Statistics of quantum resonances and fluctuations in chaotic scattering

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Outline

Two complementary viewpoints:

from 'inside'	from 'outside'
$\text{local Green's function} \sim \frac{\text{field}}{\text{current}}$	$S\ \text{matrix} \sim \frac{\text{outgoing wave}}{\text{incoming wave}}$
eigenmodes & eigenfunctions	reflection & scattering phase

Unified description: scattering theory + non-Hermitian RMT

Main object: resonances $=$ poles of S-matrix

- Universalities in open chaotic systems
- Mean resonance density, decay law & width fluctuations
- Spectral correlations
- Quasi-resonances

Application: uniform vs non-uniform absorption

Open wave-chaotic systems

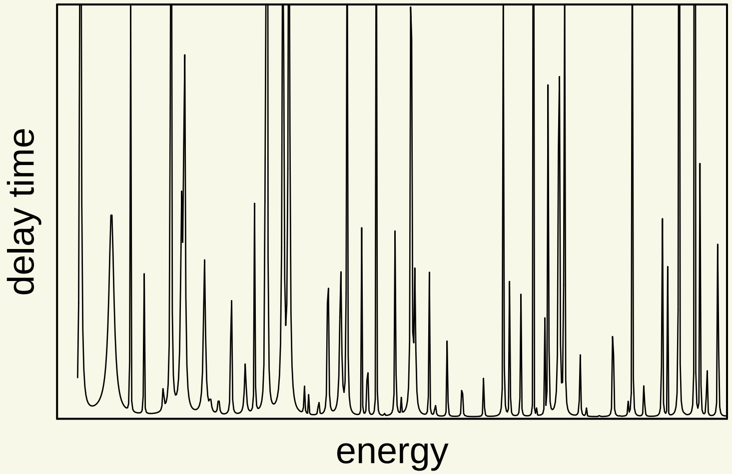
- microwave cavities / billiards

(*non-integrable shape*)

- ultrasonics on elastodynamic billiards
- light propagation in random media

(*disorder / impurities*)

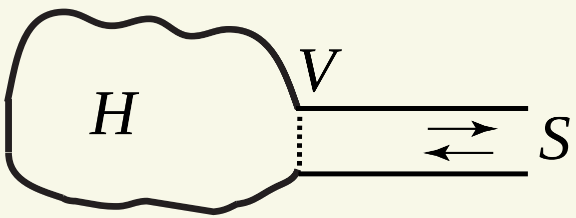
- mesoscopic quantum dots
- compound nuclei (*interactions*)



Fluctuations in scattering observables reflect statistics of resonance states.

Aim is to study their statistical properties via distribution / correlation functions.

Resonance scattering



open system $\leftrightsquigarrow$ resonances
poles of the scattering matrix

Scattering matrix $= \frac{\text{outgoing amplitude}}{\text{incoming amplitude}}$: $(\dim S = M : \#\text{channels})$

$$S_{\text{res}}(E) = 1 - iV^\dagger \frac{1}{E - \mathcal{H}_{\text{eff}}} V \,,$$

with coupling amplitudes V_n^c

Separation of energy scales: potential vs resonance scattering

Effective non-Hermitian Hamiltonian: $(\dim \mathcal{H}_{\text{eff}} = N : \#\text{resonances})$

$\mathcal{H}_{\text{eff}} = H - \frac{i}{2} V V^\dagger$, with $H^\dagger = H$ $\rightsquigarrow$ complex eigenvalues $E_n - \frac{i}{2}\Gamma_n$

Mahaux, Weidenmüller (1969); Livšic (1973)

Flux conservation (at zero absorption) $=$ S matrix is unitary (at real E):

$S_{\text{res}}(E) = \frac{1 - iK(E)}{1 + iK(E)}$, with $K(E) = \frac{1}{2} V^\dagger \frac{1}{E-H} V$ – reaction matrix

Closed chaotic cavities

Statistical approach: replace H with a random operator

Wigner, Dyson (∼'60); Bohigas, Giannoni, Schmidt (1984)

H taken from appropriate ensemble of random matrices ⇝ **RMT**
$+$ symmetry constraints on H (e.g. $H^T = H$ for time-reversal systems)

$H^\dagger = H = H^T$	$H^\dagger = H$	$H^\dagger = H = H^R$
(GOE, β=1)	(GUE, β=2)	(GSE, β=4)

Universality of spectral correlations:

