Asymptotic Results For Hermitian Line Bundles Over Complex Manifolds: The Heat Kernel Approach

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Abstract. This paper is intended to be an introduction to a heat kernel approach to some problems in complex geometry. We recall and generalize many results from a previous paper. We demonstrate their strength by giving very simple proofs of nontrivial results of global analysis on complex manifolds. Some of the new results here will be used in a joint work with A. Abbes in order to give a simple direct proof in the case of varieties over number fields of the arithmetic Hilbert-Samuel theorem due to Gillet and Soulé.

Introduction

This paper is a continuation of [B1] where we computed the zeroth order asymptotic expansion of the heat kernel associated to high tensor powers of a hermitian line bundle over a complex manifold. Our aim is to indicate an entirely new method of constructing holomorphic (or more generally harmonic) sections of vector bundles over complex manifolds by using heat kernel estimates. This will range from bounds on cohomology groups (as it is known, these can imply lower bounds on the dimension of the space of holomorphic sections under suitable hypothesis), vanishing theorems, to an explicit construction of sections of some vector bundle twisted by high powers of a positive line bundle such that their norm converges to a Dirac mass at some point on the manifold. This construction produces holomorphic sections satisfying arbitrary conditions at some point. As an application, we will derive very simple proofs of well known theorems such as Kodaira vanishing or Kodaira embedding. Our hope is to convince complex geometers that these techniques lead to a partial alternative to Hörmander's L^2 estimates. We also include some generalizations of previous results in [B1], with applications to arithmetic geometry in mind (c.f. [A-B]).

Now, we introduce our notations: X is a compact complex analytic manifold of dimension n, endowed with a hermitian metric ω and associated volume element

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$dV = \omega^n/n!$. E (*resp.* L) is a hermitian holomorphic vector bundle of rank r (*resp.* 1). For a global $\mathcal{C}^\infty$ $(0,q)$-form σ of $E(k) = E \otimes L^{\otimes k}$, $|\sigma(x)|^2$ denotes the pointwise length induced by the given metrics and $\|\sigma\|_2 = \left(\int_X |\sigma|^2 dV\right)^{1/2}$ (*resp.* $\|\sigma\|_\infty = \sup_X |\sigma|$) the L^2 norm (*resp.* the sup norm). We will be primarily concerned with the spectral distribution of the operator $\overline{\square}^q_k = (\bar\partial^* + \bar\partial)^2$ which is the $\bar\partial$-laplacian (with respect to the given metrics) acting on $(0,q)$-forms with values in $E(k)$ (the exponent q will be omitted when we deal with sections (i.e. $q=0$)). As $\overline{\square}^q_k$ is elliptic and X is compact it has a discrete spectrum which can be recovered from the *heat kernel* $K^q_k(t,x,y)$ which is the smooth kernel of the operator $e^{-\frac{2t}{k}\overline{\square}^q_k}$. It enjoys the following expansion: for $j = 0,1,\ldots$, let μ^k_j be the eigenvalues of $\overline{\square}^q_k$ (counted with multiplicities), and $(\psi_j)_j$ be an orthonormal L^2 basis of eigen-forms associated to the μ^k_j's, then

$$K^q_k(t,x,y) = \sum_{j\geq 0} e^{-\frac{2t}{k}\mu^k_j}\psi_j(x)\otimes\psi^*_j(y),$$

and is characterized by the following properties:

(1i) $K^q_k \in \mathcal{C}^\infty\left(]0,+\infty[\times X\times X, \mathrm{Hom}(\bigwedge^{0,q}T^*X\otimes E(k), \bigwedge^{0,q}T^*X\otimes E(k))\right)$;
(1ii) $\left(\frac{\partial}{\partial t} + \overline{\square}^q_k\right)K^q_k = 0$ where $\overline{\square}^q_k$ acts on the first variable;
(1iii) $K^q_k(t,x,y)\to\delta_y$ (Dirac δ-function at point y) if $t\to 0$;
(1iv) $K^q_k(t,x,y) = (K^q_k)^*(t,y,x)$.

If Ω is an open set of X, and $\overline{\square}^q_{k,\Omega}$ is the operator defined by $\overline{\square}^q_k$ on Ω, with Dirichlet condition at the boundary, we get in the same way the associated heat kernel $K^q_{k,\Omega}$ satisfying:

(2i) $K^q_{k,\Omega} \in \mathcal{C}^\infty\left(]0,+\infty[\times\overline{\Omega}\times\overline{\Omega}, \mathrm{Hom}(\bigwedge^{0,q}T^*X\otimes E(k), \bigwedge^{0,q}T^*X\otimes E(k))\right)$;
(2ii) $\left(\frac{\partial}{\partial t} + \overline{\square}^q_k\right)K^q_{k,\Omega} = 0$ inside Ω;
(2iii) $K^q_{k,\Omega}(t,\partial\Omega,y) = \{0\}$;
(2iv) $K^q_{k,\Omega} \underset{t\to 0}{\longrightarrow}\delta_y$.

