 0$, then $\nabla\cdot\omega(t) = 0$ for all $t > 0$.

Proof. We proceed as in the proof of Lemma 3.2. Let e^{tL} be the semigroup generated by L, which is given by the explicit expression (2.6). The integral equation corresponding to (1.22) is

$$\omega(t) \;=\; e^{tL}\omega_0 - \alpha\int_0^t e^{(t-s)L}\Lambda\omega(s)\,\mathrm{d}s \;=:\; (F\omega)(t)\;, \quad t > 0\;. \tag{4.4}$$

Given $k \in \mathbb{N}\setminus\{0\}$ and a sufficiently small $T \in (0,1]$, we shall solve (4.4) in the Banach space

$$\mathbb{U}_{k,T} \;=\; \Big\{\omega \in L^\infty_{loc}((0,T);\mathbb{X}(m)) \cap C([0,T];\mathbb{X}^p_{loc}(m)) \;\Big|\; \|\omega\|_{k,T} < \infty\Big\}\;,$$

equipped with the norm

$$\|\omega\|_{k,T} \;=\; \sum_{|\beta|\le k}\Big(\sup_{0<t<T} a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}\|\partial_x^\beta\omega(t)\|_{\mathbb{X}(m)} + \sup_{0<t<T} a(t)^{\frac{|\beta|}{2}}\|\partial_x^\beta\omega(t)\|_{\mathbb{X}^p(m)}\Big)\;,$$

where $a(t) = 1-e^{-t}$. If $\omega_0 \in \mathbb{X}^p(m)$, we know from Proposition 2.1 that the map $t \mapsto e^{tL}\omega_0$ belongs to $\mathbb{U}_{k,T}$ for any $T > 0$, and that $\|e^{tL}\omega_0\|_{k,T} \le C_1\|\omega_0\|_{\mathbb{X}^p(m)}$ for some $C_1 > 0$ depending only on k, m, p.

Given $\omega \in \mathbb{U}_{k,T}$, we now estimate the integral term in (4.4). Using Proposition 2.1 and Lemma 2.5, we find

$$\begin{aligned}\|\partial_x^\beta e^{(t-s)L}\Lambda\omega(s)\|_{\mathbb{X}(m)} \;&\le\; \frac{C\|\Lambda\omega(s)\|_{\mathbb{X}^p(m)}}{a(t-s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}} \;\le\; \frac{C\sum_{|\tilde\beta|\le 1}\|\partial_x^{\tilde\beta}\omega(s)\|_{\mathbb{X}^p(m)}}{a(t-s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\\ &\le\; \frac{C\|\omega\|_{k,T}}{a(t-s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}\,a(s)^{\frac{1}{2}}}\;, \qquad 0 < s < t\;.\end{aligned} \tag{4.5}$$

Similarly we have $\|\partial_x^\beta e^{(t-s)L}\Lambda\omega(s)\|_{\mathbb{X}^p(m)} \le Ca(t-s)^{-\frac{|\beta|}{2}}\, a(s)^{-\frac{1}{2}}\|\omega\|_{k,T}$ for $0 < s < t$. In the particular case where $\beta = 0$, it follows that

$$\Big\|\int_0^t e^{(t-s)L}\Lambda\omega(s)\,\mathrm{d}s\Big\|_{\mathbb{X}(m)} \;\le\; Ca(t)^{1-\frac{1}{p}}\|\omega\|_{k,T}\ , \tag{4.6}$$

$$\Big\|\int_0^t e^{(t-s)L}\Lambda\omega(s)\,\mathrm{d}s\Big\|_{\mathbb{X}^p(m)} \;\le\; Ca(t)^{\frac{1}{2}}\|\omega\|_{k,T}\ , \qquad 0 < t \le T\ . \tag{4.7}$$

Assume now that $1 \le |\beta| \le k$. If $\beta' \le \beta$ and $|\beta'| = |\beta| - 1$, we have from Lemma 2.5

$$\|\partial_x^{\beta'}\Lambda\omega(s)\|_{\mathbb{X}(m)} \;\le\; C\sum_{|\tilde\beta|\le|\beta|}\|\partial_x^{\tilde\beta}\omega(s)\|_{\mathbb{X}(m)} \;\le\; \frac{C\|\omega\|_{k,T}}{a(s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\ , \qquad 0 < s \le T\ .$$

Thus, writing $\partial_x^\beta e^{(t-s)L} = \partial_x^{\beta-\beta'}\partial_x^{\beta'}e^{(t-s)L} = \partial_x^{\beta-\beta'}e^{(\frac{\beta_1'+\beta_2'}{2}-\beta_3')t}e^{(t-s)L}\partial_x^{\beta'}$, and using Proposition 2.1 again, we obtain

$$\|\partial_x^\beta e^{(t-s)L}\Lambda\omega(s)\|_{\mathbb{X}(m)} \;\le\; C\|\partial_x^{\beta-\beta'}e^{(t-s)L}\partial_x^{\beta'}\Lambda\omega(s)\|_{\mathbb{X}(m)} \;\le\; \frac{C\|\omega\|_{k,T}}{a(t-s)^{\frac{1}{2}}\,a(s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\ , \tag{4.8}$$

for $0 < s < t$. Similarly, we have $\|\partial_x^\beta e^{(t-s)L}\Lambda\omega(s)\|_{\mathbb{X}^p(m)} \le a(t-s)^{-\frac{1}{2}}\,a(s)^{-\frac{|\beta|}{2}}\|\omega\|_{k,T}$. Combining (4.5) and (4.8), we obtain the following estimate

$$\begin{aligned}
&\Big\|\partial_x^\beta\int_0^t e^{(t-s)L}\Lambda\omega(s)\,\mathrm{d}s\Big\|_{\mathbb{X}(m)}\\
&\quad\le C\Big(\int_0^{\frac{t}{2}} a(t-s)^{-\frac{1}{p}+\frac{1}{2}-\frac{|\beta|}{2}}\,a(s)^{-\frac{1}{2}}\,\mathrm{d}s + \int_{\frac{t}{2}}^t a(t-s)^{-\frac{1}{2}}\,a(s)^{-\frac{1}{p}+\frac{1}{2}-\frac{|\beta|}{2}}\,\mathrm{d}s\Big)\|\omega\|_{k,T} \qquad (4.9)\\
&\quad\le Ca(t)^{1-\frac{1}{p}-\frac{|\beta|}{2}}\|\omega\|_{k,T}\ , \qquad 0 < t \le T\ ,
\end{aligned}$$

which generalizes (4.6). Similarly, the generalization of (4.7) is

$$\Big\|\partial_x^\beta\int_0^t e^{(t-s)L}\Lambda\omega(s)\,\mathrm{d}s\Big\|_{\mathbb{X}^p(m)} \;\le\; Ca(t)^{\frac{1}{2}-\frac{|\beta|}{2}}\|\omega\|_{k,T}\ , \qquad 0 < t \le T\ . \tag{4.10}$$

Summarizing, we have shown that the linear map F defined by (4.4) satisfies the estimate

$$\|F\omega\|_{k,T} \;\le\; C_1\|\omega_0\|_{\mathbb{X}^p(m)} + \tilde C|\alpha|T^{\frac{1}{2}}\|\omega\|_{k,T}\ , \qquad \text{if} \quad 0 < T \le 1\ ,$$

where $\tilde C > 0$ depends only on k, m and p. Arguing as in [10, Corollary A.7 and Remark A.8], it is also straightforward to verify that $F\omega \in C([0,T];\mathbb{X}^p_{loc}(m))$ if $\omega \in \mathbb{U}_{k,T}$. If we now assume that $T \le C_2(1+|\alpha|^2)^{-1}$, where $C_2 = 1/(4\tilde C^2)$, we see that F is a strict contraction in $\mathbb{U}_{k,T}$. As a consequence, the integral equation (4.4) has a unique fixed point $\omega \in \mathbb{U}_{k,T}$, which satisfies $\|\omega\|_{k,T} \le 2C_1\|\omega_0\|_{\mathbb{X}^p(m)}$. This proves that equation (1.22) is locally well-posed in $\mathbb{X}^p(m)$, and that the solutions satisfy (4.3). Moreover, as the local existence time T is independent of the initial data, the solutions can be extended globally in time. Finally, since both operators L and Λ preserve the divergence-free condition, it is easy to check that, if $\nabla\cdot\omega_0 = 0$, the solution ω of (1.22) satisfies $\nabla\cdot\omega(t) = 0$ for all $t > 0$. This completes the proof. $\square$

4.2 Decay estimates for the vertical derivatives

Proposition 4.2 shows that the linearized equation (1.22) is globally well-posed in the space $\mathbb{X}(m)$ for $m > 1$, but does not provide accurate estimates on the solution $\omega(t) = e^{t(L-\alpha\Lambda)}\omega_0$ for large times. In this section, we focus on the derivatives of $\omega(t)$ with respect to the vertical variable x_3. Using identity (1.23), we shall show that $\partial_{x_3}^k \omega(t)$ decays exponentially as $t \to \infty$, provided $k \in \mathbb{N}$ is large enough depending on $|\alpha|$. Albeit elementary, this observation plays a crucial role in the proof of Proposition 4.1, because it will allow us to simplify the study of the semigroup associated to $L - \alpha\Lambda$ by disregarding most of the terms involving a vertical derivative.