In the RMT limit $N \to \infty$, local fluctuations at the scale of mean level spacing Δ are universal and described by those in Gaussian ensembles:

$$\langle(\cdots)\rangle = \mathrm{const} \int dH\,(\cdots)\,\exp\{-\frac{N\beta}{4}\mathrm{Tr}H^2\}\,, \quad dH = \prod dH_{nm}$$

Examples: mean density (global, non-universal) and 2-point correlator (local, universal)

$$\langle\rho(E)\rangle = \langle\textstyle\sum_n \delta(E-E_n)\rangle = -\frac{1}{\pi}\mathrm{Im\,Tr}\langle\frac{1}{E-H}\rangle = (N/\pi)\sqrt{1-(E/2)^2}$$

$1-\Delta^2\langle\rho(E_1)\rho(E_2)\rangle = Y_{2\beta}(\omega)$ with $\omega = (E_2-E_1)/\Delta$ ⇝ enough considering $E = \frac{E_1+E_2}{2} = 0$

Open chaotic cavities

$\mathcal{H}_{\text{eff}} = H - \frac{i}{2}VV^\dagger$ requires statistical assumptions on coupling amplitudes

Fixed ('f-case')

with 'orthogonality' condition

$\sum_{n=1}^{N} V_n^a V_n^b = 2\gamma_a \delta^{ab}$

Verbaarschot, Weidenmüller, Zirnbauer (1984)

Random ('r-case')

gaussian, uncorrelated

$\langle V_n^a V_m^b \rangle = 2(\gamma_a/N)\delta^{ab}\delta_{nm}$

Sokolov, Zelevinsky (1988)

Direct reaction absent: $\langle S_{ab}(E)\rangle = \delta^{ab}\frac{1-\gamma_a g(E)}{1+\gamma_a g(E)}$, $\quad a = 1, \ldots, M$

Global E-dependence of $g(E)$ not essential for local fluctuations at $E = 0$
Dependence of scattering observables via transmission coefficients:

$$T_a = 1 - |\langle S_{aa}\rangle|^2 = \frac{4\gamma_{\text{eff}}}{(1+\gamma_{\text{eff}})^2} \text{ with } \gamma_{\text{eff}} = \gamma_a g(0)$$

Universality (model-independence): Lehmann, Saher, Sokolov, Sommers (1995)

▷ 'quantum' case of finite M $\quad (\gamma_{\text{eff}} = \gamma_a)$

▷ 'semiclassical' case of $M, N \to \infty$ with fixed $m = M/N \ll 1$ $\quad (\gamma_{\text{eff}} \approx \gamma_a)$

Qualitatively similar results for moderate $m < 1$

Isolated resonances

Porter-Thomas distribution appears at both $\gamma \ll 1$ and $\gamma \gg 1$ limits

Case $\gamma \ll 1$: $\mathcal{H}_{\rm eff} = \varepsilon_n \delta_{nm} - \frac{i}{2}(VV^\dagger)_{nm}$ and treat $VV^\dagger$ as a perturbation

$\hookrightarrow E_n \approx \varepsilon_n$ (GβE) and $\Gamma_n \approx (VV^\dagger)_{nn} = \sum_i^{M\beta} v_i^2$

Distribution of widths $\mathcal{P}(\Gamma)$ is a $\chi^2_{M\beta}$ distribution

$$\mathcal{P}(\Gamma) \propto \left(\frac{\Gamma}{\langle\Gamma\rangle}\right)^{M\beta/2-1} \exp(-\frac{M\beta}{2}\frac{\Gamma}{\langle\Gamma\rangle}) \text{ with } \langle\Gamma\rangle = 2\gamma M/N$$

$\hookrightarrow$ noting $4\gamma \approx T$ gives Weisskopf width $\Gamma_W = MT\Delta/2\pi$

Case $\gamma \gg 1$: ‘doorway’ representation in the eigenbasis of $VV^\dagger$

Dynamical reorganization of resonance states: Sokolov, Zelevinsky (1989)

- ▷ M collective states $\Gamma_{coll} \sim (1 - \frac{1}{\gamma_2})2\gamma \gg \Delta$
- ▷ $N - M$ trapped states $\Gamma_n \sim \frac{1}{\gamma_2} 2\gamma \frac{M}{N-M} \approx (2/\gamma)M/N \ll \Delta$