For the sake of convenience, we also introduce the following notations: $e^q_k(t,x)$ (resp. $e^q_{k,\Omega}(t,x)$) is the fiberwise trace of $K^q_k(t,x,x)$ (resp. $K^q_{k,\Omega}(t,x,x)$) considered as an endomorphism of the fiber $E(k)_x$. We shall often call abusively e^q_k the heat kernel of $\overline{\square}^q_k$. From the formula above we get:

$$e^q_k(t,x) = \sum_{j\geq 0} e^{-\frac{2t}{k}\mu^k_j}|\psi_j(x)|^2. \qquad (*)$$

Our point here is to illustrate how this "heat kernel" technique works out in a number of classical theorems of complex geometry, and even allows to refine them by a more local control on the holomorphic (*resp.* harmonic) objects they deal with. The present paper is thus both of an expository and research nature, the new results discussed in sections 2 or 3 being slight extensions of known results; however, the point of view, and most of our proofs, are new. Further developments of the theory are to be found in the papers [A-B] and [B4], the underlying Riemannian situation is made explicit in [B5]. In the first section of this paper, we state the main theorem from [B1] giving an asymptotic estimation for e_k^q when $k \to +\infty$ and show how it yields fine control on the cohomology groups of $E(k)$. The second section is devoted to estimates and applications of the so-called envelope and distortion functions which give some taste of how the global sections of $E(k)$ behave at a given point of X. Most of the results there are new for $r \geq 2$, $q > 0$. The third section illustrates our strategy of producing holomorphic sections with heat kernel estimates, and includes a self-contained proof of the Kodaira embedding theorem. Throughout the paper, the letter C will denote any constant not depending on x, t nor k.

1. Asymptotic bounds on cohomology

We denote by $\alpha_1, \ldots, \alpha_n$ the eigenvalues of the curvature of L with respect to the metric ω. For a multi-index J, we put $\bar{\alpha}_J = \sum_{j \notin J} \alpha_j - \sum_{j \in J} \alpha_j$. We set also the function $\frac{\alpha}{\sinh \alpha t}$ to be $\frac{1}{t}$ when $\alpha = 0$. Let us call

$$e_\infty^q(t,x) = r(4\pi)^{-n}\Big(\sum_{|J|=q} e^{t\bar{\alpha}_J}\Big)\prod_{j=1}^{n} \frac{\alpha_j(x)}{\sinh \alpha_j(x)t}.$$

Then we get from [B1] the

Theorem 1.1.

(1a) When $k \to +\infty$, the function $k^{-n}e_k^q(t,x)$ converges to $e_\infty^q(t,x)$ uniformly with respect to $x \in X$ and $t \in]0, k^\varepsilon]$ for a given $\varepsilon > 0$, not depending on k.

(1b) More precisely, if we fix $x_0 \in X$ and some $\eta \in]0, \frac{1}{6}[$, for any sequence of real numbers r_k such that $r_k k^{-\frac{1}{2}+\eta}$ is bounded above and below by positive constants, and for a sequence of open subsets Ω_k of X containing for all k the geodesic ball of center x_0 and radius r_k, the following convergence occurs

$$k^{-n}e_{k,\Omega_k}^q(t,x_0) \to e_\infty^q(t,x_0)$$

uniformly with respect to $t \in]0, k^\varepsilon]$ for any $\varepsilon < \eta$.

Remarks. Notice that the rank r of E was omitted in Theorems 1 and 2 of [B1], which was a typo. The precise sense of uniformity here is stated as Theorem 2 in [B1]: it means that the function denoted by the Landau symbol $o(k^n)$ is bounded independently of x and t in the given domain. (1a) is thus only the reformulation of Theorems 1 and 2 (using the remark before Corollary 1). Although (1b) does not appear there, it is also a direct consequence of the whole method of the paper, because (1a) is proven by localizing the estimates of e_k^q at x_0 to those of e_{k,B_k}^q

where B_k is a geodesic ball of center x_0 and whose radius satisfies conditions weaker than those on r_k. Since this paper was written, his author has improved some of the results it presents by using Eq. (1b) in full details (see [B4]).

Theorem 1.1 (1a) has first been proven by J.-M. Bismut [Bi] by using probabilistic methods, then generalized by E. Getzler to Heisenberg manifolds. There exists also a simple analytic approach of Demailly [D2] (all these without uniformity with respect to t outside bounded intervals). The original motivation for this theorem is that it implies Demailly's holomorphic strong Morse inequalities [D1]. In fact, it can be viewed as a *local* version of them (by analogy to the expression "local index theorems": it is a pointwise asymptotic estimation of the functions that give, after integration, the inequalities of Demailly). We shall be a little more precise: it is immediate from the infinite sum expansion of the heat kernel $(*)$ given in the introduction (and the Hodge identification between cohomology and harmonic forms) that

$$\dim H^i(X, E(k)) \leq \int_X e_k^i(k^\varepsilon, x)\, dV.$$

According to theorem 1.1 Eq. (1a), e_∞^q is the uniform limit on X of $k^{-n} e_k^q(t,x)$. If we divide the inequality above by k^n and let k tend to $+\infty$, we obtain as limit of the right hand side the integral of e_∞^q. As the left hand side does not depend on t, we can bound it by taking the right hand side's limit when t goes to infinity. It is then only a matter of simple computations to get the "weak Morse inequalities" of Demailly [D1] (see [B2] for the detailed derivation of the strong form):