Proposition 4.3 *Fix $m \in (1, \infty]$. There exist positive constants C_3, C_4 such that, for all $\alpha \in \mathbb{R}$, all $k \in \mathbb{N}$, and all $\omega_0 \in \mathbb{X}(m)$ with $\partial_{x_3}^k \omega_0 \in \mathbb{X}(m)$, the following estimate holds:*

$$\|\partial_{x_3}^k e^{t(L-\alpha\Lambda)}\omega_0\|_{\mathbb{X}(m)} \;\le\; C_3\, e^{(C_4(|\alpha|^2+1)-k)t}\|\partial_{x_3}^k\omega_0\|_{\mathbb{X}(m)}\ , \qquad t \ge 0\ . \tag{4.11}$$

Proof. In view of (1.23), it is sufficient to prove (4.11) for $k = 0$. If $\omega_0 \in \mathbb{X}(m)$, we know from Proposition 4.2 that there exist constants $C_1 \ge 1$ and $C_2 > 0$, depending only on m, such that the solution $\omega(t) = e^{t(L-\alpha\Lambda)}\omega_0$ of (1.22) satisfies $\|\omega(t)\|_{\mathbb{X}(m)} \le C_1\|\omega_0\|_{\mathbb{X}(m)}$ for $t \in (0, t_0]$, where $t_0 = C_2/(|\alpha|^2 + 1)$. Using the semigroup property, we can iterate this bound, and we easily obtain

$$\|e^{t(L-\alpha\Lambda)}\omega_0\|_{\mathbb{X}(m)} \;\le\; C_3\, e^{C_4(|\alpha|^2+1)t}\|\omega_0\|_{\mathbb{X}(m)}\ , \qquad t \ge 0\ ,$$

where $C_3 = C_1$ and $C_4 = C_2^{-1}\log(C_1)$. This concludes the proof. $\square$

4.3 Decomposition of the linearized operator

Motivated by Proposition 4.3, we now decompose the linear operator $L - \alpha\Lambda$ as follows:

$$L - \alpha\Lambda \;=\; \mathscr{L}_\alpha + \mathcal{L}_3 - \alpha H\ , \tag{4.12}$$

where $\mathscr{L}_\alpha$ is defined in (3.3) and $\mathcal{L}_3 = \partial_{x_3}^2 - x_3\partial_{x_3}$. We recall that the operator $\mathscr{L}_\alpha$ does not involve any derivative with respect to the vertical variable x_3, and does not couple the horizontal and vertical components of $\omega = (\omega_h, \omega_3)^\top$. In view of (3.1)–(3.3), the last term in (4.12) has the following expression:

$$H \;=\; \Lambda_3 - \tilde{\Lambda}_3 - \Lambda_4\ ,$$

where Λ_3, Λ_4 are defined in (3.2) and $\tilde{\Lambda}_3$ after (3.3). More explicitly, we have

$$H\omega \;=\; \begin{pmatrix} H_h\omega \\ H_3\omega \end{pmatrix} \;=\; \begin{pmatrix} -g(K_{3D} * \partial_{x_3}\omega)_h \\ ((K_{3D} * \omega)_h - K_{2D} \star \omega_3, \nabla_h)g - g(K_{3D} * \partial_{x_3}\omega)_3 \end{pmatrix}\ , \tag{4.13}$$

where K_{3D}, K_{2D} are the Biot-Savart kernels (1.6), (1.7), and g is defined in (1.8). Here $\star$ denotes the convolution with respect to the horizontal variables, so that

$$(K_{2D} \star \omega_3)(x_h, x_3) \;=\; \int_{\mathbb{R}^2} K_{2D}(x_h - y_h)\,\omega_3(y_h, x_3)\,\mathrm{d}y_h\ .$$

Thus, unlike $\mathscr{L}_\alpha$, the operator H involves vertical derivatives, and couples the horizontal and vertical components of ω. As was already observed in Section 3, we have $H\omega = 0$ whenever $\partial_{x_3}\omega = 0$, see Proposition 4.5 below.

Let $R_\alpha(t)$ denote the semigroup generated by the linear operator $\mathscr{L}_\alpha + \mathcal{L}_3$. In analogy with (2.5), we have the following representation:

$$(R_\alpha(t)\omega)(x) \;=\; \frac{1}{\sqrt{2\pi a(2t)}}\int_{\mathbb{R}} e^{-\frac{|x_3e^{-t} - y_3|^2}{2a(2t)}}\Big(e^{t\mathscr{L}_\alpha}\omega(\cdot, y_3)\Big)(x_h)\,\mathrm{d}y_3\ , \quad t > 0\ , \tag{4.14}$$

where $a(t) = 1 - e^{-t}$ and $e^{t\mathscr{L}_\alpha}$ is the semigroup generated by $\mathscr{L}_\alpha$. Since $R_\alpha(t)$ does not couple the horizontal and vertical components of ω, we can write

$$R_\alpha(t)\omega \;=\; \begin{pmatrix} R_{\alpha,h}(t)\omega_h \\ R_{\alpha,3}(t)\omega_3 \end{pmatrix} ,$$

where $R_{\alpha,h}(t)$ and $R_{\alpha,3}(t)$ are the semigroups generated by $\mathscr{L}_{\alpha,h}+\mathcal{L}_3$ and $\mathscr{L}_{\alpha,3}+\mathcal{L}_3$, respectively. Using the results of Section 3, we obtain the following estimates:

Proposition 4.4 *Fix $m \in (1,\infty]$, $\alpha \in \mathbb{R}$, $\mu \in (1,\frac{3}{2})$, and take $\eta \in (0,\frac{1}{2}]$ such that $2\eta < m-1$. Then there exists $C_5 > 0$ such that*

$$\|R_{\alpha,h}(t)\omega_h\|_{X(m)^2} \;\le\; C_5\, e^{-\mu t}\|\omega_h\|_{X(m)^2} \;, \tag{4.15}$$

$$\|R_{\alpha,3}(t)\omega_3\|_{X(m)} \;\le\; C_5\, e^{-\eta t}\|\omega_3\|_{X(m)} \;, \tag{4.16}$$

for all $\omega \in \mathbb{X}(m)$ and all $t \ge 0$.

Proof. Both estimates follow from the representation (4.14), Proposition 3.1, and estimate (2.11). The calculations are straightforward, and can be omitted here. We just remark that, even if $\nabla \cdot \omega = 0$, the map $x_h \mapsto \omega_h(x_h, x_3)$ usually has a nonzero divergence for all values of $x_3 \in \mathbb{R}$. This is why Proposition 3.1, hence also Proposition 4.4, was established without imposing any divergence-free condition. $\square$

We conclude this section with a useful bound on the linear operator H.

Proposition 4.5 *Fix $m \in (1,\infty]$ and $\gamma \in (0,1)$. There exists $C_6 > 0$ such that, for all $\omega \in \mathbb{X}(m)$ with $\partial_{x_3}\omega \in \mathbb{X}(m)$, one has*

$$\|H_h\omega\|_{X(m)^2} \;\le\; C_6\|\partial_{x_3}\omega\|_{\mathbb{X}(m)} \;, \tag{4.17}$$

$$\|H_3\omega\|_{X(m)} \;\le\; C_6(\|\partial_{x_3}\omega\|_{\mathbb{X}(m)} + \|\omega_h\|_{X(m)^2}^{\gamma}\|\partial_{x_3}\omega_h\|_{X(m)^2}^{1-\gamma}) \;. \tag{4.18}$$

Proof. We use the expression (4.13) of the linear operator H. Since $\partial_{x_3}\omega \in \mathbb{X}(m)$, we know from Proposition 2.3 that $\partial_{x_3}u \equiv K_{3D} * \partial_{x_3}\omega \in X^4(0)$. Thus, using Hölder's inequality, we obtain

$$\|g\,\partial_{x_3}u\|_{\mathbb{X}(m)} \;\le\; \|\partial_{x_3}u\|_{X^4(0)}\Big(\int_{\mathbb{R}^2}\rho_m(|x_h|^2)^2 g(x_h)^4\,\mathrm{d}x_h\Big)^{1/4} \;\le\; C\|\partial_{x_3}\omega\|_{\mathbb{X}(m)} \;.$$

In particular, we have $\|H_h\omega\|_{X(m)^2} \le C\|\partial_{x_3}\omega\|_{\mathbb{X}(m)}$, which is (4.17).