‘Overlapping’ is weaker than ‘interference’!
Example: Absorption limit $T \to 0$ and $M \to \infty$ with fixed $MT = 2\pi\Gamma_{\rm abs}/\Delta$

Mean resonance density

Idea: electrostatic analogy — Sommers, Crisanty, Somplinsky, Stein (1988)

$\hookrightarrow$ average Green's function as a 2D field — Sokolov, Zelevinsky (1988)

$$g(z) = \frac{1}{N}\langle \mathrm{Tr}\frac{1}{z-\mathcal{H}_{\mathrm{eff}}}\rangle = \Re\, g(x,y) + i\Im\, g(x,y)$$

- Maxwell eqs = Cauchy-Riemann for $\rho(x,y) \equiv 0$
- 'charge' density: $\rho(E,\Gamma) = -\frac{1}{4\pi}(\partial_x^2 + \partial_y^2)\Phi(x,y)|_{x=E,y=-\Gamma/2}$

'Electrostatic' potential $\Phi(x,y) = \langle \ln \mathrm{Det}[(z-\mathcal{H}_{\mathrm{eff}})^{\dagger}(z-\mathcal{H}_{\mathrm{eff}}) + \delta^2]\rangle$

$\hookrightarrow$ relation to a 2-point correlator problem

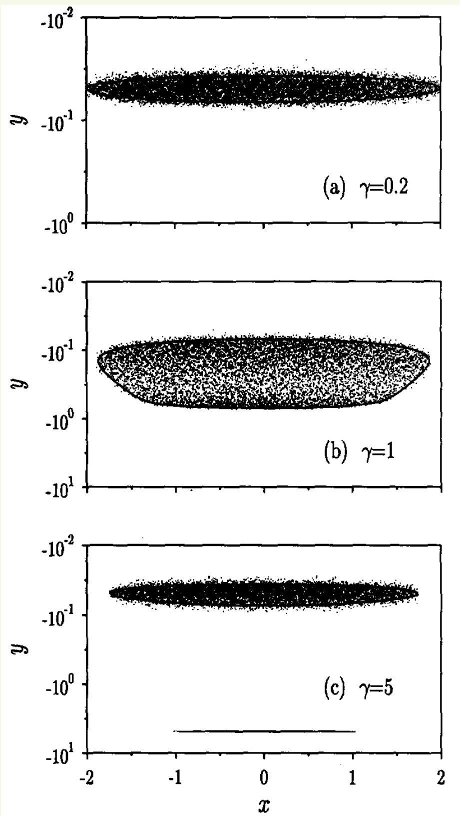
<u>perturbative</u>	<u>non-perturbative</u>
'strong' non-Hermiticity	'weak' non-Hermiticity
mean-field approach $\langle \ln(\ldots)\rangle = \ln\langle(\ldots)\rangle$	SUSY calculation $\mathcal{Z} = \langle \frac{\det[...]}{\det[...]}\rangle$
no 'soft' mode	saddle-point manifold appears

Strongly overlapping resonances

Formation of the gap Γ_g in the spectrum

Haake et al. (1992)

Nonzero density $\rho(x,y)=\rho_{\mathrm{r,f}}(y)$ (universal at $m\ll 1$):



Lehmann, Saher, Sokolov, Sommers (1995)

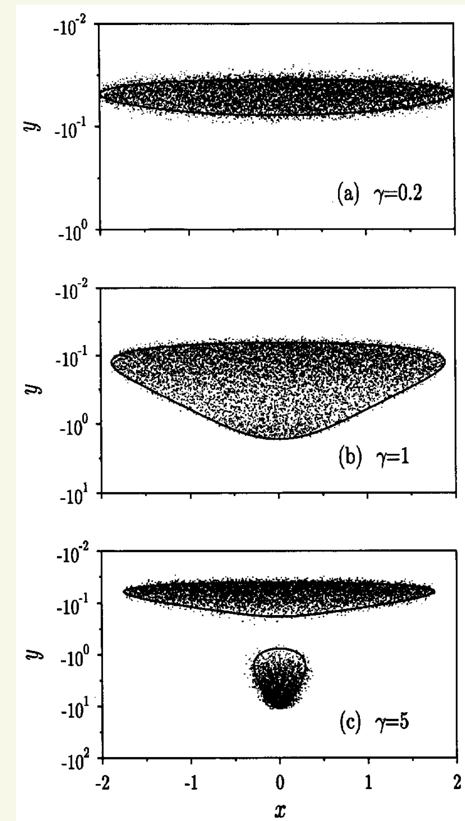
Redistribution of states at $\gamma\sim 1$