Theorem 1.2. *For any $q = 0, \ldots, n$*

$$\dim H^q(X, E(k)) \leq (-1)^q r \frac{k^n}{n!} \int_{X(q)} \Big(\frac{i}{2\pi} c(L)\Big)^n + \mathrm{o}(k^n).$$

Unfortunately, these inequalities fail to be sharp when the order of growth of the cohomology groups is not maximum (a polynomial expansion would be much better although it seems quite out of reach by now: we only have been able to bound the heat kernel to a lower order in k in the case of degenerate curvature, under some additional assumptions, c.f. section 3 of [B3]). The point we want to stress now is that Theorem 1.1 can be much sharper in such a situation. In fact, we are able to prove the

Theorem 1.3 (generalized Kodaira vanishing). *If the curvature of L has at least $n - q + 1$ positive eigenvalues everywhere on X, then*

$$H^i(X, E(k)) = 0 \quad \text{for} \quad i \geq q$$

as soon as k is sufficiently large.

Proof. In our setting, the strategy of the proof is quite simple: we are going to show that the integral over X of the heat kernel $e_k^i(k^\varepsilon, x)$ tends to zero. Therefore it must be less than 1 for k large, but this integral bounds the dimension of $H^i(X, E(k))$, and we are done. We shall now suppose that the eigenvalues of $ic(L)$

are ordered (i.e. $\alpha_1 \leq \ldots \leq \alpha_n$) and denote $\alpha_0 = \inf_X \alpha_q$ which is positive by our hypothesis on L and X. With our notations, we have, for $|J| = i$, $\bar{\alpha}_J \leq -\alpha_1 - \cdots - \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$. For any j, according to the discussion which led to theorem 1.2, $\alpha_j e^{k^\varepsilon \alpha_j} / \sinh k^\varepsilon \alpha_j$ is a bounded function whatever the sign of α_j is. Especially, if $\alpha_j > 0$, $\alpha_j e^{-k^\varepsilon \alpha_j} / \sinh k^\varepsilon \alpha_j \leq C e^{-2k^\varepsilon \alpha_j}$ for some constant C. But the point is that, if $i \geq q$, $\alpha_i \geq \alpha_0 > 0$; thus $e^{\bar{\alpha}_J} t / \prod_{j=1}^n \sinh \alpha_j t \leq C e^{-k^\varepsilon \alpha_0}$. Now we use the strong uniformity of (1a) to see that $e_k^i(k^\varepsilon, x) \leq C k^n e_\infty^i(k^\varepsilon, x)$. Integrating this last inequality over X yields $\dim H^i(X, E(k)) \leq C \operatorname{Vol}(X) k^n e^{-k^\varepsilon \alpha_0}$ thanks to the previous one. This proves the theorem.

Remarks. The note [B4] presents an enhanced version of this result, which does not require strict positivity. The same proof yields the vanishing in degree less or equal to q if $ic(L)$ has at least $q+1$ negative eigenvalues, which also follows from Serre duality. This remark will be used in theorem 2.1. The Bochner technique proof of theorem 1.3 involves a careful construction of an adapted metric ω for which the control of the curvature eigenvalues is possible. What is remarkable in our heat equation proof is that we just don't have to care about ω. Before ending this section, we would like to ask:

QUESTION. *Can one prove* precise *vanishing theorems by a heat equation method?*

For instance, it is clear that one should be able to prove Andreotti-Grauert vanishing for vector bundles on q-complete manifolds. See [B4] for another answer to this question, which is however not specific to the heat equation approach.

2. The envelope and distortion functions

We now turn our attention to generalizations and applications of the "main theorem" in [B1]. We first make the following general observation: the spaces $H^q(X, E(k))$ are finite dimensional vector spaces, thus any two norms on it must be equivalent. When identified with the space $\mathcal{H}^q(X, E(k))$ of harmonic $(0,q)$-forms, it is naturally endowed with the L^2 and sup norms. The first inequality between these two norms: $\|\sigma\|_2^2 \leq \operatorname{Vol}(X)\|\sigma\|_\infty^2$ is obvious and does not depend on k. The second one is more subtle, and must depend on k, for reasons which will soon be clear. Notice that it is quite unnatural to bound the sup norm by an integrated one, except when, for instance, the function $|\sigma|^2$ is plurisubharmonic, but this would mean that L has negative curvature and so no cohomology (at least for $q < n$)! Anyway, one can look for the constant C_k giving the other inequality, one can even be more optimistic and seek a description of what we shall call the *envelope function* for the L^2 norm.