We next consider the two-dimensional vector $I = (K_{3D} * \omega)_h - K_{2D} \star \omega_3$ and estimate the term $(I,\nabla_h)g$. Using the definitions (1.6), (1.7), it is straightforward to verify that $I(x) = I_1(x) + I_2(x)$, where

$$I_1(x) \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x_h - y_h)^\perp}{|x-y|^3}\,(\omega_3(y_h,y_3) - \omega_3(y_h,x_3))\,\mathrm{d}y \;,$$

$$I_2(x) \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x_3 - y_3}{|x-y|^3}\,(\omega_h(y_h,y_3) - \omega_h(y_h,x_3))^\perp\,\mathrm{d}y \;.$$

Here we have used the identities

$$\int_{\mathbb{R}}\frac{1}{|x-y|^3}\,\mathrm{d}y_3 \;=\; \frac{2}{|x_h - y_h|^2} \;, \qquad \text{and} \qquad \int_{\mathbb{R}}\frac{x_3 - y_3}{|x-y|^3}\,\mathrm{d}y_3 \;=\; 0 \;.$$

Since $\nabla_h g(x_h) = -g(x_h)x_h/2$ and $|x_h \cdot (x_h - y_h)^\perp| \le |x_h||x_h - y_h|^{1-\sigma}|y_h|^\sigma$ for any $\sigma \in [0,1]$, we can bound

$$\begin{aligned}|(I_1,\nabla_h)g(x)| \;&\le\; Cg(x_h)|x_h|\int_{|x_3-y_3|\ge 1}\frac{|y_h|^\sigma}{|x-y|^{2+\sigma}}\,|\omega_3(y_h,y_3)-\omega_3(y_h,x_3)|\,\mathrm{d}y\\ &\quad + Cg(x_h)|x_h|\int_{|x_3-y_3|<1}\frac{1}{|x-y|^2}\,|\omega_3(y_h,y_3)-\omega_3(y_h,x_3)|\,\mathrm{d}y\;.\end{aligned}$$

We now proceed like in the proof of Lemma 2.2. Integrating first with respect to the horizontal variable $y_h \in \mathbb{R}^2$ and applying Hölder's inequality, we obtain

$$\begin{aligned}|(I_1,\nabla_h)g(x)| \;&\le\; Cg(x_h)|x_h|\int_{|x_3-y_3|\ge 1}\frac{1}{|x_3-y_3|^{2+\sigma}}\||\cdot|^\sigma\{\omega_3(\cdot,y_3)-\omega_3(\cdot,x_3)\}\|_{L^1(\mathbb{R}^2)}\,\mathrm{d}y_3\\ &\quad + Cg(x_h)|x_h|\int_{|x_3-y_3|<1}\frac{1}{|x_3-y_3|}\|\omega_3(\cdot,y_3)-\omega_3(\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\,\mathrm{d}y_3\;.\end{aligned}$$

Assuming $0<\sigma<m-1$, we have the estimate $\||\cdot|^\sigma f\|_{L^1(\mathbb{R}^2)} \le C\|f\|_{L^2(m)}$ for any $f \in L^2(m)$, hence

$$\||\cdot|^\sigma\{\omega_3(\cdot,y_3)-\omega_3(\cdot,x_3)\}\|_{L^1(\mathbb{R}^2)} + \|\omega_3(\cdot,y_3)-\omega_3(\cdot,x_3)\|_{L^2(\mathbb{R}^2)} \;\le\; C|x_3-y_3|\|\partial_{x_3}\omega_3\|_{X(m)}\;.$$

We conclude that

$$\begin{aligned}|(I_1,\nabla_h)g(x)| \;&\le\; Cg(x_h)|x_h|\Big(\int_{|x_3-y_3|\ge 1}\frac{1}{|x_3-y_3|^{1+\sigma}}\,\mathrm{d}y_3 + \int_{|x_3-y_3|<1}\mathrm{d}y_3\Big)\|\partial_{x_3}\omega_3\|_{X(m)}\\ &\le\; Cg(x_h)|x_h|\|\partial_{x_3}\omega_3\|_{X(m)}\;,\end{aligned}$$

hence $\|(I_1,\nabla)g\|_{X(m)} \le C\|\partial_{x_3}\omega_3\|_{X(m)}$.

Finally we consider the term $(I_2,\nabla_h)g$. Using again Hölder's inequality, we obtain

$$\begin{aligned}|I_2(x)| \;&\le\; C\int_{|x_3-y_3|\ge 1}\frac{1}{|x-y|^2}|\omega_h(y_h,y_3)-\omega_h(y_h,x_3)|\,\mathrm{d}y\\ &\quad + C\int_{|x_3-y_3|<1}\frac{1}{|x-y|^2}|\omega_h(y_h,y_3)-\omega_h(y_h,x_3)|\,\mathrm{d}y\\ &\le\; C\int_{|x_3-y_3|\ge 1}\frac{1}{|x_3-y_3|^2}\|\omega_h(\cdot,y_3)-\omega_h(\cdot,x_3)\|_{L^1(\mathbb{R}^2)}\,\mathrm{d}y_3\\ &\quad + \int_{|x_3-y_3|<1}\frac{1}{|x_3-y_3|}\|\omega_h(\cdot,y_3)-\omega_h(\cdot,x_3)\|_{L^2(\mathbb{R}^2)}\,\mathrm{d}y_3\;.\end{aligned}$$

Since $L^2(m) \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in [1,2]$, we have $\|\omega_h(\cdot,y_3)-\omega_h(\cdot,x_3)\|_{L^p(\mathbb{R}^2)^2} \le 2\|\omega_h\|_{X(m)^2}$ and $\|\omega_h(\cdot,y_3)-\omega_h(\cdot,x_3)\|_{L^p(\mathbb{R}^2)^2} \le |x_3-y_3|\|\partial_{x_3}\omega_h\|_{X(m)^2}$. In particular, for any $\gamma \in (0,1)$,

$$\|\omega_h(\cdot,y_3)-\omega_h(\cdot,x_3)\|_{L^p(\mathbb{R}^2)^2} \;\le\; 2^\gamma|x_3-y_3|^{1-\gamma}\|\omega_h\|_{X(m)^2}^\gamma\|\partial_{x_3}\omega_h\|_{X(m)^2}^{1-\gamma}\;.$$

Thus we obtain

$$\|I_2\|_{L^\infty(\mathbb{R}^3)^2} \;\le\; C\|\omega_h\|_{X(m)^2}^\gamma\|\partial_{x_3}\omega_h\|_{X(m)^2}^{1-\gamma}\;,$$

and conclude that $\|(I_2,\nabla_h)g\|_{X(m)} \le C\|\omega_h\|_{X(m)^2}^\gamma\|\partial_{x_3}\omega_h\|_{X(m)^2}^{1-\gamma}$. This proves (4.18). □

4.4 Large time estimates

In this section we complete the proof of Proposition 4.1. Fix $m \in (1,\infty]$, $\alpha \in \mathbb{R}$, and assume that $\omega_0 \in \mathbb{X}^p(m)$ for some $p \in [1,2]$. Let $\omega(t) = e^{t(L-\alpha\Lambda)}\omega_0$ be the solution of the linearized equation (1.22) given by Proposition 4.2. Take any $k \in \mathbb{N}$ such that $k > C_4(|\alpha|^2+1)+1/2$, where C_4 is as in Proposition 4.3, and choose $t_0 > 0$ small enough so that estimate (4.3) holds for all $t \in (0,t_0]$ and all $\beta \in \mathbb{N}^3$ with $|\beta| \le k$. Our goal is to control the solution $\omega(t)$ for $t \ge t_0$ and to establish the decay estimates (4.1), (4.2).

To this end, we first observe that $\omega(t)$ satisfies the integral equation

$$\omega(t) \;=\; R_\alpha(t-t_0)\omega(t_0) - \alpha\int_{t_0}^t R_\alpha(t-s)H\omega(s)\,\mathrm{d}s\;, \quad t \ge t_0\;, \tag{4.19}$$

where $R_\alpha(t)$ is the semigroup defined by (4.14). Fix $\bar\eta \in (0,1/2)$ such that $2\bar\eta < m-1$. By Proposition 4.4, we have

$$\|\omega(t)\|_{\mathbb{X}(m)} \;\le\; C_5\,e^{-\bar\eta(t-t_0)}\|\omega(t_0)\|_{\mathbb{X}(m)} + C_5|\alpha|\int_{t_0}^t e^{-\bar\eta(t-s)}\|H\omega(s)\|_{\mathbb{X}(m)}\,\mathrm{d}s\;. \tag{4.20}$$

To estimate the term $\|H\omega(s)\|_{\mathbb{X}(m)}$, we first apply Proposition 4.5 with $\gamma = 1/2$, and then the classical interpolation inequality

$$\|\partial_{x_3}\omega\|_{\mathbb{X}(m)} \;\le\; C\|\omega\|_{\mathbb{X}(m)}^{1-1/k}\,\|\partial_{x_3}^k\omega\|_{\mathbb{X}(m)}^{1/k}\;.$$

Using in addition Young's inequality, we conclude that, given any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$C_5|\alpha|\,\|H\omega(s)\|_{\mathbb{X}(m)} \;\le\; \epsilon\|\omega(s)\|_{\mathbb{X}(m)} + C_\epsilon\|\partial_{x_3}^k\omega(s)\|_{\mathbb{X}(m)}\;. \tag{4.21}$$

On the other hand, since $k > C_4(|\alpha|^2+1)+1/2$, it follows from (4.11) that

$$\|\partial_{x_3}^k\omega(s)\|_{\mathbb{X}(m)} \;\le\; C_3\,e^{-(s-t_0)/2}\|\partial_{x_3}^k\omega(t_0)\|_{\mathbb{X}(m)}\;, \quad s \ge t_0\;. \tag{4.22}$$

Replacing (4.21) and (4.22) into (4.20), we easily obtain

$$\|\omega(t)\|_{\mathbb{X}(m)} \;\le\; \Big(C_5\|\omega(t_0)\|_{\mathbb{X}(m)} + C'_\epsilon\|\partial_{x_3}^k\omega(t_0)\|_{\mathbb{X}(m)}\Big)\,e^{-\bar\eta(t-t_0)} + \epsilon\int_{t_0}^t e^{-\bar\eta(t-s)}\|\omega(s)\|_{\mathbb{X}(m)}\,\mathrm{d}s\;,$$

for some $C'_\epsilon > 0$. Applying now Gronwall's lemma, and using (4.3) to bound $\|\omega(t_0)\|_{\mathbb{X}(m)}$ and $\|\partial_{x_3}^k\omega(t_0)\|_{\mathbb{X}(m)}$ in terms of ω_0, we see that $\|\omega(t)\|_{\mathbb{X}(m)} \le C\,e^{-\eta t}\|\omega_0\|_{\mathbb{X}^p(m)}$ for $t \ge t_0$, where $\eta = \bar\eta - \epsilon$. Finally, using (4.3) again to control the solution for $t < t_0$, we conclude that there exists $C_7 > 0$ such that

$$\|\omega(t)\|_{\mathbb{X}(m)} \;\equiv\; \|e^{t(L-\alpha\Lambda)}\omega_0\|_{\mathbb{X}(m)} \;\le\; \frac{C_7\,e^{-\eta t}}{a(t)^{\frac{1}{p}-\frac{1}{2}}}\,\|\omega_0\|_{\mathbb{X}^p(m)}\;, \tag{4.23}$$

for all $t > 0$. Since $\epsilon > 0$ was arbitrary, estimate (4.23) holds for any $\eta \in (0,1/2)$ such that $2\eta < m-1$.