$$\gamma_{cr\,1}=1-\tfrac{1}{2}m^{1/3},\qquad m\ll 1$$

$$\gamma_{cr\,2}=1+\tfrac{3}{2}m^{1/3},\qquad m\ll 1$$

density inside upper cloud

$$\rho(y)=\frac{1}{4\pi}\frac{m}{y^2}$$



- $\Gamma_g=\Gamma_{\mathrm{corr}}$ correlation length of fluctuations in scattering $(\neq\Gamma_W$!)
  - ▷ S-matrix correlator $=\left|\frac{i\Gamma(\epsilon)}{\epsilon+i\Gamma(\epsilon)}\frac{T(\epsilon)}{T(\gamma_{\mathrm{eff}})}\right|^2=\frac{\Gamma^2_{\mathrm{corr}}}{\epsilon^2+\Gamma_{\mathrm{corr}}}$ at $\epsilon\ll 1$
  - ▷ time-delay correlator $=\frac{\Gamma^2_{\mathrm{corr}}-\epsilon^2}{(\epsilon^2+\Gamma^2_{\mathrm{corr}})^2}$ Lehmann, Savin, Sokolov, Sommers (1995)

Width distribution

Exact GUE result valid at any $T_a, a = 1, \ldots, M$ Fyodorov, Sommers (1997)

Equivalent channels, $g = 2/T - 1 \geq 1$:

$$\mathcal{P}(y) = \frac{(-1)^M}{\Gamma(M)} y^{M-1} \frac{d^M}{dy^M} \left(e^{-gy} \frac{\sinh y}{y} \right), \quad y = \pi\Gamma/\Delta$$

Limiting cases of isolated and many strongly overlapping resonances:

- $T \ll 1$: then $y \sim T \ll 1$ so $\frac{\sinh y}{y} \approx 1 \leftrightsquigarrow \chi^2_{2M}$ (Porter-Thomas)
- $M \gg 1$: $\mathcal{P}(\Gamma) = M/(2y^2)$ only for $\frac{1}{2}MT < y < \frac{MT}{2(1-T)}$

  cloud $\nearrow$ with upper bound $\to \infty$ at $T = 1$
- Moldauer-Simonius relation as a consequence of y^{-2} tail

  $\langle \Gamma \rangle = -\frac{\Delta}{2\pi} \sum_a \ln(1 - T_a)$

GOE result is also known Sommers, Fyodorov, Titov (1999)

Decay law

... is directly related to fluctuations of the widths!

Gap in spectrum shows up as classical (exponential) decay
When (and how) does quantum (power law) decay appear?

The 'norm-leakage' decay function: Savin, Sokolov (1997)

$$P(t) = \overline{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{N}\langle \mathrm{Tr} e^{i\mathcal{H}_{\text{eff}}^\dagger t} e^{-i\mathcal{H}_{\text{eff}} t}\rangle$$

$P_{\text{closed}}(t) \equiv 1$ $\rightsquigarrow$ time-dependence is due to the openness only

Consider the eigenbasis of $\mathcal{H}_{\text{eff}}$

$\mathcal{H}_{\text{eff}}|n\rangle = \mathcal{E}_n|n\rangle$ and $\langle\tilde{n}|\mathcal{H}_{\text{eff}} = \mathcal{E}_n\langle\tilde{n}|$

$\langle\tilde{n}|m\rangle = \delta_{nm}$ but $\langle\tilde{n}| \neq |n\rangle^\dagger$ (bi-orthogonal)

$\hookrightarrow U_{nm} = \langle n|m\rangle$ non-orthogonality matrix Bell, Steinberger (1959)

Express $P(t)$ in terms of resonances:

$$P(t) = \frac{1}{N}\langle\sum U_{nn}^2 e^{-\Gamma_n t}\rangle + \frac{1}{N}\langle{\sum}' U_{nm}^2 e^{i(E_n-E_m)t} e^{-(\Gamma_n+\Gamma_m)t/2}\rangle$$

Hierarchy of time scales

Qualitative: diagonal approximation

$$P_d(t) = \frac{1}{N}\langle\sum e^{-\Gamma_n t}\rangle = \int_0^\infty d\Gamma e^{-\Gamma t}\mathcal{P}(\Gamma)$$ (exact at $t \to \infty$)

$$= \frac{1}{T}\int_0^{T/(1-T)} \frac{d\xi}{(1+\xi)^2} \exp[-M\ln(1+\frac{1+\xi}{M}\Gamma_W t)]$$