DEFINITION. *Let the family* $h_1, \dots, h_m$ *be an* L^2 *orthonormal basis of* $\mathcal{H}^q(X, E(k))$*, then the distortion function is defined by:*

$$b_k^q(x) = \sum_{j=1}^{m} |h_j(x)|^2.$$

The envelope function is:

$$s_k^q(x) = \sup_{\|h\|_2=1} |h(x)|^2.$$

It is clear that the constant C_k is simply the supremum of the envelope function whereas the distortion function is exactly the zero-eigenvalue part of the expansion of the heat kernel (notice also that the distortion function does not depend on the choice of the orthonormal basis (h_j)). The terminology for b_k^0 has the following geometrical interpretation generalizing the one given in [B1] following G. Kempf, when E is the trivial line bundle. When k is large, the global sections of the bundle $E(k)$ define an embedding in a Grassmannian such that $E(k)$ is the pull-back of the universal quotient bundle Q. As Q is a quotient bundle of a trivial one, it has a natural metric whose pull-back on $E(k)$ is proportional to the initial metric. The function b_k^0 is precisely the pointwise ratio (or "distortion") between the initial metric and the pulled-back one. The connection between these two functions is made precise in the

PROPOSITION 2.1. *We have the two inequalities*

$$b_k^q(x) \le e_k^q(t, x) \quad \text{for any} \quad t > 0 \tag{2a}$$

$$s_k^q \le b_k^q \le M(x) s_k^q \tag{2b}$$

where $M(x)$ *is the maximum rank of the subspace of the fiber* $\bigwedge^{0,q} T^*X \otimes E(k) \restriction_x$ *generated by the global harmonic* $(0, q)$*-forms.*

Proof. (2a) needs no proof. The first inequality in (2b) follows simply from the fact that for any global form h of unit L^2 norm you can find an orthonormal basis containing it, then compute b_k^q with this basis. To prove the second inequality, we construct a L^2 orthonormal basis of $\mathcal{H}^q(X, E(k))$ $h_1, \dots, h_m$ such that $h_1(x), \dots, h_M(x)$ form an orthogonal basis of the fiber $\bigwedge^{0,q} T^*X \otimes E(k)) \restriction_x$, and $h_j(x) = 0$ for $j > M(x)$. This is easily done by induction: we first define the h_j's for $j > M$ as an orthonormal basis of the space of global harmonic forms vanishing at x, then we pick one unit norm element h_M of the orthogonal (which is precisely of dimension M), then choose h_{M-1} in the same space such that its value at x is orthogonal (in the fiber) to that of h_M, and so on. Now, if we take any h of unit norm, it can be written $h = \sum_{j=1}^{m} \gamma_j h_j$ with $\sum |\gamma_j|^2 = 1$, thus $|h(x)|^2 = \sum_{j=1}^{M} |\gamma_j|^2 |h_j(x)|^2 \le \max_{1 \le j \le M} |h_j(x)|^2$. This proves that, up to some reordering, $s_k^q(x) = |h_1(x)|^2$. The proof is complete.

Proposition (2b) is completely satisfactory in case $r = 1$, $q = 0$ because it asserts that the two functions are the same. Nevertheless, the case $q = 0$ only has important geometric sense. In fact, the second inequality in (2b) should be an equality in many cases (e. g. if Aut(E) is transitive on the fibers): we shall show that it is the limiting case when L is positive (for $q = 0$). We now turn to answer (asymptotically) the question we raised at the beginning of this section.

Corollary 2.1. *Let $C_k = \sup_{\|h\|_2=1} \|h\|_\infty^2$ for $h \in \mathcal{H}^q(X, E(k))$ then*

$$C_k \leq r \sup_X \left|\det_\omega \frac{i}{2\pi} c(L)\right| k^n + o(k^n).$$

Proof. In fact, using (2a) and the estimates of e_∞^q, we get:

$$s_k^q(x) \leq e_k^q(k^\varepsilon, x) \leq r\left|\det_\omega \frac{i}{2\pi} c(L)_x\right| k^n + o(k^n).$$

This proves that, if the determinant is nonzero somewhere on X, we can choose a point on X and k sufficiently large in order that the corresponding term of order k^n dominates e_k^q everywhere on X, whence the result. Here, however, $o(k^n)$ is definitely not under control.

Corollary 2.1 is an improvement of an earlier result of Gromov, Gillet and Soulé [G-S] which only asserted the order of growth of C_k for $q = 0$ under the unnecessary hypothesis that L should be positive. If we require some positivity, we can obtain a much more precise statement generalizing the main theorem of [B1]:

Theorem 2.1. *If the curvature of L has constant signature q (i.e. $X(q) = X$) then the following asymptotic estimation holds:*

$$b_k^q(x) \sim r(-1)^q \det_\omega \frac{i}{2\pi} c(L)_x (k^n + o(k^n))$$

uniformly over X.

Remarks. Notice that we know by theorem 1.3 and Serre duality that the cohomology in all other degrees vanish. Of course, by integrating the above relation, one obtains an estimation for the dimension of the only nonvanishing group, which already follows from the Riemann-Roch theorem in this context. The case $q = 0$ (L positive) is specially interesting because it allows to construct many holomorphic sections of $E(k)$, as will be seen in the next section. The theorem of [B1] treats the case $q = 0$, and E trivial. In this case, there is another approach by Tian [T] based on L^2 estimates. We also point out that, according to proposition 2.1, corollary 2.1 is sharp in degree 0 when L is positive and $r = 1$.