To conclude the proof, it remains to find the optimal decay rates for $\|\omega_h(t)\|_{\mathbb{X}(m)}$, $\|\omega_3(t)\|_{\mathbb{X}(m)}$ (including the value $\eta = 1/2$ if $m > 2$), and to establish (4.1), (4.2) for $\beta \neq 0$ too. In view of Proposition 4.2, we can assume here without loss of generality that $\partial_{x_3}\omega_0 \in \mathbb{X}(m)\cap\mathbb{X}^p(m)$. First, combining (1.23), (4.23), we immediately obtain

$$\|\partial_{x_3}\omega(t)\|_{\mathbb{X}(m)} \;\equiv\; \|\partial_{x_3}e^{t(L-\alpha\Lambda)}\omega_0\|_{\mathbb{X}(m)} \;\le\; C\,e^{-(\eta+1)t}\|\partial_{x_3}\omega_0\|_{\mathbb{X}(m)}\;, \tag{4.24}$$

for all $t \ge 0$. Moreover, if $m > 2$, we know from Proposition 4.4 that (4.20) holds with $\bar\eta = 1/2$. Thus, applying Proposition 4.5 with $\gamma \le 1/2$ to estimate $\|H\omega(s)\|_{\mathbb{X}(m)}$ and using (4.23), (4.24), we find that $\|\omega(t)\|_{\mathbb{X}(m)}$ decays like $e^{-t/2}$ as $t \to \infty$, hence (4.23) holds with $\eta = 1/2$ if $m > 2$.

Next, to obtain a faster decay estimate for the horizontal component ω_h, we use (4.15) and (4.17). Instead of (4.20), we find

$$\|\omega_h(t)\|_{X(m)^2} \;\le\; C\,e^{-\mu(t-t_0)}\|\omega_h(t_0)\|_{X(m)^2} + C|\alpha|\int_{t_0}^t e^{-\mu(t-s)}\|\partial_{x_3}\omega(s)\|_{\mathbb{X}(m)}\,\mathrm{d}s\;, \tag{4.25}$$

for any $\mu \in (1, \frac{3}{2})$. Invoking (4.24), we conclude that $\|\omega_h(t)\|_{X(m)^2}$ decays like $e^{-\mu t}$ as $t \to \infty$, provided $\mu < 1 + \eta$. In other words, if $\mu \in (1, \frac{3}{2})$ satisfies $2\mu < m + 1$, we have

$$\|\omega_h(t)\|_{X(m)^2} \;\equiv\; \|(e^{t(L-\alpha\Lambda)}\omega_0)_h\|_{X(m)^2} \;\le\; C\,e^{-\mu t}(\|(\omega_0)_h\|_{X(m)^2} + \|\partial_{x_3}\omega_0\|_{\mathbb{X}(m)})\;, \tag{4.26}$$

for all $t \ge 0$. Using the arguments leading to (4.25) and proceeding as in Proposition 4.2, we can also derive the following short time estimate, which complements (4.3):

$$\|\partial_x^\beta\omega_h(t)\|_{X(m)^2} \;\le\; \frac{C_1}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\Big(\|(\omega_0)_h\|_{X^p(m)^2} + \|\partial_{x_3}\omega_0\|_{\mathbb{X}^p(m)}\Big)\;, \quad 0 < t \le \frac{C_2}{|\alpha|^2+1}\;. \tag{4.27}$$

Finally, to obtain decay estimates for the derivative $\partial_x^\beta\omega(t)$, where $\beta \in \mathbb{N}^3$, we can restrict ourselves to $t \ge 2t_1$, where $t_1 > 0$ is small enough so that the short time estimates (4.3), (4.27) hold for $0 < t \le 2t_1$. In view of (1.23), we have the identity

$$\partial_x^\beta e^{t(L-\alpha\Lambda)}\omega_0 \;=\; e^{-\beta_3(t-t_1)}\;\partial_{x_h}^{\beta_h}e^{t_1(L-\alpha\Lambda)}\;e^{(t-2t_1)(L-\alpha\Lambda)}\;\partial_{x_3}^{\beta_3}e^{t_1(L-\alpha\Lambda)}\omega_0\;.$$

Using the short time estimates (4.3), (4.27) with $p = 2$ to bound the first operator $\partial_{x_h}^{\beta_h}e^{t_1(L-\alpha\Lambda)}$, then the long-time estimates (4.23), (4.24) or (4.26) to treat the middle term $e^{(t-2t_1)(L-\alpha\Lambda)}$, and finally (4.3) again to bound the last term $\partial_{x_3}^{\beta_3}e^{t_1(L-\alpha\Lambda)}\omega_0$, we easily obtain (4.1) and (4.2), together with the following estimate

$$\|\partial_x^\beta(e^{t(L-\alpha\Lambda)}\omega_0)_h\|_{X(m)^2} \;\le\; \frac{C\,e^{-(\mu+\beta_3)t}}{a(t)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\Big(\|(\omega_0)_h\|_{X^p(m)^2} + \|\partial_{x_3}\omega_0\|_{\mathbb{X}^p(m)}\Big)\;, \quad t > 0, \tag{4.28}$$

which will also be used in the next section. This concludes the proof of Proposition 4.1. □

5 Nonlinear stability

In this section we consider the nonlinear stability of the Burgers vortex and prove Theorems 1.2 and 1.3. Our starting point is the perturbation equation (1.10), which is equivalent to the integral equation

$$\omega(t) \;=\; e^{t(L-\alpha\Lambda)}\omega_0 + \sum_{j=1}^{2}\int_0^t e^{(t-s)(L-\alpha\Lambda)}N_j(\omega(s),\omega(s))\,\mathrm{d}s\;, \qquad t \ge 0\;, \tag{5.1}$$

where $N_1(v, w) = -(K_{3D} * v, \nabla)w$, $N_2(v, w) = (v, \nabla)K_{3D} * w$, and K_{3D} is the Biot-Savart kernel (1.6). We first establish the following result, which already implies Theorem 1.2.

Proposition 5.1 *Fix $m \in (1,\infty]$, $\alpha \in \mathbb{R}$, and take $\eta \in (0,\frac{1}{2}]$ such that $2\eta < m-1$. Then there exist $\delta = \delta(\alpha,m,\eta) > 0$ and $C = C(\alpha,m,\eta) > 0$ such that, for any $\omega_0 \in \mathbb{X}(m)$ with $\nabla\cdot\omega_0 = 0$ and $\|\omega_0\|_{\mathbb{X}(m)} \le \delta$, Eq. (5.1) has a unique solution $\omega \in L^\infty(\mathbb{R}_+;\mathbb{X}(m)) \cap C([0,\infty);\mathbb{X}_{loc}(m))$, which satisfies*

$$\|\partial_x^\beta \omega(t)\|_{\mathbb{X}(m)} \le \frac{C\|\omega_0\|_{\mathbb{X}(m)}}{a(t)^{\frac{|\beta|}{2}}}\, e^{-\eta t}\ , \qquad t > 0\ , \tag{5.2}$$

for any multi-index $\beta \in \mathbb{N}^3$ of length $|\beta| \le 1$.

Proof. Let $\mathbb{U}$ be the Banach space of all $\omega \in L^\infty(\mathbb{R}_+;\mathbb{X}(m)) \cap C([0,\infty);\mathbb{X}_{loc}(m))$ such that $\nabla\cdot\omega(t) = 0$ for all $t > 0$ and $\|\omega\|_{\mathbb{U}} < \infty$, where

$$\|\omega\|_{\mathbb{U}} = \sum_{|\beta|\le 1} \sup_{t>0} a(t)^{\frac{|\beta|}{2}} e^{\eta t} \|\partial_x^\beta \omega(t)\|_{\mathbb{X}(m)}\ .$$

Given $\omega_0 \in \mathbb{X}(m)$ such that $\nabla\cdot\omega_0 = 0$, we denote by $\Phi : \mathbb{U} \to \mathbb{U}$ the nonlinear map defined by

$$\Phi(\omega)(t) = e^{t(L-\alpha\Lambda)}\omega_0 + \sum_{j=1}^{2} \Phi_j(\omega,\omega)(t)\ , \quad t > 0\ , \tag{5.3}$$

where Φ_1, Φ_2 are the following bilinear operators:

$$\Phi_j(\omega,\tilde\omega)(t) = \int_0^t e^{(t-s)(L-\alpha\Lambda)} N_j(\omega(s),\tilde\omega(s))\,\mathrm{d}s\ , \quad j = 1,2\ . \tag{5.4}$$

If $\|\omega_0\|_{\mathbb{X}(m)}$ is sufficiently small, we shall show that the map Φ is a strict contraction in the ball $B_K = \{\omega \in \mathbb{U}\,|\,\|\omega\|_{\mathbb{U}} \le K\}$ for some suitable $K > 0$. It will follow that Φ has a unique fixed point ω in B_K, which by construction is the desired solution of (5.1).