$\mathcal{P}(\Gamma)$

formation of the gap

Semiclassical regime of $M \gg 1$ strongly overlapping resonances:

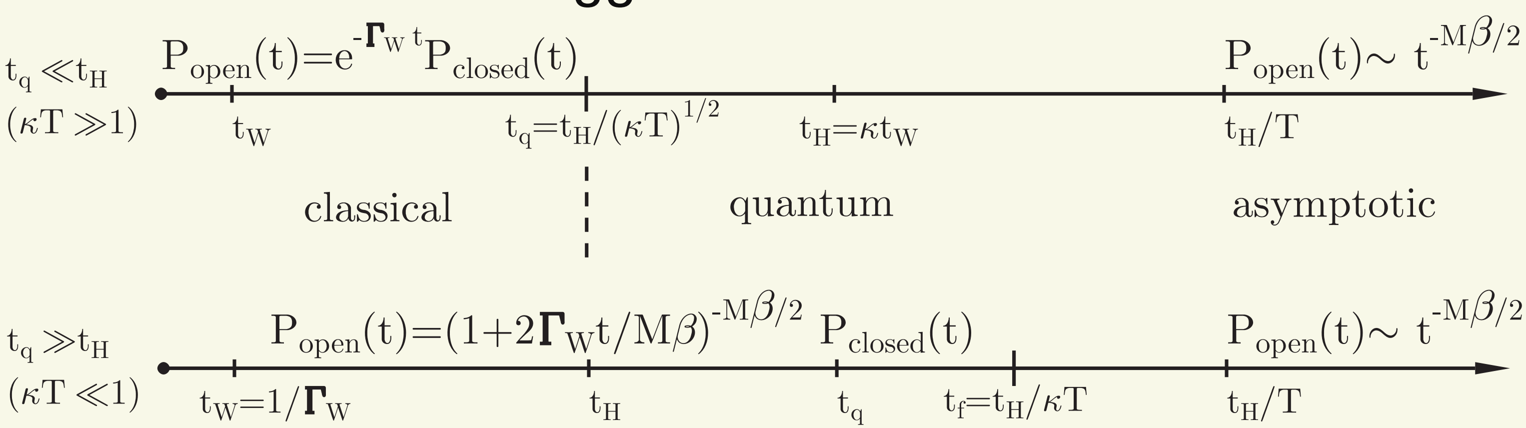
$\kappa = MT \gg 1$

$P(t)$

exponential decay

Sub-gap resonances slow down decay at $t_q = \sqrt{M}t_{cl} = \sqrt{\frac{\kappa}{T}}t_{cl} = \frac{t_H}{\sqrt{\kappa T}}$

Exact: SUSY calculation suggests



Spectral correlations

Consider $\{\varepsilon_n\}$ (GβE) and $\{\gamma_n\}$ (Porter-Thomas). Then N complex eigenvalues depend on $(M-1)(N-\frac{M}{2})$ extra parameters (angles)

$M=1$ case is special: Sokolov, Zelevinsky (1989)

$P(\{E_n\},\{\Gamma_n\}) = J(\ldots)p(\{\varepsilon_n\},\{\gamma_n\})$ Stöckmann, Šeba (1998)

$$\propto \prod_{m<n} \frac{(E_m-E_n)^2+\frac{1}{4}(\Gamma_n-\Gamma_m)^2}{\sqrt{(E_m+E_n)^2+\frac{1}{4}(\Gamma_n+\Gamma_m)^2}} \prod_m \frac{1}{\sqrt{\Gamma_m}} e^{-\frac{N}{4}(\sum E_n^2+\frac{1}{2}\sum\Gamma_n\Gamma_m+\frac{1}{\gamma}\sum\Gamma_n)}$$

$M>1$: Arbitrary correlators derived for GUE Fyodorov, Khoruzhenko (1999)

- ▷ Determinantal structure: $R_n(x+\frac{z_1}{N},\ldots,x+\frac{z_n}{N}) = \det[K(z_i,z_k^*)]$
- ▷ Example: mean density $\rho(x,y)=|K(z,z^*)|$