Proof. Under our hypothesis, the estimate of theorem 2.1 is the limit of the one for e_k^q when k goes to infinity. The only thing we have to check is that the part of the expansion of $e_k^q(k^\varepsilon, x)$ corresponding to nonzero eigenvalues converges uniformly to zero when $k \to +\infty$. This is done by observing that this part injects *via* $\bar\partial$ (*resp.* $\bar\partial^*$) in the corresponding part of e_k^{q+1} (*resp.* e_k^{q-1}). But, after the proof of theorem 1.3, these two heat kernels tend uniformly to zero at (k^ε, x, x) when $k \to +\infty$. In fact, if $\alpha_0 = \inf_X(|\alpha_q|, \alpha_{q+1})$, we have $e_k^{q+1}(k^\varepsilon, x) \le Ck^n e^{-2\alpha_0 t}$ and the same bound applies for e_k^{q-1}. On the other hand, the eigen-forms in degree q are controlled by the convergence of their heat kernel, namely: $|\psi_j(x)|^2 \le e^{\frac{2t}{k}\mu_j^k} e_k^q(t,x) \le Ck^n e^{\frac{2t}{k}\mu_j^k}$. Using this last inequality for a t strictly smaller than k^ε and combining it with the previous one finishes the proof (which is entirely similar to the one of [B1] except for invoking $\bar\partial^*$).

If X is a submanifold in a manifold Y of dimension m, there is another norm comparison similar to the one we outlined at the beginning of this section: suppose L is defined and positive on the whole of Y, then you would like to compare the L^2 norm on $H^0(X, E(k))$ with the quotient L^2 norm inherited from $H^0(Y, E(k))$ when k is sufficiently large (which yields the surjectivity of the restriction map). When X is the zero locus of a section of a vector bundle on Y, such results are well-known [O], [M], but do not lead to optimal constants (with respect to the k-dependency). However, once again, there must be some constant for linear algebraic reasons in *any* case, and we can hope to have the following asymptotic answer:

CONJECTURE 2.1. *Let Y (resp. X) be a complex-analytic manifold (resp. submanifold of Y) of dimension m (resp. n), E and L as above (L positive). Let $s \in H^0(X, E(k))$, and $\|s\|_{2,X}$ (resp. $\|s\|_{2,Y}$) denote the L^2 norm (resp. quotient L^2 norm) of s. Put $c_k^1 = \inf_{\|s\|_{2,Y}=1} \|s\|_{2,X}^2$ and $c_k^2 = \sup_{\|s\|_{2,Y}=1} \|s\|_{2,X}^2$, then there exists positive constants c_1, c_2 such that*

$$c_1 k^{n-m} \le c_k^1 \le c_k^2 \le c_2 k^{n-m}.$$

More precisely, we conjecture that $k^{m-n} c_k^i$ converge to the same limit for $i = 1, 2$, and that this limit only depends on the metric on Y and the curvature of L.

A solution to this conjecture could lead to a drastic improvement in the paper [A-B]. Some evidence for it can be found in the study of the case $Y = \mathbb{P}^m$ and $L = \mathcal{O}(1)$ where the expected limiting constant is the degree of the projective manifold X (i.e. its volume with respect to the $ic(L)^n$ volume element). The results of section 3 also lead to its complete verification in the case $n = 0$ (X a point).

3. Construction of peak sections when L is positive

We conclude this paper by results of existence of sections of $E(k)$ using the main theorems of sections 1 and 2. We suppose from now on that L is positive and X is endowed with the Kähler metric $\omega = ic(L)$.

THEOREM 3.1. *Fix $x_0 \in X$. Then, for any $k \ge 0$ there exists a global section σ_k of $E(k)$ of unit L^2 norm such that for any geodesic ball of radius r_k such that*

$r_k\sqrt{k} \to +\infty$, the following bound holds

$$\int_{B(x_0,r_k)} |\sigma_k|^2 dV \geq 1 - o(1). \tag{3}$$

Where $o(1)$ means some quantity tending to zero when $k \to +\infty$. Moreover, $|\sigma_k(x)|^2 = o(k^n)$ uniformly on compact subsets of $X \setminus \{x_0\}$.

Remarks. Theorem 3.1 generalizes slightly Lemma 2.1 of [T] which was proven by using L^2 methods. In fact, his lemma produces sections of L^k with prescribed Taylor expansion of order p and mass almost concentrated at x_0, called *peak* sections. This is the same as producing sections of a power of the line bundle over a blown-up manifold we shall construct in the proof of the Kodaira embedding theorem, where we will see that it is positive. The first proof of theorem 3.1 we shall give is in some sense the converse to the one of the lemma of Tian. The reason is that Tian obtains as a consequence of his lemma a special case of our theorem 2.1 (see lemma 3.2 of [T]). Here, we use theorem 2.1 to prove theorem 3.1. In both cases, the heart of the proof is the computation of the same integral.