Since $\omega_0 \in \mathbb{X}(m)$ and $\nabla\cdot\omega_0 = 0$, Proposition 4.1 shows that the map $t \mapsto e^{t(L-\alpha\Lambda)}\omega_0$ belongs to $\mathbb{U}$, and satisfies the estimate

$$\|e^{t(L-\alpha\Lambda)}\omega_0\|_{\mathbb{U}} \le C_1\|\omega_0\|_{\mathbb{X}(m)}\ ,$$

for some $C_1 > 0$ (depending on m, α, η). On the other hand, if $v, w \in \mathbb{X}(m)$ and $\nabla w \in \mathbb{X}(m)$, Corollary 2.4 implies that $N_1(v,w)$ and $N_2(v,w)$ belong to $X^p(m)^3$ for any $p \in (1,2)$, and satisfy the bound

$$\|N_1(v,w)\|_{X^p(m)^3} + \|N_2(v,w)\|_{X^p(m)^3} \le C\|v\|_{\mathbb{X}(m)}\|\nabla w\|_{\mathbb{X}(m)}\ ,$$

for some $C > 0$ (depending on m and p). If in addition $\nabla\cdot v = 0$, then denoting $u = K_{3D} * v$ we find

$$\int_{\mathbb{R}^2} (N_1(v,v) + N_2(v,v))_3\,\mathrm{d}x_h = \int_{\mathbb{R}^2} \nabla_h\cdot(v_h u_3 - u_h v_3)\,\mathrm{d}x_h = 0\ , \tag{5.5}$$

for all $x_3 \in \mathbb{R}$, hence $N_1(v,v) + N_2(v,v) \in \mathbb{X}^p(m)$. As a consequence, if $\omega, \tilde\omega \in \mathbb{U}$, we have $N_j(\omega(t),\tilde\omega(t)) \in X^p(m)^3$ for $j = 1,2$ and all $t > 0$, and using Proposition 4.1 again we obtain the following estimate for the bilinear operators Φ_j:

$$
\begin{aligned}
\Big\|\sum_{j=1}^{2}\partial_x^\beta\Phi_j(\omega,\tilde\omega)(t)\Big\|_{\mathbb{X}(m)} &\le \sum_{j=1}^{2}\int_0^t \|\partial_x^\beta e^{(t-s)(L-\alpha\Lambda)}N_j(\omega(s),\tilde\omega(s))\|_{\mathbb{X}(m)}\,\mathrm{d}s \\
&\le C\sum_{j=1}^{2}\int_0^t \frac{e^{-\eta(t-s)}}{a(t-s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|N_j(\omega(s),\tilde\omega(s))\|_{X^p(m)^3}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-\eta(t-s)}}{a(t-s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}}\|\omega(s)\|_{\mathbb{X}(m)}\|\nabla\tilde\omega(s)\|_{\mathbb{X}(m)}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-\eta(t-s)}e^{-2\eta s}}{a(t-s)^{\frac{1}{p}-\frac{1}{2}+\frac{|\beta|}{2}}a(s)^{\frac{1}{2}}}\,\mathrm{d}s\,\|\omega\|_{\mathbb{U}}\|\tilde\omega\|_{\mathbb{U}} \le \frac{Ce^{-\eta t}}{a(t)^{\frac{1}{p}+\frac{|\beta|}{2}-1}}\|\omega\|_{\mathbb{U}}\|\tilde\omega\|_{\mathbb{U}}\,.
\end{aligned}
$$

Since we also know that $N_1(\omega(t),\omega(t))+N_2(\omega(t),\omega(t))$ belongs to $\mathbb{X}^p(m)$ for all $t>0$ and is divergence-free, we have shown that Φ maps $\mathbb{U}$ into $\mathbb{U}$, and that there exists $C_2>0$ (depending on $|\alpha|$, m, and η) such that

$$
\|\Phi(\omega)\|_{\mathbb{U}} \le C_1\|\omega_0\|_{\mathbb{X}(m)}+C_2\|\omega\|_{\mathbb{U}}^2\,,\quad \|\Phi(\omega)-\Phi(\tilde\omega)\|_{\mathbb{U}} \le C_2(\|\omega\|_{\mathbb{U}}+\|\tilde\omega\|_{\mathbb{U}})\|\omega-\tilde\omega\|_{\mathbb{U}}\,,
$$

for all $\omega,\tilde\omega\in\mathbb{U}$. We now take $K>0$ such that $2C_2K<1$, and assume that $\|\omega_0\|_{\mathbb{X}(m)}\le K/(2C_1)$. Then the estimates above show that Φ is a strict contraction in the ball B_K, hence has a unique fixed point $\omega\in B_K$ which, of course, satisfies (5.1). Moreover $\|\omega\|_{\mathbb{U}}\le 2C_1\|\omega_0\|_{\mathbb{X}(m)}$, hence (5.2) holds with $C=2C_1$. This concludes the proof. □

Remark. The size δ of the local basin of attraction of the Burgers vortex αG in $\mathbb{X}(m)$ depends a priori on α, m, and η. However, as announced in Theorem 1.3, the dependence on the decay rate η can easily be removed by the following (standard) argument. Given $m>1$, we first choose $\eta=\bar\eta(m)=\min(\frac{1}{2},\frac{m-1}{4})$ and apply Proposition 5.1 with that value of η. We thus obtain a constant $\bar\delta>0$ depending only on α and m such that, for any $\omega_0\in\mathbb{X}(m)$ with $\nabla\cdot\omega_0=0$ and $\|\omega_0\|_{\mathbb{X}(m)}\le\bar\delta$, Eq. (5.1) has a unique solution $\omega\in L^\infty(\mathbb{R}_+;\mathbb{X}(m))\cap C([0,\infty);\mathbb{X}_{loc}(m))$, which converges exponentially to zero as $t\to\infty$. In particular, given any $\eta\in(0,\frac{1}{2}]$ such that $2\eta<m-1$, there exists $T=T(\eta)>0$ such that $\|\omega(t)\|_{\mathbb{X}(m)}\le\delta$ for all $t\ge T$, where $\delta=\delta(\alpha,m,\eta)$ is the constant given by Proposition 5.1. By uniqueness of the solution, we conclude that ω satisfies (5.2) for any admissible value of η.

In view of Proposition 5.1 and the remark that follows, the proof of Theorem 1.3 will be complete once we have established the improved decay estimate (1.20) for the horizontal component ω_h. A convenient way to do so is to repeat the proof of Proposition 5.1 using a different function space, which incorporates a faster decay rate as $t\to\infty$. Given $\mu\in(1,1+\eta)$, where $\eta\in(0,\frac{1}{2}]$ is as in Proposition 5.1, we introduce the space $\mathbb{V}\subset\mathbb{U}$ defined by the norm

$$
\|\omega\|_{\mathbb{V}} = \sum_{k=0,1}\sum_{|\beta|\le1}\Big(\sup_{t>0}a(t)^{\frac{k}{2}}e^{(\mu+k\eta)t}\|\partial_{x_3}^k\partial_x^\beta\omega_h(t)\|_{X(m)^2}+\sup_{t>0}a(t)^{\frac{k}{2}}e^{(\eta+k)t}\|\partial_{x_3}^k\partial_x^\beta\omega_3(t)\|_{X(m)}\Big)\,.
$$

In view of (5.2), we can assume here (without loss of generality) that $\|\partial_x^\beta\omega_0\|_{\mathbb{X}(m)}$ is finite and arbitrarily small, for all $\beta\in\mathbb{N}^3$ with $|\beta|\le1$. Using Proposition 4.1, we thus obtain

$$
\|e^{t(L-\alpha\Lambda)}\omega_0\|_{\mathbb{V}} \le C_3\sum_{|\beta|\le1}\|\partial_x^\beta\omega_0\|_{\mathbb{X}(m)}\,,
$$

for some $C_3 > 0$. On the other hand, if $v, w \in \mathbb{X}(m)$, the following estimates hold for any $p \in (1,2)$:

$$
\begin{aligned}
\|N_{1,h}(v,w)\|_{X^p(m)^2} &\le C\|v\|_{\mathbb{X}(m)}\|\nabla w_h\|_{X(m)^2}\ , \\
\|N_2(v,w)\|_{X^p(m)^3} &\le C(\|v_h\|_{X(m)^2}\|\nabla_h w\|_{\mathbb{X}(m)} + C\|v_3\|_{X(m)}\|\partial_{x_3} w\|_{\mathbb{X}(m)})\ , \\
\|\partial_{x_3} N_j(v,w)\|_{X^p(m)^3} &\le C(\|\partial_{x_3} v\|_{\mathbb{X}(m)}\|\nabla w\|_{\mathbb{X}(m)} + \|v\|_{\mathbb{X}(m)}\|\partial_{x_3}\nabla w\|_{\mathbb{X}(m)})\ .
\end{aligned}
$$

We now estimate the bilinear operators $\Phi_j(\omega,\tilde\omega)$ for $\omega,\tilde\omega \in \mathbb{V}$. First, using (4.28), we find for $t \ge 1$:

$$
\begin{aligned}
\|\partial_x^\beta \Phi_{1,h}(\omega,\tilde\omega)(t)\|_{X(m)^2} &\le \int_0^t \|\partial_x^\beta\{e^{(t-s)(L-\alpha\Lambda)}N_1(\omega(s),\tilde\omega(s))\}_h\|_{X(m)^2}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-\mu(t-s)}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}}(\|N_{1,h}(\omega(s),\tilde\omega(s))\|_{X^p(m)^2} + \|\partial_{x_3}N_1(\omega(s),\tilde\omega(s))\|_{X^p(m)^3})\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-\mu(t-s)}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}}(\|\omega(s)\|_{\mathbb{X}(m)}\|\nabla\tilde\omega_h(s)\|_{(X(m))^2} \\
&\qquad + \|\partial_{x_3}\omega(s)\|_{\mathbb{X}(m)}\|\nabla\tilde\omega(s)\|_{\mathbb{X}(m)} + \|\omega(s)\|_{\mathbb{X}(m)}\|\partial_{x_3}\nabla\tilde\omega(s)\|_{\mathbb{X}(m)})\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-\mu(t-s)}e^{-(\mu+\eta)s}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}a(s)^{\frac12}}\,\mathrm{d}s\ \|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}} \le Ce^{-\mu t}\|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}}\ .
\end{aligned} \tag{5.6}
$$

In the last inequality, we have used the definition of the norm in $\mathbb{V}$ and the fact that $\mu+\eta < 1+2\eta$. The bound (5.6) also holds for $t < 1$, and can easily be established using (4.1) instead of (4.28).