Universal regimes of ‘weak’ and ‘strong’ non-Hermiticity identified

- ▷ $M\gg 1$ and $MT\gg 1$: Ginibre-like statistics

$$K(z_1,z_2)=\rho(z)e^{-(\pi/2)\rho(z)|z_1-z_2|^2} \quad \text{with} \quad \rho(z)=\frac{M}{4\pi(\operatorname{Im} z)^2}$$

Quasi-resonances

- Stroboscopic dynamics: map $\Psi(n+1) = U\Psi(n)$ with unitary U

  Decay via sub-unitary contraction: $\Psi(n+1) = A\Psi(n)$, $A = U\sqrt{1-\tau\tau^\dagger}$
  where $\tau_{nm} = \delta_{nm}\sqrt{T_m}$, $\quad 1 < n < N, 1 < m < M$ $\qquad$ ($M < N$)

- Input-output signals at frequency ω related by

  $S(\omega) = \sqrt{1-\tau^\dagger\tau} - \tau^\dagger \frac{1}{e^{-i\omega}-A} U\tau$, $\qquad$ transmission coefficients $T_m \leq 1$

  Universal statistics of sub-unitary matrices $\qquad$ Fyodorov, Sommers (2000/3)

- Physical realisation: ‘Bloch particle’ in
  a constant force with periodic driving $\qquad$ Glück, Kolovsky, Korsch (1999)

$T = 1$: Truncation of random unitary matrices $\qquad$ Zyczkowski, Sommers (2000)

mean density $p(r) = \frac{2r}{N-M} \frac{(1-x)^{M-1}}{(M-1)} \frac{d^M}{dx^M} \frac{1-x^N}{1-x}$ $\qquad$ $x = r^2 = |z|^2$

▷ $N \to \infty$ and fixed $\frac{M}{N} = m$: $\quad$ gap and Ginibre-like correlations

▷ $N \to \infty$ and fixed M: $\quad$ universal resonance-width statistics

Finite absorption

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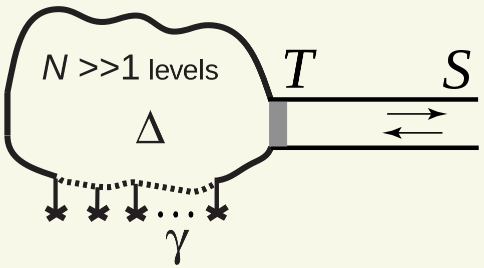
Modelling absorption: dissipation, exponential in time



uniform absorption $=$ imaginary shift $E \to E + \frac{i}{2}\Gamma$

absorption width $\nearrow$

Justified here by E-dependence via Green's function $(E - \mathcal{H}_{\rm eff})^{-1}$only:

$$E - H + \frac{i}{2}(VV^\dagger + \sum_{\rm wall} W^w W^{w\,\dagger}) \to E - (H - \frac{i}{2}VV^\dagger) + \frac{i}{2}\Gamma$$

S matrix with absorption: $S \equiv S(E + \frac{i}{2}\Gamma) = \frac{1 - iK}{1 + iK}$

R matrix ('impedance'): $K = \|V\|^2 \left(\frac{1}{E + \frac{i}{2}\Gamma - H}\right)_{11}$ $\rightsquigarrow$ local Green's function

coupling strength $\nearrow$

- Obvious effect on correlations (acquire additional $e^{-\Gamma t}$ in time domain)
- Nontrivial distributions of $K = u - iv$ and $S = \sqrt{r}e^{i\theta}$ derived at arbitrary absorption and coupling (generally in GOE-GUE crossover)

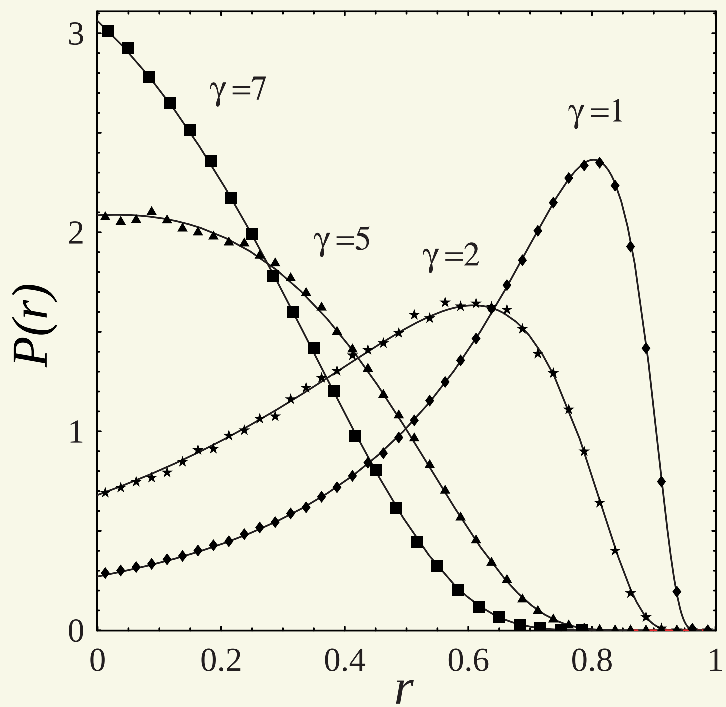
Fyodorov, Savin, Sommers (2005)