Proof. Let U be a small open subset of X containing x_0 on which there exist holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x_0, and a local frame ℓ (*resp.* $(e_1, \ldots, e_r)$) of L (*resp.* E) such that

$$\begin{aligned} < e_\lambda(z), e_\mu(z) > &= \delta_{\lambda\mu} + O(|z|) \\ |\ell(z)|^2 &= 1 - |z|^2 + O(|z|^4) \\ \omega_z &= i\partial\bar{\partial}|z|^2 + O(|z|^2) \end{aligned}$$

where $|z|$ denotes the Euclidean norm in $\mathbb{C}^n$. Now, we define σ_k to be one unit norm generator of the orthogonal to the space of global sections vanishing at x_0 such that $|\sigma_k(x_0)|^2 = s_k(x_0)$ (this was shown to be possible during the proof of proposition 2.1). Then we know that $|\sigma_k(x_0)|^2 = s_k(x_0) = \left(\frac{k}{2\pi}\right)^n + o(k^n)$. Define $f_k = (f_k^1, \ldots, f_k^r)$ to be the holomorphic function on U representing σ_k, namely $\sigma_k = \sum_\lambda f_k^\lambda \, e_\lambda \otimes \ell^k$. Because of the mean inequality for holomorphic functions in $\mathbb{C}^n$ we have:

$$|f_k(0)|^2 \leq \frac{1}{\mathrm{Vol}(S(0,t))} \int_{S(0,t)} |f_k(z)|^2 d\sigma(z) \tag{4}$$

where $d\sigma$ is the canonical volume element of the Euclidean sphere of radius t in $\mathbb{C}^n$. We denote $a(n) = \mathrm{Vol}(S(0,1)) = 2\pi^n/(n-1)!$. Multiplying (4) by $(1-t^2)^k$ and integrating with respect to t over $]0, r_k]$ we get

$$\begin{aligned} |f_k(0)|^2 a(n) \int_0^{r_k} t^{2n-1}(1-t^2)^k dt &\leq \int_0^{r_k} dt \int_{S(0,t)} |f_k(z)|^2 (1-|z|^2)^k d\sigma(z) \\ &= \int_{B(0,r_k)} |f_k(z)|^2 (1-|z|^2)^k d\lambda(z). \end{aligned} \tag{5}$$

Here $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$ hence $dV = \frac{\omega^n}{n!} = 2^n d\lambda + O(|z|^2)$. The real integral on the left of (5) is easily computed: $\int_0^{r_k} t^{2n-1}(1-t^2)^k dt \sim \frac{(n-1)!k!}{2(n+k)!}$ because the assumption $r_k k^{1/2} \to +\infty$ insures us that terms of the form $(1-r_k^2)^k = e^{k\log(1-r_k^2)} \leq e^{-kr_k^2}$ tend to zero. This implies that the left hand side of (5) has limit 2^{-n} when $k \to +\infty$. But this concludes the argument because the right hand side is asymptotically equal to the integral of $|\sigma_k|^2$ over $B(x_0, r_k)$ up to the constant factor 2^{-n} as soon as we choose $r_k \to 0$.

From now on, we omit q in the notation as it is always zero.

Lemma 3.1. *If L is positive*

$$s_k(x) \sim \left(\frac{k}{2\pi}\right)^n \tag{6}$$

uniformly over X.

Proof. Theorem 2.1 and the inequality $rs_k \geq b_k$ due to (2b) in proposition 2.1 yield the lower bound. Now, suppose we have at some point x_0 $\limsup k^{-n}s_k(x_0) > (2\pi)^{-n}$. For any k we can choose again σ_k to be a unit norm element of the L^2 orthogonal to the space of sections of $E(k)$ vanishing at x_0 such that $|\sigma_k(x_0)|^2 = s_k(x_0)$. By the argument above this would imply for a k large enough:

$$\int_{B(x_0,r_k)} |\sigma_k|^2 dV > 1,$$

a contradiction. The uniformity comes from the fact that all estimates which lead to the contradiction are uniform.

The last statement of theorem 3.1 comes from the same analysis. If

$$\limsup k^{-n}\sigma_k(x) > 0$$

there would be a non-negligible contribution of neighborhoods of x to the global norm of σ_k, which contradicts the previous statement. Now, we derive the

Corollary 3.1 (Kodaira embedding theorem). *If L is positive, the map $\Phi_{kL} : X \to \mathbb{P}\big(H^0(X, L^k)^*\big)$ defined by the linear system $\big|kL\big|$ is a projective embedding for k sufficiently large.*

Proof. The first observation is that theorem 2.1 implies that $|kL|$ is base point free when k is large because the nonzero limit of b_k at any point forces some section to be nonzero at this point. Thus Φ_{kL} is a well-defined map on X. Moreover, it is injective because, if two points were not separated by sections of L^k, the orthogonal of the space of sections vanishing at both points would be one dimensional. But then, the proof of theorem 3.1 shows that the unit norm generator of this space should have its mass concentrated near each one of the two points, thus a total norm greater than $2-\varepsilon$. What remains to check is that the sections of L^k separate infinitely near points. The derivation of this result from a precise vanishing theorem goes back to Kodaira himself who proved his embedding theorem as a consequence of his precise vanishing for positive line bundles together with the "quadratic transform" technique (blow-up of a point). If we restrict ourselves to the heat kernel method, thus to Theorem 1.3, the same technique applies and leads in turn to prove m-jet ampleness rather than very ampleness. We sketch hereafter this standard argument. Let us choose a point x on X and call $\widetilde{X} \xrightarrow{\pi} X$ the blow-up of X at x. Let us also call D the exceptional divisor, and $\mathcal{O}(D)$ the associated line bundle, whose restriction to $D \simeq \mathbb{P}(T_x^* X)$ is isomorphic to $\mathcal{O}(-1)$, thus negative. Using the fact that $\pi^* L$ is positive on any tangent vector to $\widetilde{X}$ except those tangent to D, it is easy to check that the line bundle $\widetilde{L} = \pi^* L^{k_0} \otimes \mathcal{O}(-D)$ is positive for some $k_0 \geq 0$. Theorem 1.3 yields $H^1(\widetilde{X}, \widetilde{L}^m) = 0$ for some large $m \geq 2$. If we call $\mathcal{I}_x$ the maximal ideal sheaf associated to x, the space of m-jets of L^{mk_0} at x is $H^0(X, L^{mk_0} \otimes \mathcal{O}_X / \mathcal{I}_x^m)$. Using the exact sequence