Next, to bound $\partial_{x_3}\Phi_{1,h}(\omega,\tilde\omega)$, we recall that $\partial_{x_3}e^{t(L-\alpha\Lambda)} = e^{-t}e^{t(L-\alpha\Lambda)}\partial_{x_3}$. Applying (4.1), we find

$$
\begin{aligned}
\|\partial_{x_3}\partial_x^\beta \Phi_{1,h}(\omega,\tilde\omega)(t)\|_{X(m)^2} &\le \int_0^t e^{-(t-s)}\|\partial_x^\beta\{e^{(t-s)(L-\alpha\Lambda)}\partial_{x_3}N_1(\omega(s),\tilde\omega(s))\}_h\|_{X(m)^2}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-(\mu+1)(t-s)}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}}\|\partial_{x_3}N_1(\omega(s),\tilde\omega(s))\|_{X^p(m)^3}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-(\mu+1)(t-s)}e^{-(\mu+\eta)s}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}a(s)^{\frac12}}\,\mathrm{d}s\ \|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}} \le \frac{Ce^{-(\mu+\eta)t}}{a(t)^{\frac1p+\frac{|\beta|}{2}-1}}\|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}}\ .
\end{aligned}
$$

Similarly, for $k = 0, 1$, we can estimate $\partial_{x_3}^k\Phi_{2,h}(\omega,\tilde\omega)$ as follows:

$$
\begin{aligned}
\|\partial_{x_3}^k\partial_x^\beta \Phi_{2,h}(\omega,\tilde\omega)(t)\|_{X(m)^2} &\le \int_0^t e^{-k(t-s)}\|\partial_x^\beta\{e^{(t-s)(L-\alpha\Lambda)}\partial_{x_3}^k N_2(\omega(s),\tilde\omega(s))\}_h\|_{X(m)^2}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-(\mu+k)(t-s)}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}}\|\partial_{x_3}^k N_2(\omega(s),\tilde\omega(s))\|_{X^p(m)^3}\,\mathrm{d}s \\
&\le C\int_0^t \frac{e^{-(\mu+k)(t-s)}e^{-(\mu+\eta)s}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}a(s)^{\frac{k}{2}}}\,\mathrm{d}s\ \|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}} \le \frac{Ce^{-(\mu+k\eta)t}}{a(t)^{\frac1p+\frac{|\beta|}{2}+\frac{k}{2}-\frac32}}\|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}}\ .
\end{aligned}
$$

Finally, using (4.2), we obtain for the vertical components of $\Phi_j(\omega,\tilde\omega)$:

$$
\begin{aligned}
\|\partial_{x_3}^k\partial_x^\beta\Phi_{j,3}(\omega,\tilde\omega)(t)\|_{X(m)} \;&\le\; \int_0^t e^{-k(t-s)}\|\partial_x^\beta\{e^{(t-s)(L-\alpha\Lambda)}\partial_{x_3}^k N_j(\omega(s),\tilde\omega(s))\}_3\|_{X(m)}\,\mathrm{d}s \\
&\le\; C\int_0^t \frac{e^{-(\eta+k)(t-s)}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}}\|\partial_{x_3}^k N_j(\omega(s),\tilde\omega(s))\|_{X^p(m)^3}\,\mathrm{d}s \\
&\le\; C\int_0^t \frac{e^{-(\eta+k)(t-s)}e^{-(k+2\eta)s}}{a(t-s)^{\frac1p-\frac12+\frac{|\beta|}{2}}a(s)^{\frac k2}}\,\mathrm{d}s\;\|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}} \;\le\; \frac{Ce^{-(\eta+k)t}}{a(t)^{\frac1p+\frac{|\beta|}{2}+\frac k2-\frac32}}\|\omega\|_{\mathbb{V}}\|\tilde\omega\|_{\mathbb{V}}\;.
\end{aligned}
$$

Summarizing, we have shown that Φ defined by (5.3) maps $\mathbb{V}$ into $\mathbb{V}$ and satisfies the following bounds:

$$
\begin{aligned}
\|\Phi(\omega)\|_{\mathbb{V}} \;&\le\; C_3\sum_{|\beta|\le1}\|\partial_x^\beta\omega_0\|_{\mathbb{X}(m)} + C_4\|\omega\|_{\mathbb{V}}^2\;,\\
\|\Phi(\omega)-\Phi(\tilde\omega)\|_{\mathbb{V}} \;&\le\; C_4(\|\omega\|_{\mathbb{V}}+\|\tilde\omega\|_{\mathbb{V}})\|\omega-\tilde\omega\|_{\mathbb{V}}\;,
\end{aligned}
$$

for all $\omega,\tilde\omega\in\mathbb{V}$. If $K = 2C_3\sum_{|\beta|\le1}\|\partial_x^\beta\omega_0\|_{\mathbb{X}(m)}$ is sufficiently small, it follows that Φ is a strict contraction in the ball $\tilde B_K = \{\omega\in\mathbb{V}\,|\,\|\omega\|_{\mathbb{V}}\le K\}$, hence has a unique fixed point there. Denoting by $\omega(t)$ the solution of (5.1) given by Proposition 5.1, this implies that $t\mapsto\omega(t+T)$ belongs to $\tilde B_K$ if $T>0$ is sufficiently large. In particular, $\omega(t)$ satisfies (1.20) for some suitable $C>0$. The proof of Theorem 1.3 is now complete. $\square$

6 Appendix

6.1 Proof of Lemma 1.1

Let $\chi\in C_0^\infty(\mathbb{R}^2)$ be a cut-off function such that $\chi(x_h)=1$ if $|x_h|\le1$ and $\chi(x_h)=0$ if $|x_h|\ge2$. Given $R>0$, we denote $\chi_R(x_h)=\chi(x_h/R)$, so that $|\nabla_h\chi_R(x_h)|\le C/R$. For any $x_3\in\mathbb{R}$, we define

$$
f(x_3) \;=\; \int_{\mathbb{R}^2}\tilde\omega_3(x_h,x_3)\,\mathrm{d}x_h\;, \qquad f_R(x_3) \;=\; \int_{\mathbb{R}^2}\tilde\omega_3(x_h,x_3)\chi_R(x_h)\,\mathrm{d}x_h\;.
$$

Since $\tilde\omega_3\in X(m)$ for some $m>1$, it is easy to verify that $\|f-f_R\|_{L^\infty(\mathbb{R})}\to0$ as $R\to\infty$. On the other hand, for any test function $\psi\in C_0^\infty(\mathbb{R})$, we have

$$
\Big|\int_{\mathbb{R}} f(x_3)\frac{\mathrm{d}\psi}{\mathrm{d}x_3}(x_3)\,\mathrm{d}x_3\Big| \;\le\; \Big|\int_{\mathbb{R}} f_R(x_3)\frac{\mathrm{d}\psi}{\mathrm{d}x_3}(x_3)\,\mathrm{d}x_3\Big| + \|f-f_R\|_{L^\infty(\mathbb{R})}\Big\|\frac{\mathrm{d}\psi}{\mathrm{d}x_3}\Big\|_{L^1(\mathbb{R})}\;. \tag{6.1}
$$

The last term in the right-hand side converges to zero as $R\to\infty$. To treat the other term, we observe that

$$
\int_{\mathbb{R}} f_R(x_3)\frac{\mathrm{d}\psi}{\mathrm{d}x_3}(x_3)\,\mathrm{d}x_3 \;=\; \int_{\mathbb{R}^3}\tilde\omega_3(x_h,x_3)\chi_R(x_h)\frac{\mathrm{d}\psi}{\mathrm{d}x_3}(x_3)\,\mathrm{d}x_h\,\mathrm{d}x_3 \;=\; \langle\tilde\omega_3\,,\,\frac{\partial\phi_R}{\partial x_3}\rangle\;,
$$

where $\phi_R(x_h,x_3)=\chi_R(x_h)\psi(x_3)$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $\mathcal{D}'(\mathbb{R}^3)$ and $C_0^\infty(\mathbb{R}^3)$. Now, since $\nabla\cdot\tilde\omega=0$ in the sense of distributions, we have

$$
\langle\tilde\omega_3\,,\,\frac{\partial\phi_R}{\partial x_3}\rangle \;=\; -\langle\frac{\partial\tilde\omega_3}{\partial x_3}\,,\,\phi_R\rangle \;=\; \langle\nabla_h\cdot\tilde\omega_h\,,\,\phi_R\rangle \;=\; -\langle\tilde\omega_h\,,\,\nabla_h\phi_R\rangle\;,
$$

so that

$$
\int_{\mathbb{R}} f_R(x_3)\frac{\mathrm{d}\psi}{\mathrm{d}x_3}(x_3)\,\mathrm{d}x_3 \;=\; -\int_{\mathbb{R}^3}\tilde\omega_h(x_h,x_3)\cdot\nabla_h\chi_R(x_h)\psi(x_3)\,\mathrm{d}x_h\,\mathrm{d}x_3\;.
$$

