Character deformations of the zero's of Selberg's zeta fct. for $\Gamma_0(4)$ via transfer operators

(M. Fraczek, D.M., DFG-project*)

- Spectral properties of χ-automorphic Laplace-Beltrami operators

- Selberg's zeta fct. and the transfer operator for $\Gamma_0(4)$ with Selberg's character χ_α

- some numerical results

* "Transfer operator approach to the Phillips-Sarnak Conjecture"

Spectral properties of χ-automorphic Laplacian

Γ cofinite Fuchsian group, $\Gamma \subset PSL(2,\mathbb{R})$)

$gz = \frac{az+b}{cz+d}$, $g \in \Gamma$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z \in \mathbb{H} = \{x+iy : y>0\}$

$ds^2 = \frac{dx^2+dy^2}{y^2}$, $d\mu(z) = \frac{dx\,dy}{y^2}$

$\Delta_{LB} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

F_Γ fundamental domain, $\mu(F_\Gamma) < \infty$

$\chi : \Gamma \to \mathbb{C}$ character (1-dim. unit. repr.)

Hilbert space of χ-automorphic fct's:

$$\mathcal{H}_{\chi,\Gamma} = \{ f : \mathbb{H} \to \mathbb{C} : f(gz) = \chi(g) f(z) \ \forall g \in \Gamma$$
$$\int_F |f(z)|^2 d\mu(z) < \infty \}$$

$\sigma(\Delta_\chi)$ spectrum of Δ on $\mathcal{H}_{\chi,\Gamma}$

$z_1, \dots, z_k$ inequivalent cusps of Γ

$T_i z_i = z_i$, T_i primitive parab. element in Γ

<u>Def</u>: χ singular in cusp $z_i \Leftrightarrow \chi(T_i) = 1$

$\kappa(\chi) := \#\{i : \chi(T_i) = 1\}$ degree of singularity of χ.

χ non-singular in $z_i \Leftrightarrow \chi(T_i) \neq 1$
("cusp z_i is closed")

If $\chi(T_i) \neq 1 \ \forall \ 1 \le i \le k \Rightarrow \Delta_\chi$ has spectrum like cocompact case, only eigenvalues obeying Weyl's law: $N(\lambda, \chi) = \#\{\lambda_i < \lambda\} \underset{\lambda \to \infty}{\sim} \frac{\mu(F)}{4\pi} \lambda$
(Selberg)

If $\kappa(\chi) > 0 \Rightarrow \Delta_\chi$ has continuous spectrum $[\frac{1}{4}, \infty)$ of multiplicity $\kappa(\chi)$
generalized eigenfcts are the analytically continued Eisenstein-Maass series $E_l(\beta, z, \chi)$ for $\beta = \frac{1}{2} + it$, $l = 1, \dots, \kappa(\chi)$

<u>Problem</u>: do there exist eigenvalues and cusp forms for general cofinite Γ?

- yes for congruence subgroups $\Gamma \subseteq SL(2, \mathbb{Z})$ and $\chi \equiv 1$ (Selberg), Weyl-law

Sarnak-Phillips conjecture: (trivial character)
Weyl law valid only for arithmetic groups
(besides certain arithmetic non-gruence subgroups)

singular character:
studied only for special groups like congruence
subgroups and special characters χ:
Kernel of character again congruence subgroups

Sarnak-Phillips: non-singular perturbation
in Teichmüller space

Wolpert: singular perturbation in T-space

Sarnak-Phillips: singular character deformation
for $\Gamma(2)$

Balslev-Venkov: singular and nonsingular
character deformation for several
congruence subgroups

Method: perturbation theory of automorphic
Laplacian
"Fermi's golden rule"

The Selberg zeta fct. via the transfer operator

$$Z_{\Gamma,\chi}(\beta) = \prod_{\gamma} \prod_{k=0}^{\infty} \left(1 - \chi(g_\gamma) e^{-(\beta+k) l(\gamma)}\right), \operatorname{Re}\beta > 1$$

γ primitive closed orbit of geodesic flow on $S\Gamma\backslash\mathbb{H}$

$l(\gamma)$ period, $g_\gamma \in \Gamma$ hyperbolic elem. with $g_\gamma \gamma = \gamma$.