$$0 \to \mathcal{I}_x^m \to \mathcal{O}_X \to \mathcal{O}_X / \mathcal{I}_x^m \to 0,$$

and blowing it up at x, we obtain the diagram

$$\begin{array}{ccccc} H^0(\widetilde{X}, \pi^* L^{mk_0}) & \to & H^0(D, \pi^* L^{mk_0} \otimes \mathcal{O}_{\widetilde{X}} / \mathcal{O}_{\widetilde{X}}(-mD)) & \to & H^1(\widetilde{X}, \widetilde{L}^m) = 0 \\ \pi^* \uparrow & & \pi^* \uparrow & & \\ H^0(X, L^{mk_0}) & \to & H^0(X, L^{mk_0} \otimes \mathcal{O}_X / \mathcal{I}_x^m) & & \end{array}$$

where vertical arrows are isomorphisms. This yields the result. □

Remarks. After corollary 3.1 we know that X is an algebraic projective manifold. Thus, any coherent sheaf on X admits a global syzygy. A simple induction on the length of the syzygy shows that the vanishing theorem 1.3 is valid for any coherent sheaf E. This is known as theorem A (see for instance [G-H]). Our results of the previous section also lead to an easy direct proof of the so-called theorem B in the case of vector bundles (twisted by ideal sheaves of zero-dimensional subschemes):

Corollary 3.2. *If L is positive and E is any vector bundle of rank r, let $Z \subset X$ be a finite set, and $x \in X \setminus Z$. Then the fiber $E(k)_x$ is generated by global sections of $E(k)$ vanishing on Z for k sufficiently large (k uniform with respect to x in compact subsets of $X \setminus Z$).*

Proof. We treat the case $Z = \{x_0\}$ for simplicity, the general case being similar. The pointwise norm at $x \neq x_0$ of the peak section σ_k associated to x_0 is negligible relatively to $k^{\frac{n}{2}}$, and thus does not alter the estimate for b_k nor s_k at x (This statement is uniform for x outside any ball B of center x_0). Therefore, if we call $b'_k(x) = b_k(x) - |\sigma_k(x)|^2$ we have $b'_k \sim b_k$ uniformly outside B. But b'_k is the sum of the squares of the pointwise norms of an orthonormal basis of the space of sections of $E(k)$ vanishing at x_0 (because σ_k is orthogonal to this space). Denote by $M'(x)$ the dimension of the linear subspace of $E(k)_x$ generated by global section vanishing at x_0. The above remark, lemma 3.1 and (an obvious adaptation of) proposition 2.1 imply: $rs_k \sim b'_k \leq M'(x) s_k \leq rs_k$. As $M'(x)$ and r are integers, it is clear that they must be equal for k large, which precisely means the announced property.

An alternative proof of Theorem 3.1. We would like to finish this paper with a second proof of theorem 3.1 which illustrates a possible use of (1b) in theorem 1.1. It can also be seen as an introduction to the delicate localization procedure we use in our joint paper [A-B]. Let us first introduce some notations: given an open subset Ω of X and some $\mu > 0$, we shall denote $\mathcal{H}_k(\Omega, \mu)$ the direct sum of the eigen-spaces of $\overline{\square}_{k,\Omega}$ corresponding to eigenvalues $\mu_j^k(\Omega) \leq k\mu$. Now, if we denote (somewhat abusively) μ_1^k the first nonzero eigenvalue of $\overline{\square}_k$ on X, we have the well known

LEMMA 3.2. *If α_0 is the infimum of the curvature eigenvalues on X, we have the lower bound:*

$$\liminf k^{-1}\mu_1^k \geq \alpha_0.$$

Proof. This fact follows immediately from the proof of theorem 2.1 because we showed there that

$$\sum_{j\geq 1} e^{-\frac{2\mu_j^k}{k}t} \leq \int_X e_k^1(t,x)dV \leq Ck^n e^{-2\alpha_0 t}$$

if $t \leq k^\varepsilon$. $e^{-\frac{2\mu_1^k}{k}t}$ has thus the same upper bound. Take the Log of this inequality and let t tend slowly to infinity. □