Using the inclusion $L^2(m) \hookrightarrow L^1(\mathbb{R}^2)$ and the definition (1.15) of the space $X(m)$, we thus find

$$\Big|\int_{\mathbb{R}} f_R(x_3)\frac{\mathrm{d}\psi}{\mathrm{d}x_3}(x_3)\,\mathrm{d}x_3\Big| \;\le\; \frac{C}{R}\,\|\tilde{\omega}_h\|_{X(m)^2}\|\psi\|_{L^1(\mathbb{R})} \xrightarrow[R\to\infty]{} 0\;.$$

Returning to (6.1), we conclude that the left-hand side vanishes for all $\psi \in C_0^\infty(\mathbb{R})$, hence $\frac{\mathrm{d}f}{\mathrm{d}x_3} = 0$ in the sense of distributions. Since $f \in BC(\mathbb{R})$, it follows that f is identically constant, which is the desired result. □

Remark. If $\omega(x,t)$ is any solution of (1.10) which is integrable with respect to the horizontal variables, we can define

$$\phi(x_3,t) \;=\; \int_{\mathbb{R}^2}\omega_3(x_h,x_3,t)\,\mathrm{d}x_h\;, \qquad x_3\in\mathbb{R}\;,\quad t\ge 0\;.$$

As was observed in [10], this quantity satisfies a remarkably simple equation

$$\partial_t\phi(x_3,t) + x_3\partial_{x_3}\phi(x_3,t) \;=\; \partial_{x_3}^2\phi(x_3,t)\;, \tag{6.2}$$

which can be solved explicitly. However, if $\omega(\cdot,t)\in X(m)^3$ for some $m>1$ and if $\nabla\cdot\omega(\cdot,t)=0$, Lemma 1.1 shows that $\phi(x_3,t)$ does not depend on x_3, and (6.2) then implies that $\phi(x_3,t)$ is also independent of t. Thus, as was already mentioned, we can restrict ourselves to the particular case where $\phi\equiv 0$ without loss of generality. Being unaware of this simple observation, the authors of [10] have stated their stability result in a seemingly more general form, allowing (apparently) for nontrivial functions $\phi(x_3,t)$, but thanks to Lemma 1.1 (which also holds in the slightly different functional setting of [10]) the simpler presentation adopted here in Theorem 1.2 is exactly as general.

6.2 Proof of Proposition 3.4.

This final section is devoted to the proof of Proposition 3.4, which shows that eigenfunctions of $\mathscr{L}_{\alpha,h}$ corresponding to eigenvalues outside the essential spectrum have a Gaussian decay at infinity. For the nonlocal operator $\mathscr{L}_{\alpha,3}$, the same result was established in [9, Lemma 4.5] using ODE techniques, but we prefer using here a more flexible method based on weighted L^2 estimates. In fact, we shall consider a more general elliptic problem of the form

$$-\mathcal{L}f + F(x,f,\nabla f) + \lambda f \;=\; h\;, \qquad x\in\mathbb{R}^n\;, \tag{6.3}$$

where the unknown is the vector-valued function $f=(f_1,\dots,f_N)^\top$. Here and below we denote by $\mathcal{L} = \Delta + \frac{x}{2}\cdot\nabla + \frac{n}{2}$ the analog of operator (2.2) in dimension n. The data of the problem are the functions $F:\mathbb{R}^n\times\mathbb{C}^N\times\mathbb{C}^{nN}\to\mathbb{C}^N$ and $h:\mathbb{R}^n\to\mathbb{C}^N$, and the complex number λ.

For $m\in[0,\infty]$, we denote by $L^2(m)$, $H^1(m)$ the following complex Hilbert spaces on $\mathbb{R}^n$:

$$\begin{aligned} L^2(m) \;&=\; \Big\{f\in L^2(\mathbb{R}^n,\mathbb{C})\;\Big|\;\int_{\mathbb{R}^n}|f(x)|^2\rho_m(|x|^2)\,\mathrm{d}x<\infty\Big\}\;,\\ H^1(m) \;&=\; \Big\{f\in L^2(m)\;\Big|\;\partial_{x_j}f\in L^2(m)\quad\text{for } j=1,\dots,n\Big\}\;, \end{aligned}$$

where ρ_m is the weight function defined by (1.12). Our main result is:

Proposition 6.1 *Let $m\in[0,\infty)$, $\lambda\in\mathbb{C}$, $h\in L^2(\infty)^N$, and assume that F is a continuous function satisfying*

$$|F(x,p,Q)| \;\le\; A(x)|p| + B(x)|Q|\;, \qquad \textit{for all } (x,p,Q)\in\mathbb{R}^n\times\mathbb{C}^N\times\mathbb{C}^{nN}\;, \tag{6.4}$$

where A and B are bounded, nonnegative functions such that

$$\lim_{R\to\infty} \sup_{|x|\geq R} A(x) \;=\; \lim_{R\to\infty} \sup_{|x|\geq R} B(x) \;=\; 0\;. \tag{6.5}$$

If $\mathrm{Re}\,\lambda > \frac{n}{4} - \frac{m}{2}$, *then any solution* $f \in H^1(m)^N$ *of* (6.3) *satisfies* $f \in H^1(\infty)^N$.

Proof. The proof is a simple modification of [15, Proposition 12], which in turn is inspired by a recent work of Fukuizumi and Ozawa [5] where decay estimates are obtained for solutions of the Haraux-Weissler equation. For $k \geq 1$, $\epsilon > 0$, and $\theta \in [0, m]$, we define the weight functions

$$\xi_{k,\epsilon}(x) \;=\; e^{\frac{(1-\epsilon)k|x|^2}{4k+|x|^2}}\;, \qquad \zeta_\theta(x) \;=\; (1+|x|^2)^\theta\;, \qquad x \in \mathbb{R}^n\;. \tag{6.6}$$

Multiplying both sides of (6.3) by $\zeta_\theta \xi_{k,\epsilon} \bar{f}$ and integrating by parts the real part of the resulting expression, we obtain the identity

$$\begin{aligned} &\int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |\nabla f|^2 \,\mathrm{d}x + \mathrm{Re} \int_{\mathbb{R}^n} \bar{f} \cdot (\nabla(\zeta_\theta \xi_{k,\epsilon}), \nabla) f \,\mathrm{d}x + \int_{\mathbb{R}^n} |f|^2 \frac{x}{4} \cdot \nabla(\zeta_\theta \xi_{k,\epsilon}) \,\mathrm{d}x \\ &\qquad = -\,\mathrm{Re} \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} \bar{f} \cdot F(x, f(x), \nabla f(x)) \,\mathrm{d}x + \Big(\frac{n}{4} - \mathrm{Re}\,\lambda\Big) \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |f|^2 \,\mathrm{d}x \\ &\qquad\quad + \mathrm{Re} \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} \bar{f} \cdot h \,\mathrm{d}x\;. \end{aligned} \tag{6.7}$$

Clearly,

$$\nabla \xi_{k,\epsilon}(x) \;=\; \frac{8(1-\epsilon)k^2 x}{(4k+|x|^2)^2}\, \xi_{k,\epsilon}(x)\;, \qquad \nabla \zeta_\theta(x) \;=\; \frac{2\theta x}{1+|x|^2}\, \zeta_\theta(x)\;. \tag{6.8}$$

Thus, the second term in the left-hand side of (6.7) can be written in the following way:

$$\begin{aligned} &\mathrm{Re} \int_{\mathbb{R}^n} \bar{f} \cdot (\xi_{k,\epsilon} \nabla \zeta_\theta, \nabla) f \,\mathrm{d}x + \mathrm{Re} \int_{\mathbb{R}^n} \bar{f} \cdot (\zeta_\theta \nabla \xi_{k,\epsilon}, \nabla) f \,\mathrm{d}x \\ &\quad = -\int_{\mathbb{R}^n} |f|^2\, \nabla \cdot \Big(\frac{\theta x \zeta_\theta \xi_{k,\epsilon}}{1+|x|^2}\Big) \,\mathrm{d}x + \mathrm{Re} \int_{\mathbb{R}^n} \bar{f} \cdot (\zeta_\theta \nabla \xi_{k,\epsilon}, \nabla) f \,\mathrm{d}x \\ &\quad = -\int_{\mathbb{R}^n} |f|^2\, \xi_{k,\epsilon}\, x \cdot \nabla \frac{\theta \zeta_\theta}{1+|x|^2} \,\mathrm{d}x - \int_{\mathbb{R}^n} |f|^2 \frac{\theta \zeta_\theta}{1+|x|^2}\, x \cdot \nabla \xi_{k,\epsilon} \,\mathrm{d}x \\ &\qquad - n\theta \int_{\mathbb{R}^n} \frac{\zeta_\theta \xi_{k,\epsilon}}{1+|x|^2} |f|^2 \,\mathrm{d}x + \mathrm{Re} \int_{\mathbb{R}^n} \frac{8(1-\epsilon)k^2 \zeta_\theta \xi_{k,\epsilon}}{(4k+|x|^2)^2}\, \bar{f} \cdot (x, \nabla) f \,\mathrm{d}x\;. \end{aligned}$$