Zero's of $Z_{\Gamma,\chi}(\beta)$:

- trivial zeros (or poles) at $s = 0, -1, -2, \dots$
- non-trivial zero's on $\operatorname{Re}\beta = \frac{1}{2}$ with $\lambda = \beta(1-\beta)$ EV of $-\Delta_\chi$ corr. to cusp form
- non-trivial zero's in $\operatorname{Re}\beta < \frac{1}{2}$ resonances = complex poles of scattering determinant
- finitely many zero's in $(\frac{1}{2}, 1]$ small eigenvalues of $-\Delta_\chi$, residues of poles of Eisenstein series in $(\frac{1}{2}, 1)$
- poles at $\beta = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots$ order $\kappa(\chi)$.

$\Gamma = \Gamma_0(4) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c = 0 \bmod 4 \right\}$

$[SL(2,\mathbb{Z}) : \Gamma_0(4)] = 6$, $\Gamma_0(4) \sim \Gamma(2)$ princ. c. subgr.

generators of $\Gamma_0(4)$: parabolic

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$

relation: $ABS = \mathbb{1}$

inequiv. cusps: $z_1 = \infty$, $z_3 = \frac{1}{2}$, $z_2 = 0$

$Az_1 = z_1$, $Bz_2 = z_2$, $Sz_3 = z_3$

$\Gamma_0(4)$ freely generated by A, B

$\Gamma_0(4) \ni g = A^{n_1} B^{m_1} \cdots A^{n_k} B^{m_k}$, $n_i, m_i \in \mathbb{Z}\ \forall i$

<u>Selberg's character</u> χ_α for $\Gamma_0(4)$, $0 \leq \alpha \leq 1$

$\chi_\alpha(g) := e^{2\pi i \alpha P_A(g)}$

$P_A(g) = \sum_{i=1}^{k} n_i$

$\chi_\alpha(A) = e^{2\pi i \alpha}$, $\chi_\alpha(B) = 1$, $\chi_\alpha(S) = e^{-2\pi i \alpha}$

$\Rightarrow k(\chi_0) = 3 \rightarrow k(\chi_\alpha) = 1$ for $0 < \alpha < 1$

$\alpha = 0 \longrightarrow \alpha > 0$

multipl. of cont. spectrum: $3 \rightarrow 1$

<u>Selberg</u>: • zeros of $Z_{\Gamma_0(4),\chi_\alpha}$ with $\mathrm{Re}\,\beta < \frac{1}{2}$ accumulate at $\mathrm{Re}\,\beta = \frac{1}{2}$ (resonances accumulate at cont. spectrum) for $\alpha \to 0$!
multiplicity changes from 3 to 1: Selberg zero's should appear in pairs

• for $\alpha \to \frac{1}{4} - \varepsilon$ some zero's tend to $-\infty$

on the other hand (Phillips-Sarnak, Balslev-Venkov)

for $\alpha_j = \frac{j}{8}$, $j = 0, 1, \dots, 7$ is χ_{α_j} arithmetic
$\Longrightarrow$ all nontrivial zero's of $Z_{\Gamma_0(4),\chi_{\alpha_j}}$ are on $\mathrm{Re}\,\beta = \frac{1}{2}$, $\mathrm{Re}\,\beta = \frac{1}{4}$ (Riemann) and $\mathrm{Re}\,\beta = 0$.

$$Z_{\Gamma_8}(\beta) = \prod_{i=0}^{7} Z_{(\Gamma_0(4),\chi_{\frac{i}{8}})}(\beta)$$

$\Gamma_8 \subset \Gamma_0(4)$ congruence subgroup of $SL(2,\mathbb{Z})$ normal in $\Gamma_0(4)$ with $[\Gamma_0(4) : \Gamma_8] = 8$.

$$\Gamma_8 = \{ g \in \Gamma_0(4) : 8 \mid P_A(g) \}$$

$\Gamma_0(4)/\Gamma_8 = \{1, A, A^2, \dots, A^7\}$ finite, cyclic group

The transfer operator for $\Gamma_0(4)$ with character χ_α

- $Z_{(\Gamma_0(4),\chi_\alpha)}(\beta) = Z_{(SL(2,\mathbb{Z}),\chi_\alpha^{ind},)}(\beta)$