Reflection distribution: exact GOE result

Explicit expression for the integrated probability of $x = \frac{1+r}{1-r}$:

$$W(x) = -\frac{x^2-1}{2\pi}\frac{d}{dx}F(x) = \int_x^\infty dx'\, P_0(x')$$

$$= \frac{x+1}{4\pi}\Big[f_1(w)g_2(w) + f_2(w)g_1(w) + h_1(w)j_2(w) + h_2(w)j_1(w)\Big]_{w=\frac{x-1}{2}}$$



$$f_1(w) = \int_w^\infty dt \sqrt{t|t-w|}\,\frac{e^{-\gamma t/2}}{(1+t)^{3/2}}\left[1 - e^{-\gamma} + \frac{1}{t}\right]$$

$$g_1(w) = \int_w^\infty dt\, \frac{1}{\sqrt{t|t-w|}}\,\frac{e^{-\gamma t/2}}{(1+t)^{3/2}}$$

$$h_1(w) = \int_w^\infty dt\, \frac{\sqrt{|t-w|}\,e^{-\gamma t/2}}{\sqrt{t(1+t)}}\left[\gamma+(1-e^{-\gamma})(\gamma t-2)\right]$$

$$j_1(w) = \int_w^\infty dt\, \frac{1}{\sqrt{t|t-w|}}\,\frac{e^{-\gamma t/2}}{\sqrt{1+t}}$$

and $f_2(w) = \int_0^w dt\,(\dots)$ etc.

Perfect agreement with impedance and reflection experiments found

Non-uniform absorption

Experiment: Barthelemy, Legrand, Mortessagne (2005)

- microwave cavity at room temperature in tunneling regimes
- homogenous and inhomogeneous contribution to $\Gamma_{\rm abs} \gg \Gamma_{\rm escape}$
- complexness of modes $q^2 = \frac{\langle {\rm Im}\psi^2\rangle}{\langle {\rm Re}\psi^2\rangle} \sim \Gamma_{\rm inh}^2$

Model: Savin, Legrand, Mortessagne (2006)

- coupling $V = \{A_n^a, B_n^b, C_n^c\}$ to antennas, 'bulk' and 'contour' channels

$$M_b \sim (\tfrac{L}{\lambda})^2 \gg (\tfrac{L}{\lambda}) \sim M_c \quad \rightsquigarrow \quad \mathcal{H}_{\rm eff} = H - \tfrac{i}{2}(AA^\dagger + CC^\dagger) - \tfrac{i}{2}\Gamma_{\rm hom}$$

- limit of weak coupling to antenna

$$S = 1 - iA^\dagger \frac{1}{E+\frac{i}{2}\Gamma_{\rm hom}-\mathcal{H}'_{\rm eff}} A, \quad \mathcal{H}'_{\rm eff} = H - \tfrac{i}{2}CC^\dagger$$

- pole representation $\rightsquigarrow$ complex (biorthogonal) modes $\phi_n^a = A^a|n\rangle$

$$q^2 \propto \frac{1}{M_c}\Gamma_{\rm inh}^2 = {\rm var}(\Gamma_{\rm inh}) \quad (M_c \gg 1)$$

Open questions

- Within RMT:
  - ▷ distribution of transmission amplitudes S_{ab}
  - ▷ 4-point (and higher order) correlation functions (cross-sections)
  - ▷ statistics of bi-orthogonal resonance states
  - ▷ other symmetry classes (internal symmetries of H)
- Beyond RMT:
  - ▷ Disordered systems in d-D
  - ▷ Effects of Anderson localisation and absorption
- Semiclassics: access to the above
  - ▷ resonance density? wave functions? etc...