To bound this quantity from below, we observe that

$$\int_{\mathbb{R}^n} |f|^2\, \xi_{k,\epsilon} x \cdot \nabla \frac{\theta \zeta_\theta}{1+|x|^2} \,\mathrm{d}x \leq 2\theta^2 \int_{\mathbb{R}^n} \frac{\zeta_\theta \xi_{k,\epsilon}}{1+|x|^2} |f|^2 \,\mathrm{d}x\;.$$

Moreover, for each $\eta_1 > 0$,

$$\begin{aligned} -\,\mathrm{Re} \int_{\mathbb{R}^n} \frac{8(1-\epsilon)k^2 \zeta_\theta \xi_{k,\epsilon}}{(4k+|x|^2)^2}\, \bar{f} \cdot (x, \nabla) f \,\mathrm{d}x \;&\leq\; \int_{\mathbb{R}^n} \frac{2(1-\epsilon)k \zeta_\theta \xi_{k,\epsilon}}{4k+|x|^2} |xf||\nabla f| \,\mathrm{d}x \\ &\leq\; (1-\eta_1) \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |\nabla f|^2 \,\mathrm{d}x + \frac{(1-\epsilon)^2}{1-\eta_1} \int_{\mathbb{R}^n} \frac{k^2 \zeta_\theta \xi_{k,\epsilon} |xf|^2}{(4k+|x|^2)^2} \,\mathrm{d}x\;. \end{aligned}$$

Thus, using the expression (6.8) of $\nabla \xi_{k,\epsilon}$, we find

$$\begin{aligned} \mathrm{Re} \int_{\mathbb{R}^n} \bar{f} \cdot (\nabla(\zeta_\theta \xi_{k,\epsilon}), \nabla) f \,\mathrm{d}x \;&\geq\; -C \int_{\mathbb{R}^n} \frac{\zeta_\theta \xi_{k,\epsilon}}{1+|x|^2} |f|^2 \,\mathrm{d}x - \int_{\mathbb{R}^n} \frac{8(1-\epsilon)\theta k^2 \zeta_\theta \xi_{k,\epsilon} |xf|^2}{(4k+|x|^2)^2 (1+|x|^2)} \,\mathrm{d}x \\ &\quad -\; (1-\eta_1) \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |\nabla f|^2 \mathrm{d}x - \frac{(1-\epsilon)^2}{1-\eta_1} \int_{\mathbb{R}^n} \frac{k^2 \zeta_\theta \xi_{k,\epsilon} |xf|^2}{(4k+|x|^2)^2} \,\mathrm{d}x\;, \end{aligned} \tag{6.9}$$

where $C = n\theta + \theta^2$ does not depend on k and ϵ. We next consider the third term in the left-hand side of (6.7), which satisfies

$$\int_{\mathbb{R}^n} |f|^2 \frac{x}{4} \cdot \nabla(\zeta_\theta \xi_{k,\epsilon}) \,\mathrm{d}x \;=\; \frac{\theta}{2} \int_{\mathbb{R}^n} \frac{\zeta_\theta \xi_{k,\epsilon}}{1+|x|^2} \,|xf|^2 \,\mathrm{d}x + 2(1-\epsilon) \int_{\mathbb{R}^n} \frac{k^2 \zeta_\theta \xi_{k,\epsilon} |xf|^2}{(4k+|x|^2)^2} \,\mathrm{d}x \;. \tag{6.10}$$

To estimate the right-hand side of (6.7), we use (6.4) and obtain, for each $\eta_2 > 0$,

$$\begin{aligned} -\operatorname{Re} \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} \bar{f} \cdot F(x, f(x), \nabla f(x)) \,\mathrm{d}x \;&\le\; \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} A |f|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} B |f| |\nabla f| \,\mathrm{d}x \\ &\le\; \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} \Big(A + \frac{B^2}{4\eta_2} \Big) |f|^2 \,\mathrm{d}x + \eta_2 \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |\nabla f|^2 \,\mathrm{d}x \;. \end{aligned} \tag{6.11}$$

Finally, for each $\eta_3 > 0$, we have

$$\operatorname{Re} \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} \bar{f} \cdot h \,\mathrm{d}x \;\le\; \eta_3 \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |f|^2 \,\mathrm{d}x + \frac{1}{4\eta_3} \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |h|^2 \,\mathrm{d}x \;. \tag{6.12}$$

Substituting (6.9)–(6.12) into (6.7), we arrive at our basic inequality:

$$\begin{aligned} (\eta_1 - \eta_2) \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |\nabla f|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} \frac{(1-\epsilon) k^2 \zeta_\theta \xi_{k,\epsilon} |xf|^2}{(4k+|x|^2)^2} \Big(\frac{1 - 2\eta_1 + \epsilon}{1 - \eta_1} - \frac{8\theta}{1+|x|^2} \Big) \,\mathrm{d}x \\ \le\; \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} \Big\{ \Big(\frac{C}{1+|x|^2} + \frac{n}{4} - \operatorname{Re}\lambda + A + \frac{B^2}{4\eta_2} + \eta_3 - \frac{\theta}{2} \Big) |f|^2 + \frac{1}{4\eta_3} |h|^2 \Big\} \,\mathrm{d}x \;. \end{aligned} \tag{6.13}$$

To exploit (6.13), we first take $\eta_1 = \eta_2 = \frac{1}{2}$ and $\theta = m$. Using (6.5) and the assumption that $\operatorname{Re}\lambda > \frac{n}{4} - \frac{m}{2}$, we see that there exists $R > 0$ independent of $k \ge 1$ such that, if $\eta_3 > 0$ is sufficiently small, the following inequality holds:

$$\epsilon(1-\epsilon) \int_{\mathbb{R}^n} \frac{k^2 \zeta_\theta \xi_{k,\epsilon} |xf|^2}{(4k+|x|^2)^2} \,\mathrm{d}x \le C \int_{|x| \le R} \zeta_\theta \xi_{k,\epsilon} |f|^2 \,\mathrm{d}x + \frac{1}{4\eta_3} \int_{\mathbb{R}^n} \zeta_\theta \xi_{k,\epsilon} |h|^2 \,\mathrm{d}x \;,$$

where the constant $C > 0$ is independent of $k \ge 1$. Thus, taking the limit $k \to \infty$ and using Fatou's lemma, we obtain

$$\frac{\epsilon(1-\epsilon)}{16} \int_{\mathbb{R}^n} (1+|x|^2)^m e^{\frac{1-\epsilon}{4}|x|^2} |xf|^2 \,\mathrm{d}x \;\le\; C(R) \int_{|x| \le R} |f|^2 \,\mathrm{d}x + \frac{1}{4\eta_3} \int_{\mathbb{R}^n} (1+|x|^2)^m e^{\frac{1-\epsilon}{4}|x|^2} |h|^2 \,\mathrm{d}x \;,$$

which shows that $e^{\frac{1-\epsilon}{8}|x|^2} f \in L^2(\mathbb{R}^2)$ for any $\epsilon > 0$. Next we choose $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{8}$, $\eta_3 = 1$, and $\theta = 0$ in (6.13). Taking again the limit $k \to \infty$ and using Lebesgue's dominated convergence theorem, we find

$$\frac{1}{8} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |\nabla f|^2 \,\mathrm{d}x + \frac{1-\epsilon}{24} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |xf|^2 \,\mathrm{d}x \le C \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |f|^2 \,\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |h|^2 \,\mathrm{d}x \;,$$

where the constant $C > 0$ does not depend on $\epsilon > 0$. This inequality shows that

$$\frac{1}{8} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |\nabla f|^2 \,\mathrm{d}x + \frac{1-\epsilon}{48} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |xf|^2 \,\mathrm{d}x \;\le\; C \int_{|x| \le R'} e^{\frac{1-\epsilon}{4}|x|^2} |f|^2 \,\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |h|^2 \,\mathrm{d}x \,,$$

for some $R' > 0$ independent of $\epsilon > 0$. Taking now the limit $\epsilon \to 0$, we conclude that $f \in H^1(\infty)$, which is the desired result. $\square$

Proof of Proposition 3.4. We consider the eigenvalue equation (3.16), which can be written in the form

$$-\mathcal{L}_h\omega_h + \alpha\Lambda_1\omega_h - \alpha\tilde{\Lambda}_2\omega_h + \Big(\lambda + \frac{3}{2}\Big)\omega_h \;=\; 0\;, \tag{6.14}$$

where $\mathcal{L}_h$ is given by (2.2) and the operators Λ_1, $\tilde{\Lambda}_2$ are defined at the beginning of Section 3. We recall that $|\Lambda_1\omega_h| \le |U_h^G||\nabla_h\omega_h|$ and $|\tilde{\Lambda}_2\omega_h| \le |\nabla_h U_h^G||\omega_h|$, where the velocity profile U_h^G satisfies (3.8). Assume that $\mathrm{Re}\,\lambda > -\frac{m}{2} - 1$ and let $\omega_h \in H^1(m)^2$ be a solution to (6.14). Applying Proposition 6.1 with $n = N = 2$, $F(x,f,\nabla f) = \alpha\Lambda_1 f - \alpha\tilde{\Lambda}_2 f$, and $h = 0$, we obtain $\omega_h \in H^1(\infty)^2$. This completes the proof of Proposition 3.4. □

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