Now, we claim that

LEMMA 3.3. *For $\mu < \alpha_0$ and $k \gg 0$, the map*

$$\Psi_{k,\mu} : \mathcal{H}_k(\Omega, \mu) \to H^0(X, L^k)$$

which is simply the orthogonal projection is injective. Moreover, it is not far from being an isometry: For $u \in \mathcal{H}_k(\Omega, \mu)$ one has the estimate

$$\|\Psi_{k,\mu}(u) - u\|_2^2 \leq (k\mu/\mu_1^k)\|u\|_2^2.$$

Proof. Define $H_\Omega(u) = \int_\Omega < \frac{1}{k}\overline{\square}_k u, u > dV$ (*resp.* $H(u) = \int_X < \frac{1}{k}\overline{\square}_k u, u > dV$). For $u \in \mathcal{H}_k(\Omega, \mu)$ we have $H(u) = H_\Omega(u) \leq \mu\|u\|_2^2$ thus, if $u \neq 0$ $\Psi(u)$ cannot be zero because this would imply $H(u) \geq (\mu_1^k/k)\|u\|_2^2 \geq (\alpha_0 - o(1))\|u\|_2^2 > \mu\|u\|_2^2$. This proves the first assertion while the second boils down to the Pythagorean theorem: if $u = \Psi(u) + u_1$ is the orthogonal decomposition, we have $\|u\|_2^2 = \|\Psi(u)\|_2^2 + \|u_1\|_2^2$ with $H(u_1) = H(u) \leq \mu\|u\|_2^2$ and $H(u_1) \geq (\mu_1^k/k)\|u_1\|_2^2$. □

Now, our construction goes like this: for any $\varepsilon > 0$ choose a $\mu < \alpha_0\varepsilon$ and any unit norm section τ_k of $\mathcal{H}_k(B_k, \mu)$ where $B_k = B(x_0, r_k)$. This is possible because theorem 1.1 (1b) implies that the first eigenvalue of $\frac{1}{k}\overline{\square}_{k,B_k}$ tends to zero. In fact, if it was not the case, one can find a subsequence k_ν such that $e_{k_\nu,B_{k_\nu}}(k_\nu^\varepsilon, x_0, x_0) \to 0$, which contradicts (1b) (the estimates are the same as for the proof of theorem 2.1). Thus, for any $\mu > 0$ the space $\mathcal{H}_k(B_k, \mu)$ is not trivial when k is large. We put $\hat{\sigma}_k = \Psi(\tau_k)$ and we obtain $\hat{\sigma}_k + \tilde{\sigma}_k = \tau_k$ with $\|\tilde{\sigma}_k\|_2^2 \leq \varepsilon$. Whence $\|\hat{\sigma}_k\|_2^2 = \|\tau_k\|_2^2 - \|\tilde{\sigma}_k\|_2^2 \geq 1 - \varepsilon$ and $\varepsilon \geq \int_X |\hat{\sigma}_k - \tau_k|^2 dV \geq \int_{X \setminus B_k} |\sigma_k|^2 dV$. So the section $\sigma_k = \frac{\hat{\sigma}_k}{\|\hat{\sigma}_k\|_2^2}$ is the global holomorphic section we are looking for.

Concluding remarks. It has been shown that many important results in analytic geometry can be derived from Eq. (1a). One can even improve some of these results by giving them some flavor of what they should mean at a point or, to say it in another manner, what is their local analogue. Anyway, the global results we have been able to prove in this paper are somewhat outdated. This is because strict positivity was a necessary condition in order to get rid of the non-harmonic part in the heat kernel (because strict positivity forces the first eigenvalue of $\overline{\Box}_k$ to go to infinity with k). There are thus two possible ways of development for this theory. One should be able, following a conjecture of Siu [S], to find estimates for the first eigenvalue of $\overline{\Box}_k$ under weaker conditions than positivity. (We shall discuss in the forthcoming paper [B4] what we can do in this direction with the tools developed here: this is quite far from what could be expected!) For instance, if L has semi-positive curvature everywhere, and positive definite at some point, many of the results we discussed have weaker analogues (e.g. Grauert-Riemenschneider theorems). More generally (and we come to the second point), the same question holds for nef and big line bundles: the overlap of L^2 methods in this case is due to the use of singular metrics. It would be quite marvelous to obtain Shokurov's base point freeness *via* an analogue of theorem 2.1. The other point is thus: is it possible to develop these devices for singular metrics?

The last remark we would like to write is due to Y.-T. Siu. Since our main interest is in holomorphic (or harmonic) objects, why should we work with the whole spectrum of $\overline{\Box}_k^q$, and then try to get rid of its greater part rather than work directly with holomorphic objects like the Bergman kernel? For instance, if you develop in a power series along the fibers the Bergman kernel of the unit ball in the total space L^*, the value along the diagonal of the coefficient of the k^{th} power should look much like our distortion function b_k. The (technical) reason for which we worked with the heat kernel is that it is a very supple function, which allows lots of manipulations (among which the localization properties as in (1b) is not the least): it is less rigid than the Bergman kernel. Moreover, results like theorem 1.2 or the second proof of theorem 3.1 show that a little walk outside holomorphy can lead to powerful controls of holomorphic objects. Anyway, a vector-bundle approach to the Bergman kernel is certainly a worthy direction for future research.

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