χ_α^{ind} is repres. of $SL(2,\mathbb{Z})$ induced from character χ_α of $\Gamma_0(4)$

$[SL(2,\mathbb{Z}):\Gamma_0(4)] = 6$

$\chi_\alpha^{ind}: SL(2,\mathbb{Z}) \to \mathrm{end}(\mathbb{C}^6)$

$SL(2,\mathbb{Z}) = \bigcup_{i=1}^{6} \Gamma_0(4) g_i$

$$(U_\alpha(g))_{i,j} = \delta_{\Gamma_0(4)}(g_i g g_j^{-1})\, \chi_\alpha(g_i g g_j^{-1}),\ 1\le i,j\le 6$$

$\alpha = 0$ permutation matrix

$\alpha > 0$ monomial matrix

$$\mathcal{L}_{\beta,\alpha} := \mathcal{L}_{\beta,\chi_\alpha^{ind}} = \begin{pmatrix} 0 & \mathcal{L}_{\beta,\alpha}^{(+)} \\ \mathcal{L}_{\beta,\alpha}^{(-)} & 0 \end{pmatrix}$$

$$(\mathcal{L}_{\beta,\alpha}^{(\varepsilon)} \vec{f})(z) \overset{"}{=} \sum_{k=0}^{\infty} \sum_{m=1}^{8} \frac{U_\alpha(ST^{m\varepsilon})}{8^{2\beta+k}} \cdot$$

$$\cdot \zeta_M\left(U_\alpha(T^{8\varepsilon}), 2\beta+k, \frac{z+m}{8}\right) \frac{\vec{f}^{(k)}(0)}{k!}$$

with $\zeta_M(A,\beta,z) := \sum_{n=0}^{\infty} A^n (z+n)^{-\beta}$, A unitary, matrix valued Lerch zeta fct. ($\mathrm{Re}\,\beta > 1$) has meromorphic contin. to entire β-plane with pole at $\beta = 1$.

$\Rightarrow$ when defined on appropriate B-space of vector valued holom. fcts is $\mathcal{L}^{(\varepsilon)}_{\beta,\alpha}$ meromorphic, nuclear operator in β-plane

$$Z_{(\Gamma_0(4),\chi_\alpha)}(\beta) = \det(1 - \mathcal{L}_{\beta,\alpha})$$

Since $Z_{(\Gamma_0(4),\chi_\alpha)} = Z_{(\Gamma_0(4),\chi_{1-\alpha})}$

restrict to $0 \le \alpha \le \frac{1}{2}$

Numerical calculation of $Z_{\Gamma_0(4),\chi_\alpha}$ for β's with $\mathrm{Im}\,\beta$ small

Approximation of transfer-operator $\mathcal{L}_{\beta,\alpha}$:

Truncate Taylor expansion at N-th term leads to $2\mu N \times 2\mu N$ matrix, whose EV have to be determined: $(\mu = 6)$

$$\left(\left(M^{(\varepsilon)}_{\beta,\alpha}\right)_{s,k}\right)_{i,j} = \frac{(-1)^s}{s!} \prod_{p=0}^{s-1} (2\beta + k + p)\, 4^{-(2\beta+k+s)} \cdot$$

$$\cdot \sum_{m=1}^{4} \left(U_\alpha(ST^{m\varepsilon})\right)_{i,j} \zeta_L\left(P_A(g_i T^{4\varepsilon} g_j^{-1}), 2\beta+k+s, \tfrac{m}{4}\right)$$

$1 \leq s \leq N$, $1 \leq i,j \leq 6 = [SL(2,\mathbb{Z}) : \Gamma_0(4)]$

with $\zeta_L(\lambda, \beta, z) = \sum_{n=0}^{\infty} e^{2\pi i \lambda n} \left(\frac{1}{z+n}\right)^s$

$$\mathcal{L}_{\beta,\alpha} \sim \begin{pmatrix} 0 & M^{(+)}_{\beta,\alpha} \\ M^{(-)}_{\beta,\alpha} & 0 \end{pmatrix}$$

$$Z_{(\Gamma_0(4),\chi_\alpha)}(\beta) \sim \det\left(1 - \begin{pmatrix} 0 & M^{(+)}_{\beta,\alpha} \\ M^{(-)}_{\beta,\alpha} & 0 \end{pmatrix}\right)$$